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Generalized Iterative Methods for Caputo Fractional Differential Equations via Coupled Lower and Upper Solutions with Superlinear Convergence

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Abstract: Existence of coupled lower and upper solutions for nonlinear differential equations guarantees the existence as well as interval of existence of the solution. In this work, a methodology has been developed to compute coupled lower and upper solutions using natural lower and upper solutions by iterative methods. Further, using the computed lower and upper solutions, sequences are developed which converge uniformly and monotonically to the unique solution. In addition, it has been shown that the convergence of these sequences is superlinear. Further the convergence of the sequences is accelerated by Gauss-Seidel method. Finally, some numerical examples are presented.

Keywords: *Caputo fractional differential equation; superlinear convergence.*

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1 Introduction

It is well-known that qualitative and quantitative properties of fractional differential equations are very useful in applications. In addition, fractional differential equations in several situations have proved to be better and more economical models than their counterpart with integer derivatives. For details see [5, 9, 11] and the references therein. In the past thirty years there has been a rapid development in the qualitative study of fractional differential equation such as existence, uniqueness and stability results due to its applications. In particular, it has been very useful in biological sciences such

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as population models. However, most of the existence and uniqueness results for fractional differential equations are obtained by some type of fixed point theorem approach. See [1,2,17]. Unfortunately, these methods do not provide the interval of existence of the solution as well as a methodology to compute solutions. The method of lower and upper solutions and the method of coupled lower and upper solutions which guarantees the interval of existence, and is well-known for ordinary differential equations have now been extended to Riemann-Liouville and Caputo fractional differential equations in [4, 13]. Monotone method combined with lower and upper solutions provides both theoretical and constructive method of existence of the minimal and maximal solution or the unique solution if the uniqueness conditions are satisfied. See [6] for details. Monotone method works only when the nonlinear function is either increasing or could be made increasing by adding a linear term. Monotone method yields alternating sequences when the nonlinear function is decreasing with an additional assumption. In [18] and the references therein they have developed generalized monotone method for scalar first order ordinary differential equations. Generalized monotone method uses coupled lower and upper solutions and the method is very convenient to use when the nonlinear function is the sum of an increasing and decreasing functions. Furthermore, we do not need an additional assumption which we need when the nonlinear function is decreasing when we use an appropriate type of coupled lower and upper solutions, namely of type I. Generalized monotone method has been extended to scalar and system of Caputo fractional differential equations in [10, 16]. Generalized monotone method with coupled lower and upper solutions has an added advantage for fractional differential equations, since it avoids the computation of Mittag-Leffler function. The disadvantage of the generalized monotone method is the computation of coupled lower and upper solutions of type I on the interval of existence. The computation of coupled lower and upper solution is not a trivial matter. Using the generalized monotone method as a tool, both the theoretical and the numerical results for computing coupled lower and upper solutions for scalar and system of ordinary differential equations can be found in [15]. Computation of coupled lower and upper solution to any desired interval using generalized monotone method as a tool and the corresponding numerical results for scalar and system of Caputo fractional differential equations are developed in [11] and [14] respectively. However, the rate of convergence of the sequences is linear. In [13] generalized quasilinearization method was developed using coupled lower and upper solutions when the nonlinear function is the sum of a convex and a concave function. The method of generalized quasilinearization yields sequences which converge uniformly to the unique solution and the rate of convergence is quadratic. The complexity of this method is that the sequences are solutions of two systems of coupled linear equations. The solutions of these two systems are difficult even with constant coefficients for fractional differential equations. To overcome this difficulty, in this work we have taken the nonlinear function as the sum of a convex function and a non-increasing function. We have combined the method of generalized quasilinearization for the convex function and generalized monotone method for the non-increasing function. We compute the sequences as two systems of Caputo fractional differential equations which are decoupled. The method yields superlinear convergence. See [13] for details. In this work, we provide a methodology to compute coupled lower and upper solutions of type I, to any desired interval by using the mixed generalized quasilinearization method and generalized monotone method. The convergence is superlinear. Further we can accelerate the convergence by using Gauss-Seidel accelerated convergence. We have applied our theoretical results to the logistic equation. The first set of iterates is in terms of the Mittag-Leffler function. Computation of further iterates has led to interesting open problems, since it requires the exponential formula related to Mittag-Leffler function. The exponential properties of the Mittag-Leffler function are yet to be established. This has been addressed in our conclusion.

2 Preliminary and Auxiliary Results

In this section, we recall known results, some definitions which are needed for our main results.

Definition 2.1 Caputo fractional derivative of order q is given by:

$$^cD^q u(t) = \frac{1}{\Gamma(1-q)} \int_0^t (t-s)^{-q} u'(s)ds,$$

where $0 < q < 1$ and $\Gamma(q)$ is the Gamma function.

Although in this work, we study Caputo fractional differential equations, our comparison results follow from the relation between Riemann-Liouville derivative and Caputo fractional derivative. Hence the next definition is for the Riemann-Liouville derivative.

Definition 2.2 Riemann-Liouville fractional derivative of order q with respect to t is defined by:

$$D^q u(t) = \frac{1}{\Gamma(m-q)} \frac{d^m}{dt^m} \int_0^t (t-s)^{m-q-1} f(s)ds,$$

where $m-1 < q < m$.

In particular, if $0 < q < 1$, then

$$D^q u(t) = \frac{1}{\Gamma(1-q)} \frac{d}{dt} \int_0^t (t-s)^{-q} f(s)ds.$$

Here, and throughout this work, we will consider fractional differential equations of order q, where, $0 < q < 1$.

Consider the nonlinear Caputo fractional differential equation with initial condition of the form:

$$^cD^q u(t) = f(t, u(t)), \quad u(0) = u_0, \tag{1}$$

where $f \in C[J \times \mathbb{R}, \mathbb{R}]$ and $J = [0, T]$. The integral representation of (1) is given by:

$$u(t) = u_0 + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, u(s))ds. \tag{2}$$

The sequences we develop are always solutions of linear Caputo fractional differential equation. In order to compute the solution of the linear fractional differential equation with constant coefficients we need Mittag-Leffler function.

Definition 2.3 Mittag-Leffler function of two parameters q, r is given by

$$E_{q,r}(\lambda(t-t_0)^q) = \sum_{k=0}^{\infty} \frac{(\lambda(t-t_0)^q)^k}{\Gamma(qk+r)},$$

where $q, r > 0$. Also, for $t_0 = 0$ and $r = 1$, we get

$$E_{q,1}(\lambda t^q) = \sum_{k=0}^{\infty} \frac{(\lambda t^q)^k}{\Gamma(qk+1)},$$

where $q > 0$.

Also, consider linear Caputo fractional differential equation

$$^cD^q u(t) = \lambda u(t) + f(t), \quad u(0) = u_0, \ \ on \ \ J, \tag{3}$$

where $J = [0, T]$, λ is a constant and $f(t) \in C[J, \mathbb{R}]$. The solution of (3) exists and is unique. The explicit solution of (3) is given by:

$$u(t) = u_0 E_{q,1}(\lambda t^q) + \int_0^t (t-s)^{q-1} E_{q,q}(\lambda(t-s)^q) f(s) ds. \tag{4}$$

See [7] for details. In particular, if $\lambda = 0$, the solution $u(t)$ is given by:

$$u(t) = u_0 + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s) ds. \tag{5}$$

Also we recall known results related to scalar Caputo nonlinear fractional differential equations of the following form

$$^cD^q u(t) = f(t, u) + g(t, u), \quad u(0) = u_0 \ \ on \ \ J = [0, T], \tag{6}$$

where $0 < q < 1$. Results when $q = 1$ is proved in [18]. Here $f, g \in C(J \times \mathbb{R}, \mathbb{R})$, $f(t, u)$ is non-decreasing in u on J and $g(t, u)$ is non-increasing in u on J.

In order to prove the comparison result relative to coupled lower and upper solutions of (6) we recall a basic lemma relative to the Riemann-Liouville fractional derivative.

Lemma 2.1 *Let $m(t) \in C_p[J, \mathbb{R}]$ (where $J = [0, T]$) be such that for some $t_1 \in (0, T]$, $m(t_1) = 0$ and $m(t) \leq 0$, on $(0, T]$. Then $D^q m(t_1) \geq 0$.*

Proof. See [4,7] for details. Note that the above result has been proved in [4] without using the Hölder continuity assumption of $m(t)$. □

The above lemma is true for Caputo derivative also, using the relation $^cD^q x(t) = D^q(x(t) - x(0))$ between the Caputo derivative and the Riemann-Liouville derivative. This is the version we will be using to prove our comparison results.

We recall the following known definitions which are needed for our main results.

Definition 2.4 The functions $\alpha_0, \beta_0 \in C^1(J, \mathbb{R})$ are called natural lower and upper solutions of (6) if :

$$\begin{cases} ^cD^q \alpha_0(t) \leq f(t, \alpha_0) + g(t, \alpha_0), & \alpha_0(0) \leq u_0, \\ ^cD^q \beta_0(t) \geq f(t, \beta_0) + g(t, \beta_0), & \beta_0(0) \geq u_0. \end{cases}$$

Definition 2.5 The functions $\alpha_0, \beta_0 \in C^1(J, \mathbb{R})$ are called coupled lower and upper solutions of (6) of type I if :

$$\begin{cases} ^cD^q \alpha_0(t) \leq f(t, \alpha_0) + g(t, \beta_0), & \alpha_0(0) \leq u_0, \\ ^cD^q \beta_0(t) \geq f(t, \beta_0) + g(t, \alpha_0), & \beta_0(0) \geq u_0. \end{cases}$$

See [10] for other types of coupled lower and upper solutions relative to (6).

Denoting $F(t,u) = f(t,u) + g(t,u)$, we state the next comparison result.

Theorem 2.1 *Let α, β be natural lower and upper solutions of (6), respectively. Suppose that $F(t,\beta) - F(t,\alpha) \leq L(\beta - \alpha)$ whenever $\beta \geq \alpha$, where L is a constant such that $L > 0$, then $\alpha(0) \leq \beta(0)$ implies that $\alpha(t) \leq \beta(t)$, $t \in J$.*

Proof. See [7] for details. □

Also, see [10, 16] for comparison result for coupled lower and upper solution of type I. Next, we recall a corollary of Theorem 2.1, which is useful in our main result.

Corollary 2.1 *Let $p \in C^1[J,\mathbb{R}]$. ${}^cD^q p(t) \leq Lp(t)$, where $L \geq 0$ and $p(0) \leq 0$. Then $p(t) \leq 0$ on J.*

We define the following sector Ω for convenience. That is,
$\Omega = [(t,u) : \alpha(t) \leq u(t) \leq \beta(t),\ t \in J]$.

Theorem 2.2 *Suppose $\alpha, \beta \in C^1[J,\mathbb{R}]$ are coupled lower and upper solutions of type I of (6) such that $\alpha(t) \leq \beta(t)$ on J and $F \in C(\Omega,\mathbb{R})$. Further, if $g(t,u)$ is decreasing in u, on J, then there exists a solution $u(t)$ of (6) such that $\alpha(t) \leq u(t) \leq \beta(t)$ on J, provided $\alpha(0) \leq u_0 \leq \beta(0)$.*

Proof. The proof follows from the scalar version of the result of [13]. □

Note that from the hypotheses of the above theorem, it follows that coupled lower and upper solution of type I are also natural lower and upper solutions.

The next results give the uniqueness theorem.

Theorem 2.3 *Let $\alpha, \beta \in C^1[J,\mathbb{R}]$, where α, β are coupled lower and upper solutions of (6) of type I, with $\alpha(t) \leq \beta(t)$ on J. If $f(t,u)$ is convex in u and $g(t,u)$ is decreasing in u, the hypotheses of Theorem 2.1 are satisfied. Then,(6) has a unique solution.*

The next result is useful in proving the equicontinuity of the sequences we develop in the next two theorems.

Theorem 2.4 *Let $\alpha_n(t)$ be a family of continuous functions on $[0,T]$, for each $n > 0$, where ${}^cD^q\alpha_n(t) = f(t,\alpha_n(t))$, $\alpha_n(0) = u_0$ and $|f(t,\alpha_n(t))| \leq M$ for $0 \leq t \leq T$. Then, the family $\{\alpha_n(t)\}$ is equicontinuous on $[0,T]$.*

Proof. See [7, 13] for details. □

Next, we provide two results relative to (6) where in the first result we assume f is convex in u and g is concave in u, and in the second result we assume f is convex in u and g is non-increasing in u. The first result we provide is related to the generalized quasilinearization method of (6) using coupled lower and upper solutions of type I.

Theorem 2.5 *Assume that*

(i) $\alpha_0, \beta_0 \in C^1[J,\mathbb{R}]$ are coupled lower solutions of type I, for (6) with $\alpha_0 \leq \beta_0$ on J,

(ii) $f, g \in C[\Omega,\mathbb{R}], f_u, g_u, f_{uu}$, and g_{uu} exist, are continuous and satisfy $f_{uu}(t,u) \geq 0, g_{uu}(t,u) \leq 0$ for $(t,u) \in \Omega$,

(iii) $g_u(t,u) \leq 0$ on Ω.

Then there exit monotone sequences $\{\alpha_n(t\}, \{\beta_n(t)\}$ which converge uniformly and monotonically to the unique solution of (6) and the convergence is quadratic.

Proof. See [13] for details. □

The next theorem is proved under the weaker assumption on $g(t,u)$. Also, this result mixes generalized quasilinearization method relative to the convex function $f(t,u)$ and generalized monotone method relative to the nonincreasing function $g(t,u)$ for $t \in J$.

Theorem 2.6 *Assume that*

(i) $\alpha_0, \beta_0 \in C^1[J,\mathbb{R}]$ *are coupled lower and upper solutions of type I, for (6) with* $\alpha_0 \leq \beta_0$ *on* J,

(ii) $f, g \in C[\Omega, \mathbb{R}], f_u, g_u,$ *and* f_{uu} *exist, are continuous and satisfy* $f_{uu}(t,u) \geq 0$ *for* $(t,u) \in \Omega$,

(iii) $g_u(t,u) \leq 0$ *on* Ω.

Then there exit monotone sequences $\{\alpha_n(t)\}, \{\beta_n(t)\}$ *which converge uniformly to the unique solution of (6) and the convergence is superlinear.*

Proof. See [13] for details. □

3 Main Results

In this section we will provide a method to compute coupled lower and upper solutions on any desired interval when we have the natural lower and upper solutions. Natural lower and upper solutions are relatively easy to compute. For example, equilibrium solutions are natural solutions. In the next result we use the superlinear convergence scheme as in Theorem 2.6, using natural lower and upper solutions. However, when we use natural lower and upper solutions, the results of Theorem 2.6 are true only when $\alpha_0 \leq \alpha_1$ and $\beta_0 \geq \beta_1$. This, in general, will not be true on the interval J, namely, the interval of existence of the solution. In the next result, monotone sequences constructed will converge to coupled minimal and maximal solutions as well as they are coupled lower and upper solutions on the interval of existence J.

Theorem 3.1 *Assume that*

(i) $\alpha_0, \beta_0 \in C^1[J,\mathbb{R}]$, α_0 *and* β_0 *are natural lower and upper solutions of (6) on* J *with* $\alpha_0 \leq \beta_0$ *on* J,

(ii) $f, g \in C[\Omega, \mathbb{R}], f_u, g_u, and f_{uu}$ *exist, are continuous and satisfy* $f_{uu}(t,u) \geq 0$ *for* $(t,u) \in \Omega$,

(iii) $g_u(t,u) \leq 0$ *on* Ω.

Then there exit monotone sequences $\{\alpha_n(t)\}, \{\beta_n(t)\}$ *which converge uniformly to the coupled lower and upper solution of (6). Here the sequences* $\{\alpha_n(t)\}$ *and* $\{\beta_n(t)\}$ *are computed using the following iterative scheme*

$$^cD^q\alpha_{n+1} = f(t,\alpha_n) + f_u(t,\alpha_n)(\alpha_{n+1} - \alpha_n) + g(t,\beta_n), \alpha_{n+1}(0) = u_0, \tag{7}$$

$$^cD^q\beta_{n+1} = f(t,\beta_n) + f_u(t,\alpha_n)(\beta_{n+1} - \beta_n) + g(t,\alpha_n), \beta_{n+1}(0) = u_0. \tag{8}$$

Proof. From the first iteration we will have $\alpha_0(t) \leq \alpha_1(t)$ on $[0,t_1]$ and $\beta_1(t) \leq \beta_0(t)$ on $[0,\bar{t}_1\ ]$. If $t_1 \geq T$, and $\bar{t}_1 \geq T$ there is nothing to prove, since one can use Theorem 2.6 to compute coupled minimal and maximal solutions. If not, certainly $t_1 < T$ and $\bar{t}_1 < T$. Also $\alpha_1(t_1) = \alpha_0(t_1)$. and $\beta_1(\bar{t}_1) = \beta_0(\bar{t}_1)$. We will now redefine $\alpha_1(t)$, and $\beta_1(t)$ on $[0,T]$ as follows:

$$^cD^q\alpha_1(t) = f(t,\alpha_0) + f_u(t,\alpha_0)(\alpha_1 - \alpha_0) + g(t,\beta_0), \quad \alpha_1(0) = u_0 \ \ on \ \ [0,t_1],$$

$$^cD^q\beta_1(t) = f(t,\beta_0) + f_u(t,\alpha_0)(\beta_1-\beta_0) + g(t,\alpha_0), \quad \beta_1(0) = u_0 \ \ on \ \ [0,\bar{t}_1\,],$$

and

$$\alpha_1(t) = \alpha_0(t) \ \ on \ \ [t_1,T],$$
$$\beta_1(t) = \beta_0(t) \ \ on \ \ [\,\bar{t}_1,T].$$

Proceeding in this manner, we will have $\alpha_n(t_n) = \alpha_0(t_n)$, and $\beta_n(\bar{t}_n) = \beta_0(\bar{t}_n)$. Now we can redefine α_n, β_n as follows.

$$^cD^q\alpha_n(t) = f(t,\alpha_{n-1}) + f_u(t,\alpha_{n-1})(\alpha_n-\alpha_{n-1}) + g(t,\beta_{n-1}), \quad v_n(0) = u_0 \ \ on \ \ [0,t_n],$$
$$\alpha_n(t) = \alpha_0(t) \quad on \quad [t_n,T].$$

Similarly,

$$^cD^q\beta_n(t) = f(t,\beta_{n-1}) + f_u(t,\alpha_{n-1})(\beta_n-\beta_{n-1}) + g(t,\alpha_{n-1}), \quad \beta_n(0) = u_0 \ \ on \ \ [0,\bar{t}_n],$$
$$\beta_n(t) = \beta_0(t) \quad on \quad [\,\bar{t}_n,T],$$

where α_n,β_n intersect α_0 and β_0 at t_n, $\bar{t}_n$ respectively. If $t_n \geq T$, and $\bar{t}_n \geq T$ we can stop the process. Certainly $\alpha_n \leq \beta_n$ and α_n and β_n are coupled minimum and maximum solutions of (6) respectively.

Now we can show that the sequences $\{\alpha_n(t)\}$ and $\{\beta_n(t)\}$ constructed above are equicontinuous and uniformly bounded on J. Hence by Arzelá-Ascoli theorem, a subsequence converges uniformly and monotonically. Since the sequences are monotone, the entire sequence converges uniformly and monotonically to α and β respectively.

It is easy to observe that

$$^cD^q\alpha_n(t) = f(t,\alpha_{n-1}) + f_u(t,\alpha_{n-1})(\alpha_n-\alpha_{n-1}) + g(t,\beta_{n-1}), \ \ \alpha_n(0) = u_0 \ \ on \quad [0,t_n],$$
$$\alpha_n(t) = \alpha_0(t) \quad on \quad [t_{n-1},T], \ \ such \ \ that \ \ \alpha_n(t_{n-1}) = \alpha_0(t_n),$$

and

$$^cD^q\beta_n(t) = f(t,\beta_{n-1}) + f_u(t,\alpha_{n-1})(\beta_n-\beta_{n-1}) + g(t,\alpha_{n-1}), \quad \beta_n(0) = u_0 \ \ on \ \ [0,\bar{t}_n],$$
$$\beta_n(t) = \beta_0(t) \quad on \quad [\,\bar{t}_{n-1},T], \ \ such \ \ that \ \ \beta_n(\,\bar{t}_n) = \beta_0(\,\bar{t}_{n-1}),$$

for all $n \geq 1$.

As $n\to\infty$, $t_n,\bar{t}_n \to T$, $\alpha_n(t)\to\alpha(t)$, and $\beta_n(t)\to\beta(t)$, uniformly and monotonically.

Further,

$$^cD^q\alpha(t) = f(t,\alpha) + g(t,\beta), \quad \alpha(0) = u_0 \quad on \quad J,$$

and

$$^cD^q\beta(t) = f(t,\beta) + g(t,\alpha), \quad \beta(0) = u_0 \quad on \quad J.$$

Hence α,β are coupled lower and upper solutions of (6) such that $\alpha\leq\beta$ on J. This concludes the proof. □

Theorem 3.2 *Assume that*

(i) $\alpha_0,\beta_0 \in C^1[J,\mathbb{R}]$, α_0 *and* β_0 *are natural lower and upper solutions of (6) on* J *with* $\alpha_0\leq\beta_0$ *on* J,

(ii) $f,g \in C[\Omega,\mathbb{R}], f_u, g_u, f_{uu}$ *exist, are continuous and satisfy* $f_{uu}(t,u) \geq 0$ *for* $(t,u)\in\Omega$,

(iii) $g_u(t,u)\leq 0$ *on* Ω.

Then there exit monotone sequences $\{\alpha_n(t)\},\{\beta_n(t)\}$ *which converge uniformly to the unique solution of (6) and the convergence is superlinear.*

Proof. Theorem 3.1 proves that, there exit monotone sequences $\{\alpha_n(t)\}, \{\beta_n(t)\}$ such that $\{\alpha_n(t)\} \longrightarrow \alpha(t)$ and $\{\beta_n(t)\} \longrightarrow \beta(t)$ uniformly and monotonically and (α, β) are coupled lower and upper solutions of type I for (6) respectively on J. However, it is easy to observe that each pair of $\alpha_n(t), \beta_n(t)$ computed are also coupled lower and upper solutions of (6) on the common interval of $[0, t_n]$ and $[0, \bar{t}_n]$. Suppose that for some $n = k$ both t_k and $\bar{t}_k$ are $\geq T$, then the computation of $\alpha_{k+1}(t), \beta_{k+1}(t)$ will no longer need $\alpha_0(t), \beta_0(t)$. Then it is easy to observe that $\alpha_{k+1}(t), \beta_{k+1}(t)$ will be coupled lower and upper solutions of type I for (6) respectively on J. Also this sequence will converge uniformly and monotonically to α, β using Theorem 3.1. This implies that $\alpha \leq \beta$ on J. By hypotheses and using Theorem 2.3, it can be shown that $\alpha \equiv \beta \equiv u$, where u is the unique solution of (6) on J. In order to prove superlinear convergence we let $p_n(t) = u(t) - \alpha_n(t)$ and $q_n(t) = \beta_n(t) - u(t)$. It is easy to see that $p_n(0) = 0$, $q_n(0) = 0$. Using Gronwall type of Lemma and the estimate on f_{uu} and g_u on J, we can prove that $\max_J |p_n + q_n| \leq \max_J (|(p_{n-1} + q_{n-1})|^2 + |(p_{n-1} + q_{n-1})|)$ which proves superlinear convergence. See [13]for details. □

Note that if $g(t, u)$ is non-increasing in u on J, then α, β constructed above are also natural lower and upper solutions. By the existence theorem, there exists a solution of (6) on J such that $\alpha \leq u \leq \beta$ provided, $\alpha(0) \leq u_0 \leq \beta(0)$.

Remark 3.1 Note that Theorem 3.1 provides coupled lower and upper solutions of (6) on J. Now we can develop sequences $\{\alpha_n\}$ and $\{\beta_n\}$ using Theorem 2.6. These sequences converge uniformly and monotonically to coupled minimal and maximal solutions. Further if uniqueness condition is satisfied, the sequences converge to the unique solution of (6). Further we can apply Gauss-Seidel method such that the sequences converge faster. This is what we have proved in the next result.

Theorem 3.3 *Let all the hypotheses of Theorem 2.6 hold with the iterative scheme given by*

$$^cD^q\alpha^*_{n+1} = f(t, \alpha^*_n) + f_u(t, \alpha^*_n)(\alpha^*_{n+1} - \alpha^*_n) + g(t, \beta^*_n), \alpha^*_{n+1}(0) = u_0, \tag{9}$$

$$^cD^q\beta^*_{n+1} = f(t, \beta^*_n) + f_u(t, \alpha^*_{n+1})(\beta^*_{n+1} - \beta^*_n) + g(t, \alpha^*_n), \beta^*_{n+1}(0) = u_0. \tag{10}$$

starting with $\alpha^*_0 = \alpha_1$ on J. Then there exist monotone sequences $\{\alpha_n\}$ and $\{\beta_n\}$, which converge uniformly to the unique solution of (6) and the convergence is faster than superlinear.

Proof. We provide a brief proof. Initially compute α_1 using $^cD^q\alpha_1 = f(t, \alpha_0) + f_u(t, \alpha_0)(\alpha_1 - \alpha_0) + g(t, \beta_0), \alpha_1(0) = u_0$. Relabel $\alpha_1 = \alpha^*_0$. Now compute β_1 using $\beta_0$and α^*_0. That is $^cD^q\beta_1 = f(t, \beta_0) + f_u(t, \alpha^*_0)(\beta_1 - \beta_0) + g(t, \alpha^*_0), \beta_1(0) = u_0$. One can easily see that $\alpha_0(t) \leq \alpha_1(t)$ on J. Now it is enough if we prove that $\beta^*_0 \leq \beta_1$.

$$\begin{gathered}
\text{Let } p(t) = \beta^*_0 - \beta_1, \quad p(0) = 0. \\
^cD^q p(t) = {}^cD^q\beta_0{}^* - {}^cD^q\beta_1 \\
= f(t, \beta_0) + f_u(t, \alpha_1)(\beta_1 - \beta_0) + g(t, \alpha_1) - (f(t, \beta_0) + f_u(t, \alpha_0)(\beta_1 - \beta_0) + g(t, \alpha_0)) \\
= (f_u(t, \alpha_1) - f_u(t, \alpha_0))(\beta_1 - \beta_0) + g(t, \alpha_1) - g(t, \alpha_0) \\
\leq 0, \quad \text{since } \alpha_1(t) \geq \alpha_0(t) \text{ on } J.
\end{gathered}$$

This implies $p(t) \leq 0$ on J, using Corollary 2.1. That is $\beta^*_0 \leq \beta_1$ on J. Continuing the process, we can show that that the sequences $\{\alpha^*_n\}$ and $\{\beta^*_n\}$ converge faster than the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ which are computed using Theorem 3.1.

4 Numerical Results

In this section, we provide a numerical example as an application of our main results. We take a simple logistic equation and apply Theorem 3.1. In order to apply Theorem 3.1, we assume that α_1 and β_1 should satisfy $\alpha_0 \leq \alpha_1$, $\beta_1 \leq \beta_0$ on $[0,T]$. If $q=1$, the solution of the logistic equation can be computed explicitly. However, if $0<q<1$, we cannot compute the solution explicitly. Method of lower and upper solution guarantees the interval of existence. The equilibrium solutions play the role of lower and upper solutions.

Consider the example

$$ {}^cD^q u(t) = u - u^2, \quad u(0) = \frac{1}{2}, \quad t \in [0,T], \quad T \geq 1. \tag{11}$$

It is easy to observe that $\alpha_0(t)=0$ and $\beta_0(t)=1$ are natural lower and upper solutions respectively of (11) such that $\alpha_0 \leq \beta_0$ on $[0,T]$. Here $f(t,u)=u$ and $g(t,u)=-u^2$.

Using the iterative schemes as in Theorem 3.1 we obtain

$$ {}^cD^q\alpha_1(t) = \alpha_1 - \beta_1^2 \quad \textit{and} \quad {}^cD^q\beta_1(t) = \beta_1 - \alpha_1^2. $$

Solving for α_1 and β_1, we arrive at

$$ \alpha_1 = 1 - \tfrac{1}{2}E_{q,1}(t^q) \text{ and } \beta_1 = \tfrac{1}{2}E_{q,1}(t^q) $$

Similarly, the next iteration gives rise to

$$ {}^cD^q\alpha_2(t) = \alpha_2 - \beta_1^2 \text{ and } {}^cD^q\beta_2(t) = \beta_2 - \alpha_1^2 $$
$$ {}^cD^q\alpha_2(t) = \alpha_2 - (\tfrac{1}{2}E_{q,1}(t^q))^2 \text{ and } {}^cD^q\beta_2(t) = \beta_2 - (1-\tfrac{1}{2}E_{q,1}(t^q))^2. $$

In order to compute α_2 and β_2, we use (3) with $\lambda = 1$, and $f(t)$ as $-(\frac{1}{2}E_{q,1}(t^q))^2$ and $-(1-\frac{1}{2}E_{q,1}(t^q))^2$ respectively. Here, we have computed $(\frac{1}{2}E_{q,1}(t^q))^2$ and $(1-\frac{1}{2}E_{q,1}(t^q))^2$ using the product formula. The product formula is given by

$$ E_{q,1}(\lambda(t-t_0)^q) * E_{q,1}(\mu(t-t_0)^q) = \sum_{k=0}^{\infty} \frac{(t-t_0)^{qk}}{\Gamma(qk+1)} (\lambda+\mu)_{q,1}^k, $$

where

$$ (\lambda+\mu)_{q,1}^k = \sum_{l=0}^{k} \frac{\lambda^l \mu^{k-l}\Gamma(qk+1)}{\Gamma(ql+1)\Gamma(q(k-l)+1)}, $$

which is the generalized binomial formula. Further we need to multiply this by $E_{q,q}((t-s)^q)$ as in formula (4) to compute α_2 and β_2. Computing α_2 and β_2, we arrive at

$$ \alpha_2 = \tfrac{1}{2}E_{q,1}(t^q) - \tfrac{1}{4}s_1 \text{ and } \beta_2 = 1 - \tfrac{1}{2}E_{q,1}(t^q) - \tfrac{1}{4}s_1 + s_2, $$

where

$$ s_1 = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{k} \frac{t^{q+jq+kq}\Gamma(1+kq)}{\Gamma(lq+1)\Gamma(kq-lq+1)\Gamma(q+jq+kq+1)}, $$

$$ s_2 = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{t^{q+jq+kq}}{\Gamma(q+jq+kq+1)}. $$

The graphs of α_1, β_1 and α_2, β_2 have been drawn in Figure 1 where $q=\frac{1}{2}, t_0 = 0$.

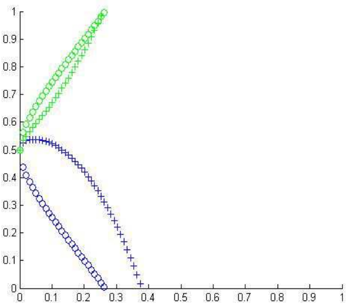

Figure 1: Coupled Lower and Upper Solutions of (11) with $q = 1/2$ using Theorem 3.1.

5 Conclusion

In this work we have mixed generalized quasilinearization method and generalized monotone method to compute the coupled lower and upper solution of type I on the desired interval. In addition, the method also provides the unique solution of the nonlinear problem. This mixed method yields superlinear convergence. Computation of the solution of the coupled lower and upper solutions numerically involves the generalized Mittag-Leffler function which involves the generalized binomial coefficients. In Figure 1, we can see that $\overline{t_2} \not> \overline{t_1}$, since the evaluation of β_2 is not accurate. This is due to the lack of knowledge of product of Mittag-Leffler function and its accurate computation. We plan to develop the necessary properties of the Mittag-Leffler function in our future work and obtain better estimates for the sequences $\{\alpha_n\}$ and $\{\beta_n\}$.

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