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Parabolic Equations with Measure Data and Three Unbounded Nonlinearities in Weighted Sobolev Spaces

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Abstract: In this work, we study the degenerated problem

$$\begin{gathered} \frac{\partial b(x,u)}{\partial t} + \mathrm{div}(a(x,t,u,Du)) + H(x,t,u,Du) = \mu \quad \text{in } Q, \\ u = 0 \quad \text{on } \partial\Omega \times (0,T), \\ b(x,u)(t=0) = b(x,u_0) \quad \text{on } \Omega, \end{gathered} \tag{1}$$

in the framework of weighted Sobolev space. The main contribution of our work is to prove the existence of a renormalized solution without the sign condition and the coercivity condition on $H(x,t,u,Du)$. The critical growth condition on H is with respect to Du and no growth with respect to u. The datum μ is assumed in $L^1(Q) + L^{p'}(0,T;W^{-1,p'}(\Omega,w^*))$ and $b(x,u_0) \in L^1(\Omega)$.

Keywords: *nonlinear parabolic equation; weighted Sobolev spaces; renormalized solutions.*

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1 Introduction

Let Ω be a bounded open set of $\mathbb{R}^N$, p be a real number such that $2 < p < \infty$, $Q = \Omega \times [0,T]$ and $w = \{w_i(x) : 0 \leq i \leq N\}$ be a vector of weight functions (i.e., every component $w_i(x)$ is a measurable almost everywhere strictly positive function on Ω), satisfying some integrability conditions (see Section 2). Let $Au = -\operatorname{div}(a(x,t,u,Du))$ be a Leray-Lions operator defined from the weighted Sobolev space $L^p(0,T;W_0^{1,p}(\Omega,w))$ into its dual $L^{p'}(0,T;W^{-1,p'}(\Omega,w^*))$.

Now, we consider the degenerated parabolic problem associated with the differential equation

$$\begin{gathered}\frac{\partial b(x,u)}{\partial t} + Au + H(x,t,u,Du) = \mu \quad \text{in } Q,\\ u = 0 \quad \text{on } \partial\Omega\times]0,T[,\\ b(x,u)(t=0) = b(x,u_0) \quad \text{on } \Omega.\end{gathered} \tag{2}$$

In problem (2), the data μ and $b(x,u_0)$ are in $L^1(Q) + L^{p'}(0,T;W^{-1,p'}(\Omega,w^*))$ and $L^1(\Omega)$. The operator $-\operatorname{div}(a(x,t,u,Du))$ is a Leray-Lions operator which is coercive, $b(x,u)$ is unbounded function on u, H is a nonlinear lower order term and $\mu = f - \operatorname{div}F$ with $f \in L^1(Q)$, $F \in \prod_{i=1}^{N} L^{p'}(Q,w_i^*)$.

Problem (2) is studied in [2] with $\mu \in L^{p'}(0,T;W^{-1,p'}(\Omega,w^*))$ and under the strong hypothesis relatively to H, more precisely they supposed that $b(x,u) = u$ and the non-linearity H satisfying the sign condition

$$H(x,t,s,\xi)s \geq 0, \tag{3}$$

and the growth condition of the form

$$|H(x,t,s,\xi)| \leq b(s)\Big(\sum_{i=1}^{N} w_i(x)|\xi_i|^p + c(x,t)\Big). \tag{4}$$

In the case where the second member $f \in L^1(Q)$, (2) is studied in [2].

It is our purpose to prove the existence of renormalized solution for (2) in the setting of the weighted Sobolev space without the sign condition (3), and without the following coercivity condition

$$|H(x,t,s,\xi)| \geq \beta \sum_{i=1}^{N} w_i(x)|\xi_i|^p \quad \text{for } |s| \geq \gamma, \tag{5}$$

our growth condition on H is simpler than (4) it is a growth with respect to Du and no growth condition with respect to u (see assumption (H3) below), the second term μ belongs to $L^1(Q) + L^{p'}(0,T;W^{-1,p'}(\Omega,w^*))$. Note that our paper generalizes [2].

In the case of $H(x,t,u,Du) = div(\phi(u))$ is studied by H. Redwane in the classical Sobolev spaces $W^{1,p}(\Omega)$ and Orlicz spaces see [18, 20].

The notion of renormalized solution was introduced by DiPerna and Lions [11] in their study of the Boltzmann equation. This notion was then adapted to an elliptic version of (2) by Boccardo et al [7] when the right hand side is in $W^{-1,p'}(\Omega)$, by Rakotoson [18] when the right hand side is in $L^1(\Omega)$, and finally by Dal Maso, Murat, Orsina and Prignet [10] for the case of the right hand side being general measure data. Our paper can be considered as a continuation of [3–5] in the case where $F = 0$.

2 Preliminaries

Let Ω be a bounded open set of $\mathbb{R}^N$, p be a real number such that $2 < p < \infty$ and $w = \{w_i(x),\, 0 \le i \le N\}$ be a vector of weight functions; i.e., every component $w_i(x)$ is a measurable function which is strictly positive a.e. in Ω. Further, we suppose in all our considerations that, there exists

$$r_0 > \max(N,p) \quad \text{such that } w_i^{\frac{-r_0}{r_0-p}} \in L^1_{\text{loc}}(\Omega), \tag{6}$$

$$w_i \in L^1_{\text{loc}}(\Omega), \tag{7}$$

$$w_i^{\frac{-1}{p-1}} \in L^1_{\text{loc}}(\Omega), \tag{8}$$

for any $0 \le i \le N$. We denote by $W^{1,p}(\Omega,w)$ the space of real-valued functions $u \in L^p(\Omega, w_0)$ such that the derivatives in the sense of distributions fulfill

$$\frac{\partial u}{\partial x_i} \in L^p(\Omega, w_i) \quad \text{for } i = 1, \ldots, N.$$

Which is a Banach space under the norm

$$\|u\|_{1,p,w} = \Big[\int_\Omega |u(x)|^p w_0(x)\,dx + \sum_{i=1}^N \int_\Omega |\frac{\partial u(x)}{\partial x_i}|^p w_i(x)\,dx\Big]^{1/p}. \tag{9}$$

Condition (7) implies that $C_0^\infty(\Omega)$ is a space of $W^{1,p}(\Omega,w)$ and consequently, we can introduce the subspace $V = W_0^{1,p}(\Omega,w)$ of $W^{1,p}(\Omega,w)$ as the closure of $C_0^\infty(\Omega)$ with respect to the norm (9). Moreover, condition (8) implies that $W^{1,p}(\Omega,w)$ as well as $W_0^{1,p}(\Omega,w)$ are reflexive Banach spaces.

We recall that the dual space of weighted Sobolev spaces $W_0^{1,p}(\Omega,w)$ is equivalent to $W^{-1,p'}(\Omega,w^*)$, where $w^* = \{w_i^* = w_i^{1-p'},\, i = 0,\ldots,N\}$ and where p' is the conjugate of p; i.e., $p' = \frac{p}{p-1}$, (see [13]).

3 Basic Assumptions

Assumption (H1)

For $2 \le p < \infty$, we assume that the expression

$$|||u|||_V = \Big(\sum_{i=1}^N \int_\Omega |\frac{\partial u(x)}{\partial x_i}|^p w_i(x)\,dx\Big)^{1/p} \tag{10}$$

is a norm defined on V which is equivalent to the norm (9), and there exists a weight function σ on Ω such that, $\sigma \in L^1(\Omega)$ and $\sigma^{-1} \in L^1(\Omega)$. We assume also the Hardy inequality

$$\Big(\int_\Omega |u(x)|^p \sigma\,dx\Big)^{1/q} \le c\Big(\sum_{i=1}^N \int_\Omega |\frac{\partial u(x)}{\partial x_i}|^p w_i(x)\,dx\Big)^{1/p} \tag{11}$$

holds for every $u \in V$ with a constant $c > 0$ independent of u, and moreover, the imbedding

$$W^{1,p}(\Omega,w) \hookrightarrow L^p(\Omega,\sigma), \tag{12}$$

expressed by the inequality (11) is compact. Notice that $(V, |||\cdot|||_V)$ is a uniformly convex (and thus reflexive) Banach space.

Remark 3.1 If we assume that $w_0(x) \equiv 1$ and in addition the integrability condition: There exists $\nu \in]\frac{N}{p}, +\infty[\cap[\frac{1}{p-1}, +\infty[$ such that

$$w_i^{-\nu} \in L^1(\Omega) \quad \text{and} \quad w_i^{\frac{N}{N-1}} \in L^1_{\text{loc}}(\Omega) \quad \text{for all } i = 1, \dots, N. \tag{13}$$

Notice that the assumptions (7) and (13) imply

$$|||u||| = \Big(\sum_{i=1}^{N} \int_{\Omega} |\frac{\partial u}{\partial x_i}|^p w_i(x)\, dx\Big)^{1/p}, \tag{14}$$

which is a norm defined on $W_0^{1,p}(\Omega, w)$ and its equivalent to (9) and that, the imbedding

$$W_0^{1,p}(\Omega, w) \hookrightarrow L^q(\Omega) \tag{15}$$

is compact for all $1 \leq q \leq p_1^*$ if $p\nu < N(\nu + 1)$ and for all $q \geq 1$ if $p\nu \geq N(\nu + 1)$ where $p_1 = \frac{p\nu}{\nu+1}$ and p_1^* is the Sobolev conjugate of p_1; see [12, pp. 30-31].

Assumption (H2)

$$b : \Omega \times \mathbb{R} \to \mathbb{R} \quad \text{is a Carathéodory function} \tag{16}$$

such that for every $x \in \Omega$, $b(x, .)$ is a strictly increasing C^1-function with $b(x, 0) = 0$. Next, for any $k > 0$, there exists $\lambda_k > 0$ and functions $A_k \in L^\infty(\Omega)$ and $B_k \in L^p(\Omega)$ such that

$$\lambda_k \leq \frac{\partial b(x, s)}{\partial s} \leq A_k(x) \quad \text{and} \quad \Big|D_x\Big(\frac{\partial b(x, s)}{\partial s}\Big)\Big| \leq B_k(x) \tag{17}$$

for almost every $x \in \Omega$, for every s such that $|s| \leq k$, we denote by $D_x\big(\frac{\partial b(x,s)}{\partial s}\big)$ the gradient of $\frac{\partial b(x,s)}{\partial s}$ defined in the sense of distributions. For $i = 1, \dots, N$,

$$|a_i(x, t, s, \xi)| \leq \beta w_i^{1/p}(x)[k(x, t) + \sigma^{1/p'}|s|^{q/p'} + \sum_{j=1}^{N} w_j^{1/p'}(x)|\xi_j|^{p-1}], \tag{18}$$

for a.e. $(x, t) \in Q$,all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$, some function $k(x, t) \in L^{p'}(Q)$ and $\beta > 0$, here σ and q are as in (H1).

$$[a(x, t, s, \xi) - a(x, t, s, \eta)](\xi - \eta) > 0 \quad \text{for all } \xi \neq \eta, \tag{19}$$

$$a(x, t, s, \xi).\xi \geq \alpha \sum_{i=1}^{N} w_i |\xi_i|^p, \tag{20}$$

where α is a strictly positive constant.

Assumption (H3)

Furthermore, let $H(x, t, s, \xi) : Q \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ be a Carathéodory function such that for a.e $(x, t) \in Q$ and for all $s \in \mathbb{R}, \xi \in \mathbb{R}^N$, the growth condition

$$|H(x, t, s, \xi)| \leq \gamma(x, t) + g(s) \sum_{i=1}^{N} w_i(x)|\xi_i|^p, \tag{21}$$

is satisfied, where $g : \mathbb{R} \to \mathbb{R}^+$ is a bounded continuous positive function that belongs to $L^1(\mathbb{R})$, while $\gamma(x,t)$ belongs to $L^1(Q)$.

We recall that, for $k > 1$ and s in $\mathbb{R}$, the truncation is defined as

$$T_k(s) = \begin{cases} s, & \text{if } |s| \le k, \\ k\frac{s}{|s|}, & \text{if } |s| > k. \end{cases}$$

4 Some Technical Results

Characterization of the time mollification of a function u.

In order to deal with time derivative, we introduce a time mollification of a function u belonging to a some weighted Lebesgue space. Thus we define for all $\mu \ge 0$ and all $(x,t) \in Q$,

$$u_\mu = \mu \int_\infty^t \tilde{u}(x,s)\exp(\mu(s-t))ds \quad where \quad \tilde{u}(x,s) = u(x,s)\chi_{(0,T)}(s).$$

Proposition 4.1 *[2]*
1) if $u \in L^p(Q,w_i)$ then u_μ is measurable in Q and $\frac{\partial u_\mu}{\partial t} = \mu(u - u_\mu)$ and

$$\|u_\mu\|_{L^p(Q,w_i)} \le \|u\|_{L^p(Q,w_i)}.$$

2) If $u \in W_0^{1,p}(Q,w)$, then $u_\mu \to u$ in $W_0^{1,p}(Q,w)$ as $\mu \to \infty$.
3) If $u_n \to u$ in $W_0^{1,p}(Q,w)$, then $(u_n)_\mu \to u_\mu$ in $W_0^{1,p}(Q,w)$.

Some weighted embedding and compactness results.

In this section we establish some embedding and compactness results in weighted Sobolev spaces, some trace results, Aubin's and Simon's results [21].

Let $V = W_0^{1,\,p}(\Omega,w)$, $H = L^2(\Omega,\sigma)$ and let $V^* = W^{-1,p'}$ with $(2 \le p < \infty)$.

Let $X = L^p(0,T;W_0^{1,\,p}(\Omega,w))$. The dual space of X is $X^* = L^{p'}(0,T,V^*)$ where $\frac{1}{p} + \frac{1}{p'} = 1$ and denoting the space $W_p^1(0,T,V,H) = \{v \in X : \; v' \in X^*\}$ endowed with the norm

$$\|u\|_{W_p^1} = \|u\|_X + \|u'\|_{X^*},$$

which is a Banach space. Here u' stands for the generalized derivative of u, i.e.,

$$\int_0^T u'(t)\varphi(t)dt = -\int_0^T u(t)\varphi'(t)dt \quad for \quad all \quad \varphi \in C_0^\infty(0,T).$$

Lemma 4.1 *[19]*
1)The evolution triple $V \subseteq H \subseteq V^*$ is verified.
2) The imbedding $W_p^1(0,T,V,H) \subseteq C(0,T,H)$ is continuous.
3)The imbedding $W_p^1(0,T,V,H) \subseteq L^p(Q,\sigma)$ is compact.

Lemma 4.2 *[2] Let $g \in L^r(Q,\gamma)$ and let $g_n \in L^r(Q,\gamma)$, with $\|g_n\|_{L^r(Q,\gamma)} \le C$, $1 < r < \infty$. If $g_n(x) \to g(x)$ a.e in Q, then $g_n \rightharpoonup g$ in $L^r(Q,\gamma)$*

Lemma 4.3 *[2]. Assume that*

$$\frac{\partial v_n}{\partial t} = \alpha_n + \beta_n \quad in \quad D'(Q),$$

where α_n and β_n are bounded respectively in X^* and in $L^1(Q)$. If v_n is bounded in $L^p(0,T;W_0^{1,\,p}(\Omega,w))$, then $v_n \to v$ in $L^p_{loc}(Q,\sigma)$. Further $v_n \to v$ strongly in $L^1(Q)$.

Definition 4.1 Let $f \in L^1(Q)$, $F \in \prod_{i=1}^{N} L^{p'}(Q,w_i^*)$ and $b(x,u_0) \in L^1(\Omega)$. A real-valued function u defined on Q is a renormalized solution of problem (2) if

$$T_k(u) \in L^p(0,T;W_0^{1,\,p}(\Omega,w)) \text{ for all } k \geq 0 \;\; and \;\; b(x,u) \in L^\infty(0,T;L^1(\Omega)), \tag{22}$$

$$\int_{\{m\leq|u|\leq m+1\}} a(x,t,u,Du)Dudxdt \to 0 \quad as \quad m \to +\infty, \tag{23}$$

$$\begin{aligned}&\frac{\partial B_S(x,u)}{\partial t} - div\,(S'(u)a(x,t,u,Du)) + S''(u)a(x,t,u,Du)Du \\ &+ H(x,t,u,Du)S'(u) = fS'(u) - div\,(S'(u)F) + S''(u)FDu \quad in \quad D'(Q),\end{aligned} \tag{24}$$

for all functions $S \in W^{2,\,\infty}(\mathbb{R})$ which is piecewise C^1 and such that S' has a compact support in $\mathbb{R}$, where $B_S(x,z) = \int_0^z \frac{\partial b(x,r)}{\partial r} S'(r)dr$ and

$$B_S(x,u)(t=0) = B_S(x,u_0) \quad in \quad \Omega. \tag{25}$$

Remark 4.1 Equation (24) is formally obtained through pointwise multiplication of equation (2) by $S'(u)$. However, while $a(x,t,u,Du)$ and $H(x,t,u,Du)$ do not in general make sense in (2), all the terms in (2) have a meaning in $D'(Q)$. Indeed, if M is such that $suppS' \subset [-M,M]$, the following identifications are made in (24):

- $S(u)$ belongs to $L^\infty(Q)$ since S is a bounded function.
- $S'(u)a(x,t,u,Du)$ identifies with $S'(u)a(x,t,T_M(u),DT_M(u))$ a.e in Q.

Since $|T_M(u)| \leq M$ a.e in Q and $S'(u) \in L^\infty(Q)$, we obtain from (18) and (22) that

$$S'(u)a(x,t,T_M(u),DT_M(u)) \in \prod_{i=1}^{N} L^{p'}(Q,w_i^*).$$

- $S''(u)a(x,t,u,Du)Du$ identifies with $S''(u)a(x,t,T_M(u),DT_M(u))DT_M(u)$ and

$$S''(u)a(x,t,T_M(u),DT_M(u))DT_M(u) \in L^1(Q).$$

- $S'(u)H(x,t,u,Du)$ identifies with $S'(u)H(x,t,T_M(u),DT_M(u))$ a.e in Q. Since $|T_M(u)| \leq M$ a.e in Q and $S'(u) \in L^\infty(Q)$, we obtain from (18) and (21) that

$$S'(u)H(x,t,T_M(u),DT_M(u)) \in L^1(Q).$$

- $S'(u)f$ belongs to $L^1(Q)$ while $S'(u)F$ belongs to $\prod_{i=1}^{N} L^{p'}(Q,w_i^*)$.
- $S''(u)FDu$ identifies with $S''(u)FDT_k(u)$ which belongs to $L^1(Q)$.

The above considerations show that equation (24) holds in $D'(Q)$ and that

$$\frac{\partial B_S(x,u)}{\partial t} \in L^{p'}(0,T;W^{-1,\,p'}(\Omega,w^*)) + L^1(Q).$$

Due to the properties of S and (24), $\frac{\partial S(u)}{\partial t} \in L^{p'}(0,T;W^{-1,\,p'}(\Omega,w^*)) + L^1(Q)$, which implies that $S(u) \in C^0([0,T];L^1(\Omega))$ so that the initial condition (25) makes sense, since, due to the properties of S (increasing) and (17), we have

$$|B_S(x,r) - B_S(x,r')| \leq A_k(x)\,|S(r) - S(r')| \quad for \quad all \quad r,r' \in \mathbb{R}. \tag{26}$$

5 Existence Results

In this section we establish the following existence theorem.

Theorem 5.1 *Let $f \in L^1(Q)$, $F \in \prod_{i=1}^{N} L^{p'}(Q,w_i^*)$ and u_0 is a measurable function such that $b(x,u_0) \in L^1(\Omega)$. Assume that (H1) and (H2) hold true. Then, there exists at least a renormalized solution u of the problem (2) in the sense of Definition 4.1.*

Proof. **Step 1: Approximate problem and a priori estimates.**
For $n > 0$, let us define the following approximation of b, H, f and u_0;

$$b_n(x,r) = b(x,T_n(r)) + \frac{1}{n}r \quad \text{for } n > 0. \tag{27}$$

In view of (27), b_n is a Carathéodory function and satisfies (17), there exist $\lambda_n > 0$ and functions $A_n \in L^1(\Omega)$ and $B_n \in L^p(\Omega)$ such that

$$\lambda_n \leq \frac{\partial b_n(x,s)}{\partial s} \leq A_n(x) \quad \text{and} \quad \Big|D_x\Big(\frac{\partial b_n(x,s)}{\partial s}\Big)\Big| \leq B_n(x)$$

a.e. in Ω, $s \in \mathbb{R}$.

$$H_n(x,t,s,\xi) = \frac{H(x,t,s,\xi)}{1 + \frac{1}{n}|H(x,t,s,\xi)|}\chi_{\Omega_n}.$$

Note that Ω_n is a sequence of compacts covering the bounded open set Ω and χ_{Ω_n} is its characteristic function.

$$f_n \in L^{p'}(Q), \quad \text{and} \quad f_n \to f \quad \text{a.e. in } Q \text{ and strongly in } L^1(Q) \text{ as } n \to +\infty, \tag{28}$$

$$u_{0n} \in D(\Omega), \quad \|b_n(x,u_{0n})\|_{L^1} \leq \|b(x,u_0)\|_{L^1}, \tag{29}$$

$$b_n(x,u_{0n}) \to b(x,u_0) \quad \text{a.e. in } \Omega \text{ and strongly in } L^1(\Omega). \tag{30}$$

Let us now consider the approximate problem:

$$\begin{gathered} \frac{\partial b_n(x,u_n)}{\partial t} - \operatorname{div}(a(x,t,u_n,Du_n)) + H_n(x,t,u_n,Du_n) = f_n - div(F) \quad \text{in } D'(Q), \\ u_n = 0 \quad \text{in } (0,T)\times\partial\Omega, \\ b_n(x,u_n(t=0)) = b_n(x,u_{0n}). \end{gathered} \tag{31}$$

Note that $H_n(x,t,s,\xi)$ satisfies the following conditions

$$|H_n(x,t,s,\xi)| \leq H(x,t,s,\xi) \quad \text{and} \quad |H_n(x,t,s,\xi)| \leq n.$$

For all $u, v \in L^p(0,T; W_0^{1,p}(\Omega, w))$,

$$\Big|\int_Q H_n(x,t,u,Du)v\,dx\,dt\Big| \leq \Big(\int_Q |H_n(x,t,u,Du)|^{q'}\sigma^{-\frac{q'}{q}}\,dx\,dt\Big)^{1/q'}\Big(\int_Q |v|^q\sigma\,dx\,dt\Big)^{1/q}$$

$$\leq n\int_0^T \Big(\int_{\Omega_n}\sigma^{1-q'}dx\Big)^{1/q'} dt\,\|v\|_{L^q(Q,\sigma)} \leq C_n\,\|v\|_{L^p(0,T;W_0^{1,p}(\Omega,w))}.$$

Moreover, since $f_n \in L^{p'}(0,T; W^{-1,p'}(\Omega, w^*))$, proving existence of a weak solution $u_n \in L^p(0,T; W_0^{1,p}(\Omega, w))$ of (31) is an easy task (see e.g. [15], [2]).

Let $\varphi \in L^p(0,T; W_0^{1,\,p}(\Omega, w)) \cap L^\infty(Q)$ with $\varphi > 0$, choosing $v = \exp(G(u_n))\varphi$ as test function in (31) where $G(s) = \int_0^s \frac{g(r)}{\alpha}dr$ (the function g appears in (21)), we have

$$\int_Q \frac{\partial b_n(x,u_n)}{\partial t}\exp(G(u_n))\varphi dxdt + \int_Q a(x,t,u_n,Du_n)D(\exp(G(u_n))\varphi)dxdt$$
$$+\int_Q H_n(x,t,u_n,Du_n)\exp(G(u_n))\varphi dxdt = \int_Q f_n\exp(G(u_n))\varphi dxdt$$
$$+\int_Q FD(\exp(G(u_n))\varphi)dxdt.$$

In view of (21) and (20) we obtain

$$\int_Q \frac{\partial b_n(x,u_n)}{\partial t}\exp(G(u_n))\varphi dxdt + \int_Q a(x,t,u_n,Du_n)\exp(G(u_n))D\varphi dxdt$$
$$\leq \int_Q \gamma(x,t)\exp(G(u_n))\varphi dxdt + \int_Q f_n\exp(G(u_n))\varphi dxdt$$
$$+\int_Q F\exp(G(u_n))D\varphi dxdt + \int_Q FD(\exp(G(u_n)))\varphi dxdt, \tag{32}$$

for all $\varphi \in L^p(0,T; W_0^{1,\,p}(\Omega, w)) \cap L^\infty(Q)$ with $\varphi > 0$.

On the other hand, taking $v = \exp(-G(u_n))\varphi$ as test function in (31) we deduce as in (32) that,

$$\int_Q \frac{\partial b_n(x,u_n)}{\partial t}\exp(-G(u_n))\varphi dxdt + \int_Q a(x,t,u_n,Du_n)\exp(-G(u_n))D\varphi dxdt$$
$$+\int_Q \gamma(x,t)\exp(-G(u_n))\varphi dxdt \geq \int_Q f_n\exp(-G(u_n))\varphi dxdt$$
$$+\int_Q F\exp(-G(u_n))D\varphi dxdt + \int_Q FD(\exp(-G(u_n)))\varphi dxdt, \tag{33}$$

for all $\varphi \in L^p(0,T; W_0^{1,\,p}(\Omega, w)) \cap L^\infty(Q)$ with $\varphi > 0$.

For every $\tau \in]0,T[$, let $\varphi = T_k(u_n)^+\chi_{(0,\tau)}$ in (32) we have

$$\int_\Omega B_{k,G}^n(x,u_n(\tau))dx + \int_{Q_\tau} a(x,t,u_n,Du_n)\exp(G(u_n))DT_k(u_n)^+dxdt$$

$$\leq \int_{Q_\tau} \gamma(x,t) \exp(G(u_n)) T_k(u_n)^+ dxdt + \int_{Q_\tau} f_n \exp(G(u_n)) T_k(u_n)^+ dxdt$$
$$+ \int_Q F D(T_k(u_n)^+) \exp(G(u_n)) dxdt + \int_Q F T_k(u_n)^+ \exp(G(u_n)) D u_n \frac{g(u_n)}{\alpha} dxdt \quad (34)$$
$$+ \int_\Omega B^n_{k,G}(x, u_{0n}) dx,$$

where $B^n_{k,G}(x,r) = \displaystyle\int_0^r \frac{\partial b_n(x,s)}{\partial s} T_k(s)^+ \exp(G(s)) ds$. Due to the definition of $B^n_{k,G}$ and $|G(u_n)| \leq \exp\left(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\right)$ we have

$$0 \leq \int_\Omega B^n_{k,G}(x, u_{0n}) dx \leq k \exp\left(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\right) \|b(x,u_0)\|_{L^1(\Omega)}. \quad (35)$$

Using (35), $B^n_{k,G}(x,u_n) \geq 0$, Young's inequality and (20) we obtain

$$\alpha\Big(\frac{p-1}{p}\Big) \int_{Q_\tau} \sum_{i=1}^N \left|\frac{\partial T_k(u_n)^+}{\partial x_i}\right|^p w_i \exp(G(u_n)) dxdt \quad (36)$$
$$\leq k \exp\left(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\right) \Big(\|f_n\|_{L^1(Q)} + \|\gamma\|_{L^1(Q)} + c\,\|F\|^{p'}_{\prod_{i=1}^N L^{p'}(Q,w_i^*)} + \|b_n(x,u_{0n})\|_{L^1(\Omega)} \Big)$$
$$+ \frac{1}{\alpha} \int_{Q_\tau} F g(u_n) \exp(G(u_n)) D u_n \chi_{\{u_n>0\}} dxdt.$$

Let us observe that, if we take $\varphi = \rho(u_n) = \displaystyle\int_0^{u_n} g(s)\chi_{\{s>0\}} ds$ in (32) and using (20) we obtain

$$\left[\int_\Omega B^n_g(x,u_n) dx\right]_0^T + \alpha \int_Q \sum_{i=1}^N \left|\frac{\partial u_n}{\partial x_i}\right|^p w_i g(u_n) \chi_{\{u_n>0\}} \exp(G(u_n)) dxdt$$
$$\leq \left(\int_0^\infty g(s) ds\right) \exp\left(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\right) \Big(\|\gamma\|_{L^1(Q)} + \|f_n\|_{L^1(Q)}\Big)$$
$$+ \int_Q F D u_n g(u_n) \chi_{\{u_n>0\}} \exp(G(u_n)) dxdt$$
$$+ \left(\int_0^\infty g(s) ds\right) \int_Q \Big|F D u_n\Big| \frac{g(u_n)}{\alpha} \exp(G(u_n)) \chi_{\{u_n>0\}} dxdt,$$

where $B^n_g(x,r) = \displaystyle\int_0^r \frac{\partial b_n(x,s)}{\partial s} \rho(s) \exp(G(s)) ds$, which implies, since $B^n_g(x,r) \geq 0$ and Young's inequality,

$$\alpha \int_{\{u_n>0\}} \sum_{i=1}^N \left|\frac{\partial u_n}{\partial x_i}\right|^p w_i g(u_n) \exp(G(u_n)) dxdt$$
$$\leq \exp\left(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\right) \Big(\|\gamma\|_{L^1(Q)} + \|f\|_{L^1(Q)} + \|b(x,u_0)\|_{L^1(\Omega)}\Big)$$

$$+C_1 \|g\|_\infty \exp\left(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\right) \int_Q \sum_{i=1}^N |F_i|^{p'} w_i^* \, dx\, dt$$

$$+\frac{\alpha}{2p} \int_Q \sum_{i=1}^N \left|\frac{\partial u_n}{\partial x_i}\right|^p w_i \frac{g(u_n)}{\alpha} \exp(G(u_n))\chi_{\{u_n>0\}} dxdt$$

$$+C_2 \int_0^\infty g(s)\, ds \, \|g\|_\infty \exp\left(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\right) \int_Q |F|^{p'} w^* \, dx\, dt$$

$$+\frac{\alpha}{2p} \int_Q \sum_{i=1}^N \left|\frac{\partial u_n}{\partial x_i}\right|^p w_i \frac{g(u_n)}{\alpha} \exp(G(u_n))\chi_{\{u_n>0\}} dxdt$$

we obtain

$$\int_{\{u_n>0\}} g(u_n) \sum_{i=1}^N \left|\frac{\partial u_n}{\partial x_i}\right|^p w_i \exp(G(u_n)) dxdt \leq C_3.$$

Similarly, let $\varphi = \displaystyle\int_{u_n}^0 g(s)\chi_{\{s<0\}} ds$ as a test function in (33), we conclude that

$$\int_{\{u_n<0\}} g(u_n) \sum_{i=1}^N \left|\frac{\partial u_n}{\partial x_i}\right|^p w_i \exp(G(u_n)) dxdt \leq C_4.$$

Consequently,

$$\int_Q g(u_n) \sum_{i=1}^N \left|\frac{\partial u_n}{\partial x_i}\right|^p w_i \exp(G(u_n)) dxdt \leq C_5. \tag{37}$$

where $C_1, \cdots,\ C_5$ are constants independent of n. We deduce that

$$\int_Q \sum_{i=1}^N \left|\frac{\partial T_k(u_n)^+}{\partial x_i}\right|^p w_i dxdt \leq C_6\ k. \tag{38}$$

Similarly to (38) we take $\varphi = T_k(u_n)^- \chi_{(0,\tau)}$ in (33) we deduce that

$$\int_Q \sum_{i=1}^N \left|\frac{\partial T_k(u_n)^-}{\partial x_i}\right|^p w_i dxdt \leq C_7\ k. \tag{39}$$

Combining (38) and (39) we conclude that

$$\|T_k(u_n)\|^p_{L^p(0,T;W_0^{1,\ p}(\Omega,w))} \leq C_8\ k, \tag{40}$$

where $C_6,\ C_7,\ C_8$ are constants independent of n.

Then, $T_k(u_n)$ is bounded in $L^p(0,T;W_0^{1,p}(\Omega,w))$, and $T_k(u_n)$ converges to v_k weakly in $L^p(0,T;W_0^{1,p}(\Omega,w))$, and by the compact imbedding (15) gives

$$T_k(u_n) \to v_k \quad \text{strongly in } L^p(Q,\sigma) \text{ and a.e. in } Q.$$

We deduce from the above inequalities (34), (35) and (40) that

$$\int_\Omega B^n_{k,G}(x, u_n(\tau)) dx \leq C\ k. \tag{41}$$

Let $k > 0$ be large enough and B_R be a ball of Ω, we have

$$\begin{aligned}
& k\,\mathrm{meas}(\{|u_n| > k\} \cap B_R \times [0,T]) \\
&= \int_0^T \int_{\{|u_n|>k\}\cap B_R} |T_k(u_n)|\,dx\,dt \\
&\le \int_0^T \int_{B_R} |T_k(u_n)|\,dx\,dt \\
&\le \Big(\int_Q |T_k(u_n)|^p \sigma\,dx\,dt\Big)^{1/p} \Big(\int_0^T \int_{B_R} \sigma^{1-p'}\,dx\,dt\Big)^{1/p'} \\
&\le T c_R \Big(\int_Q \sum_{i=1}^N w_i(x) \Big|\frac{\partial T_k(u_n)}{\partial x_i}\Big|^p\,dx\,dt\Big)^{1/p} \\
&\le c k^{1/p},
\end{aligned}$$

which implies

$$\mathrm{meas}(\{|u_n| > k\} \cap B_R \times [0,T]) \le \frac{c_1}{k^{1-\frac{1}{p}}}, \quad \forall k \ge 1.$$

So, we have

$$\lim_{k\to+\infty} (\mathrm{meas}(\{|u_n| > k\} \cap B_R \times [0,T])) = 0.$$

Now we turn to prove the almost everywhere convergence of u_n and $b_n(x,u_n)$. Consider now a function non decreasing $g_k \in C^2(\mathbb{R})$ such that $g_k(s) = s$ for $|s| \le \frac{k}{2}$ and $g_k(s) = k$ for $|s| \ge k$. Multiplying the approximate equation by $g'_k(u_n)$, we get

$$\begin{aligned}
&\frac{\partial B_k^n(x,u_n)}{\partial t} - div(a(x,t,u_n,Du_n)g'_k(u_n)) + a(x,t,u_n,Du_n)g''_k(u_n)Du_n \\
&+ H_n(x,t,u_n,Du_n)g'_k(u_n) = f_n g'_k(u_n) - div(F g'_k(u_n)) + F g''_k(u_n)Du_n, \qquad (42)
\end{aligned}$$

where $B_k^n(x,z) = \displaystyle\int_0^z \frac{\partial b_n(x,s)}{\partial s} g'_k(s)ds$.

As a consequence of (40), we deduce that $g_k(u_n)$ is bounded in $L^p(0,T;W_0^{1,\,p}(\Omega,w))$ and $\frac{\partial B_k^n(x,u_n)}{\partial t}$ is bounded in $L^1(Q) + L^{p'}(0,T;W^{-1,p'}(\Omega,w^*))$. Due to the properties of g_k and (17), we conclude that $\frac{\partial g_k(u_n)}{\partial t}$ is bounded in $L^1(Q) + L^{p'}(0,T;W^{-1,p'}(\Omega,w^*))$, which implies that $g_k(u_n)$ is compact in $L^1(Q)$.

Hence Lemma 4.3 allows us to conclude that $g_k(u_n)$ is compact in $L^p_{loc}(Q,\sigma)$. Thus, for a subsequence, it also converges in measure and almost everywhere in Q (since we have, for every $\lambda > 0$,)

$$\begin{aligned}
&meas(\{|u_n - u_m| > \lambda\} \cap B_R \times [0,T]) \le meas(\{|u_n| > k\} \cap B_R \times [0,T]) \\
&+meas(\{|u_m| > k\} \cap B_R \times [0,T]) + meas(\{|g_k(u_n) - g_k(u_m)| > \lambda\}).
\end{aligned}$$

Let $\varepsilon > 0$, then, there exist $k(\varepsilon) > 0$ such that,

$$meas(\{|u_n - u_m| > \lambda\} \cap B_R \times [0,T]) \le \varepsilon \quad for \;\; all \;\; n,m \ge n_0(k(\varepsilon),\lambda,R).$$

This proves that (u_n) is a Cauchy sequence in measure in $B_R \times [0,T])$, thus converges almost everywhere to some measurable function u. Then for a subsequence denoted again u_n, we have

$$u_n \to u \quad a.e \quad in \quad Q, \tag{43}$$

and from (40) we deduce

$$b_n(x, u_n) \to b(x, u) \quad a.e \quad in \quad Q, \tag{44}$$

$$T_k(u_n) \rightharpoonup T_k(u) \quad weakly \quad in \quad L^p(0, T; W_0^{1,\,p}(\Omega, w)) \tag{45}$$

and then, the compact imbedding (12) gives,

$$T_k(u_n) \to T_k(u) \text{ strongly in } L^q(Q, \sigma) \text{ and a.e in } Q.$$

Which implies, by using (18), for all $k > 0$ that there exists a function $\Lambda_k \in \prod_{i=1}^{N} L^{p'}(Q, w_i^*)$, such that

$$a(x, t, T_k(u_n), DT_k(u_n)) \rightharpoonup \Lambda_k \quad weakly \quad in \quad \prod_{i=1}^{N} L^{p'}(Q, w_i^*). \tag{46}$$

We now establish that $b(x, u)$ belongs to $L^\infty(0, T; L^1(\Omega))$. Using (43) and passing to the limit-inf in (41) as n tends to $+\infty$, we obtain that $\frac{1}{k}\int_\Omega B_{k,G}(x, u(\tau))dx \le C$, for almost any τ in $(0, T)$. Due to the definition of $B_{k,G}(x, s)$ and the fact that $\frac{1}{k}B_{k,G}(x, u)$ converges pointwise to $\int_0^u sgn(s)\frac{\partial b(x, s)}{\partial s}\exp(G(s))ds \ge |b(x, u)|$, as k tends to $+\infty$, shows that $b(x, u)$ belongs to $L^\infty(0, T; L^1(\Omega))$.

Lemma 5.1 *Let u_n be a solution of the approximate problem (31). Then*

$$\lim_{m\to\infty} \limsup_{n\to\infty} \int_{\{m\le|u_n|\le m+1\}} a(x, t, u_n, Du_n)Du_n dxdt = 0. \tag{47}$$

Proof. Considering the following function $\varphi = T_1(u_n - T_m(u_n))^+ = \alpha_m(u_n)$ in (32) this function is admissible since $\varphi \in L^p(0, T; W_0^{1,\,p}(\Omega, w))$ and $\varphi \ge 0$. Then by Young's inequality, we have

$$\int_\Omega B_{n,G}^m(x, u_n)(T)dx + \int_{\{m\le u_n\le m+1\}} a(x, t, u_n, Du_n)Du_n \exp(G(u_n))dxdt$$

$$\le \exp\left(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\right)\left[\int_{\{|u_n|>m\}} |f_n|\,dxdt + \int_{\{|u_n|>m\}} |\gamma|\,dxdt + \int_{\{|u_{n0}|>m\}} |b_n(x, u_{0n})|\,dx\right]$$

$$+C_1\int_{\{u_n\ge m\}} \sum_{i=1}^{N} |F_i|^{p'} w_i^* dxdt + \frac{\alpha}{p}\int_{\{m\le u_n\le m+1\}} \sum_{i=1}^{N} \left|\frac{\partial u_n}{\partial x_i}\right|^p w_i \exp(G(u_n))dxdt$$

$$+C_2\int_{\{u_n\ge m\}} \sum_{i=1}^{N} |F_i|^{p'} w_i^* dxdt + C_3\int_{\{u_n\ge m\}} \sum_{i=1}^{N} \left|\frac{\partial u_n}{\partial x_i}\right|^p w_i g(u_n)\exp(G(u_n))dxdt,$$

where $B_{n,G}^m(x, r) = \int_0^r \frac{\partial b_n(x, s)}{\partial s}\exp(G(s))\alpha_m(s)ds.$

Using (20) and since $B^m_{n,G}(x,u_n)(T) > 0$, we obtain

$$\left(\frac{p-1}{p}\right)\int_{\{m\le u_n\le m+1\}} a(x,t,u_n,Du_n)Du_n\exp(G(u_n))dxdt$$

$$\le \exp\left(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\right)\left[\int_{\{|u_n|>m\}}(|f_n|+|\gamma|)dxdt+\int_{\{|u_{n0}|>m\}}|b_n(x,u_{0n})|\,dx\right]$$

$$+C_4\int_{\{u_n\ge m\}}\sum_{i=1}^N |F_i|^{p'}\,w_i^*dxdt+C_5\int_{\{u_n>m\}} g(u_n)\exp(G(u_n))\sum_{i=1}^N\left|\frac{\partial u_n}{\partial x_i}\right|^p w_idxdt. \quad (48)$$

Take $\varphi=\rho_m(u_n)=\int_0^{u_n} g(s)\chi_{\{s>m\}}ds$ as test function in (32), we obtain

$$\left[\int_\Omega B^n_m(x,u_n)dx\right]_0^T+\int_Q a(x,t,u_n,Du_n)Du_ng(u_n)\chi_{\{u_n>m\}}\exp(G(u_n))dxdt$$

$$\le\left(\int_m^\infty g(s)\chi_{\{s>m\}}ds\right)\exp\left(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\right)\left(\|\gamma\|_{L^1(Q)}+\|f_n\|_{L^1(Q)}\right)$$

$$+\int_Q FDu_ng(u_n)\chi_{\{u_n>m\}}\exp(G(u_n))dxdt$$

$$+\left(\int_m^\infty g(s)ds\right)\int_Q FDu_n\frac{g(u_n)}{\alpha}\exp(G(u_n))\chi_{\{u_n>m\}}dxdt,$$

where $B^n_m(x,r)=\int_0^r\frac{\partial b_n(x,s)}{\partial s}\rho_m(s)\exp(G(s))ds$, which implies, since $B^n_m(x,r)\ge 0$, (20) and Young's inequality,

$$\frac{\alpha(p-1)}{p}\int_{\{u_n>m\}}\sum_{i=1}^N\left|\frac{\partial u_n}{\partial x_i}\right|^p w_ig(u_n)\exp(G(u_n))dxdt \quad (49)$$

$$\le\left(\int_m^\infty g(s)ds\right)\exp\left(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\right)$$

$$\cdot\left(\|\gamma\|_{L^1(Q)}+\|f_n\|_{L^1(Q)}+\|b_n(x,u_{0n})\|_{L^1(\Omega)}+C\,\|F\|^{p'}_{\prod_{i=1}^N L^{p'}(Q,w_i^*)}\right).$$

Using (49) and the strong convergence of f_n in $L^1(\Omega)$ and $b_n(x,u_{0n})$ in $L^1(\Omega)$, $\gamma\in L^1(\Omega)$, $g\in L^1(\mathbb{R})$ and $F\in\prod_{i=1}^N L^{p'}(Q,w_i^*)$, by Lebesgue's theorem, passing to the limit in (48), we conclude that

$$\lim_{m\to\infty}\limsup_{n\to\infty}\int_{\{m\le u_n\le m+1\}} a(x,t,u_n,Du_n)Du_ndxdt=0. \quad (50)$$

On the other hand, let $\varphi=T_1(u_n-T_m(u_n))^-$ as test function in (33) and reasoning as in the proof of (50) we deduce that

$$\lim_{m\to\infty}\limsup_{n\to\infty}\int_{\{-(m+1)\le u_n\le -m\}} a(x,t,u_n,Du_n)Du_ndxdt=0. \quad (51)$$

Thus (47) follows from (50) and (51).

Step 2: Almost everywhere convergence of the gradients.

This step is devoted to introduce for $k \geq 0$ a fixed time regularization of the function $T_k(u)$ in order to perform the monotonicity method. Let $\psi_i \in D(\Omega)$ be a sequence which converges strongly to u_0 in $L^1(\Omega)$. Set $w^i_\mu = (T_k(u))_\mu + e^{-\mu t}T_k(\psi_i)$ where $(T_k(u))_\mu$ is the mollification with respect to time of $T_k(u)$. Note that w^i_μ is a smooth function having the following properties:

$$\frac{\partial w^i_\mu}{\partial t} = \mu(T_k(u) - w^i_\mu), \quad w^i_\mu(0) = T_k(\psi_i), \quad \left|w^i_\mu\right| \leq k, \tag{52}$$

$$w^i_\mu \to T_k(u) \ \text{ in } \ L^p(0,T;W_0^{1,\,p}(\Omega,w)), \ \text{as } \mu \to \infty. \tag{53}$$

We will introduce the following function of one real variable s, which is defined as:

$$h_m(s) = \begin{cases} 1, & \text{if } |s| \leq m, \\ 0, & \text{if } |s| \geq m+1, \\ m+1+|s|, & \text{if } m \leq |s| \leq m+1. \end{cases}$$

For $m > k$, let $\varphi = (T_k(u_n) - w^i_\mu)^+ h_m(u_n) \in L^p(0,T;W_0^{1,\,p}(\Omega,w)) \cap L^\infty(Q)$ and $\varphi \geq 0$, then taking this function in (32), we obtain

$$\begin{aligned}
&\int_{\{T_k(u_n)-w^i_\mu \geq 0\}} \frac{\partial b_n(x,u_n)}{\partial t} \exp(G(u_n))(T_k(u_n) - w^i_\mu)h_m(u_n)dxdt \\
&\quad + \int_{\{T_k(u_n)-w^i_\mu \geq 0\}} a(x,t,u_n,Du_n)D(T_k(u_n) - w^i_\mu)h_m(u_n)dxdt \\
&- \int_{\{m \leq |u_n| \leq m+1\}} \exp(G(u_n))a(x,t,u_n,Du_n)Du_n(T_k(u_n) - w^i_\mu)^+ dxdt \\
&\quad \leq \int_Q (\gamma(x,t) + f_n)\exp(G(u_n))(T_k(u_n) - w^i_\mu)^+ h_m(u_n)dxdt \\
&\quad + \int_Q FDu_n \frac{g(u_n)}{\alpha} \exp(G(u_n))(T_k(u_n) - w^i_\mu)^+ h_m(u_n)dxdt \\
&\quad + \int_{\{T_k(u_n)-w^i_\mu \geq 0\}} \exp(G(u_n))FD(T_k(u_n) - w^i_\mu)h_m(u_n)dxdt \\
&\quad - \int_{\{m \leq |u_n| \leq m+1\}} \exp(G(u_n))FDu_n(T_k(u_n) - w^i_\mu)^+ dxdt.
\end{aligned} \tag{54}$$

Observe that

$$\begin{aligned}
&\left| \int_{\{m \leq |u_n| \leq m+1\}} \exp(G(u_n))a(x,t,u_n,Du_n)Du_n(T_k(u_n) - w^i_\mu)^+ dxdt \right| \\
&\quad \leq 2k \exp\left(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\right) \int_{\{m \leq u_n \leq m+1\}} a(x,t,u_n,Du_n)Du_n dxdt,
\end{aligned}$$

and

$$\left| \int_{\{m \leq |u_n| \leq m+1\}} \exp(G(u_n))FDu_n(T_k(u_n) - w^i_\mu)^+ dxdt \right|$$

$$\leq 2k \exp\left(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\right) \frac{\|F\|_{\prod_{i=1}^{N} L^{p'}(Q,w_i^*)}}{\alpha^{\frac{1}{p}}} \left(\int_{\{m\leq|u_n|\leq m+1\}} a(x,t,u_n,Du_n)Du_n dxdt\right)^{\frac{1}{p}}.$$

Thanks to (47) the third integral and fourth integral of the right hand side tend to zero as n and m tend to infinity, and by Lebesgue's theorem and $F \in \prod_{i=1}^{N} L^{p'}(Q, w_i^*)$, we deduce that the right hand side converges to zero as n, m and μ tend to infinity. Since

$$(T_k(u_n) - w_\mu^i)^+ h_m(u_n) \rightharpoonup (T_k(u) - w_\mu^i)^+ h_m(u) \;\; weakly* \;\; in \;\; L^\infty(Q), \;\; as \;\; n \to \infty$$

and strongly in $L^p(0,T;W_0^{1,\,p}(\Omega,w))$ and $(T_k(u) - w_\mu^i)^+ h_m(u) \rightharpoonup 0$ weakly* in $L^\infty(Q)$ and strongly in $L^p(0,T;W_0^{1,\,p}(\Omega,w))$ as $\mu \to \infty$. Let $\varepsilon_l(n,m,\mu,i)$: $l = 1, ...,$ are various functions tending to zero as n, m, i and μ tend to infinity.

The very definition of the sequence w_μ^i makes it possible to establish the following lemma.

Lemma 5.2 *For $k \geq 0$ we have*

$$\int_{\{T_k(u_n)-w_\mu^i\geq 0\}} \frac{\partial b_n(x,u_n)}{\partial t} \exp(G(u_n))(T_k(u_n) - w_\mu^i) h_m(u_n) dxdt \geq \varepsilon(n,m,\mu,i). \quad (55)$$

Proof. (see [19]).

Similarly to [3, 4] for the second term of the left hand side of (54) we conclude

$$\lim_{n\to\infty} \int_Q [a(x,t,T_k(u_n),DT_k(u_n)) - a(x,t,T_k(u_n),DT_k(u))] \times [DT_k(u_n) - DT_k(u)]\, dxdt = 0. \quad (56)$$

Which implies that

$$T_k(u_n) \to T_k(u) \;\; strongly \;\; in \;\; L^p(0,T;W_0^{1,\,p}(\Omega,w)) \;\; \forall k. \quad (57)$$

Now, observe that we have, for every $\sigma > 0$

$$meas\Big\{(x,t) \in \Omega\times[0,T] : |Du_n - Du| > \sigma\Big\} \leq meas\Big\{(x,t) \in \Omega\times[0,T] : |Du_n| > k\Big\}$$
$$+meas\Big\{(x,t) \in \Omega\times[0,T] : |u| > k\Big\}$$
$$+meas\Big\{(x,t) \in \Omega\times[0,T] : |DT_k(u_n) - DT_k(u)| > \sigma\Big\}$$

then as a consequence of (57) we also have, that Du_n converges to Du in measure and therefore, always reasoning for subsequence,

$$Du_n \to Du \;\; a.e \;\; in \;\; Q. \quad (58)$$

Which implies that

$$a(x,t,T_k(u_n),DT_k(u_n)) \rightharpoonup a(x,t,T_k(u),DT_k(u)) \;\; in \;\; \prod_{i=1}^{N} L^{p'}(Q,w_i^*). \quad (59)$$

Step 3: Equi-integrability of the nonlinearity sequence.
We shall now prove that $H_n(x,t,u_n,Du_n) \to H(x,t,u,Du)$ strongly in $L^1(Q)$ by using Vitali's theorem. Since $H_n(x,t,u_n,Du_n) \to H(x,t,u,Du)$ *a.e in* Q, consider now $\varphi = \rho_h(u_n) = \int_0^{u_n} g(s)\chi_{\{s>h\}}ds$ as test function in (32), we obtain

$$\left[\int_\Omega B_h^n(x,u_n)dx\right]_0^T + \int_Q a(x,t,u_n,Du_n)Du_n g(u_n)\chi_{\{u_n>h\}}\exp(G(u_n))dxdt$$
$$\leq \left(\int_h^\infty g(s)\chi_{\{s>h\}}ds\right)\exp\left(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\right)\left(\|\gamma\|_{L^1(Q)} + \|f_n\|_{L^1(Q)}\right)$$
$$+ \int_Q FDu_n g(u_n)\chi_{\{u_n>h\}}\exp(G(u_n))dxdt$$
$$+ \left(\int_h^\infty g(s)\chi_{\{s>h\}}ds\right)\int_Q |FDu_n|\frac{g(u_n)}{\alpha}\exp(G(u_n))\chi_{\{u_n>h\}}dxdt,$$

where $B_h^n(x,r) = \int_0^r \frac{\partial b_n(x,s)}{\partial s}\rho_h(s)\exp(G(s))ds$, which implies, since $B_h^n(x,r) \geq 0$, (20) and Young's inequality,

$$\frac{\alpha(p-1)}{p}\int_{\{u_n>h\}}\sum_{i=1}^N\left|\frac{\partial u_n}{\partial x_i}\right|^p w_i g(u_n)\exp(G(u_n))dxdt$$
$$\leq \left(\int_h^\infty g(s)ds\right)\exp\left(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\right)$$
$$\left(\|\gamma\|_{L^1(Q)} + \|f_n\|_{L^1(Q)} + \|b_n(x,u_{0n})\|_{L^1(\Omega)} + C\,\|F\|^{p'}_{\prod_{i=1}^N L^{p'}(Q,w_i^*)}\right),$$

we conclude that

$$\lim_{h\to\infty}\sup_{n\in\mathbb{N}}\int_{\{u_n>h\}}\sum_{i=1}^N\left|\frac{\partial u_n}{\partial x_i}\right|^p w_i g(u_n)dxdt = 0.$$

Consequently,

$$\lim_{h\to+\infty}\sup_{n\in\mathbb{N}}\int_{\{|u_n|>h\}} g(u_n)\sum_{i=1}^N\left|\frac{\partial u_n}{\partial x_i}\right|^p w_i dxdt = 0,$$

which implies, for h large enough and for a subset E of Q,

$$\lim_{meas(E)\to 0}\int_E g(u_n)\sum_{i=1}^N\left|\frac{\partial u_n}{\partial x_i}\right|^p w_i dxdt \leq \|g\|_\infty \lim_{meas(E)\to 0}\int_E\sum_{i=1}^N\left|\frac{\partial T_h(u_n)^+}{\partial x_i}\right|^p w_i dxdt$$
$$+ \int_{\{|u_n|>h\}} g(u_n)\sum_{i=1}^N\left|\frac{\partial u_n}{\partial x_i}\right|^p w_i dxdt$$

then we deduce that $g(u_n)\sum_{i=1}^N\left|\frac{\partial u_n}{\partial x_i}\right|^p w_i$ is equi-integrale. Thus we have obtained that $g(u_n)\sum_{i=1}^N\left|\frac{\partial u_n}{\partial x_i}\right|^p w_i$ converge to $g(u)\sum_{i=1}^N\left|\frac{\partial u}{\partial x_i}\right|^p w_i$ strongly in $L^1(Q)$. Consequently, by using (21), we conclude that

$$H_n(x,t,u_n,Du_n) \to \; H(x,t,u,Du) \;\; strongly \;\; in \;\; L^1(Q). \tag{60}$$

Step 4: In this step we prove that u satisfies (23).
Observe that for any fixed $m \geq 0$ one has

$$\int_{\{m\leq|u_n|\leq m+1\}} a(x,t,u_n,Du_n)Du_n = \int_Q a(x,t,u_n,Du_n)(DT_{m+1}(u_n) - DT_m(u_n))$$

$$= \int_Q a(x,t,T_{m+1}(u_n),DT_{m+1}(u_n))DT_{m+1}(u_n) - \int_Q a(x,t,T_m(u_n),DT_m(u_n))DT_m(u_n).$$

According to (59) and (57), one is at liberty to pass to the limit as $n \to \; +\infty$ for fixed $m \geq 0$ and to obtain

$$\begin{aligned} &\lim_{n\to+\infty} \int_{\{m\leq|u_n|\leq m+1\}} a(x,t,u_n,Du_n)Du_n dxdt \\ &= \int_Q a(x,t,T_{m+1}(u),DT_{m+1}(u))DT_{m+1}(u)dxdt - \int_Q a(x,t,T_m(u),DT_m(u))DT_m(u)dxdt \\ &= \int_{\{m\leq|u|\leq m+1\}} a(x,t,u,Du)Dudxdt. \end{aligned} \tag{61}$$

Taking the limit as $m \to +\infty$ in (61) and using the estimate (47) show that u satisfies (24).

Step 5: In this step we show that u satisfies (24) and (25). Let S be a function in $W^{2,\infty}(\mathbb{R})$ such that S' has a compact support. Let M be a positive real number such that supp$(S') \subset [-M,M]$. Pointwise multiplication of the approximate equation (31) by $S'(u_n)$ leads to

$$\begin{aligned} &\frac{\partial B_S^n(x,u_n)}{\partial t} - div[S'(u_n)a(x,t,u_n,Du_n)] + S''(u_n)a(x,t,u_n,Du_n)Du_n \\ &+ S'(u_n)H_n(x,t,u_n,Du_n = fS'(u_n) - div(FS'(u)) + S''(u)FDu \;\; in \;\; D'(Q). \end{aligned} \tag{62}$$

In what follows we pass to the limit as in (62) n tends to $+\infty$.
• Limit of $\frac{\partial B_S^n(x,u_n)}{\partial t}$.
Since S is bounded and continuous, $u_n \to u \;\; a.e \;\; in \;\; Q$ implies that $B_S^n(x,u_n)$ converges to $B_S(x,u)$ a.e in Q and $L^\infty \;\; weak - *$. Then $\frac{\partial B_S^n(x,u_n)}{\partial t}$ converges to $\frac{\partial B_S(x,u)}{\partial t}$ in $D'(Q)$ as n tends to $+\infty$.
• Limit of $-div[S'(u_n)a_n(x,t,u_n,Du_n)]$.
Since supp$(S') \subset [-M,M]$, we have for $n \geq M$

$$S'(u_n)a_n(x,t,u_n,Du_n) = S'(u_n)a(x,t,T_M(u_n),DT_M(u_n)) \;\; a.e \;\; in \;\; Q.$$

The pointwise convergence of $u_n \;\; to \;\; u$ and (59) as n tends to $+\infty$ and the bounded character of S' permit us to conclude that

$$S'(u_n)a_n(x,t,u_n,Du_n) \rightharpoonup S'(u)a(x,t,T_M(u),DT_M(u)) \;\; in \;\; \prod_{i=1}^N L^{p'}(Q,w_i^*), \tag{63}$$

as n tends to $+\infty$. $S'(u)a(x,t,T_M(u),DT_M(u))$ has been denoted by $S'(u)a(x,t,u,Du)$ in equation (24).

• Limit of $S''(u_n)a(x,t,u_n,Du_n)Du_n$.
As far as the 'energy' term

$$S''(u_n)a(x,t,u_n,Du_n)Du_n = S''(u_n)a(x,t,T_M(u_n),DT_M(u_n))DT_M(u_n) \ \ a.e \ \ in \ \ Q.$$

The pointwise convergence of $S'(u_n)$ *to* $S'(u)$ and (59) as n tends to $+\infty$ and the bounded character of S'' permit us to conclude that

$$S''(u_n)a_n(x,t,u_n,Du_n)Du_n \rightharpoonup S''(u)a(x,t,T_M(u),DT_M(u))DT_M(u) \ \ weakly \ \ in \ \ L^1(Q). \tag{64}$$

Recall that $S''(u)a(x,t,T_M(u),DT_M(u))DT_M(u) = S''(u)a(x,t,u,Du)Du \ \ a.e \ \ in \ \ Q.$

• Limit of $S'(u_n)H_n(x,t,u_n,Du_n)$.
Since supp$(S') \subset [-M,M]$ and (60), we have

$$S'(u_n)H_n(x,t,u_n,Du_n) \to S'(u)H(x,t,u,Du) \ \ strongly \ \ in \ \ L^1(Q), \tag{65}$$

as n tends to $+\infty$.

• Limit of $S'(u_n)f_n$.
Since $u_n \to u$ a.e in Q, we have $S'(u_n)f_n \to S'(u)f$ strongly in $L^1(Q)$ as $n \to +\infty$.

• Limit of $div(S'(u_n)F)$.
The fact that $S'(u_n)$ is bounded and converges to $S'(u)$ a.e in Q as n tends to $+\infty$ makes it possible to obtain that $div(S'(u_n)F) \to div(S'(u)F)$ strongly in $L^{p'}(0,T;W^{-1,p'}(\Omega,w^*))$ as $n \to +\infty$.

• Limit of $S''(u_n)FDu_n$.
This term is equal to $FDS'(u_n)$. Since $DS'(u_n)$ converges to $DS'(u_n)$ weakly in $\prod_{i=1}^{N} L^{p'}(Q,w_i^*)$ as n tends to $+\infty$, we obtain $S''(u_n)FDu_n = FDS'(u_n) \rightharpoonup FDS'(u)$ weakly in $L^1(Q)$ as $n \to +\infty$. The term $FDS'(u)$ identifies with $S''(u)FDu$.

As a consequence of the above convergence result, we are in a position to pass to the limit as n tends to $+\infty$ in equation (62) and to conclude that u satisfies (24). It remains to show that $B_S(x,u)$ satisfies the initial condition (25). To this end, firstly remark that, S being bounded, $B_S^n(x,u_n)$ is bounded in $L^\infty(Q)$. Secondly, (62) and the above considerations on the behavior of the terms of this equation show that $\frac{\partial B_S^n(x,u_n)}{\partial t}$ is bounded in $L^1(Q) + L^{p'}(0,T;W^{-1,p'}(\Omega,w^*))$. As a consequence, an Aubin's type lemma (see, e.g, [21]) implies that $B_S^n(x,u_n)$ lies in a compact set of $C^0([0,T],L^1(\Omega))$. It follows that on the one hand, $B_S^n(x,u_n)(t=0) = B_S^n(x,u_0^n)$ converges to $B_S(x,u)(t=0)$ strongly in $L^1(\Omega)$. On the other hand, the smoothness of S implies that $B_S(x,u)(t=0) = B_S(x,u_0)$ in Ω. As a conclusion of step 1 to step 5, the proof of Theorem 5.1 is complete.

6 Example

Let us consider the following special case: $b(x,s) = Z(x)C(s)$ where $Z \in W^{1,\ p}(\Omega, w)$, $Z(x) \geq \alpha > 0$ and $C \in C^1(\mathbb{R})$ such that $\forall\ \ k > 0\ :\ \ 0 < \lambda_k \equiv \inf_{|s|\leq k} C'(s)$ and $C(0) = 0$.

$$0 < \lambda_k \leq \frac{\partial b(x,s)}{\partial s} \leq A_k(x) \quad \text{and} \quad \left|\nabla_x \left(\frac{\partial b(x,s)}{\partial s}\right)\right| \leq B_k(x) \quad \forall\ |s| \leq k, \tag{66}$$

$$H(x,t,s,\xi) = \frac{-2s}{1+s^4} \sum_{i=1}^{N} w_i(x) \left|\xi_i\right|^p \quad \text{and} \quad a_i(x,t,s,d) = w_i(x) \left|d_i\right|^{p-2} d_i, \quad i = 1, ..., N, \tag{67}$$

with $w_i(x)$ a weight function strictly positive. Then, we can consider the Hardy inequality in the form

$$\left(\int_\Omega |u(x)|^p \, \sigma(x) dx\right)^{\frac{1}{p}} \leq c \left(\int_\Omega |Du(x)|^p \, w(x) dx\right)^{\frac{1}{p}}.$$

It is easy to show that the $a_i(t,x,s,d)$ are Caratheodory functions satisfying the growth condition (18), the coercivity (20) and the monotonicity condition.

While the Carathéodory function $H(x,t,s,\xi)$ satisfies the condition (21), indeed $|H(x,t,s,\xi)| \leq \frac{2|s|}{1+s^4} \sum_{i=1}^{N} w_i(x) \left|\xi_i\right|^p = g(s) \sum_{i=1}^{N} w_i(x) \left|\xi_i\right|^p$ where $g(s) = \frac{2|s|}{1+s^4}$ is a function bounded positive continuous which belongs to $L^1(\mathbb{R})$. Note that $H(x,t,s,\xi)$ does not satisfy the sign condition (3) and the coercivity condition. In particular, let us use special weight function, w, expressed in terms of the distance to the bounded $\partial\Omega$. Denote $d(x) = dist(x, \partial\Omega)$ and set $w(x) = d^\lambda(x)$, $\sigma(x) = d^\mu(x)$. Finally, the hypotheses of Theorem 5.1 are satisfied. Therefore, the following problem:

$$\begin{cases} b(x,u) \in L^\infty([0,T]; L^1(\Omega)) \quad \text{and} \quad T_k(u) \in L^p(0,T; W_0^{1,\ p}(\Omega, w)), \\ \lim_{m \to +\infty} \int_{\{m \leq |u| \leq m+1\}} a(x,t,u,Du) Du dx dt = 0, \\ \frac{\partial B_S(x,u)}{\partial t} - div\left[S'(u) a(x,t,u,Du)\right] + S''(u) \sum_{i=1}^{N} w_i \left|\frac{\partial u}{\partial x_i}\right|^p, \\ -\frac{2u}{1+u^4} \sum_{i=1}^{N} w_i \left|\frac{\partial u}{\partial x_i}\right|^p S'(u) = f S'(u) - div(S'(u) F) + F S''(u) Du, \\ B_S(x,u)(t=0) = B_S(x,u_0) \quad in \quad \Omega, \\ \forall \quad S \in W^{2,\infty}(\mathbb{R}) \ \ with \ \ S' \ \ has \ \ a \ \ compact \ \ support \ \ in \ \ \mathbb{R}, \\ and \ B_S(x,r) = \int_0^r \frac{\partial b(x,\sigma)}{\partial \sigma} S'(\sigma) d\sigma, \end{cases} \tag{68}$$

has at least one renormalised solution.

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Asymptotic Stability Conditions for Some Classes of Mechanical Systems with Switched Nonlinear Force Fields

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Abstract: Certain classes of switched mechanical systems with nonlinear potential and dissipative forces are studied. By the use of the differential inequalities method and multiple Lyapunov functions, conditions on switching law guaranteeing the asymptotic stability of the trivial equilibrium position of the considered systems are obtained. An example and the results of a computer simulation are presented to demonstrate the effectiveness of the proposed approaches.

Keywords: *mechanical systems; switched force fields; asymptotic stability; multiple Lyapunov functions; differential inequalities; dwell-time.*

Mathematics Subject Classification (2010): 34A34, 34A38, 93D20, 93D30.

1 Introduction

Stability of switched systems has attracted an increasing attention during last decades, mainly due to the numerous applications of these systems in engineering, technological processes, mechanics, population dynamics, chemistry and economics, see, e.g., [1, 7, 9, 10, 12, 16, 17, 20] and the references cited therein. A switched system is a particular kind of hybrid dynamical system that consists of a family of subsystems and a switching law determining at each time instant which subsystem is active.

There are two principal approaches to the stability analysis of switched systems. The first one is based on the constructing of a common Lyapunov function for the family of

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subsystems corresponding to a switched system [6, 12, 13, 19]. The existence of a such function guarantees the stability of the considered system for any admissible switching law. In the situations where we cannot prove the existence of a common Lyapunov function, the stability of a switched system can be provided by means of additional restrictions on the switching law (dwell-time approach) [8, 9, 13, 19, 21]. It is known that, under the suitable assumptions on the system investigated, the stability is ensured if the intervals between consecutive switching times are sufficiently large [13, 19]. However, it should be noted that these approaches are well-developed mostly for linear switched systems.

The problem of stability analysis of hybrid systems is especially difficult for mechanical systems with switched force fields. In numerous applications, mechanical systems are described by nonlinear differential equations of the second order. This results in the appearance of certain special properties of motions and essentially complicates the investigation of systems dynamics [2, 3, 9, 16]. In particular, well-known approaches developed for switched systems of general form might be inefficient or even inapplicable for mechanical systems, see [3].

In the present paper, certain classes of switched mechanical systems with nonlinear potential and dissipative forces are studied. By the use of the differential inequalities method and multiple Lyapunov functions, conditions on switching law guaranteeing the asymptotic stability of the trivial equilibrium position of the considered systems are obtained.

2 Statement of the Problem

Let the family of systems

$$\ddot{\mathbf{x}} + \mathbf{D}_s(\mathbf{x})\dot{\mathbf{x}} + \frac{\partial \Pi_s(\mathbf{x})}{\partial \mathbf{x}} = \mathbf{0}, \quad s = 1, \ldots, N, \tag{1}$$

be given. Here $\mathbf{x} \in \mathbb{R}^n$; $\Pi_s(\mathbf{x})$ are continuously differentiable for $\mathbf{x} \in \mathbb{R}^n$ homogeneous of the order $\mu + 1$ functions, $\mu \geq 1$; entries of the matrices $\mathbf{D}_s(\mathbf{x})$ are continuous for $\mathbf{x} \in \mathbb{R}^n$ homogeneous of the order ν functions, $\nu > 0$. Systems from the family (1) are vector type Lienard equations, see [18]. They can be used for the modelling of mechanical systems with potential and essentially nonlinear velocity forces.

Switched system generated by the family (1) and a switching law σ is

$$\ddot{\mathbf{x}} + \mathbf{D}_\sigma(\mathbf{x})\dot{\mathbf{x}} + \frac{\partial \Pi_\sigma(\mathbf{x})}{\partial \mathbf{x}} = \mathbf{0}. \tag{2}$$

Here $\sigma = \sigma(t) : \ [0, +\infty) \to \{1, \ldots, N\}$ is a piecewise constant function. Without loss of generality, consider the only case where the interval $(0, +\infty)$ contains the infinite number of switching instants. Let θ_i, $i = 1, 2, \ldots$, be the switching times, $0 < \theta_1 < \theta_2 < \ldots$, and $\theta_0 = 0$. Assume that the function $\sigma(t)$ is right-continuous, and the sequence $\{\theta_i\}_{i=0}^{\infty}$ is a minimal one ($\sigma(\theta_i) \neq \sigma(\theta_{i+1})$, $i = 0, 1, \ldots$). Hereinafter, we consider non Zeno sequences [12, 13], i.e., sequences that switch at most a finite number of times in any finite time interval. This kind of switching law is called admissible one.

Systems (1) and system (2) admit the trivial equilibrium position $\mathbf{x} = \dot{\mathbf{x}} = \mathbf{0}$. Assume that, for every system from the family (1), the equilibrium position is asymptotically stable. Let us determine conditions under which the equilibrium position $\mathbf{x} = \dot{\mathbf{x}} = \mathbf{0}$ of switched system (2) is also asymptotically stable.

The problem of the construction of a common Lyapunov function for the family of systems of the form (1) was studied in [5, 15]. As it was mentioned in the Introduction, the existence of a such function guarantees the asymptotic stability of (2) for any admissible switching law.

In this paper, it is assumed that we failed to prove the existence of a common Lyapunov function for (1). We will look for conditions on switching law guaranteeing asymptotic stability of the equilibrium position.

It should be noted that such conditions were obtained in [4] for system (2) with constant matrices $\mathbf{D}_1, \ldots, \mathbf{D}_N$. The goal of the present paper is extension of the results of [4] to the case of essentially nonlinear velocity forces. We will assume that the forces $\mathbf{F}_s(\mathbf{x}, \dot{\mathbf{x}}) = -\mathbf{D}_s(\mathbf{x})\dot{\mathbf{x}}$, $s = 1, \ldots, N$, are dissipative ones and consider two types of such forces. It is worth mentioning that asymptotic stability conditions will depend not only on the type of the dissipative forces but also on the information available on the switching law.

3 The First Type of Dissipative Forces

3.1 Stability analysis via multiple Lyapunov functions

First, consider the case when the switching instants θ_i, $i = 1, 2, \ldots$, are given, while the order of switching between the systems from (1) might be unknown.

Let us impose additional restrictions on the functions $\Pi_1(\mathbf{x}), \ldots, \Pi_N(\mathbf{x})$ and the matrices $\mathbf{D}_1(\mathbf{x}), \ldots, \mathbf{D}_N(\mathbf{x})$.

Assumption 3.1 Functions $\Pi_1(\mathbf{x}), \ldots, \Pi_N(\mathbf{x})$ are positive definite.

Assumption 3.2 For any fixed $\mathbf{x} \neq \mathbf{0}$, the matrices $\mathbf{D}_s(\mathbf{x}) + \mathbf{D}_s^T(\mathbf{x})$, $s = 1, \ldots, N$, are positive definite.

Remark 3.1 Taking into account homogeneity of $\mathbf{D}_1(\mathbf{x}), \ldots, \mathbf{D}_N(\mathbf{x})$, we obtain, see [22], that Assumption 3.2 implies that the estimates

$$\mathbf{z}^T \mathbf{D}_s(\mathbf{x}) \mathbf{z} \geq c_s \|\mathbf{x}\|^{\nu} \|\mathbf{z}\|^2, \qquad s = 1, \ldots, N,$$

hold for all $\mathbf{x}, \mathbf{z} \in \mathbb{R}^n$. Here $c_1, \ldots, c_N$ are positive constants, and $\|\cdot\|$ denotes the Euclidean norm of a vector.

Remark 3.2 It is known, see [18, 22], that if Assumptions 3.1 and 3.2 are fulfilled, then, for any system from the family (1), the equilibrium position $\mathbf{x} = \dot{\mathbf{x}} = \mathbf{0}$ is asymptotically stable.

For every s in $\{1, \ldots, N\}$, choose a Lyapunov function for the s-th system from (1) in the form

$$V_s(\mathbf{x}, \dot{\mathbf{x}}) = \Pi_s(\mathbf{x}) + \frac{1}{2}\dot{\mathbf{x}}^T\dot{\mathbf{x}} - \gamma_{1s}\|\dot{\mathbf{x}}\|^{\beta-1}\mathbf{x}^T\dot{\mathbf{x}} + \gamma_{2s}\|\mathbf{x}\|^{k-1}\mathbf{x}^T\dot{\mathbf{x}}, \tag{3}$$

where $\gamma_{1s} > 0$, $\gamma_{2s} > 0$, $\beta \geq 1$, $k \geq 1$. Differentiating function (3) with respect to the s-th system, we obtain

$$\dot{V}_s\big|_{(s)} = -\gamma_{2s}(\mu+1)\|\mathbf{x}\|^{k-1}\Pi_s(\mathbf{x}) - \gamma_{1s}\|\dot{\mathbf{x}}\|^{\beta+1} - \dot{\mathbf{x}}^T\mathbf{D}_s(\mathbf{x})\dot{\mathbf{x}}$$

$$-\gamma_{1s}\mathbf{x}^T\frac{\partial\left(\|\dot{\mathbf{x}}\|^{\beta-1}\dot{\mathbf{x}}\right)}{\partial\dot{\mathbf{x}}}\left(-\frac{\partial\Pi_s(\mathbf{x})}{\partial\mathbf{x}}-\mathbf{D}_s(\mathbf{x})\dot{\mathbf{x}}\right)$$

$$+\gamma_{2s}\dot{\mathbf{x}}^T\frac{\partial\left(\|\mathbf{x}\|^{k-1}\mathbf{x}\right)}{\partial\mathbf{x}}\dot{\mathbf{x}}-\gamma_{2s}\|\mathbf{x}\|^{k-1}\mathbf{x}^T\mathbf{D}_s(\mathbf{x})\dot{\mathbf{x}}.$$

Hence, the estimates

$$a_{1s}\left(\|\dot{\mathbf{x}}\|^2+\|\mathbf{x}\|^{\mu+1}\right)-\left(\gamma_{1s}\|\mathbf{x}\|\|\dot{\mathbf{x}}\|^{\beta}+\gamma_{2s}\|\mathbf{x}\|^{k}\|\dot{\mathbf{x}}\|\right)\leq V_s(\mathbf{x},\dot{\mathbf{x}})$$

$$\leq a_{2s}\left(\|\dot{\mathbf{x}}\|^2+\|\mathbf{x}\|^{\mu+1}\right)+\left(\gamma_{1s}\|\mathbf{x}\|\|\dot{\mathbf{x}}\|^{\beta}+\gamma_{2s}\|\mathbf{x}\|^{k}\|\dot{\mathbf{x}}\|\right),$$

$$\dot{V}_s\big|_{(s)}\leq -a_{3s}\left(\gamma_{2s}\|\mathbf{x}\|^{k+\mu}+\gamma_{1s}\|\dot{\mathbf{x}}\|^{\beta+1}+\|\mathbf{x}\|^{\nu}\|\dot{\mathbf{x}}\|^2\right)$$

$$+a_{4s}\left(\gamma_{1s}\|\mathbf{x}\|^{\mu+1}\|\dot{\mathbf{x}}\|^{\beta-1}+\gamma_{1s}\|\mathbf{x}\|^{\nu+1}\|\dot{\mathbf{x}}\|^{\beta}+\gamma_{2s}\|\mathbf{x}\|^{k-1}\|\dot{\mathbf{x}}\|^2+\gamma_{2s}\|\mathbf{x}\|^{k+\nu}\|\dot{\mathbf{x}}\|\right)$$

hold for $\mathbf{x},\dot{\mathbf{x}}\in\mathbb{R}^n$. Here $a_{1s},\ldots,a_{4s}$ are positive constants.

By the use of generalized homogeneous functions properties [22], it is easy to verify that, if

$$k=\max\{\mu-\nu;\ \nu+1\},\qquad \beta=1+\max\left\{\frac{2\nu}{\mu+1};\ \frac{2(k-1)}{k+\mu-\nu}\right\},\tag{4}$$

then there exist positive numbers $\gamma_{11},\ldots,\gamma_{1N}$, $\gamma_{21},\ldots,\gamma_{2N}$, b_1, b_2, α and H such that the inequalities

$$b_1r(\mathbf{x},\dot{\mathbf{x}})\leq V_s(\mathbf{x},\dot{\mathbf{x}})\leq b_2r(\mathbf{x},\dot{\mathbf{x}}),\qquad s=1,\ldots,N,\tag{5}$$

$$\dot{V}_s\big|_{(s)}\leq-\alpha V_s^{1+\xi}(\mathbf{x},\dot{\mathbf{x}}),\qquad s=1,\ldots,N,\tag{6}$$

are valid for $r(\mathbf{x},\dot{\mathbf{x}})<H$. Here $r(\mathbf{x},\dot{\mathbf{x}})=\|\dot{\mathbf{x}}\|^2+\|\mathbf{x}\|^{\mu+1}$, and $\xi=(k-1)/(\mu+1)$.

Find $\omega\geq1$, such that

$$V_s(\mathbf{x},\dot{\mathbf{x}})\leq\omega V_l(\mathbf{x},\dot{\mathbf{x}}),\qquad s,l=1,\ldots,N,\tag{7}$$

for $r(\mathbf{x},\dot{\mathbf{x}})<H$.

Denote $h=\omega^{-\xi}$; $\tau_i=\theta_i-\theta_{i-1}$, $i=1,2,\ldots$; $\psi(m,1)=0$, and $\psi(m,p)=\sum_{i=1}^{p-1}\tau_{m+i}h^{p-i}$ for $p=2,3,\ldots$, $m=1,2,\ldots$.

Theorem 3.1 *Let Assumptions 3.1 and 3.2 be fulfilled, and for family* (1) *the Lyapunov functions* $V_1(\mathbf{x},\dot{\mathbf{x}}),\ldots,V_N(\mathbf{x},\dot{\mathbf{x}})$ *be constructed satisfying the estimates* (5), (6) *and* (7). *If*

$$\psi(m,p)\to+\infty\quad as\quad p\to\infty\tag{8}$$

for any positive integer m, *then the equilibrium position* $\mathbf{x}=\dot{\mathbf{x}}=\mathbf{0}$ *of system* (2) *is asymptotically stable. In the case when the tendency* (8) *is uniform with respect to* $m=1,2,\ldots$, *the equilibrium position is uniformly asymptotically stable.*

Proof. By the use of the partial Lyapunov functions $V_1(\mathbf{x},\dot{\mathbf{x}}),\ldots,V_N(\mathbf{x},\dot{\mathbf{x}})$, construct the multiple Lyapunov function $V_{\sigma(t)}(\mathbf{x},\dot{\mathbf{x}})$ corresponding to the switching law $\sigma(t)$.

Choose $\varepsilon\in(0,H)$ and $t_0\geq0$. Consider a solution $\mathbf{x}(t)$ of (2) with initial conditions satisfying the inequalities $0<r(\mathbf{x}(t_0),\dot{\mathbf{x}}(t_0))<\varepsilon$. Find the positive integer m such that $t_0\in[\theta_{m-1},\theta_m)$.

Assume that $r(\mathbf{x}(t),\dot{\mathbf{x}}(t)) < \varepsilon$ for $t \in [t_0,\tilde{t}]$. If $t_0 < \tilde{t} \le \theta_m$ then, integrating the corresponding differential inequality from (6), we obtain that the estimate

$$V^{-\xi}_{\sigma(\theta_{m-1})}(\mathbf{x}(\tilde{t}),\dot{\mathbf{x}}(\tilde{t})) \ge \alpha\xi(\tilde{t}-t_0) + V^{-\xi}_{\sigma(\theta_{m-1})}(\mathbf{x}(t_0),\dot{\mathbf{x}}(t_0)) \tag{9}$$

is valid.

In the case when $\tilde{t} \ge \theta_m$, there exists a positive integer p such that $\theta_{m+p-1} \le \tilde{t} < \theta_{m+p}$. It should be noted that $p \to \infty$ as $\tilde{t} \to +\infty$. Integrating successively the corresponding differential inequalities from family (6) on the intervals $[\theta_{m+p-1},\tilde{t}]$, $[\theta_{m+p-2},\theta_{m+p-1}]$, ..., $[t_0,\theta_m]$ and taking into account inequalities (7), we obtain

$$\begin{aligned} V^{-\xi}_{\sigma(\theta_{m+p-1})}(\mathbf{x}(\tilde{t}),\dot{\mathbf{x}}(\tilde{t})) &\ge \alpha\xi\left(\tilde{t}-\theta_{m+p-1}\right) + V^{-\xi}_{\sigma(\theta_{m+p-1})}(\mathbf{x}(\theta_{m+p-1}),\dot{\mathbf{x}}(\theta_{m+p-1})) \\ &\ge hV^{-\xi}_{\sigma(\theta_{m+p-2})}(\mathbf{x}(\theta_{m+p-1}),\dot{\mathbf{x}}(\theta_{m+p-1})) + \alpha\xi\left(\tilde{t}-\theta_{m+p-1}\right) \ge \dots \\ &\ge h^pV^{-\xi}_{\sigma(\theta_{m-1})}(\mathbf{x}(t_0),\dot{\mathbf{x}}(t_0)) + \alpha\xi\left(\left(\tilde{t}-\theta_{m+p-1}\right) + \psi(m,p) + h^p\left(\theta_m - t_0\right)\right). \end{aligned} \tag{10}$$

From (5), (9) and (10) it follows that

$$r(\mathbf{x}(\tilde{t}),\dot{\mathbf{x}}(\tilde{t})) \le b_1^{-1}\Big(b_2^{-\xi}r^{-\xi}(\mathbf{x}(t_0),\dot{\mathbf{x}}(t_0)) + \alpha\xi\left(\tilde{t}-t_0\right)\Big)^{-\frac{1}{\xi}}$$

for $\tilde{t} \in [t_0,\theta_m)$, and

$$\begin{aligned} r(\mathbf{x}(\tilde{t}),\dot{\mathbf{x}}(\tilde{t})) \le b_1^{-1}\Big(&h^pb_2^{-\xi}r^{-\xi}(\mathbf{x}(t_0),\dot{\mathbf{x}}(t_0)) \\ &+\alpha\xi\big(\left(\tilde{t}-\theta_{m+p-1}\right) + \psi(m,p) + h^p\left(\theta_m - t_0\right)\big)\Big)^{-\frac{1}{\xi}} \end{aligned}$$

for $\tilde{t} \in [\theta_{m+p-1},\theta_{m+p})$, $p \ge 1$.

With the usage of these estimates the subsequent proof is similar to that of Theorem 1 in [4]. □

Corollary 3.1 *Let Assumptions 3.1 and 3.2 be fulfilled. If $\tau_i \to +\infty$ as $i \to \infty$, then the equilibrium position $\mathbf{x} = \dot{\mathbf{x}} = \mathbf{0}$ of system (2) is uniformly asymptotically stable.*

Remark 3.3 It is a fairly well-known fact, see [13, 19], that for any family consisting of a finite number of linear time invariant asymptotically stable systems there exists a number $L > 0$ (dwell time), such that the corresponding switched system is also asymptotically stable providing that the intervals between consecutive switching times are not less than L. Theorem 3.1 does not permit to obtain a similar result for the family of nonlinear systems (1). If $\tau_i = L = \text{const} > 0$, $i = 1,2,\dots$, then condition (8) is not fulfilled for any choice of L. However, for nonlinear switched system (2), a positive lower bound for the values of $\tau_1,\tau_2,\dots$ can be found guaranteeing the practical stability [11] of the equilibrium position.

Corollary 3.2 *Let Assumptions 3.1 and 3.2 be fulfilled. Then there exists a positive number Δ, such that for any $\varepsilon > 0$ one can choose $L_1 > 0$ and $L_2 > 0$ satisfying the following condition: if $\tau_i \ge L_1$, $i = 1,2,\dots$, and for a solution $\mathbf{x}(t)$ of (2) the inequalities $t_0 \ge 0$, $r(\mathbf{x}(t_0),\dot{\mathbf{x}}(t_0)) < \Delta$ are valid, then $r(\mathbf{x}(t),\dot{\mathbf{x}}(t)) < \varepsilon$ for all $t \ge t_0 + L_2$.*

3.2 Asymptotic stability conditions in the case of complete information on the switching law

Assume now that we know not only the switching instants θ_i, $i = 1, 2, \ldots$, but also the order of switching between the systems. Then another approach for the stability analysis can be used [4, 14]. Choose a system from family (1) and determine relationship between this system activity intervals and those of the remained systems under which it is possible to guarantee the asymptotic stability of the equilibrium position $\mathbf{x} = \dot{\mathbf{x}} = \mathbf{0}$ of switched system (2).

Let (for definiteness) the first system from (1) be chosen. In the present subsection, instead of Assumption 3.1, we will use a weaker assumption.

Assumption 3.3 Function $\Pi_1(\mathbf{x})$ is positive definite.

Consider the Lyapunov function

$$V_1(\mathbf{x}, \dot{\mathbf{x}}) = \Pi_1(\mathbf{x}) + \frac{1}{2}\dot{\mathbf{x}}^T\dot{\mathbf{x}} - \gamma_{11}\|\dot{\mathbf{x}}\|^{\beta-1}\mathbf{x}^T\dot{\mathbf{x}} + \gamma_{21}\|\mathbf{x}\|^{k-1}\mathbf{x}^T\dot{\mathbf{x}},$$

where $\gamma_{11} > 0$, $\gamma_{21} > 0$, and the values of the parameters β and k are defined by the formulae (4).

Denote by $\dot{V}_1\big|_{(s)}$ the derivative of $V_1(\mathbf{x}, \dot{\mathbf{x}})$ with respect to the s-th system from (1), $s = 1, \ldots, N$. We obtain

$$\dot{V}_1\big|_{(s)} = -\gamma_{21}(\mu + 1)\|\mathbf{x}\|^{k-1}\Pi_s(\mathbf{x}) - \gamma_{11}\|\dot{\mathbf{x}}\|^{\beta+1} - \dot{\mathbf{x}}^T\mathbf{D}_s(\mathbf{x})\dot{\mathbf{x}}$$

$$-\gamma_{11}\mathbf{x}^T\frac{\partial\left(\|\dot{\mathbf{x}}\|^{\beta-1}\dot{\mathbf{x}}\right)}{\partial\dot{\mathbf{x}}}\left(-\frac{\partial\Pi_s(\mathbf{x})}{\partial\mathbf{x}} - \mathbf{D}_s(\mathbf{x})\dot{\mathbf{x}}\right)$$

$$+\gamma_{21}\dot{\mathbf{x}}^T\frac{\partial\left(\|\mathbf{x}\|^{k-1}\mathbf{x}\right)}{\partial\mathbf{x}}\dot{\mathbf{x}} - \gamma_{21}\|\mathbf{x}\|^{k-1}\mathbf{x}^T\mathbf{D}_s(\mathbf{x})\dot{\mathbf{x}} + \left(\frac{\partial\Pi_1(\mathbf{x})}{\partial\mathbf{x}}\right)^T\dot{\mathbf{x}} - \left(\frac{\partial\Pi_s(\mathbf{x})}{\partial\mathbf{x}}\right)^T\dot{\mathbf{x}}.$$

Let again $r(\mathbf{x}, \dot{\mathbf{x}}) = \|\dot{\mathbf{x}}\|^2 + \|\mathbf{x}\|^{\mu+1}$, $\xi = (k-1)/(\mu+1)$. It is easy to verify that if $\mu \geq 2\nu+1$, Assumptions 3.2 and 3.3 are fulfilled, and values of γ_{11} and γ_{21} are sufficiently small, then there exists a number $H > 0$ such that the estimates

$$b_1 r(\mathbf{x}, \dot{\mathbf{x}}) \leq V_1(\mathbf{x}, \dot{\mathbf{x}}) \leq b_2 r(\mathbf{x}, \dot{\mathbf{x}}), \qquad \dot{V}_1\big|_{(s)} \leq \alpha_s V_1^{1+\xi}(\mathbf{x}, \dot{\mathbf{x}}), \quad s = 1, \ldots, N, \tag{11}$$

hold for $r(\mathbf{x}, \dot{\mathbf{x}}) < H$. Here b_1, b_2, $\alpha_1, \ldots, \alpha_N$ are constants with $b_1 > 0$, $b_2 > 0$, $\alpha_1 < 0$.

For given switching law $\sigma(t)$, define the auxiliary piecewise constant function $\eta(t)$ by the formula $\eta(t) = -\alpha_{\sigma(t)}$ for $t \geq 0$.

Theorem 3.2 *Let $\mu \geq 2\nu + 1$, Assumptions 3.2 and 3.3 be fulfilled, and for family* (1) *the Lyapunov function $V_1(\mathbf{x}, \dot{\mathbf{x}})$ be constructed satisfying the estimates* (11)*. If*

$$\int_0^t \eta(\tau)\, d\tau \to +\infty \quad as \quad t \to +\infty, \tag{12}$$

then the equilibrium position $\mathbf{x} = \dot{\mathbf{x}} = \mathbf{0}$ of system (2) *is asymptotically stable. In the case when*

$$\int_{t_0}^{t_0+t} \eta(\tau)\, d\tau \to +\infty \quad as \quad t \to +\infty \tag{13}$$

uniformly with respect to $t_0 \geq 0$, the equilibrium position $\mathbf{x} = \dot{\mathbf{x}} = \mathbf{0}$ of system (2) *is uniformly asymptotically stable.*

Proof. For given switching law $\sigma(t)$, construct the function $\eta(t)$. Let the numbers $\varepsilon > 0$ and $t_0 \geq 0$ be chosen. Without loss of generality, assume that $\varepsilon < H$.

If (12) holds, then there exists a constant ρ_0, such that $\int_{t_0}^{t} \eta(\tau)\, d\tau \geq \rho_0$ for all $t \geq t_0$. Choose $\delta > 0$ satisfying the condition

$$(b_2\delta)^{-\xi} + \xi\rho_0 > (b_1\varepsilon)^{-\xi}.$$

Consider a solution $\mathbf{x}(t)$ of system (2), such that $0 < r(\mathbf{x}(t_0), \dot{\mathbf{x}}(t_0)) < \delta$. If $r(\mathbf{x}(t), \dot{\mathbf{x}}(t)) < \varepsilon$ for $t \in [t_0, \tilde{t})$, then the differential inequality

$$\dot{V}_1(\mathbf{x}(t), \dot{\mathbf{x}}(t)) \leq -\eta(t) V_1^{1+\xi}(\mathbf{x}(t), \dot{\mathbf{x}}(t)) \tag{14}$$

is valid for $t \in [t_0, \tilde{t}]$.

With the aid of estimate (14), it is easy to show that

$$\left(b_1 r(\mathbf{x}(\tilde{t}), \dot{\mathbf{x}}(\tilde{t}))\right)^{-\xi} \geq V_1^{-\xi}(\mathbf{x}(\tilde{t}), \dot{\mathbf{x}}(\tilde{t})) \geq V_1^{-\xi}(\mathbf{x}(t_0), \dot{\mathbf{x}}(t_0)) + \xi \int_{t_0}^{\tilde{t}} \eta(\tau)\, d\tau$$

$$\geq (b_2 r(\mathbf{x}(t_0), \dot{\mathbf{x}}(t_0)))^{-\xi} + \xi \int_{t_0}^{\tilde{t}} \eta(\tau)\, d\tau \geq (b_2\delta)^{-\xi} + \xi\rho_0 > (b_1\varepsilon)^{-\xi}.$$

Hence, $r(\mathbf{x}(t), \dot{\mathbf{x}}(t)) < \varepsilon$ for all $t \geq t_0$, and $r(\mathbf{x}(t), \dot{\mathbf{x}}(t)) \to 0$ as $t \to +\infty$.

If the tendency (13) is uniform with respect to $t_0 \geq 0$, then the number δ can be chosen independent of t_0, and $r(\mathbf{x}(t), \dot{\mathbf{x}}(t)) \to 0$ as $t - t_0 \to +\infty$ uniformly with respect to $t_0 \geq 0$. □

Remark 3.4 In the proof of Theorem 3.2, we did not use the positive definiteness property of functions $\Pi_2(\mathbf{x}), \ldots, \Pi_N(\mathbf{x})$. Hence, this theorem remains valid also in the case when the equilibrium position $\mathbf{x} = \dot{\mathbf{x}} = \mathbf{0}$ is not asymptotically stable either for a part of systems numbered $2, \ldots, N$, or for all of these systems.

4 The Second Type of Dissipative Forces

Next, we will assume that in system (2) potential forces are switched, whereas dissipative forces are nonswitched, i.e., $\mathbf{D}_s(\mathbf{x}) = \mathbf{D}(\mathbf{x})$, $s = 1, \ldots, N$, where entries of the matrix $\mathbf{D}(\mathbf{x})$ are continuous for $\mathbf{x} \in \mathbb{R}^n$ homogeneous of the order ν functions, $\nu > 0$.

Moreover, we will impose an additional restriction on the structure of the matrix $\mathbf{D}(\mathbf{x})$.

Assumption 4.1 The matrix $\mathbf{D}(\mathbf{x})$ is represented in the form $\mathbf{D}(\mathbf{x}) = \partial\mathbf{G}(\mathbf{x})/\partial\mathbf{x}$, where components of the vector $\mathbf{G}(\mathbf{x})$ are continuously differentiable for $\mathbf{x} \in \mathbb{R}^n$ homogeneous of the order $\nu + 1$ functions, $\nu > 0$.

Then family (1) can be rewritten as follows

$$\dot{\mathbf{x}} = \mathbf{y} - \mathbf{G}(\mathbf{x}), \quad \dot{\mathbf{y}} = -\frac{\partial \Pi_s(\mathbf{x})}{\partial \mathbf{x}}, \quad s = 1, \ldots, N. \tag{15}$$

4.1 Stability analysis via multiple Lyapunov functions

As in the previous section, consider first the case when the switching instants θ_i, $i = 1, 2, \ldots$, are given, while the order of switching between the systems from (15) might be unknown.

Assumption 4.2 The functions $(\partial \Pi_s(\mathbf{x})/\partial \mathbf{x})^T \mathbf{G}(\mathbf{x})$, $s = 1, \ldots, N$, are positive definite.

Remark 4.1 The class of matrices $\mathbf{D}(\mathbf{x})$ defined by Assumption 3.2 differs from that defined by Assumptions 4.1 and 4.2.

Example 4.1 Let $\Pi_s(\mathbf{x}) = a_1^{(s)} x_1^{\mu+1} + \ldots + a_n^{(s)} x_n^{\mu+1}$, $s = 1, \ldots, N$. Here $\mathbf{x} = (x_1, \ldots, x_n)^T$, $\mu \geq 1$ is a rational with the odd numerator and denominator, and $a_i^{(s)}$ are positive coefficients, $i = 1, \ldots, n$; $s = 1, \ldots, N$. The functions $\Pi_1(\mathbf{x}), \ldots, \Pi_N(\mathbf{x})$ satisfy Assumption 3.1.

On the one hand, if $\mathbf{D}(\mathbf{x}) = \|\mathbf{x}\|^\nu \mathbf{A}$, where $\nu > 0$, and $\mathbf{A}$ is a constant matrix such that the matrix $\mathbf{A} + \mathbf{A}^T$ is positive definite, then Assumption 3.2 is fulfilled, whereas Assumption 4.1 is not fulfilled.

On the other hand, choose the matrix $\mathbf{D}(\mathbf{x})$ in the form $\mathbf{D}(\mathbf{x}) = \mathrm{diag}\,\{b_1 x_1^\nu, \ldots, b_n x_n^\nu\}$, where ν is a positive rational with the even numerator and the odd denominator, and b_i are positive constants, $i = 1, \ldots, n$. In this case Assumptions 4.1 and 4.2 are fulfilled (here $\mathbf{G}(\mathbf{x}) = \left(b_1 x_1^{\nu+1}, \ldots, b_n x_n^{\nu+1}\right)^T / (\nu+1)$), whereas Assumption 3.2 is not fulfilled.

Remark 4.2 It is known, see [18, 22], that under Assumptions 3.1 and 4.2 any system from the family (15) admits the asymptotically stable zero solution.

For every $s \in \{1, \ldots, N\}$, construct a Lyapunov function for the s-th system from (15) by the formula

$$\hat{V}_s(\mathbf{x}, \mathbf{y}) = \Pi_s(\mathbf{x}) + \frac{1}{2}\mathbf{y}^T\mathbf{y} - \hat{\gamma}_s \|\mathbf{y}\|^{\lambda-1}\mathbf{x}^T\mathbf{y},$$

where $\hat{\gamma}_s > 0$, $\lambda \geq 1$. We obtain

$$\dot{\hat{V}}_s\Big|_{(s)} = -\left(\frac{\partial \Pi_s(\mathbf{x})}{\partial \mathbf{x}}\right)^T \mathbf{G}(\mathbf{x}) - \hat{\gamma}_s \|\mathbf{y}\|^{\lambda+1}$$

$$+\hat{\gamma}_s \|\mathbf{y}\|^{\lambda-1}\mathbf{y}^T\mathbf{G}(\mathbf{x}) + \hat{\gamma}_s \mathbf{x}^T \frac{\partial(\|\mathbf{y}\|^{\lambda-1}\mathbf{y})}{\partial \mathbf{y}} \frac{\partial \Pi_s(\mathbf{x})}{\partial \mathbf{x}}.$$

Hence, under Assumptions 3.1 and 4.2 the estimates

$$\hat{a}_{1s}\left(\|\mathbf{x}\|^{\mu+1} + \|\mathbf{y}\|^2\right) - \hat{\gamma}_s \|\mathbf{x}\|\|\mathbf{y}\|^\lambda \leq \hat{V}_s(\mathbf{x}, \mathbf{y}) \leq \hat{a}_{2s}\left(\|\mathbf{x}\|^{\mu+1} + \|\mathbf{y}\|^2\right) + \hat{\gamma}_s \|\mathbf{x}\|\|\mathbf{y}\|^\lambda,$$

$$\dot{\hat{V}}_s\Big|_{(s)} \leq -\left(\hat{a}_{3s}\|\mathbf{x}\|^{\mu+\nu+1} + \hat{\gamma}_s \|\mathbf{y}\|^{\lambda+1}\right) + \hat{a}_{4s}\hat{\gamma}_s\left(\|\mathbf{x}\|^{\nu+1}\|\mathbf{y}\|^\lambda + \|\mathbf{x}\|^{\mu+1}\|\mathbf{y}\|^{\lambda-1}\right)$$

hold for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Here $\hat{a}_{1s}, \ldots, \hat{a}_{4s}$ are positive constants.

It is easy to verify, see [22], that if

$$\lambda = \max\left\{1 + \frac{2\nu}{\mu+1}; \frac{\mu}{\nu+1}\right\}, \tag{16}$$

then there exist positive numbers $\hat{\gamma}_1, \dots, \hat{\gamma}_N$, $\hat{b}_1$, $\hat{b}_2$, $\hat{\alpha}$ and $\hat{H}$ such that the inequalities

$$\hat{b}_1 r(\mathbf{x},\mathbf{y}) \le \hat{V}_s(\mathbf{x},\mathbf{y}) \le \hat{b}_2 r(\mathbf{x},\mathbf{y}), \qquad \dot{\hat{V}}_s\Big|_{(s)} \le -\hat{\alpha}\hat{V}_s^{1+\hat{\xi}}(\mathbf{x},\mathbf{y}), \qquad s = 1,\dots,N, \quad (17)$$

are valid for $r(\mathbf{x},\mathbf{y}) < \hat{H}$. Here $\hat{\xi} = (\lambda - 1)/2$, and $r(\mathbf{x},\mathbf{y}) = \|\mathbf{x}\|^{\mu+1} + \|\mathbf{y}\|^2$.

Find $\hat{\omega} \ge 1$, such that

$$\hat{V}_s(\mathbf{x},\mathbf{y}) \le \hat{\omega}\hat{V}_l(\mathbf{x},\mathbf{y}), \qquad s, l = 1,\dots,N, \quad (18)$$

for $r(\mathbf{x},\mathbf{y}) < \hat{H}$.

Denote $\hat{h} = \hat{\omega}^{-\hat{\xi}}$; $\hat{\psi}(m,1) = 0$, and $\hat{\psi}(m,p) = \sum_{i=1}^{p-1} \tau_{m+i}\hat{h}^{p-i}$ for $p = 2,3,\dots$, $m = 1,2,\dots$.

Theorem 4.1 *Let Assumptions 3.1, 4.1 and 4.2 be fulfilled, and for family* (15) *the Lyapunov functions* $\hat{V}_1(\mathbf{x},\mathbf{y}),\dots,\hat{V}_N(\mathbf{x},\mathbf{y})$ *be constructed satisfying the estimates* (17) *and* (18). *If*

$$\hat{\psi}(m,p) \to +\infty \quad as \quad p \to \infty \quad (19)$$

for any positive integer m, *then the equilibrium position* $\mathbf{x} = \dot{\mathbf{x}} = \mathbf{0}$ *of system* (2) *is asymptotically stable. In the case when the tendency* (19) *is uniform with respect to* $m = 1,2,\dots$, *the equilibrium position is uniformly asymptotically stable.*

The proof of the theorem is similar to that of Theorem 3.1.

Remark 4.3 For Theorem 4.1, corollaries similar to Corollaries 3.1 and 3.2 can be formulated.

4.2 Asymptotic stability conditions in the case of complete information on the switching law

Assume now that we know not only the switching instants θ_i, $i = 1,2,\dots$, but also the order of switching between the systems. Then for finding asymptotic stability conditions we can apply the approach considered in Subsection 3.2.

Choose the first system from the family (15). Instead of Assumption 4.2, we will use a weaker assumption.

Assumption 4.3 The function $(\partial\Pi_1(\mathbf{x})/\partial\mathbf{x})^T\,\mathbf{G}(\mathbf{x})$ is positive definite.

Let

$$\hat{V}_1(\mathbf{x},\mathbf{y}) = \Pi_1(\mathbf{x}) + \frac{1}{2}\mathbf{y}^T\mathbf{y} - \hat{\gamma}_1\|\mathbf{y}\|^{\lambda-1}\mathbf{x}^T\mathbf{y}.$$

Here $\hat{\gamma}_1 > 0$, and the value of the parameter λ is defined by the formula (16). Then

$$\dot{\hat{V}}_1\Big|_{(s)} = -\left(\frac{\partial\Pi_1(\mathbf{x})}{\partial\mathbf{x}}\right)^T\mathbf{G}(\mathbf{x}) - \hat{\gamma}_1\|\mathbf{y}\|^{\lambda+1} + \hat{\gamma}_1\|\mathbf{y}\|^{\lambda-1}\mathbf{y}^T\mathbf{G}(\mathbf{x})$$

$$+\hat{\gamma}_1\mathbf{x}^T\frac{\partial(\|\mathbf{y}\|^{\lambda-1}\mathbf{y})}{\partial\mathbf{y}}\frac{\partial\Pi_s(\mathbf{x})}{\partial\mathbf{x}} + \left(\frac{\partial\Pi_1(\mathbf{x})}{\partial\mathbf{x}}\right)^T\mathbf{y} - \left(\frac{\partial\Pi_s(\mathbf{x})}{\partial\mathbf{x}}\right)^T\mathbf{y}.$$

If $\mu \ge 2\nu + 1$, Assumptions 3.3 and 4.3 are fulfilled, and the value of $\hat{\gamma}_1$ is sufficiently small, then there exists a number $\hat{H} > 0$ such that the estimates

$$\hat{b}_1 r(\mathbf{x},\mathbf{y}) \le \hat{V}_1(\mathbf{x},\mathbf{y}) \le \hat{b}_2 r(\mathbf{x},\mathbf{y}), \qquad \dot{\hat{V}}_1\Big|_{(s)} \le \hat{\alpha}_s\hat{V}_1^{1+\hat{\xi}}(\mathbf{x},\mathbf{y}), \quad s = 1,\dots,N, \quad (20)$$

hold for $r(\mathbf{x},\mathbf{y}) < \hat{H}$. Here $\hat{b}_1$, $\hat{b}_2$, $\hat{\alpha}_1, \dots, \hat{\alpha}_N$ are constants with $\hat{b}_1 > 0$, $\hat{b}_2 > 0$, $\hat{\alpha}_1 < 0$, and $\hat{\xi} = (\lambda - 1)/2$.

For given switching law $\sigma(t)$, define the auxiliary piecewise constant function $\hat{\eta}(t)$ by the formula $\hat{\eta}(t) = -\hat{\alpha}_{\sigma(t)}$ for $t \geq 0$.

Theorem 4.2 *Let $\mu \geq 2\nu + 1$, Assumptions 3.3, 4.1 and 4.3 be fulfilled, and for family* (15) *the Lyapunov function $\hat{V}_1(\mathbf{x},\mathbf{y})$ be constructed satisfying the estimates* (20). *If*

$$\int_0^t \eta(\tau)\, d\tau \to +\infty \quad \text{as} \quad t \to +\infty,$$

then the equilibrium position $\mathbf{x} = \dot{\mathbf{x}} = \mathbf{0}$ of system (2) *is asymptotically stable. In the case when*

$$\int_{t_0}^{t_0+t} \eta(\tau)\, d\tau \to +\infty \quad \text{as} \quad t \to +\infty$$

uniformly with respect to $t_0 \geq 0$, the equilibrium position $\mathbf{x} = \dot{\mathbf{x}} = \mathbf{0}$ of system (2) *is uniformly asymptotically stable.*

The proof of the theorem is similar to that of Theorem 3.2.

Remark 4.4 As well as Theorem 3.2, Theorem 4.2 remains valid in the case when the zero solution is not asymptotically stable either for a part of systems from the family (15) numbered $2, \dots, N$, or for all of these systems.

5 A Numerical Example

Let family (1) consist of two systems

$$\begin{cases} \ddot{x}_1 + \sqrt{x_1^2 + x_2^2}\,(\dot{x}_1 + a_s\dot{x}_2) + c_s x_1^3 = 0, \\ \ddot{x}_2 + \sqrt{x_1^2 + x_2^2}\,(-\dot{x}_1 + b_s\dot{x}_2) + d_s x_2^3 = 0, \qquad s = 1, 2, \end{cases} \tag{21}$$

where a_s, b_s, c_s, d_s are constant coefficients. Thus, we have $n = 2$, $\mathbf{x} = (x_1, x_2)^T$, $N = 2$, $\nu = 1$, $\mu = 3$, $\Pi_s(\mathbf{x}) = (c_s x_1^4 + d_s x_2^4)/4$, and

$$\mathbf{D}_s(\mathbf{x}) = \sqrt{x_1^2 + x_2^2}\begin{pmatrix} 1 & a_s \\ -1 & b_s \end{pmatrix}, \qquad s = 1, 2.$$

The results of a numerical simulation are presented in Figs. 1–4, where for solutions of switched systems generated by the family (21) and four types of switching law the dependence of the coordinate x_1 on time is shown. The initial conditions of solutions are determined by the formulae

$$t_0 = 0, \quad x_1(0) = -0.03, \quad x_2(0) = 0.05, \quad \dot{x}_1(0) = 0.02, \quad \dot{x}_2(0) = 0.04.$$

First, the following values of coefficients were chosen: $a_1 = 0.9$, $b_1 = 0.3$, $c_1 = 1$, $d_1 = 10$, $a_2 = 0.8$, $b_2 = 0.1$, $c_2 = 10$, $d_2 = 1$. In this case Assumptions 3.1 and 3.2 are fulfilled, and both systems admit the asymptotically stable equilibrium position $\mathbf{x} = \dot{\mathbf{x}} = \mathbf{0}$.

Fig. 1 corresponds to a switching law satisfying the conditions of Theorem 3.1. Here $\tau_{2i-1} = 5$, $\tau_{2i} = 5^i$, and the first system from the family (21) is active on the intervals

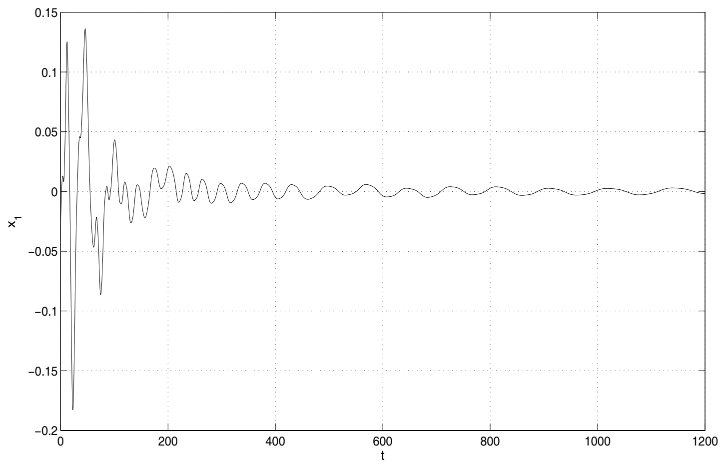

Figure 1: Switching between two asymptotically stable systems (asymptotically stable equilibrium position).

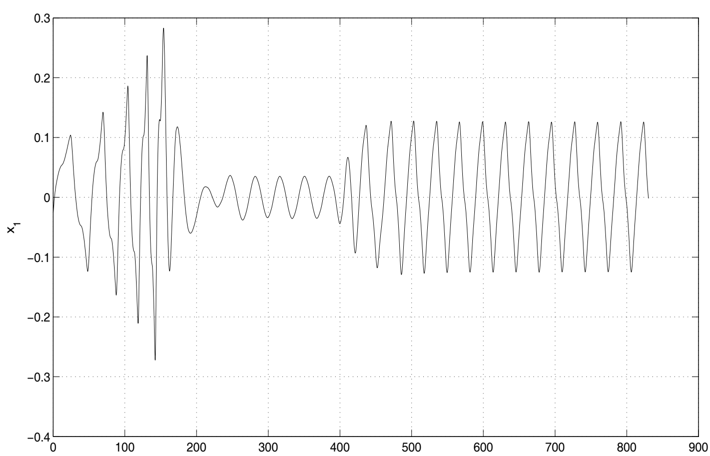

Figure 2: Switching between two asymptotically stable systems (unstable equilibrium position).

$[\theta_{2i-1}, \theta_{2i})$, whereas the second one is active on the intervals $[\theta_{2i-2}, \theta_{2i-1})$, $i = 1, 2, \dots$. For such switching law the equilibrium position is asymptotically stable.

Fig. 2 demonstrates that there exist switching laws for which the equilibrium position is unstable. Here switching from the first system to the second one occurs when $\dot{x}_2 = 0$ and $\dot{x}_1 \neq 0$, whereas switching from the second system to the first one occurs when $\dot{x}_1 = 0$. Moreover, in order to avoid Zeno type switching signal, the following additional restriction is imposed: $\tau_i \geq 4$, $i = 1, 2, \dots$.

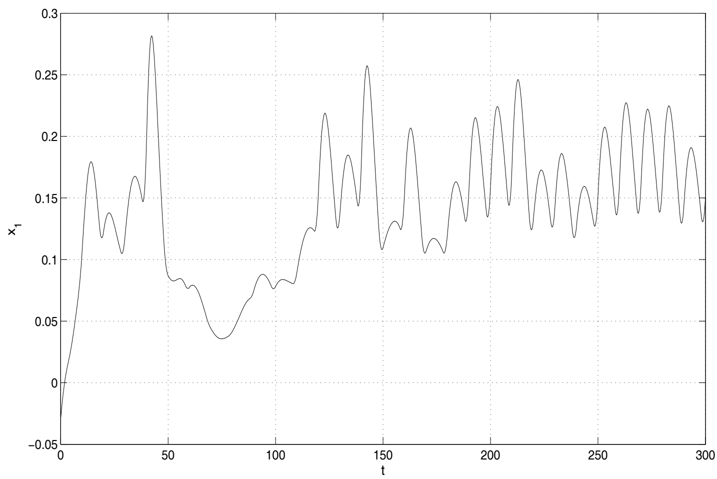

Figure 3: Switching between asymptotically stable and unstable systems (unstable equilibrium position).

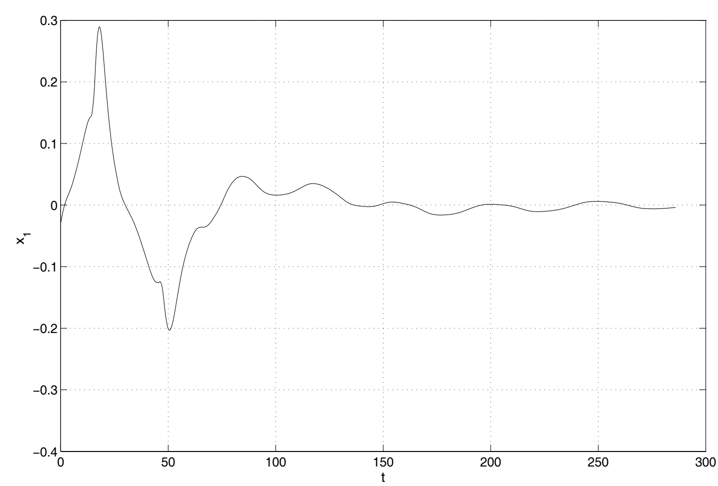

Figure 4: Switching between asymptotically stable and unstable systems (asymptotically stable equilibrium position).

Next, consider the case when $a_1 = 0.9$, $b_1 = 0.3$, $c_1 = 1$, $d_1 = 10$, $a_2 = 0.8$, $b_2 = 0.1$, $c_2 = -10$, $d_2 = -1$. Then the equilibrium position of the first system from the family (21) is asymptotically stable, and the equilibrium position of the second system is unstable. For such values of coefficients Assumptions 3.2 and 3.3 are fulfilled.

Let $\tau_{2i-1} = 2\chi$, $\tau_{2i} = 2$, where χ is a positive parameter, the first system from the family (21) be active on the intervals $[\theta_{2i-2}, \theta_{2i-1})$, and the second one be active on the intervals $[\theta_{2i-1}, \theta_{2i})$, $i = 1, 2, \ldots$. The results of numerical simulation show that if $\chi = 4$, then the equilibrium position of the corresponding switched system is unstable (see Fig. 3), whereas if $\chi = 7$, then the equilibrium position is asymptotically stable (see Fig. 4).

6 Conclusion

In the present paper, certain classes of switched mechanical systems with nonlinear dissipative and potential forces are studied. By the application of the multiple Lyapunov functions approach and the dwell time approach, we found the restrictions on the switching law guaranteeing the asymptotic stability of the trivial equilibrium position.

The obtained results can be used for the design of switched controllers providing the asymptotic stability and the practical stability of equilibrium positions for nonlinear mechanical systems.

The interesting direction for further research is the extension of the obtained results to the case when switched nonlinear dissipative forces depend on velocities and are independent of coordinates. Moreover, the impact of gyroscopis and nonconservative forces on the considered systems may be studied.

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Global Stability of Phase Synchronization in Coupled Chaotic Systems

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Abstract: In analytical or numerical synchronizations studies of coupled chaotic systems, the phase synchronizations are less considered in the leading literatures. This paper is an attempt to find a sufficient analytical condition for the stability of phase synchronization in coupled chaotic systems. The method of nonlinear feedback function and the scheme of matrix measure have been used to justify this analytical stability, and tested numerically for the existence of the phase synchronization in some coupled chaotic systems.

Keywords: *chaos; phase synchronization; stability.*

Mathematics Subject Classification (2010): 37N35, 65P20, 65P99.

1 Introduction

Sensitivity to initial conditions is a generic feature of chaotic dynamical systems. Two chaotic systems starting from slightly different initial points in the state space separate away from each other with time. Therefore, how to control two chaotic systems to be synchronized has aroused a great deal of interest.

Recently, synchronization phenomena in coupled chaotic systems have received much attention [1–17]. Pecora and Carroll have shown [1–4] that in coupled chaotic systems a complete synchronization occurs if the difference between the various states of synchronized systems converges to zero. They have also shown that synchronization stability depends upon the signs of the conditional Lyapunov exponents: i.e., if all of the Lyapunov exponents of the response system under the action of the driver are negative, then there is a complete and stable synchronization between the drive and response systems.

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Synchronization stability can also be verified using the Jacobian matrix in the linearized state difference between the drive and response chaotic systems [6]. Accordingly, despite the stability analysis in dynamical systems, if this Jacobian matrix is of full rank and all of its real parts of eigenvalues are negative, then the system will converge to zero, yielding complete synchronization.

The phenomenon of phase synchronization observed in systems of various nature [18, 19], including chemical, biological, and physiological systems, is today attracting much interest of researchers [19–21]. In this case, the Jacobian matrix has some zero eigenvalues and the difference between various states of synchronized systems may be not necessary converging to the zero, but will stay less than or equal to a constant. The main goal of this paper is to discuss the stability analysis of phase synchronization in coupled chaotic systems coupled by the nonlinear feedback function method [19]. Therefore, a brief discussion of the nonlinear coupling feedback function method is presented in Section 2, followed by the presentation of a criterion for the stability of synchronization in Section 3. In Section 4, we present some examples to corroborate our analytical assertion.

2 Description of the Method

There are different criteria for coupling two chaotic systems to be synchronized. In this paper, we apply the nonlinear coupling feedback function method introduced by Ali and Fang [19] to couple chaotic systems. Suppose $\dot{\mathbf{x}}(t) = \mathbf{F}(t, \mathbf{x}(t))$ is a chaotic system with $\mathbf{x}(t) \in \mathbb{R}^n$. Then decomposing vector-valued function $\mathbf{F}(t, \mathbf{x}(t))$ to a linear part, $\mathbf{L}(t, \mathbf{x}(t))$, and a nonlinear part, $\mathbf{N}(t, \mathbf{x}(t))$, yields

$$\mathbf{F}(t, \mathbf{x}(t)) = \mathbf{L}(t, \mathbf{x}(t)) + \mathbf{N}(t, \mathbf{x}(t)). \tag{1}$$

Now consider two chaotic systems, where their associated vector functions are decomposed as in (2) and coupled by using the nonlinear parts of their vector functions as follows:

$$\dot{\mathbf{x}}_1(t) = \mathbf{L}(t, \mathbf{x}_1(t)) - \mathbf{N}(t, \mathbf{x}_1(t)) + \alpha\left[\mathbf{N}(t, \mathbf{x}_1(t)) - \mathbf{N}(t, \mathbf{x}_2(t))\right], \tag{2}$$

$$\dot{\mathbf{x}}_2(t) = \mathbf{L}(t, \mathbf{x}_2(t)) - \mathbf{N}(t, \mathbf{x}_2(t)) + \alpha\left[\mathbf{N}(t, \mathbf{x}_2(t)) - \mathbf{N}(t, \mathbf{x}_1(t))\right]. \tag{3}$$

Here, systems (2) and (3) serve as drive and response systems, respectively, and α is the strength of their coupling. The synchronization stability of these two systems can be studied by using the evolutional equation of the difference between them, which is determined by the following linear approximation:

$$\dot{\mathbf{e}}(t) = \left[\mathbf{L}(t) + (2\alpha - 1)\frac{\partial \mathbf{N}(t, \mathbf{x}(t))}{\partial \mathbf{x}}\right]\mathbf{e}(t), \tag{4}$$

where $\mathbf{e}(t) = \mathbf{x}_1(t) - \mathbf{x}_2(t)$. Obviously, the stability type of the zero equilibrium in equation (4) shows the stability type of the synchronization between two chaotic systems. If $\mathbf{L}$ has full rank and $\alpha = 0.5$, we have

$$\dot{\mathbf{e}}(t) = \mathbf{L}(t)\mathbf{e}(t), \tag{5}$$

and then according to the stability analysis of the linear approximation in dynamical systems theory, synchronization between coupled chaotic systems (2) and (3) occurs if all eigenvalues of matrix $\mathbf{L}$ have negative real parts. Conversely, if matrix $\mathbf{L}$ does not have full rank: i.e., $\mathbf{L}$ has at least one zero eigenvalue, then we may yet have phase synchronization behavior.

3 Main Results

In this section, we present a stability criterion for synchronization. First, we introduce the concept of matrix measure. The matrix measure of a real square matrix $\mathbf{A} = (a_{ij})_{n\times n}$ is defined by

$$\mu_*(\mathbf{A}) = \lim_{\epsilon\to 0} \frac{\|\mathbf{I}+\epsilon\mathbf{A}\|_* - 1}{\epsilon},$$

where $\mathbf{I}$ is an $n\times n$ identity matrix and $\|\cdot\|_*$ is a matrix norm defined as follows:

$$\|\mathbf{A}\|_1 = \max_j \sum_{i=1}^{n} |a_{ij}|, \qquad \|\mathbf{A}\|_2 = [\lambda \max(\mathbf{A}^T\mathbf{A})]^{1/2},$$

$$\|\mathbf{A}\|_\infty = \max_i \sum_{j=1}^{n} |a_{ij}|, \qquad \|\mathbf{A}\|_\omega = \max_j \sum_{i=1}^{n} \frac{\omega_i}{\omega_j}|a_{ij}|,$$

where $\omega_i > 0$, we have the matrix measures

$$\mu_1(\mathbf{A}) = \max_j \Big\{a_{jj} + \sum_{i=1,\ i\neq j}^{n} |a_{ij}|\Big\}, \qquad \mu_2(\mathbf{A}) = \frac{1}{2}\lambda_{\max}(\mathbf{A}^T+\mathbf{A}),$$

$$\mu_\infty(\mathbf{A}) = \max_i \Big\{a_{ii} + \sum_{j=1,\ j\neq i}^{n} |a_{ij}|\Big\}, \qquad \mu_\omega(\mathbf{A}) = \max_j \Big\{a_{jj} + \sum_{i=1,\ i\neq j}^{n} \frac{\omega_i}{\omega_j}|a_{ij}|\Big\},$$

respectively.

Now suppose in error system (5), matrix $\mathbf{L}$ doesn't have a full rank and $\alpha = 0.5$. Then, as a consequence of the following theorem, we will show that under some conditions system (5) is globally asymptotically stable around a constant vector , on which $\mathbf{e}(t) = \mathbf{x}_1(t) - \mathbf{x}_2(t)$.

Theorem 3.1 *System (5) is globally asymptotically stable if there exists a non-singular time-varying matrix $\boldsymbol{B}(t)$ such that*

$$\lim_{t\to\infty} \exp\left(\int_{t_0}^{t} \mu_*(\dot{\boldsymbol{B}}\boldsymbol{B}^{-1} + \boldsymbol{B}\boldsymbol{L}\boldsymbol{B}^{-1})(s)ds\right) = 0,$$

for any $t_0 \geq 0$. Consequently, phase synchronization between systems (2) and (3) occurs which is globally asymptotically stable around a constant vector $\boldsymbol{c}$.

Proof. Let $\mathbf{e}(t)$ be a solution of error system (5) and $\mathbf{Y}(t) = \mathbf{B}(t)(\mathbf{e}(t)-\mathbf{c})$. Then for all $t \geq t_0$, we have

$$\begin{aligned}
D^+\|\mathbf{Y}(t)\|_* &= \lim_{\epsilon\to 0^+} \frac{1}{\epsilon}\left[\|\mathbf{Y}(t)+\epsilon\dot{\mathbf{Y}}(t)\|_* - \|\mathbf{Y}(t)\|_*\right] \\
&= \lim_{\epsilon\to 0^+} \frac{1}{\epsilon}\left[\left\|\mathbf{B}(t)(\mathbf{e}(t)-\mathbf{c}) + \epsilon\left(\dot{\mathbf{B}}(t)(\mathbf{e}(t)-\mathbf{c}) + \mathbf{B}\mathbf{L}(t)(\mathbf{e}(t)-\mathbf{c})\right)\right\|_*\right. \\
&- \left.\|\mathbf{B}(t)(\mathbf{e}(t)-\mathbf{c})\|_*\right] \\
&= \lim_{\epsilon\to 0^+} \frac{1}{\epsilon}\left[\left\|\mathbf{B}(t)(\mathbf{e}(t)-\mathbf{c}) + \epsilon(\dot{\mathbf{B}}\mathbf{B}^{-1}+\mathbf{B}\mathbf{L}\mathbf{B}^{-1})\mathbf{B}(t)(\mathbf{e}(t)-\mathbf{c})\right\|_*\right. \\
&- \left.\|\mathbf{B}(t)(\mathbf{e}(t)-\mathbf{c})\|_*\right] \\
&\leq \|\mathbf{B}(t)(\mathbf{e}(t)-\mathbf{c})\|_* \lim_{\epsilon\to 0^+} \frac{1}{\epsilon}\left[\|\mathbf{I}+\epsilon\left(\dot{\mathbf{B}}\mathbf{B}^{-1}+\mathbf{B}\mathbf{L}\mathbf{B}^{-1}\right)\|_* - 1\right] \\
&= \|\mathbf{Y}(t)\|_*\mu_*\left(\dot{\mathbf{B}}\mathbf{B}^{-1}+\mathbf{B}\mathbf{L}\mathbf{B}^{-1}\right).
\end{aligned}$$

By integrating both sides of $D^+\|\mathbf{Y}(t)\|_* \leq \|\mathbf{Y}(t)\|_*\mu_*\left(\dot{\mathbf{B}}\mathbf{B}^{-1}+\mathbf{B}\mathbf{L}\mathbf{B}^{-1}\right)$ from t_0 to t, we obtain

$$\|\mathbf{B}(t)(\mathbf{e}(t)-\mathbf{c})\|_* \leq \|\mathbf{B}(0)(\mathbf{e}(0)-\mathbf{c})\|_* \exp\left(\int_{t_0}^{t}\mu_*(\dot{\mathbf{B}}\mathbf{B}^{-1}+\mathbf{B}\mathbf{L}\mathbf{B}^{-1})(s)ds\right).$$

Therefore,

$$\begin{aligned}\|\mathbf{e}(t)-\mathbf{c}\|_* &= \|\mathbf{B}^{-1}(t)\mathbf{B}(t)(\mathbf{e}(t)-\mathbf{c})\|_* \leq \|\mathbf{B}^{-1}(t)\|_*\|\mathbf{B}(t)(\mathbf{e}(t)-\mathbf{c})\|_* \\ &\leq \|\mathbf{B}^{-1}(t)\|_*\|\mathbf{B}(0)(\mathbf{e}(0)-\mathbf{c})\|_* \exp\left(\int_{t_0}^{t}\mu_*(\dot{\mathbf{B}}\mathbf{B}^{-1}+\mathbf{B}\mathbf{L}\mathbf{B}^{-1})(s)ds\right).\end{aligned}$$

Therefore, $\lim_{t\to\infty}\|\mathbf{e}(t)-\mathbf{c}\|_* = 0$ since $\lim_{t\to\infty}\exp\left(\int_{t_0}^{t}\mu_*(\dot{\mathbf{B}}\mathbf{B}^{-1}+\mathbf{B}\mathbf{L}\mathbf{B}^{-1})(s)ds\right) = 0$ and $\|\mathbf{B}^{-1}\| > 0$. Therefore, system (5) is globally asymptotically stable around a constant vector $\mathbf{c}$ and note that the constant vector $\mathbf{c}$ depends upon the initial conditions. This completes the proof.

In the case when $\mathbf{B}(t)$ is a constant matrix, by Theorem 3.1, we have the following result.

Corollary 3.1 *System (5) is globally asymptotically stable if there exists a non-singular matrix $\boldsymbol{B}$ such that*

$$\int_{t_0}^{\infty}\mu_*(\boldsymbol{B}\boldsymbol{L}(s)\boldsymbol{B}^{-1})ds = -\infty,$$

for any $t_0 \geq 0$. Consequently, phase synchronization between systems (2) and (3) occurs which is globally asymptotically stable around a constant vector $\mathbf{c}$.

In Corollary 1, when $\mathbf{B}$ is an identity matrix, then the main result in [13, 23] is obtained.

Corollary 3.2 *System (5) is globally asymptotically stable if*

$$\int_{t_0}^{\infty}\mu_*(\boldsymbol{L}(s))ds = -\infty,$$

for any $t_0 \geq 0$.

4 Numerical Results

In this section, we give some examples to show the efficiency of the above theory.

Example 1. Consider the following forced Duffing system

$$\begin{cases}\dot{x} = y,\\ \dot{y} = ax - by - x^3 + c\cos(2\pi dt).\end{cases}$$

This system is chaotic for parameter values $a = c = 0.3, b = 0.35$ and $d = 0.2$. Using a nonlinear coupling function to couple two identical copies of this system yields

$$\begin{cases}\dot{x_1} = -x_1 + y_1 + x_1 + \alpha|x_2 - x_1|,\\ \dot{y_1} = ax_1 - by_1 - x_1^3 + c\cos(2\pi dt) + \alpha|x_1^3 - x_2^3|,\end{cases} \tag{6}$$

and

$$\begin{cases} \dot{x_2} = -x_2 + y_2 + x_2 + \alpha|x_1 - x_2|, \\ \dot{y_2} = ax_2 - by_2 - x_2^3 + c\cos(2\pi dt) + \alpha|x_2^3 - x_1^3|, \end{cases} \tag{7}$$

where the linear and nonlinear matrices are defined by

$$\mathbf{L} = \begin{bmatrix} -1 & 1 \\ 0.3 & -0.35 \end{bmatrix}, \quad \mathbf{N} = \begin{bmatrix} -x \\ x^3 - 0.3\cos(0.4\pi t) \end{bmatrix}.$$

By taking $\mathbf{B} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$, we have $\mathbf{B}^{-1} = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$ and $\mathbf{BLB}^{-1} = \begin{bmatrix} -0.11 & -1.1 \\ 1.35 & -2.05 \end{bmatrix}$. Now, by using matrix measure $\mu_2(\cdot)$, we have

$$\frac{1}{2}\lambda_{\max}\left((\mathbf{BLB}^{-1})^T + \mathbf{BLB}^{-1}\right) = \frac{1}{2}\lambda_{\max}\begin{bmatrix} -0.2 & 0.25 \\ 0.25 & -4.1 \end{bmatrix} = -0.09202.$$

Therefore, according to Corollary 3.1, synchronization of systems (7) and (8) is globally asymptotically stable. See Figure 1.

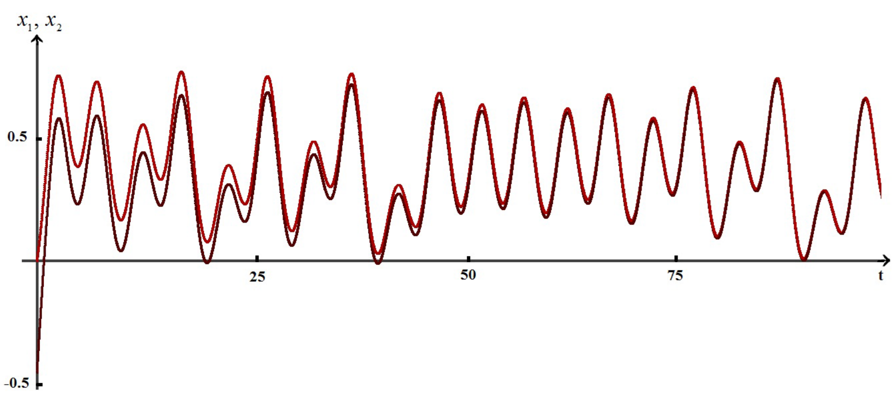

Figure 1: Global asymptotic stability of synchronization between two chaotic systems (7) and (8) in Example 1.

Remark. The above results in Corollary 3.1 and 3.2 are useful to proof the global asymptotic stability of phase synchronization in coupled chaotic systems. As discussed, this synchronization occurs whenever the maximum real part of the eigenvalues of $\mathbf{L}$ is zero. In this case, even the linear stability analysis is not useful for (local) stability analysis of phase synchronization. Nevertheless, using the results of these two corollaries, if in the hypothesis we replace $\int_{t_0}^{\infty} \mu_*(\mathbf{BL}(s)\mathbf{B}^{-1})ds = -\infty$ or $\int_{t_0}^{\infty} \mu_*(\mathbf{L}(s))ds = -\infty$ by $\int_{t_0}^{\infty} \mu_*(\mathbf{BL}(s)\mathbf{B}^{-1})ds = 0$ or $\int_{t_0}^{\infty} \mu_*(\mathbf{L}(s))ds = 0$, respectively, then the error vector in the coupled chaotic systems remains constant. That is, if there is phase synchronization between two coupled chaotic systems, then this synchronization is globally asymptotically stable.

Example 2. Consider the same Duffing system in Example 1 with parameter values $a = b = 0.35, c = 0.3$ and $d = 0.2$. Then this system is again chaotic. Now, with the same nonlinear coupling method as above, we have

$$\mathbf{L} = \begin{bmatrix} -1 & 1 \\ 0.35 & -0.35 \end{bmatrix}, \quad \mathbf{N} = \begin{bmatrix} -x \\ x^3 - 0.3\cos(0.4\pi t) \end{bmatrix}.$$

By taking identity matrix for $\mathbf{B}$ and choosing $\omega_1 = 7$ and $\omega_2 = 20$, we get $\mu_\omega(\mathbf{BLB}^{-1}) = \mu_\omega(\mathbf{L}) = 0$. Therefore, phase synchronization occurring between systems (7) and (8) is globally asymptotically stable. See Figure 2.

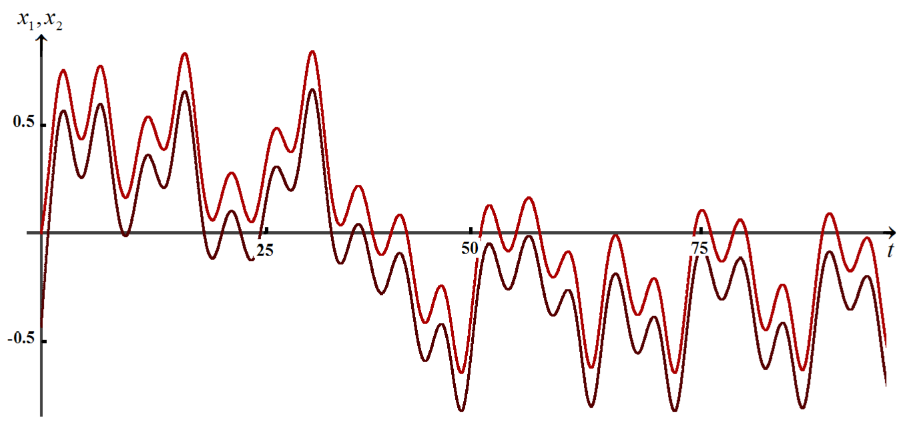

Figure 2: Global asymptotical stability of phase synchronization between two chaotic systems (7) and (8) in Example 2.

5 Conclusion

We have discussed a sufficient analytical condition for the stability of synchronization in coupled chaotic systems. As we have seen using a method of nonlinear feedback function and the scheme of matrix measure together with numerical results have justified this analytical stability. In particular, we have shown that our stability analysis is useful to proof the global asymptotic stability of phase synchronization in coupled chaotic systems.

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A Fractional Order $PI^{\alpha}D^{\beta}$ Control of the Nonlinear Systems

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Abstract: This paper studied the implementation of fractional order $PI^{\alpha}D^{\beta}$ controller for the control of an induction motor (IM). The perfection of the system performance in terms of response time and robustness is illustrated by adjusting the fractional order integral action and derivative action. A comparative study with a conventional PID controller is carried out. The observer is simple and robust, and suitable for online implementation for induction motor. Simulation tests under load disturbances and parameter uncertainties are provided to evaluate the consistency and performance of the proposed control technique.

Keywords: *conventional controller; fractional order controller; induction motor IM; electromagnetic torque and flux control.*

Mathematics Subject Classification (2010): 26A33, 93C10.

1 Introduction

The conventional PID controller is widely used in automatic and especially in industry because of its simplicity but due to the complexity of the controlled systems and parametric variations, the PID controller can not reach the desired performance control where the use of fractional order controller with integral action and derivative action, non-integer order.

The fractional order $PI^{\alpha}D^{\beta}$ controller is an improved version of the conventional PID controller. It allows two degrees of freedom to better adjust the dynamic properties

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of the system and can control non-integer order systems [1–4]. The fractional $PI^{\alpha}D^{\beta}$ controller is less sensitive to parameter variations of the system, it is a robust controller.

The fractional control was developed by mathematicians in the eighties [5, 6]. In the last decade, the calculation of fractional order is applied to each field of engineering. It made a profound impact in the theories of control [7–12].

There are several methods of approximation of the derivative and integral fractional controller [13–15]. The methods of approximations are distinguished by the entire model obtained being continuous or discrete. Researches are ongoing to improve and adjust the controller parameters to expand the scope of application of the fractional control.

In this paper, we will determine the theory of fractional $PI^{\alpha}D^{\beta}$ controller for controlling an induction machine. A parametric variation of the controller is used to determine the influence of fractional controller of control system with and without the presence of disturbance on the system [16–18].

The paper is organized as follows: In Section 2 the Induction Machine modeling is presented. In Section 3, synthesis of the IM controllers is studied. In Section 4, implementation of fractional order controller is considered. In Section 5 the simulation results are presented and discussed, and finally in Section 6 conclusions are drawn.

2 IM Modelling

Prior to the IM equating, some assumptions are considered [19, 20]:

- The gap is constant.
- The Hysteresis, the saturation and the eddy currents are neglected.
- The magneto-motive forces generated by the stator and rotor phases have a sinusoidal distribution.

(a) Mathematical model for the IM.

- *Electrical equations:*

$$\begin{array}{ll} V_{dS} = R_S I_{dS} + \frac{d\phi_{dS}}{dt} - \omega_S \phi_{qS}, & V_{qS} = R_S I_{qS} + \frac{d\phi_{qS}}{dt} + \omega_S \phi_{dS}, \\ 0 = R_r I_{dr} + \frac{d\phi_{dr}}{dt} + \omega_{Sl} \phi_{qr}, & 0 = R_r I_{qr} + \frac{d\phi_{qr}}{dt} + \omega_{Sl} \phi_{qr}, \end{array} \tag{1}$$

where

$$\begin{array}{ll} \phi_{dS} = L_S I_{dS} + L_m I_{dr}, & \phi_{qS} = L_S I_{qS} + L_m I_{qr}, \\ \phi_{dr} = L_m I_{dS} + L_r I_{dr}, & \phi_{qr} = L_m I_{qS} + L_r I_{qr}, \end{array} \tag{2}$$

$$\omega_S = 2\pi f = \frac{d\theta_S}{dt}, \tag{3}$$

$$\omega_{Sl} = \omega_S - \omega_r, \tag{4}$$

with:
L_S: Stator proper cyclical inductance,
L_r: Rotor proper cyclical inductance,
L_m: Cyclical mutual inductance between stator and rotor,
ω_S: Synchronization speed,
ω_{Sl}: Sliding angular velocity.

- *Mechanical equation:*

The mechanical equation is defined by:

$$C_{em} = \frac{3}{2}p\frac{M_{Sr}}{L_r}\left(\phi_{dr}I_{qS} - \phi_{qr}I_{dS}\right). \tag{5}$$

- *Torque equation:*

The orientation of the (dq) frame with the d axis associated with the rotor flux allows writing: $\phi_{dr} = \phi_r$ and $\phi_{qr} = 0$. Thanks to this flux orientation, which allows a high starting torque, the torque expression can be simplified as follows:

$$C_{em} = \frac{3}{2}p\frac{M_{Sr}}{L_r}\phi_{dr}I_{qS}. \tag{6}$$

3 Synthesis of the IM Controllers

The IM state equations are as follows:

$$\frac{dI_{Sd}}{dt} = -\frac{1}{\sigma L_S}\left(R_S + \frac{M_{Sr}^2 R_r}{L_r^2}\right)I_{Sd} +$$

$$\omega_S I_{Sq} + \frac{1}{\sigma L_S}\frac{M_{Sr}R_r}{L_r^2}\phi_{rd} + \frac{1}{\sigma L_S}\frac{M_{Sr}}{L_r}p\Omega_m\phi_{rq} + \frac{1}{\sigma L_S}V_{Sd}, \tag{7}$$

$$\frac{dI_{Sq}}{dt} = \frac{1}{\sigma L_S}\left(R_S + \frac{M_{Sr}^2 R_r}{L_r^2}\right)I_{Sq} +$$

$$\omega_S I_{Sd} + \frac{1}{\sigma L_S}\frac{M_{Sr}R_r}{L_r^2}\phi_{rq} - \frac{1}{\sigma L_S}\frac{M_{Sr}}{L_r}p\Omega_m\phi_{rd} + \frac{1}{\sigma L_S}V_{Sq}, \tag{8}$$

$$\frac{d\phi_{rd}}{dt} = \frac{M_{Sr}R_r}{L_r}I_{Sd} - \frac{R_r}{L_r}\phi_{rd} + \left(\omega_S - p\Omega_m\right)\phi_{rq}, \tag{9}$$

$$\frac{d\phi_{rq}}{dt} = \frac{M_{Sr}R_r}{L_r}I_{Sq} - \frac{R_r}{L_r}\phi_{rq} - \left(\omega_S - p\Omega_m\right)\phi_{rd}, \tag{10}$$

$$\frac{\Omega_m}{dt} = \frac{3}{2}\frac{M_{Sr}P}{L_r J}\left(\phi_{rd}I_{Sq} - \phi_{rq}I_{Sd}\right) - \frac{F}{J}\Omega_m - \frac{1}{J}C_r, \tag{11}$$

while: $\sigma = 1 - \frac{M_{Sr}^2 R_r}{L_S L_r}$.

(a) Control loop of the rotor flux.

The decoupling allowed by the oriented flux and the relation (3) can give

$$\frac{d\phi_{rd}}{dt} = \frac{M_{Sr}R_r}{L_r}I_{Sd} - \frac{R_r}{L_r}\phi_{rd}. \tag{12}$$

Wherein the direct stator current expression is:

$$I_{Sd} = \frac{1}{M_{Sr}}\left(\phi_{rd} + \frac{L_r}{R_r}\frac{d\phi_{rd}}{dt}\right). \tag{13}$$

Let $T_r = \frac{L_r}{R_r}$ be the rotor time constant and $T_S = \frac{L_S}{R_S}$ be the stator time one. The relations (7) and (13) can lead to:

$$V_{Sd} = \frac{R_S}{M_{Sr}}\left(\phi_{rd} + (T_S + T_r)\frac{d\phi_{rd}}{dt}\right) + \sigma T_S T_r \frac{d^2\phi_{rd}}{dt^2} - \omega_S \sigma L_S I_{Sq} = V_{Sdf} + V_{Sdc}. \quad (14)$$

To ensure the decoupling between the two axes, the term V_{Sdc} must be compensated:

$$V_{Sdf} = \frac{R_S}{M_{Sr}}\left(\phi_{rd} + (T_S + T_r)\frac{d\phi_{rd}}{dt}\right) + \sigma T_S T_r \frac{d^2\phi_{rd}}{dt^2}, \quad V_{Sdc} = -\omega_S \sigma L_S I_{Sq}. \quad (15)$$

The system transfer function is:

$$G_{flux}(p) = \frac{\phi_{rd}(p)}{V_{Sdf}(p)} = \frac{M_{Sr}}{R_S}\frac{1}{1 + (T_S + T_r)p + \sigma T_S T_r p^2}. \quad (16)$$

Let p_1 and p_2 be the denominator roots such that $p_2 \succ\succ p_1$, where $p_1 = \frac{\sigma T_S T_{Sq}}{T_S + T_{Sq} + \Delta}$, $p_2 = \frac{\sigma T_S T_{Sq}}{T_S + T_{Sq} - \Delta}$.

The flux error is $\epsilon = e_{2\_PI} = \phi_{rd\_rf} - \phi_{rd}$. The following figure shows the block diagram of the flux control loop.

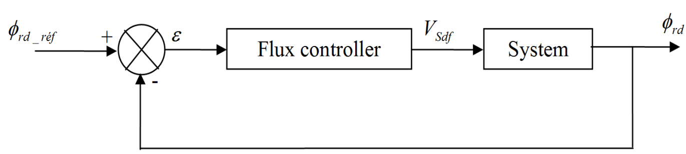

Figure 1: Flux control loop.

(b) Control loop of the electromagnetic torque.

Considering that the flux response is faster than the torque one, the flux reaches its final value $\phi_{rd} = \phi_{rd0}$, and the expression of the torque could be given by the following:

$$C_{em} = \frac{3}{2}\frac{M_{Sr}P}{L_r}\phi_{rd0}I_{Sd}. \quad (17)$$

The voltage equation V_{Sq} becomes:

$$V_{Sd} = R_S I_{Sq} + \sigma L_S \frac{dI_{Sq}}{dt} + \phi_{rd}\omega_S \frac{M_{Sr}}{L_r} + \sigma L_S \omega_S I_{Sd}. \quad (18)$$

Let

$$V_{Sq} = V_{Sqt} + V_{Sqc}. \quad (19)$$

The V_{Sqc} component represents a decoupling term that we have to compensate,

$$V_{Sqc} = \phi_{rd}\omega_S \frac{M_{Sr}}{L_r} + \sigma L_S \omega_S I_{Sd}, \quad (20)$$

$$V_{Sqt} = R_S I_{Sq} + \sigma L_S \frac{dI_{Sq}}{dt}. \quad (21)$$

The system transfer function becomes:

$$G_{cem}(p) = \frac{C_{em}(p)}{V_{Sqt}(p)} = \frac{3M_{Sr}P\phi_{rd0}}{2L_r R_S(1 + \sigma T_S p)}. \quad (22)$$

The flux error is $\epsilon = e_{1\_PI} = C_{em\_rf} - C_{em}$. The following figure shows the block diagram of the torque control loop.

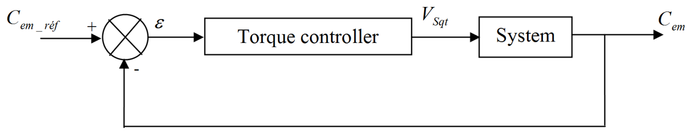

Figure 2: Torque control loop.

4 Implementation of Corrective Fractional Order

The simulation part is usually performed by integer order of finite dimension. So it is necessary to replace the transfer functions of non-integer order by the transfer functions of integer order. The methods of approximations are distinguished by the entire model obtained, being continuous or discrete.

(a) Continuous Approximation Methods: singularity function.
There are several approximation methods analog continuous (or frequency) for the fractional operators existing in the literature [21, 22]. These methods are based on the continuous model, such as the approximation of fractional order model by a continuous rational model.

The method consists in replacing the derivative operator S^n by a transmittance, where poles and zeros are related by a recurrence relation. To replace S^n by an entire model, it is necessary to apply the following approximations:

- Approximation in a frequency band $[\omega_B; \omega_H]$ of non-integer operator by a non-integral model $S^n_{[\omega_B;\omega_H]}$.
- Approximation of the non-integer model obtained by an entire model.

The approximation methods are: SFEC approximation Method (Fractional Expansion Continues), Oustaloup approximation method [23], Charef approximation method [24], other methods (Carlson, Matsuda, Roy Wang, ...). In the following we will define the Charef method as an example.

- *Approximation of fractional order integration.*

The transfer function of the fractional order integrator is given by the following irrational function [4, 25]:

$$H_1(p) = \frac{1}{P^\alpha}, \tag{23}$$

where α is a positive number $0 \prec \alpha \prec 1$ and $p = j\omega$ is the complex frequency. This operator may be approximated in a given frequency band $[\omega_B; \omega_H]$ by:

$$H_1(p) = \frac{k_1}{\left(1 + \frac{P}{\omega_C}\right)^\alpha} = k_1 \frac{\prod_{i=0}^{N-1}(1 + \frac{P}{\tau_i})}{\prod_{i=0}^{N}(1 + \frac{P}{P_i})}. \tag{24}$$

For systems with integrator: The transfer function of the fractional order integrator is given by the following irrational function [26]:

$$H_1(p) = \frac{1}{P^\alpha} = \frac{1}{P} P^{1-\alpha}. \tag{25}$$

Thus

$$H_1(p) = \frac{k_D}{P} \frac{\prod_{i=0}^{N}(1+\frac{P}{T_i})}{\prod_{i=0}^{N}(1+\frac{P}{Z_i})}, \tag{26}$$

- *Approximation of fractional order differentiation*

The transfer function of fractional order differentiator is given by the following irrational function:

$$H_D(p) = P^{\beta}, \tag{27}$$

where β is a positive number $0 \prec \beta \prec 1$ and $p = j\omega$ is the complex frequency. This operator may be approximated in a given frequency band $[\omega_B; \omega_H]$ by:

$$H_D(p) = \frac{k_D}{\left(1+\frac{P}{\omega_C}\right)^{\beta}} = k_D \frac{\prod_{i=0}^{N}(1+\frac{P}{T_i})}{\prod_{i=0}^{N}(1+\frac{P}{Z_i})}. \tag{28}$$

(b) Adjusting the parameters of the controller PI^{α}.

- *Adjustment of parameters k_p and k_i*

For flow control, we will apply the compensation method for compensating the slow term and make the system faster, hence the use of a corrector PI. This type of corrector is generally used for the first order systems such as the torque control. The adjustment of parameters k_p and of fractional order PI^{α} control is done with $\alpha = 1$, which means adjusting the parameters of a simple classical PI controller. To compensate for the dominant pole, we will use a fractional order PI^{α} controller. The shape of the fractional order PI^{α} controller, including a fractional integrator of order α, such as $0 \prec \alpha \prec 1$, see [27]. The transfer function of fractional order control is given by:

$$C(p) = k_p \left(1 + k_i \frac{1}{P^{\alpha}}\right). \tag{29}$$

- ***Flow Control:***

The transfer function of open loop flow control is:

$$H_0(p) = G_f(p)C(p) = \frac{\phi_{rd}(p)}{\epsilon(p)} = \frac{M_{Sr}}{R_S}.k_i.\frac{1}{(1+p_1p)(1+p_2p)}.\frac{1+\frac{k_p}{k_i}p}{p}. \tag{30}$$

Using the compensation method of dominant pole (offset slow time constant) is to make the system faster. The transfer function in simplified open loop is given by:

$$H_0(p) = \frac{M_{Sr}}{R_S}.k_i.\frac{1}{p(1+p_1p)}. \tag{31}$$

The transfer function of the closed loop is:

$$H_F(p) = \frac{1}{1+\frac{R_S}{M_{Sr}k_i}p+\frac{R_S}{M_{Sr}k_i}p_1p^2} = \frac{1}{1+\frac{2z}{\omega_n}p+\frac{1}{\omega_n^2}p^2} \tag{32}$$

with $k_p = k_ip_2$, $k_i = \frac{R_S}{M_{Sr}}.\frac{\omega_n}{2z}$ and $\omega_n = \frac{1}{2zp_1}$.

Choice of parameters z and ω_n.

- A good starting point is to clean the pulse ω_n equal to the open-loop process.
- The excess is determined by the value $z = 0.7$ providing a good response time.
- To have a positive adjustment we need $k_i \succ 0$.

- ***Electromagnetic torque control:***

The transfer function in open lopp is:

$$H_0(p) = G_{cem}(p)C(p) = \frac{\phi_{rd}(p)}{\epsilon(p)} = \frac{3M_{Sr}P\phi_{rd0}}{2L_rR_S}.k_i.\frac{1}{(1+\sigma T_S p)}.\frac{1+\frac{k_p}{k_i}p}{p}. \tag{33}$$

The transfer function in simplified open loop is given by:

$$H_0(p) = \frac{3M_{Sr}P\phi_{rd0}}{2L_rR_S}.k_i.\frac{1}{p}. \tag{34}$$

The transfer function of the closed loop is:

$$H_F(p) = \frac{1}{1+\frac{1}{k.k_i}p}. \tag{35}$$

Choice of parameters: k_i ***and*** τ***.***

- A good starting point is to take the constant τ equal to the process time.
- To have a positive adjustment we need $k_i \succ 0$.

- *Adjustment parameter* α

To adjust the parameters α or (β) by minimizing a performance criterion is the integral square error (ISE). The integral square error (ISE) is given by:

$$J = \int_0^\infty [e(t)]^2\, dt = \frac{1}{2\pi j}\int_{-j\infty}^{+j\infty} E(p)E(-p)dp. \tag{36}$$

The error signal $E(p)$ is obtained as:

$$E(p) = \frac{R(p)}{1+C(p)G(p)}, \tag{37}$$

where $R(p)$ is a unit step input

$$R(p) = \frac{1}{p}. \tag{38}$$

- *Hall-Sartorius method*

To calculate ISE we use the Hall-Sartorius method. It is to minimize the integral squared error of a loop with an entry level system

$$J = \frac{1}{2\pi j}\int_{-j\infty}^{+j\infty} \frac{N_E(p)N_E(-p)}{D_E(p)D_E(-p)}dp, \tag{39}$$

$$N_E(p) = b_0 + b_1 p + b_2 p^2 + \ldots + b_{n-1} p^{n-1}, \tag{40}$$

$$D_E(p) = a_0 + a_1 p + a_2 p^2 + \ldots + a_{n-1} p^{n-1} + a_n p^n, \tag{41}$$

$$N_E(p) = c_0 + c_1 p^2 + c_2 p^4 + \ldots + c_{n-1} p^{2(n-1)}, \tag{42}$$

or the general formula

$$J = \frac{(-1)^{n-1}}{2} . \frac{det(\Delta_n^N)}{det(\Delta_n^D)} \tag{43}$$

with $\Delta_n^D \in \Re^{(n+1)(n+1)}$,

$$Delta_n^D = \frac{(-1)^{n-1}}{2} . \frac{det(\Delta_n^N)}{det(\Delta_n^D)}, \tag{44}$$

$$\Delta_n^D = \begin{bmatrix} a_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_2 & a_1 & a_1 & 0 & 0 & 0 & 0 & 0 \\ . & . & . & . & . & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & 0 & a_n & a_{n-1} & a_{n-2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_n \end{bmatrix} \tag{45}$$

and $\Delta_n^N \in \Re^{(n)(n)}$. The matrix Δ_n^N is obtained by removing the last column and last row of the matrix Δ_n^N and replacing the last column of this matrix by the following vector:

$$\Delta_n^D = \begin{bmatrix} c_0 & c_1 & a_2 & \ldots & a_1 \end{bmatrix}. \tag{46}$$

The smallest index J of the criterion ISE, $J = 0.5094$ is calculated with $\alpha = 0.92$ for the flow control, and $J = 0.0054$ is obtained with $\alpha = 0.65$ for the electromagnetic torque control. The integrator and the differentiator to the fractional order controller C(p) are approximated in the frequency band $[\omega_B; \omega_H] = [0.1\omega_B; 10.\omega_H]$ with a frequency $\omega_{max} = 100\omega_h$ and an approximation error $y = 1dB$.

Hence, the controller fractional order $PI^{0.65}$ is given by:

$$C(p) = 286.308 \left(1 + \frac{1.4054}{p} . \frac{\prod_{i=0}^{3}(1 + \frac{p}{0.2215.(433.873)^i})}{\prod_{i=0}^{3}(1 + \frac{p}{1.8556.(352.1189)^i})}\right). \tag{47}$$

The controller fractional order $PI^{0.65}$ is given by:

$$C(p) = 0.0351 \left(1 + \frac{64.9076}{p} . \frac{\prod_{i=0}^{6}(1 + \frac{p}{2.2624.10^{-4}.(28.84)^i})}{\prod_{i=0}^{6}(1 + \frac{p}{2.9058.10^{-4}(22.84)^i})}\right). \tag{48}$$

Table 1 summarizes some performance characteristics of the conventional control system and fractional order in terms of the cutoff frequency $\omega_u(rad/s)$, response time $t_r(s)$, Gain Margin $GM(dB$, Phase Margin $PM(deg)$, and overshoot $D\%$.

5 Simulation Results

The following figures are determined using the Matlab / Simulink software to demonstrate the performance of the fractional order control. The performance of the control technique is defined by:

	cont	ω_u	t_r	GM	PM	$D\%5$
Flux control	PI	120	0.1805	-	65.5	4.54
	$PI^{0.65}$	120	0.1805	-	65.5	4.54
Torque control	PI	1	3.29	-	90	-
	$PI^{0.92}$	0.984	2.8	-	96.4	-

Table 1: Characteristics of performance for $(PI; PI^{\alpha})$.

- Stability in steady state.
- Response quickness.
- A relatively small static error.

The simulation is performed with unloading start, at t=60s rotation is reversed, then a load torque $C_r = 20Nm$ is introduced at t=100s.

Figure 3 represents the evolution of the electromagnetic torque considered, real and reference of the asynchronous motor in the presence of radial force $C_r = 20Nm$ t=100s. It is noted that the electromagnetic torque does not admit oscillations and reaches steady operation with a response time $tr_{PI} = 3.92s$ et $tr_{PI^{0.92}} = 2.8s$. The machine answers successfully to the inversion of its direction of rotation.

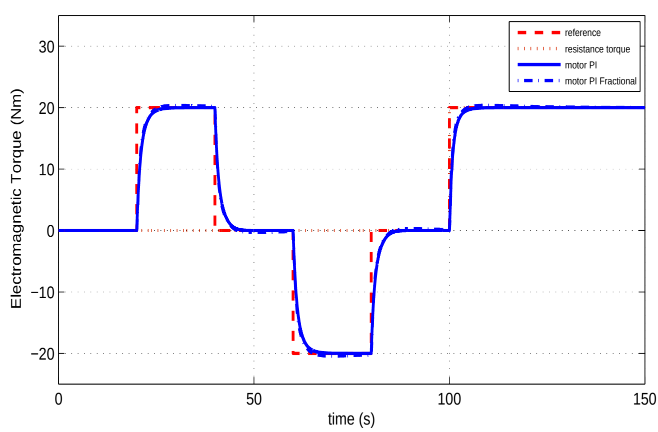

Figure 3: Evolution of the electromagnetic torque(- - PI; - $PI^{0.92}$).

Figure 4 shows the influence of controls applied on the response of flow along the two axes (d, q):

- Along the axis (d): the fractional order control is less sensitive to the reversal of direction of rotation or the introduction of load than the PI controller.
- Along the axis (q): the flow is zero regardless of the order.

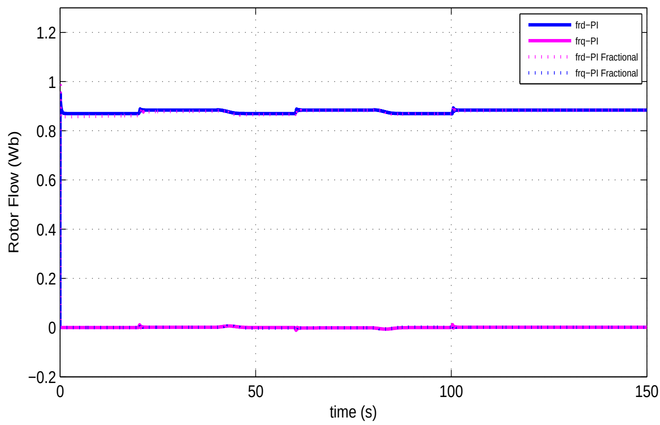

Figure 4: Rotor flux response.

Changes in the motor flux demonstrate the robustness of the control slide, it follows exactly the desired set point, with overshoot negligible, see Table 1, and without static error even for the impact load torque or reversal of direction of rotation. The evolution of direct rotor flux is not a static error with short response time.

Figure 5 is a representation of the evolution of the speed of asynchronous techniques for both commands. The response speed of the MAS shown in Figure 5 is similar to that of a first order system without overshoot, steady and stable with a response time of the order of 5.36s for the speed defined by the $PI^{0.65}$ controller and 5.63s for the speed determined by the classical PI controller. The evolution of the velocity shows at t = 100s the robustness of the fractional order control to the introduction of charging. $S_{PI^{0.65}} = (4.8\%)S_{PI}$.

To demonstrate the performance of control system by fractional order control, we will vary the time constant and process gain for the torque control in closed loop. And, we will vary the damping factor for the flux control in closed loop.

Figures 6 and 7 represent the influence of the variation of time constant. It is assumed that the gain is fixed at its nominal value K_{nom}. To study the influence of the variation of the time constant τ the parameter τ is varied around its nominal value. The results show that:

- the response time Defines by the fractional order $PI^{0.92}$ controller is still less than the response time defines by the conventional controller for different values of the time constant τ.

- the overshoot is insensitive to the variation of the time constant τ.

- the servo by the $PI^{0.92}$ controller, ensure the desired specifications with the presence of a very important property of robustness.

Figures 8 and 9 represent the influence of the variation of process gain. It is assumed that the time constant is fixed at its nominal value τ_{nom}. To study the influence of the variation of the process gain K the parameter K is varied around its nominal value. The results show that:

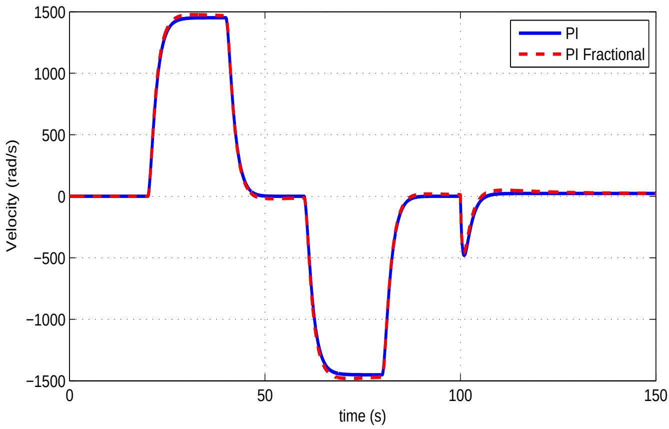

Figure 5: Evolution of the speed (- - PI; - $PI^{0.65}$).

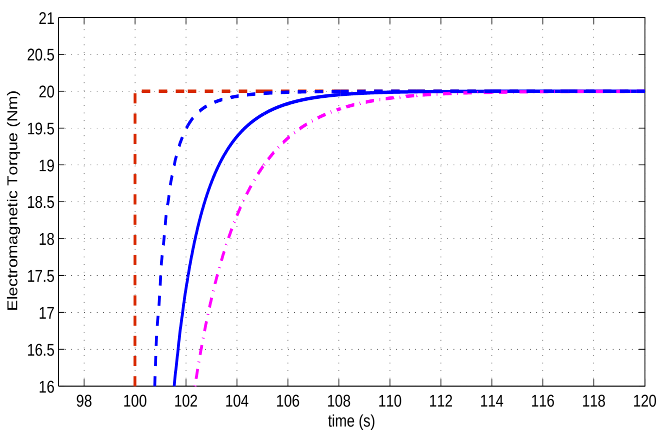

Figure 6: Evolution of the electromagnetic torque for different values of time constant τ, (- $\tau = \tau_{nom}$; -.- $\tau = 150\%\tau_{nom}$; - - $\tau = 50\%\tau_{nom}$) (conventional PI).

- the response time defined by the fractional order $PI^{0.92}$ controller is still less than the response time defined by the conventional controller for different values of the process gain K.

- the overshoot is insensitive to the variation of the process gain K.

- the servo by the $PI^{0.92}$ controller, ensures the desired specifications with the presence of a very important property of robustness.

Figures 10 and 11 show the impact of the variation of the damping factor (m) on the flux response along the axe (d). It was found that, the rise in response to the desired value, the higher the damping factor (m).

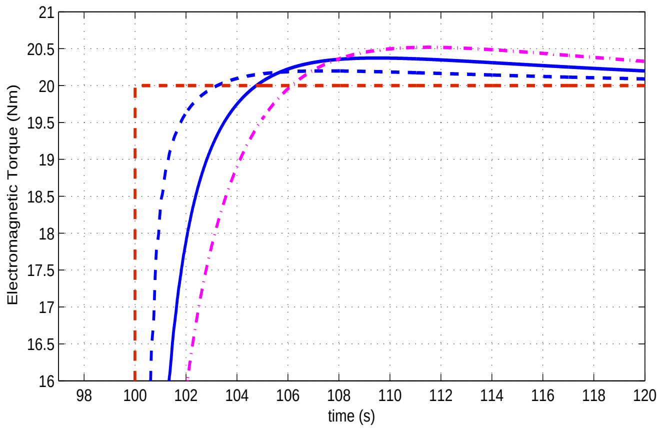

Figure 7: Evolution of the electromagnetic torque for different values of time constant τ, (- $\tau = \tau_{nom}$; -.- $\tau = 150\%\tau_{nom}$; - - $\tau = 50\%\tau_{nom}$) ($PI^{0.92}$).

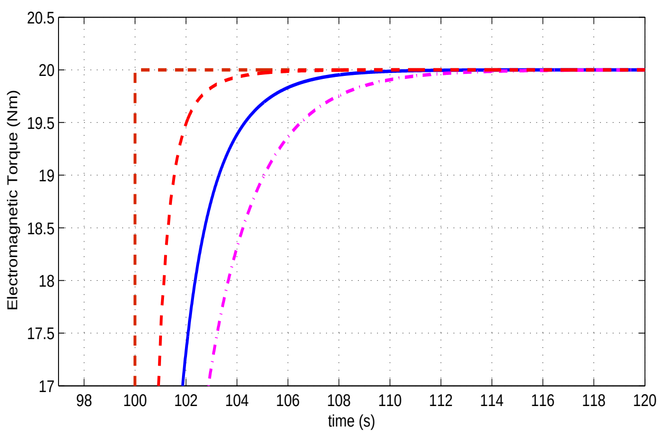

Figure 8: Evolution of the electromagnetic torque for different values of process gain K, (- $K = K_{nom}$; -.- $K = 150\%K_{nom}$; - - $K = 50\%K_{nom}$) (conventional PI).

- the response time defined by the fractional order controller is still less than the response time defined by the conventional controller for different values of the damping factor m. $tr^{PI}(m = 0.5) = 0.071s$; $tr^{PI}(m = 0.7) = 0.18s$ and $tr^{PI}(m = 1.2) = 1.797s$. $tr^{PI^{0.65}}(m = 0.5) = 0.069s$; $tr^{PI^{0.65}}(m = 0.7) = 0.156s$ and $tr^{PI^{0.65}}(m = 1.2) = 1.687s$

- the overshoot of flux defined by the fractional order $PI^{0.65}$ controller is less sensitive than the overshoot defined by the conventional controller for different values of the damping factor m. For example, m=0.5: $D^{PI}(m = 0.5) = 13.32\%$ and $D^{PI^{0.65}}(m = 0.5) = 11.53\%$

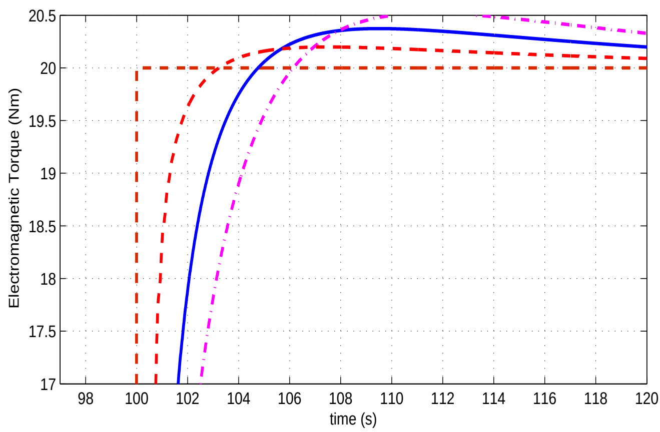

Figure 9: Evolution of the electromagnetic torque for different values of process gain K, (- $K = K_{nom}$; -.- $K = 150\% K_{nom}$; - - $K = 50\% K_{nom}$) ($PI^{0.92}$).

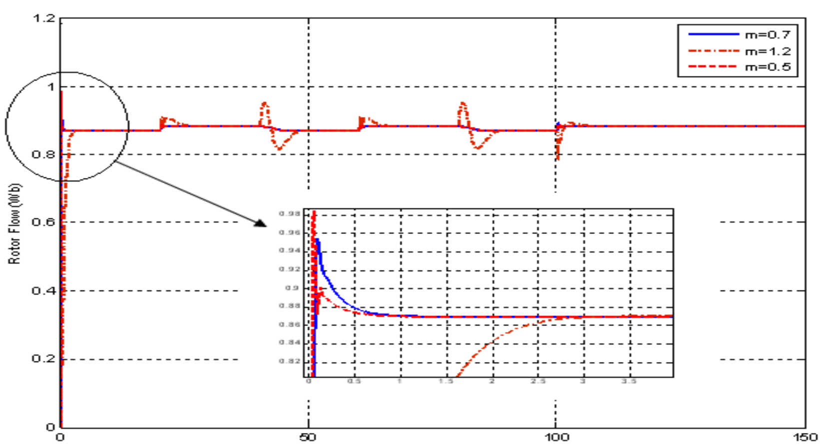

Figure 10: Response flux for different values of the damping factor m (conventional PI).

6 Conclusion

The nonlinear control system with a fractional order controller was presented in this paper, with a comparative study of the conventional controller. We define the correction order and fractional approximation of Charef to determine the rational expression of the integration and the derivation of the correction. The adjustment of the order of fractional order (α, β) is done by minimizing the control error defined by ISE using the Hall-Sartoruis method. The results obtained by simulation and comparative study demonstrate the performance of the control technique with fractional order correction in the presence of load variation and control parameters, as well as the profitability of ISE using the method of Hall-Sartoruis.

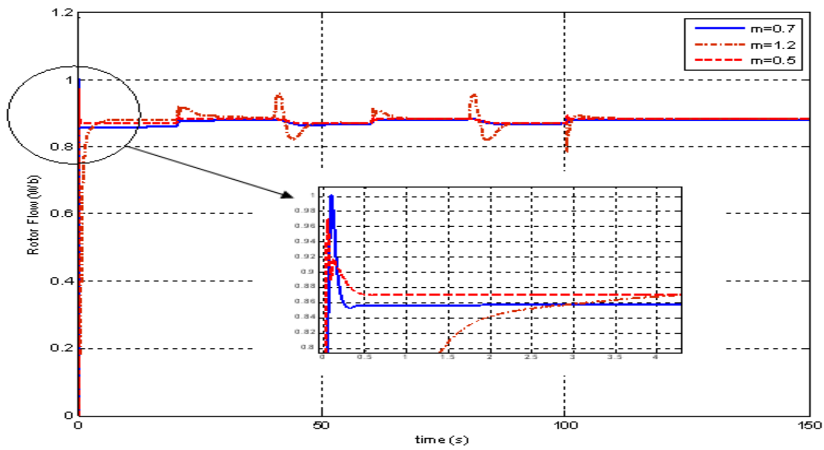

Figure 11: Response flux for different values of the damping factor m ($PI^{0.65}$).

Acknowledgment

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A New Synchronization Scheme for General 3D Quadratic Chaotic Systems in Discrete-Time

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Abstract: In this paper, a new general chaos synchronization scheme is proposed for coupled arbitrary 3-D quadratic chaotic dynamical systems in discrete-time. The proposed synchronization method, based on nonlinear controllers and Lyapunov stability theory, is theoretically rigorous. The derived synchronization criterion can be also applicable to a large class of discrete-time chaotic systems. Our control scheme is used to illustrate complete synchronization between the three-dimensional hyperchaotic discrete-time Rössler and Wang systems. Moreover numerical simulations are used to show the effectiveness and the feasibility of the proposed synchronization scheme.

Keywords: *quadratic systems; chaos synchronization; control scheme; discrete-time; Lyapunov stability.*

Mathematics Subject Classification (2010): 93C10, 93C55, 93D05.

1 Introduction

Over the last two decade, many scholars have proposed various control schemes in chaos synchronization [1–6], but the most of works have concentrated on continuous-time rather than discrete-time chaotic systems. In practice, discrete-time chaotic systems play a more important role than their continuous counterparts [7]. In fact, many mathematical models of physical processes [8], biological phenomena [10], chemical reactions [9] and economic systems [11] were defined using discrete-time chaotic systems. Many 3D chaotic and hyperchaotic dynamical systems in discrete-time are founded such as Baier-Klain map [12], Hitzl-Zele map [13], Stefanski map [14], Wang system [15], discrete-time Rössler system [16] and Grassi-Miller map [18], etc.

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Recently, synchronization in discrete-time chaotic systems attracts more and more attention in many areas of science and technology, and has been extensively studied, due to its potential applications in secure communication [19,20,22,23]. Until now, a variety of approaches have been proposed for the synchronization of chaotic systems in discrete-time [24–27] and different types of chaos synchronization have been presented [28–32].

In this paper, using new controller law and Lyapunov stability theory, a general method is proposed to guarantee global synchronization for a special class of chaotic maps. The aim of this paper is to develop a simple criterion for the synchronization between two arbitrary 3D quadratic chaotic systems in discrete-time. In order to verify the effectiveness of the new approach, the proposed scheme is applied between two 3D hyperchaotic maps: the discrete-time Rössler system and the 3D Wang system.

The rest of this paper is organized as follows. In Section 2, a description of the chaotic systems addressed in this paper is provided. In Section 3, a new chaos synchronization approach in discrete-time is introduced and new synchronization criterion is derived. In Section 4, the proposed synchronization scheme is applied to some typical 3D discrete-time hyperchaotic systems and numerical simulations are used to verify the effectiveness of the new approach. In Section 5, conclusion follows.

2 Description of Drive-response Systems

Consider the drive chaotic system in the form of

$$x_i\left(k+1\right)=\sum_{j=1}^{3}a_{ij}x_j\left(k\right)+\sum_{q=1}^{3}\sum_{p=1}^{3}\alpha_{pq}^{(i)}x_p\left(k\right)x_q\left(k\right)+c_i,\quad 1\leq i\leq 3, \tag{1}$$

where $X\left(k\right)=\left(x_i\left(k\right)\right)_{1\leq i\leq 3}\in\mathbb{R}^3$ is the state vector of the drive system, $\left(a_{ij}\right)\in\mathbb{R}^{3\times 3}$, $\left(\alpha_{pq}^{(i)}\right)\in\mathbb{R}^{3\times 3}$ $(i=1,2,3)$, and $\left(c_i\right)_{1\leq i\leq 3}$ are real numbers.

As the response chaotic system, we consider the following system

$$y_i\left(k+1\right)=\sum_{j=1}^{3}b_{ij}y_j\left(k\right)+\sum_{q=1}^{3}\sum_{p=1}^{3}\beta_{pq}^{(i)}y_p\left(k\right)y_q\left(k\right)+d_i+u_i,\quad 1\leq i\leq 3, \tag{2}$$

where $Y\left(k\right)=\left(y_i\left(k\right)\right)_{1\leq i\leq 3}\in\mathbb{R}^3$ is the state vector of the response system, $\left(b_{ij}\right)\in\mathbb{R}^{3\times 3}$, $\left(\beta_{pq}^{(i)}\right)\in\mathbb{R}^{3\times 3}$ $(i=1,2,3)$, $\left(d_i\right)_{1\leq i\leq 3}$ are real numbers and $U=\left(u_i\right)_{1\leq i\leq 3}\in\mathbb{R}^3$ is a vector controller to be determined.

Remark 2.1 3D Quadratic chaotic maps can be written under the form of (1) such as 3D Hénon-like map, Baier-Klein map, 3D generalized Hénon map, Stefanski map, discrete-time Rössler system and Wang system, etc.

Our aim is to realize synchronization between the drive system (1) and the response system (2) for arbitrary constants a_{ij}, b_{ij}, $\alpha_{pq}^{(i)}$, $\beta_{pq}^{(i)}$, c_i and d_i $(i,p,q=1,2,3)$, and to determine the controllers u_i $(1\leq i\leq 3)$, which stabilize the synchronization errors

$$e_i\left(k\right)=y_i\left(k\right)-x_i\left(k\right),\quad 1\leq i\leq 3, \tag{3}$$

then the aim of synchronization is to make $\lim_{k\to\infty}e_i\left(k\right)=0$, $(i=1,2,3)$.

3 New Chaos Synchronization Scheme in Discrete-time

The synchronization errors between the drive system (1) and the response system (2), can be derived as follows

$$e_i(k+1) = \sum_{j=1}^{3} b_{ij} e_j(k) + R_i + u_i, \quad 1 \le i \le 3, \tag{4}$$

where

$$\begin{aligned} R_i \;=\; & \sum_{j=1}^{3} (b_{ij} - a_{ij}) x_j(k) + \sum_{q=1}^{3}\sum_{p=1}^{3} \beta_{pq}^{(i)} y_p(k) y_q(k) \\ & - \sum_{q=1}^{3}\sum_{p=1}^{3} \alpha_{pq}^{(i)} x_p(k) x_q(k) + d_i - c_i, \quad 1 \le i \le 3. \end{aligned} \tag{5}$$

To achieve synchronization between systems (1) and (2), we choose the vector controller $U = (u_i)_{1\le i\le 3}$ as follows

$$\begin{aligned} u_1 &= l_1 e_1(k) + (b_{22} - b_{12} + l_2) e_2(k) - (b_{13} + b_{33} + l_3) e_3(k) - R_1, \\ u_2 &= -(b_{21} + b_{11} + l_1) e_1(k) + l_2 e_2(k) + (b_{33} - b_{23} + l_3) e_3(k) - R_2, \\ u_3 &= (b_{11} - b_{31} + l_1) e_1(k) - b_{32} e_2(k) + (b_{33} + 2l_3) e_3(k) - R_3, \end{aligned} \tag{6}$$

where $(l_i)_{1\le i\le 3}$ are control constants to be determined later. By substituting Eq. (6) into Eq. (4), the synchronization errors can be written as

$$\begin{aligned} e_1(k+1) &= (b_{11} + l_1) e_1(k) + (b_{22} + l_2) e_2(k) - (b_{33} + l_3) e_3(k), \\ e_2(k+1) &= -(b_{11} + l_1) e_1(k) + (b_{22} + l_2) e_2(k) + (b_{33} + l_3) e_3(k), \\ e_3(k+1) &= (b_{11} + l_1) e_1(k) + 2(b_{33} + l_3) e_3(k). \end{aligned} \tag{7}$$

Now, we have the following result.

Theorem 3.1 *If the control constants $(l_i)_{1\le i\le 3}$ are chosen such that*

$$\begin{cases} -b_{11} - \frac{1}{\sqrt{3}} < l_1 < -b_{11} + \frac{1}{\sqrt{3}}, \\ -b_{22} - \frac{1}{\sqrt{2}} < l_2 < -b_{22} + \frac{1}{\sqrt{2}}, \\ -b_{33} - \frac{1}{\sqrt{6}} < l_3 < \frac{1}{\sqrt{6}}, \end{cases} \tag{8}$$

then the drive system (1) and the response system (2) are globally synchronized under the controller law (6).

Proof. Let us consider the following quadratic Lyapunov function

$$V(e(k)) = \sum_{i=1}^{3} e_i^2(k), \tag{9}$$

then we obtain

$$
\begin{aligned}
\Delta V\left(e(k)\right) &= V\left(e(k+1)\right) - V\left(e(k)\right) \\
&= \sum_{i=1}^{3} e_i^2\left(k+1\right) - \sum_{i=1}^{3} e_i^2\left(k\right) \\
&= \left(3\left(b_{11}+l_1\right)^2 - 1\right) e_1^2\left(k\right) + \left(2\left(b_{22}+l_2\right)^2 - 1\right) e_2^2\left(k\right) \\
&\quad + \left(6\left(b_{33}+l_3\right)^2 - 1\right) e_3^2\left(k\right) \\
&\quad + \left[\left(b_{11}+l_1\right)\left(b_{22}+l_2\right) - \left(b_{11}+l_1\right)\left(b_{22}+l_2\right)\right] e_1\left(k\right) e_2\left(k\right) \\
&\quad + \left[-\left(b_{11}+l_1\right)\left(b_{33}+l_3\right) - \left(b_{11}+l_1\right)\left(b_{33}+l_3\right) \right. \\
&\quad \left. +2\left(b_{11}+l_1\right)\left(b_{33}+l_3\right)\right] e_1\left(k\right) e_3\left(k\right) \\
&\quad + \left[-\left(b_{22}+l_2\right)\left(b_{33}+l_3\right) + \left(b_{22}+l_2\right)\left(b_{33}+l_3\right)\right] e_2\left(k\right) e_3\left(k\right) \\
&= \left(3\left(b_{11}+l_1\right)^2 - 1\right) e_1^2\left(k\right) + \left(2\left(b_{22}+l_2\right)^2 - 1\right) e_2^2\left(k\right) \\
&\quad + \left(6\left(b_{33}+l_3\right)^2 - 1\right) e_3^2\left(k\right),
\end{aligned}
$$

and by using (8), we get: $\Delta V\left(e(k)\right) < 0$.

Thus, from the Lyapunov stability theory, it is immediate that $\lim_{k\to\infty} e_i(k) = 0$, $(i = 1, 2, 3)$. Therefore, the systems (1) and (2) are globally synchronized.

4 Illustrative Example

In this example, we consider the discrete-time Rössler system as the drive system and the controlled Wang system as the response system. The discrete-time Rössler system [16], is described by

$$
\begin{aligned}
x_1\left(k+1\right) &= \alpha x_1\left(k\right)\left(1 - x_1\left(k\right)\right) - \beta\left(x_3\left(k\right) + \gamma\right)\left(1 - 2x_2\left(k\right)\right), \\
x_2\left(k+1\right) &= \delta x_2\left(k\right)\left(1 - x_2\left(k\right)\right) + \varsigma x_3\left(k\right), \\
x_3\left(k+1\right) &= \eta\left(\left(x_3\left(k\right) + \gamma\right)\left(1 - 2x_2\left(k\right)\right) - 1\right)\left(1 - \theta x_1\left(k\right)\right),
\end{aligned} \tag{10}
$$

where $\alpha = 3.8$, $\beta = 0.05$, $\gamma = 0.35$, $\delta = 3.78$, $\varsigma = 0.2$, $\eta = 0.1$, $\theta = 1.9$. The hyperchaotic attractor of the 3D discrete-time Rössler system is shown in Fig. 1.

The controlled Wang system can be described as

$$
\begin{aligned}
y_1\left(k+1\right) &= a_3 y_2\left(k\right) + \left(a_4 + 1\right) y_1\left(k\right) + u_1, \\
y_2\left(k+1\right) &= a_1 y_1\left(k\right) + y_2\left(k\right) + a_2 y_3\left(k\right) + u_2, \\
y_3\left(k+1\right) &= \left(a_7 + 1\right) y_3\left(k\right) + a_6 y_2\left(k\right) y_3\left(k\right) + a_5 + u_3,
\end{aligned} \tag{11}
$$

where $U = \left(u_1,\ u_2,\ u_3\right)^T$ is the vector controller. The 3D hyperchaotic Wang system (i.e., the system (11) with $u_1 = 0$, $u_2 = 0$, $u_3 = 0$) is chaotic when the parameter values are taken as $\left(a_1, a_2, a_3, a_4, a_5, a_6, a_7\right) = \left(-1.9, 0.2, 0.5, -2.3, 2, -0.6, -1.9\right)$ [15]. The hyperchaotic attractor of the 3D Wang system is shown in Fig. 2. To achieve global synchronization between the discrete-time Rössler system and the controlled Wang system, according to our approach presented in Section 2, the vector controller can be

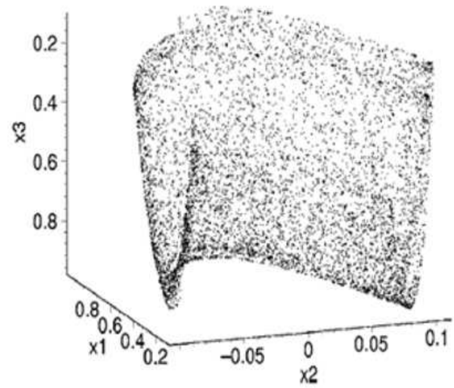

Figure 1: The hyperchaotic attractor of the discrete-time Rossler system.

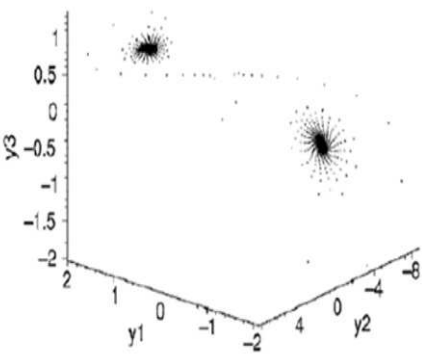

Figure 2: Hyperchaotic attractor of Wang system when $(a_1, a_2, a_3, a_4, a_5, a_6, a_7, \delta) = (-1.9, 0.2, 0.5, -2.3, 2, -0.6, -1.9, 1)$.

constructed as follows

$$\begin{aligned} u_1 &= l_1 e_1(k) + (1 - a_3 + l_2) e_2(k) - (a_7 + 1 + l_3) e_3(k) - R_1, \\ u_2 &= -(a_1 + a_4 + 1 + l_1) e_1(k) + l_2 e_2(k) + (a_7 + 1 - a_2 + l_3) e_3(k) - R_2, \\ u_3 &= (a_4 + 1 + l_1) e_1(k) + (a_7 + 1 + 2l_3) e_3(k) - R_3, \end{aligned} \tag{12}$$

where the control constants $(l_i)_{1 \le i \le 3}$ are chosen as follows

$$\begin{cases} -a_4 - 1 - \frac{1}{\sqrt{3}} < l_1 < -a_4 - 1 + \frac{1}{\sqrt{3}}, \\ -1 - \frac{1}{\sqrt{2}} < l_2 < -1 + \frac{1}{\sqrt{2}}, \\ -a_7 - 1 - \frac{1}{\sqrt{6}} < l_3 < -a_7 - 1 - \frac{1}{\sqrt{6}} \end{cases} \tag{13}$$

and

$$R_i = L_i + N_i, \quad i = 1, 2, 3, \tag{14}$$

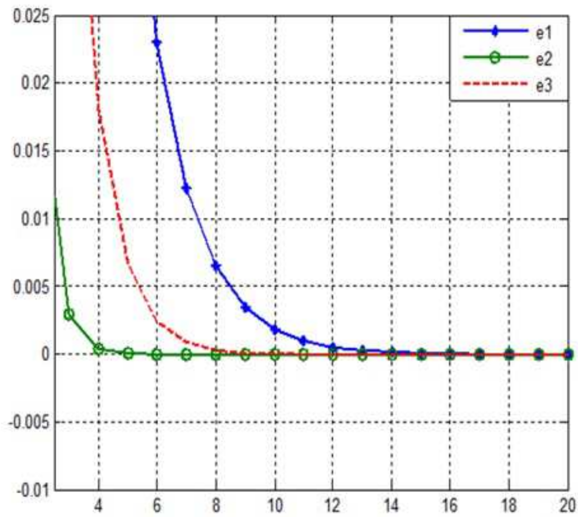

Figure 3: Time evolution of synchronization errors between the drive system (10) and the response system (11).

where

$$\begin{aligned} L_1 &= (a_4+1-\alpha)\,x_1(k)+(a_3-\beta\gamma 2)\,x_2(k)+\beta x_3(k)+\beta\gamma, \\ L_2 &= a_1x_1(k)+(1-\beta\gamma 2)\,x_2(k)+(a_2-\varsigma)\,x_3(k), \\ L_3 &= -\theta(1-\eta\gamma)\,x_1(k)+2\eta\gamma x_2(k)+(a_7+1-\eta)\,x_3(k)+a_5-\eta\gamma+1 \end{aligned} \tag{15}$$

and

$$\begin{aligned} N_1 &= \alpha x_1^2(k)-2\beta x_3(k)\,x_2(k), \\ N_2 &= \delta x_2^2(k), \\ N_3 &= a_6y_2(k)\,y_3(k)-2\eta\gamma\theta x_1(k)\,x_2(k)+\eta\theta x_1(k)\,x_3(k) \\ &\quad +2\eta x_2(k)\,x_3(k)-2\eta\theta x_1(k)\,x_2(k)\,x_3(k). \end{aligned} \tag{16}$$

It is easy to show that all conditions of Theorem 3.1 are satisfied. Therefore, the drive system (10) and the response system (11) are globally synchronized.

Using controllers (12), the error functions can be described as:

$$\begin{aligned} e_1(k+1) &= (a_4+1+l_1)\,e_1(k)+(1+l_2)\,e_2(k)-(a_7+1+l_3)\,e_3(k), \\ e_2(k+1) &= -(a_4+1+l_1)\,e_1(k)+(1+l_2)\,e_2(k)+(a_7+1+l_3)\,e_3(k), \\ e_3(k+1) &= (a_4+1+l_1)\,e_1(k)+2(a_7+1+l_3)\,e_3(k). \end{aligned} \tag{17}$$

Corollary 4.1 *For two coupled systems: the hyperchaotic discrete-time Rössler system and the hyperchaotic Wang system, if we choose the control constants* $(l_i)_{1\leq i\leq 3}$ *such that:* $l_1=1$, $l_2=-\frac{1}{2}$ *and* $l_3=0.8$. *Then, they are globally synchronized, see Fig. 3.*

5 Conclusion

In this paper, a new control scheme has been designed to achieve synchronization between 3-D quadratic drive-response chaotic systems in discrete-time. Based on nonlinear controllers and Lyapunov stability theory, a synchronization criterion has been obtained and new conditions have been derived. It was shown that the proposed controllers guarantee the asymptotic convergence to zero of the errors between the drive and the response systems. Finally, numerical example and computer simulations were used to verify the effectiveness of the proposed approach.

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Initial Trajectories of Propagation of Fatigue Cracks Under Biaxial Cyclic Loading with Phase Difference

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Abstract: This paper presents a method for predicting initial trajectories of propagation of two separate fatigue cracks, which are developed under two perpendicular cyclic loads with phase difference between them. Calculation of trajectories of these two initial cracks is the first step in prediction of trajectories and rate of propagation of long cracks. This problem is important for analysis of durability of structures subjected to biaxial loading, where it is necessary to know trajectories of cracks' propagation, stress intensity factors along the trajectories and dependence of cracks' growth rates on stress intensity factors. Existing methods, based on finite element analysis and automatic mesh generation [1,2], allow to perform such calculations only for uniaxial loading and for multi-axial proportional loading, without phase difference between applied external forces. Experiments, presented in this paper, show that under biaxial loading with phase difference between applied loads, two cracks are developed. Comparison of calculated and experimentally observed initial directions of cracks propagation shows that the calculations correctly reflect existence of two cracks and the fact that they are approximately symmetrical about the line that makes 45° with directions of applied loads. This method can become a theoretical basis for extending capabilities of existing methods, based on finite element analysis and automatic mesh generation, of predicting trajectories of fatigue cracks under complex loading conditions.

Keywords: *fatigue; biaxial loading; phase difference; trajectories of cracks propagation.*

Mathematics Subject Classification (2010): 34A34, 34D20, 70E50, 93C15.

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1 Introduction

It has been a common practice to characterize the fatigue crack growth in metals under uniaxial loading. But majority of aerospace structural components experience a combination of axial, bending, shear and torsion stresses, resulting in a complex stress state. It is thus appropriate to extend the fatigue crack growth studies to non-uniaxial loading conditions. For biaxial tension-tension loading without phase difference between applied loads, such studies were performed, for example, by Misak, Perel, Sabelkin and Mall [3]. In the present paper, this study is extended to biaxial tension-tension loading with phase difference between applied loads. Such loading results in growth of two fatigue cracks, as will be shown in this paper. The test material is aluminum alloy 7075-T6, which is widely used as a structural material in the military and civilian aircraft fleet.

Along the direction of crack propagation, the mode II stress intensity factor, K_{II}, is usually much smaller than the mode I stress intensity factor, K_I. So, in the biaxial loading, the dependence of crack growth rate $\frac{da}{dN}$ on the stress intensity factor K_{II} is small, and the dependence of $\frac{da}{dN}$ on K_I can be established experimentally, under uniaxial loading with force normal to the crack. If tips of the crack (or cracks) have different stress intensity factors at any given time instant during the loading cycle, then construction of the cracks' trajectories should be performed with account of relation between $\frac{da}{dN}$ and K_I. This means that if the cracks' trajectories are constructed by an incremental procedure, then, at each step of the procedure, the increments of the cracks' lengths are calculated from the relation between $\frac{da}{dN}$ and K_I, where K_I is a function of crack length, a. But if the tips of the crack have equal stress intensity factors, or if we have one crack originating from an edge, then in constructing the crack's trajectory incrementally, some small straight-line increments of the crack can be specified arbitrarily, without considering $\frac{da}{dN}$. In this case, a number of cycles, N, corresponding to the chosen crack length increment, can be calculated later, in the post-process stage of analysis, using the relation between $\frac{da}{dN}$ and K_I.

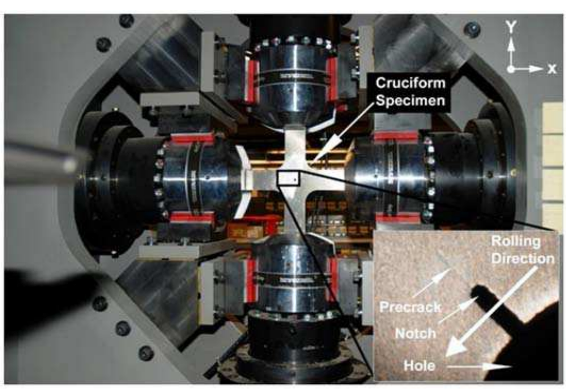

Figure 1a: Experimental setup.

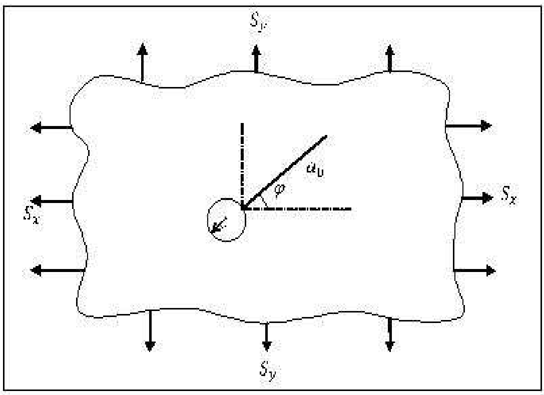

Figure 1b: Pre-crack of length a_0, originating from circular hole of radius r in a thin plate.

2 Direction of Crack Propagation

For construction of the cracks' trajectories, a formula was used for direction of initial crack propagation, based on a hypothesis that crack propagates in the direction $\theta = \Theta$ (Figure 2), in which $\sigma_{\theta\theta}(\theta)$ takes the maximum value (Erdogan and Sih [4]). This hypothesis leads to the formula

$$\Theta = 2\arctan \frac{1 - \sqrt{1 + 8\left(\frac{K_{II}}{K_I}\right)^2}}{4\frac{K_{II}}{K_I}}. \tag{1}$$

A change of shape of a macroscopic crack in one cycle of loading (or in a small number of cycles) is negligibly small, so at any time instant within one cycle, the angle $\theta = \Theta$ in eq. (1) is measured with respect to a direction of the initial crack (pre-crack), as shown in Figures 1a, 1b and 2.

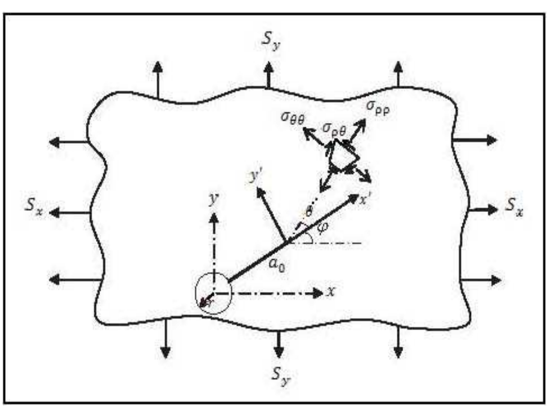

Figure 2: Global rectangular coordinate system xy, local rectangular coordinate system $x'y'$ and polar coordinate system $\rho\theta$.

3 Stress Intensity Factors

In this work, we considered a thin plate of Aluminum 7075-T6, with a pre-crack of length a_0, originating from a circular hole of radius r at angle $\varphi = 45°$ to directions of remote principal stresses S_x and S_y (Figures 1a and 1b). This pre-crack was created by applying sinusoidal loads $S_x(t) = S_y(t)$, without the phase difference between them. After the creation of the pre-crack, the phase difference γ between the loads $S_x(t)$ and $S_y(t)$ was introduced, and the loads became

$$S_x(t) = \frac{(S_x)_{\max} + (S_x)_{\min}}{2} + \frac{(S_x)_{\max} - (S_x)_{\min}}{2} \sin(2\pi\nu t), \tag{2}$$

$$S_y(t) = \frac{(S_y)_{\max} + (S_y)_{\min}}{2} + \frac{(S_y)_{\max} - (S_y)_{\min}}{2} \sin(2\pi\nu t + \gamma). \tag{3}$$

In our experiments and calculations we set

$$\frac{(S_x)_{\min}}{(S_y)_{\min}} = \frac{(S_x)_{\max}}{(S_y)_{\max}} = 1, \qquad \frac{(S_x)_{\min}}{(S_x)_{\max}} = \frac{(S_y)_{\min}}{(S_y)_{\max}} \equiv R = 0.5. \tag{4}$$

For a crack, originating from elliptical hole, at an arbitrary angle to directions of remote principal stresses, a solution for stress intensity factors is given in the paper of Kaminski and Sailov [5]. For the particular case of circular hole and the pre-crack at 45° with the principal stresses, as was the case in our experiments and calculations, this solution takes the form

$$K_I = \frac{\sqrt{\pi r}}{2\sqrt{2}} \sqrt{\frac{l_0 (l_0 + 2)^3}{(l_0 + 1)^3}} (S_x + S_y), \tag{5}$$

$$K_{II} = \frac{\sqrt{\pi r}}{2\sqrt{2}} \sqrt{\frac{l_0 (l_0 + 2)^3}{(l_0 + 1)^3}} (S_x - S_y), \tag{6}$$

where

$$l_0 = \frac{1}{2}\left(-1 + \frac{a_0}{r} + \sqrt{2\frac{a_0}{r} + \frac{a_0^2}{r^2} + 1}\right). \tag{7}$$

So, for the case of circular hole and the pre-crack at 45° with the principal stresses, we have

$$\frac{K_{II}}{K_I} = \frac{S_x - S_y}{S_x + S_y}. \tag{8}$$

4 Rate of Crack Propagation

4.1 Rate of crack growth due to cyclic variation of load

In considering a small number of loading cycles (several hundred cycles), the change of stress intensity factors is only due to the change of external load with time, since the effect of the change of the crack's shape and length on the stress intensity factors is negligibly small. According to the Dugdale hypothesis, in thin ideally elastic-plastic plates, with a through-thickness crack, plastic strains are concentrated along a narrow layer on the continuation of the crack, so that the plastic zone can be treated as a line of discontinuity of elastic displacement. Therefore, according to the Dugdale hypothesis, a solution for displacements can be sought as a discontinuous solution based on the elasticity theory. On the basis of this approach, the following formula was obtained by Cherepanov [6] for displacement of one side of the plastic yield strip:

$$v(x') = -\frac{2\sigma_Y}{\pi E}\left(2\sqrt{D(D-x')} + x'\ln\frac{\sqrt{D}-\sqrt{D-x'}}{\sqrt{D}+\sqrt{D-x'}}\right), \tag{9}$$

where

$$D = \frac{\pi K_I^2}{8\sigma_Y^2} \tag{10}$$

is size of plastic zone in the Dugdale model, σ_Y is yield stress, and E is Young's modulus. During the crack propagation, the strain energy dissipation per unit area of a newly formed surface of the crack (specific energy dissipation) is the path integral along the line of the plastic zone (Cherepanov [6]):

$$\begin{aligned}\gamma_* &= \int\limits_{\text{(plastic zone)}} \sigma_{y'y'}\,dv = \int_0^D \sigma_{y'y'}\left(\frac{\partial v}{\partial x'}dx' + \frac{\partial v}{\partial K_I}\frac{dK_I}{da}dx'\right) \\ &= \sigma_Y\int_0^d \frac{\partial v}{\partial x'}dx' + \sigma_Y\frac{dK_I}{da}\int_0^D \frac{\partial v}{\partial K_I}dx',\end{aligned} \tag{11}$$

where the term $\frac{dK_I}{da}$ is due to increase of length of plastic zone because of change of stress intensity factor (i.e. because of change of load) in a cycle. This term is not related to the growth of the crack.

Substitution of eqs. (9) and (10) into eq. (11) and performing integration gives the result

$$\gamma_* = \frac{K_I^2}{2E} - \frac{\pi}{12E\sigma_Y^2}K_I^3\frac{dK_I}{da}. \tag{12}$$

Introducing notation

$$K_* = \sqrt{2E\gamma_*}, \tag{13}$$

we receive from eq. (12)

$$|da| = \frac{\pi}{6\sigma_Y^2}\left|\frac{K_I^3}{K_*^2 - K_I^2}\,dK_I\right| \tag{14}$$

or

$$\left|\frac{da}{dt}\right| = \frac{\pi}{6\sigma_Y^2}\left|\frac{K_I^3}{K_*^2 - K_I^2}\frac{dK_I}{dt}\right|. \tag{15}$$

It should be noted that here, like in the original work of Cherepanov [6], the formula (15) should be considered as being a semi-empirical one, with K_* treated as a material constant, i.e. the derivations, leading to the formula (15), are not strict, but only such that help to guess this semi-empirical formula. The crack grows $\left(\frac{da}{dt} > 0\right)$ during the crack opening, i.e. when $K_I > 0$ and $\frac{dK_I}{dt} > 0$. Besides, usually, $K_*^2 - K_I^2 > 0$, as will be shown later (Figure 3). Therefore, eq. (15) can be written in a physically meaningful form as

$$\frac{da}{dt} = \frac{\pi}{6\sigma_Y^2}\frac{K_I^3}{K_*^2 - K_I^2}\frac{dK_I}{dt}. \tag{16}$$

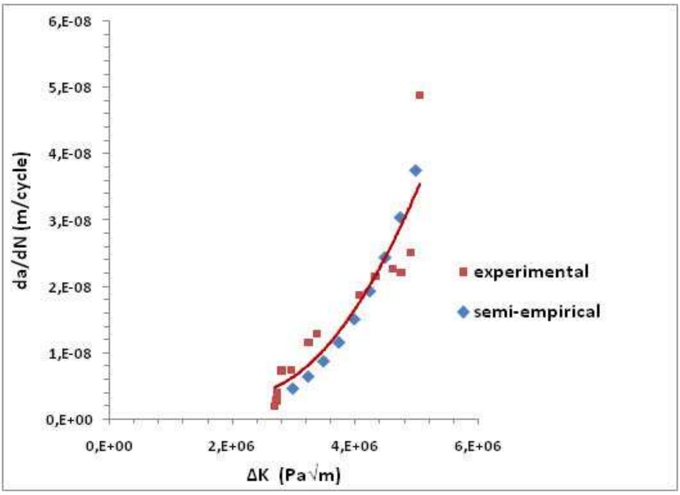

Figure 3: Uniaxial loading, force perpendicular to crack. Comparison of experimental and theoretical curves under the following choice of material constants: $K_* = 4.52459 \times 10^7 \ Pa\sqrt{m}$, $\lambda = 4.8353 \times 10^{-7} \frac{1}{Pa\sqrt{m}}$, $\nu_0 = 1.5 \times 10^{-10} \frac{m}{s}$.

4.2 Rate of crack growth due to chemical reactions

If the crack grows due to a chemical reaction, for example due to corrosion, then the crack's growth rate $\frac{da}{dt}$ is proportional to the rate of chemical reaction, and, therefore, proportional to $\exp\left(-\frac{U}{RT}\right)$, where U is activation energy of the reaction, T is temperature, and $R = 8.314\frac{J}{K \times mole}$ is universal gas constant, according to the Arrhenius equation (Arrhenius [7]; Levine [8]). The activation energy U is proportional to stress at the crack tip, and, therefore, to K_I (Cherepanov [6]). Therefore, the crack growth rate due a chemical reaction can be written as

$$\frac{da}{dt} = v_0 \exp\left(\lambda K_I\right), \tag{17}$$

where v_0 and λ are material characteristics that depend on temperature and chemical composition of environment.

4.3 Rate of crack growth due to combined effects of cyclic variation of load and chemical reactions

If the crack grows due to both cyclic variation of K_I and chemical reactions, then the right sides of eqs. (16) and (17) have to be summed up:

$$\frac{da}{dt} = \frac{\pi}{6\sigma_Y}\frac{K_I^3}{K_*^2 - K_I^2}\frac{dK_I}{dt} + v_0 \exp\left(\lambda K_I\right) \tag{18}$$

or

$$a(t) = -\frac{\pi}{12\sigma_Y^2}\left(K_I^2(t) - K_*^2\right) - \frac{\pi K_*^2}{12\sigma_Y^2}\ln\left(K_I^2(t) - K_*^2\right) + v_0\int \exp(\lambda K_I(t))\,dt + \text{const.} \tag{19}$$

5 Experimental Determination of Material Constants

Eqs. (18) and (19) contain four material characteristics, σ_Y, K_*, v_0 and λ, which need to be determined experimentally. Experimental data on fatigue crack growth rates are usually represented in the form of $\frac{da}{dN}$ versus $\Delta K_I = (K_I)_{\max} - (K_I)_{\min}$, where $(K_I)_{\max}$ and $(K_I)_{\min}$ are maximum and minimum values of K_I in a cycle of loading. So, the theoretical equation (18) or (19) needs to be rewritten in the same form, and then the material characteristics K_*, v_0 and λ in these theoretical equations can be chosen such that the theoretical plot of $\frac{da}{dN}$ versus ΔK_I is close to the experimental one.

As it was mentioned previously, the material characteristics can be established with the use of experimental data obtained in uniaxial loading with remote stress

$$S(t) = \frac{S_{\max} + S_{\min}}{2} + \frac{S_{\max} - S_{\min}}{2} \sin(2\pi\nu t) \tag{20}$$

perpendicular to the crack and the mode I stress intensity factor

$$K_I(t) = \frac{(K_I)_{\max} + (K_I)_{\min}}{2} + \frac{(K_I)_{\max} - (K_I)_{\min}}{2} \sin(2\pi\nu t). \tag{21}$$

Introducing notations

$$R \equiv \frac{S_{\min}}{S_{\max}} = \frac{(K_I)_{\min}}{(K_I)_{\max}}, \quad H \equiv \frac{1+R}{1-R}, \tag{22}$$

we can write eq. (21) as

$$K_I(t) = \frac{1}{2}(\Delta K_I)\Big(H + \sin(2\pi\nu t)\Big). \tag{23}$$

Crack growth in one cycle of loading occurs during the crack opening, i.e. from the time instant $t = \frac{3}{4\nu}$, when $K_I = (K_I)_{\min} = \frac{R}{1-R}\Delta K_I$, to the time instant $t = \frac{5}{4\nu}$, when $K_I = (K_I)_{\max} = \frac{1}{1-R}\Delta K_I$. Therefore, the increment of the crack length in one cycle of loading, $\frac{da}{dN}$, is

$$\frac{da}{dN} = a\Big|_{K_I=\Delta K_I/(1-R)} - a\Big|_{K_I=\Delta K_I R/(1-R)} = a\Big|_{t=5/(4\nu)} - a\Big|_{t=3/(4\nu)}. \tag{24}$$

Substituting eqs. (19) and (23) into eq. (24), we obtain

$$\begin{aligned}\frac{da}{dN} = -\frac{\pi K_*^2}{12\sigma_Y^2}\left(H\frac{(\Delta K_I)^2}{K_*^2} + \ln\frac{(1-R)^2 K_*^2 - (\Delta K_I)^2}{(1-R)^2 K_*^2 - R^2(\Delta K_I)^2}\right) \\ + v_0 \exp(0.5\lambda H\ \Delta K_I)\int_{\frac{3}{4\nu}}^{\frac{5}{4\nu}} \exp(0.5\lambda\ \Delta K_I\ \sin 2\pi\nu t)\ dt.\end{aligned} \tag{25}$$

In our experiments,

$$\nu = 10Hz, \quad R = 0.5, \quad H = \frac{1+R}{1-R} = 3. \tag{26a}$$

Besides, if our material, Aluminum 7075-T6, is modeled as ideally elastic-plastic, then the yield stress can be taken as

$$\sigma_Y = 4.08249 \times 10^9 Pa. \tag{26b}$$

We need to choose such numerical values of the material constants K_*, v_0 and λ that the graph of $\frac{da}{dN}$ versus ΔK_I, obtained from the semi-empirical formula (25) with numerical values (26), is close to the experimental one. We will try to use the following values

$$v_0 = 1.5 \times 10^{-10}\,\frac{m}{s}, \quad \lambda = 4.8353 \times 10^{-7}\frac{1}{Pa\sqrt{m}}, \quad K_* = 4.52459 \times 10^7 Pa\sqrt{m}. \tag{27}$$

Substituting numerical values from eqs. (26) and (27) into eq. (25), we receive

$$\begin{aligned} \frac{da}{dN} &= -\left(4.71237 \times 10^{-20}\right)\left(\Delta K_I\right)^2 \\ &- \left(3.21571 \times 10^{-5}\right)\,\ln\frac{5.11798 \times 10^{14} - \left(\Delta K_I\right)^2}{5.11798 \times 10^{14} - 0.25\left(\Delta K_I\right)^2} \\ &+ \left(1.5 \times 10^{-10}\right)\,\exp\left(7.25295 \times 10^{-7}\Delta K_I\right) \\ &\times \int_{0.075}^{0.125} \exp\left(2.41765 \times 10^{-7}\Delta K_I \sin 62.8319t\right)\,dt. \end{aligned} \tag{28}$$

Formula (28) gives the correspondence between numerical values of $\frac{da}{dN}$ and ΔK_I as shown in Table 1.

$\Delta K_I\,(Pa\sqrt{m})$	$\frac{da}{dN}\left(\frac{m}{cycle}\right)$
3×10^6	4.79467×10^{-9}
3.25×10^6	6.60663×10^{-9}
3.5×10^6	8.89594×10^{-9}
3.75×10^6	1.17424×10^{-8}
4×10^6	1.52328×10^{-8}
4.25×10^6	1.94615×10^{-8}
4.5×10^6	2.45306×10^{-8}
4.75×10^6	3.05502×10^{-8}
5×10^6	3.76391×10^{-8}

Table 1:

The plot of data in Table 1, together with experimental plot of $\frac{da}{dN}$ versus ΔK_I for uniaxial loading, is shown in Figure 3. These plots are close to each other. Therefore, numerical values of the material constants v_0, λ and K_* in eq. (27) are chosen correctly.

6 Trajectories of Cracks

Parametric equations of trajectories of cracks, in the local coordinate system $x'y'$ (Figure 2), with axis x' aligned with the pre-crack,

$$x' = x'(t), \quad y' = y'(t) \tag{29}$$

can be written as follows

$$\frac{dx'}{dt} = \frac{da}{dt}\cos\Theta, \quad \frac{dy'}{dt} = \frac{da}{dt}\sin\Theta. \tag{30}$$

Experiments show that under biaxial tensile and compressive loading with phase difference between applied loads (Figures 1a and 1b), two cracks originate from the pre-crack, and their trajectories are approximately symmetrical about the line along the pre-crack (Figures 4b, 5b, 6b). Therefore, it is assumed that in the first half-cycle of loading, one of the cracks grows starting from the edge of the pre-crack; and in the second half-cycle of loading, the second crack grows starting from the same location, i.e. from the edge of the pre-crack. So, initial conditions for the first half-period of loading are

$$x'(0) = 0, \quad y'(0) = 0, \tag{31}$$

and initial conditions for the second half-period are

$$x'\left(\frac{T}{2}\right) = 0, \quad y'\left(\frac{T}{2}\right) = 0, \tag{32}$$

where T is the time duration of one cycle of loading. A solution of differential equations (30) for the first half-period of loading (for crack branch 1), i.e. a solution with initial conditions (31), is

$$x'(t) = \int_0^t \frac{da}{dt}\cos\Theta, \quad y'(t) = \int_0^t \frac{da}{dt}\sin\Theta\, dt. \tag{33}$$

A solution of differential equations (30) for the second half-period of loading (for crack branch 2), i.e. a solution with initial conditions (32), is

$$x'(t) = \int_{T/2}^t \frac{da}{dt}\cos\Theta\, dt, \quad y'(t) = \int_{T/2}^t \frac{da}{dt}\sin\Theta\, dt. \tag{34}$$

If a small number of cycles is considered, during which the effect of change of cracks' shapes and lengths on values of the stress intensity factors is negligibly small (several hundred cycles), then a complete system of equations, leading to calculation of the cracks' trajectories, is

$$S_x(t) = \frac{(S_x)_{\max} + (S_x)_{\min}}{2} + \frac{(S_x)_{\max} - (S_x)_{\min}}{2}\sin(2\pi\nu t), \qquad \text{eq. (2)}$$

$$S_y(t) = \frac{(S_y)_{\max} + (S_y)_{\min}}{2} + \frac{(S_y)_{\max} - (S_y)_{\min}}{2}\sin(2\pi\nu t + \gamma), \qquad \text{eq. (3)}$$

$$\frac{K_{II}}{K_I} = \frac{S_x - S_y}{S_x + S_y}, \qquad \text{eq. (8)}$$

$$\Theta = 2\arctan\frac{1 - \sqrt{1 + 8\left(\frac{K_{II}}{K_I}\right)^2}}{4\frac{K_{II}}{K_I}}, \qquad \text{eq. (1)}$$

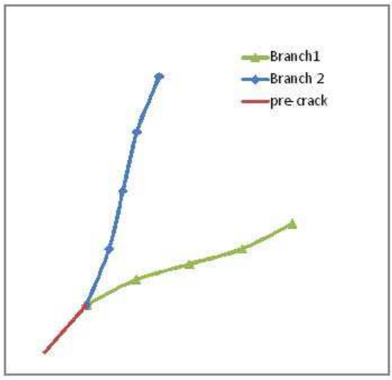

Figure 4a: Calculated trajectories of propagation of cracks in the first cycle of loading, when phase difference between applied loads was 180°.

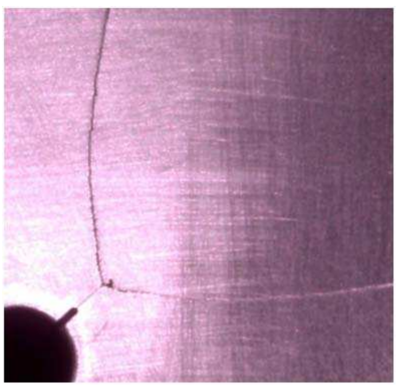

Figure 4b: Trajectories of propagation of cracks, observed in experiment, when phase difference between applied loads was 180°.

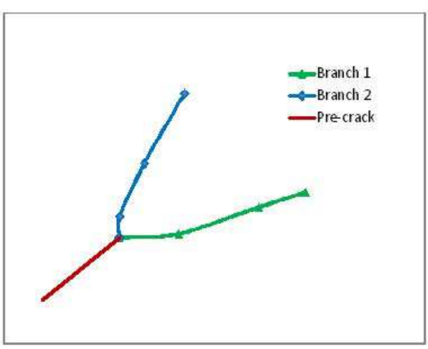

Figure 5a: Calculated trajectories of propagation of cracks in the first cycle of loading, when phase difference between applied loads was 90°.

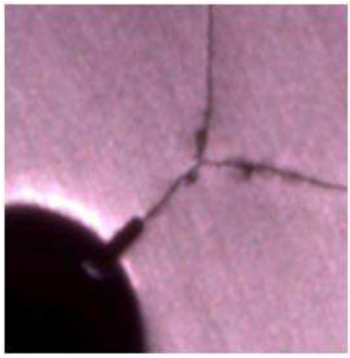

Figure 5b: Trajectories of propagation of cracks, observed in experiment, when phase difference between applied loads was 90°.

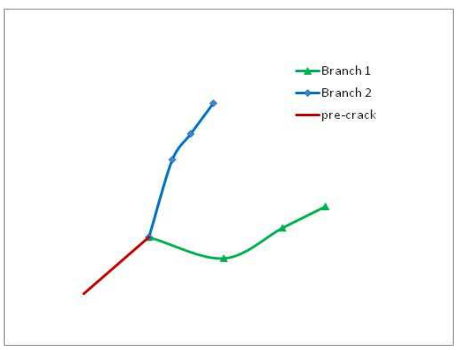

Figure 6a: Calculated trajectories of propagation of cracks in the first cycle of loading, when phase difference between applied loads was 45°.

$$l_0 = \frac{1}{2}\left(-1 + \frac{a_0}{r} + \sqrt{2\frac{a_0}{r} + \frac{a_0^2}{r^2} + 1}\right), \qquad \text{eq. (7)}$$

$$K_I = \frac{\sqrt{\pi r}}{2\sqrt{2}}\sqrt{\frac{l_0(l_0+2)^3}{(l_0+1)^3}}(S_x + S_y), \qquad \text{eq. (5)}$$

$$\frac{da}{dt} = \frac{\pi}{6\sigma_Y^2}\frac{K_I^3}{K_{Ic}^2 - K_I^2}\frac{dK_I}{dt} + v_0 \exp\left(\lambda K_I\right), \qquad \text{eq. (18)}$$

$$x'(t) = \int_0^t \frac{da}{dt}\cos\Theta\, dt, \quad y'(t) = \int_0^t \frac{da}{dt}\sin\Theta\, dt \quad \text{for branch 1}, \qquad \text{eq. (33)}$$

$$x'(t) = \int_{T/2}^t \frac{da}{dt}\cos\Theta\, dt, \quad y'(t) = \int_{T/2}^t \frac{da}{dt}\sin\Theta\, dt \quad \text{for branch 2}. \qquad \text{eq. (34)}$$

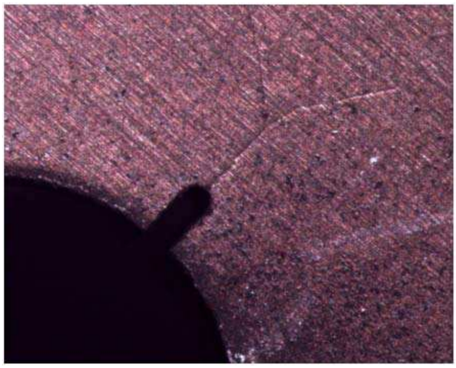

Figure 6b: Trajectories of propagation of cracks, observed in experiment, when phase difference between applied loads was 45°.

Coordinates of points of the cracks' trajectories in the global coordinate system xy can be calculated by formulas

$$\begin{aligned} x &= x' \cos\varphi - y' \sin\varphi, \\ y &= x' \sin\varphi + y' \cos\varphi, \end{aligned} \tag{35}$$

where φ is angle between axes x' and x, i.e. angle between the pre-crack and the axis x (Figure 2).

Alternatively, the cracks' trajectories can be calculated by incremental procedure, which can be written briefly as follows:

$$t_0 = 0, \quad t_m = \frac{m}{M}\frac{T}{2}, \tag{36a}$$

$$m = 1, 2, \ldots, M \quad \text{for first half-cycle}, \tag{36b}$$

$$m = M + 1, \ldots, 2M \quad \text{for second half-cycle}, \tag{36c}$$

where $2M$ is a number of equal sub-intervals into which time interval of one cycle of loading, $[0, T]$, is divided;

$$x'_m = x'(t_m), \quad y'_m = y'(t_m), \quad \Theta_m = \Theta(t_m), \quad a_m = a(t_m), \quad \Delta a_m = a_m - a_{m-1}, \tag{37}$$

$$x'_0 = 0, \quad y'_0 = 0, \quad \Theta_0 = 0, \tag{38}$$

$$x'_m = x'_{m-1} + (\Delta a_m)\cos\Theta_m, \tag{39}$$

$$y'_m = y'_{m-1} + (\Delta a_m)\sin\Theta_m. \tag{40}$$

The calculated trajectories of cracks' propagation in the first cycle of loading, with phase differences 180°, 90° and 45°, are shown in Figures 4a, 5a and 6a accordingly. The corresponding experimentally observed initial directions of cracks' propagation are shown in Figures 4b, 5b and 6b accordingly. Comparison of the calculated and experimentally observed initial directions shows that the calculations correctly reflect existence of two initial cracks and the fact that they are approximately symmetrical about the line along the pre-crack.

7 Conclusion

It should be noted again that calculations of cracks' trajectories, presented in this paper, can be valid only for a small number of cycles (several hundred cycles), during which the stress intensity factors are not significantly affected by change of cracks' shapes and lengths. Therefore, calculations in this paper can be used for prediction of cracks' trajectories only during a small initial number of cycles. But if the cracks' trajectories need to be calculated for a larger number of cycles, then values of stress intensity factors should be recalculated after every several thousand cycles with the use of the finite element method, to take account of effect of change of the cracks' shapes and lengths on the stress intensity factors. Besides, in calculations of the cracks' trajectories over intervals of large number of cycles, the angle Θ, given by eq. (1), should be treated as an angle between the current direction of the crack propagation and the direction before the latest block of several thousand cycles was applied. In other words, for long cracks, Θ should be treated not as an angle between a direction of the tangent to the crack at its tip and the direction of the straight-line pre-crack (as it was done in this paper), but as an increment of this angle in the current sub-interval of loading. Such calculations of trajectories of long cracks under biaxial loading with phase difference can be a subject of future work.

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Delay Independent Stability of Co-operative and Supportive Neural Networks

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Abstract: In this paper a cooperative and supportive neural network proposed recently is considered. Time delays both in transmission of information from subsystems to main system as well as processing of information in subsystem itself are introduced into the network. Criteria on parameters of the system are obtained that establish the stability of the system independent of time delays. Examples are provided for illustration of results.

Keywords: *cooperative and supportive networks; time delays; equilibria; global stability.*

Mathematics Subject Classification (2010): 34D23, 34K20, 92B20, 93D20.

1 Introduction

Neural networks has been a subject of research for decades with growing popularity ([2], [6-11]), for its extensive application in several real world situations ([1], [3], [12-17], [21]). In [20], a new class of networks designated as co-operative and supportive neural network (CSNN, for short) was introduced. The model is suitable for explaining the dynamics of systems exhibiting hierarchy in which the collective capabilities of components involved are utilized for better performance of the system. Such systems find application in industrial information management, financial and economic systems which involve distribution

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and monitoring of various tasks. They are also useful in solving complex network problems [2], classification and clustering problems, in data mining and financial engineering [5]. They are also utilized for parameter estimation of auto regressive signals and to decompose complex classification tasks into simpler subtasks and puzzle them out. In particular, the network considered in the present study is utilized for estimation of key parameters in infectious disease models [18]. The reliability aspects of this network are studied in [14].

The model comprises two neuronal fields say F_x and F_y. Each neuron in F_x is denoted by x_i, $i = 1, 2, ..., n$ and is connected to other neurons x_j, $j = 1, 2, ..., n$ in the same field F_x. Also each x_i is connected to r_i number of neurons in the neuronal field F_y. These are denoted by y_{i_k}, $k = 1, 2, ..., r_i$, $1 \leq r_i \leq n$. These y_{i_k} support x_i in the sense that they coordinate and cooperate with it so that any task assigned to them by x_i will be attended to. The dynamics of the model is described by the following system of equations

$$x_i' = -a_i x_i + \sum_{j=1}^{n} b_{ij} f_j(x_j) + \sum_{k=1}^{r_i} c_{ii_k} g_{i_k}(x_i, y_{i_k}) + I_i, i = 1, 2, ..., n,$$

$$y_{i_k}' = -c_{i_k} y_{i_k} + \sum_{l=1}^{r_i} d_{i_l} h_{i_l}(y_{i_l}) + J_{i_k}, \quad k = 1, 2, ..., r_i, \quad 1 \leq r_i \leq n. \tag{1}$$

In (1), x_i, i=1,2,...,n denotes a typical neuron in F_x and $y_{i_k}, k = 1, 2, ...r_i$ denotes a subgroup of neurons in F_y attached to x_i. $' = \frac{d}{dt}$ denotes the derivative with respect to time variable t. a_i and c_{i_k} are positive constants known as decay rates and b_{ij}, d_{i_l} are the synaptic connection weights for all $i, j = 1, 2, \cdots, n$, $k = 1, 2, ..., r_i$ and are assumed to be real or complex constants. c_{ii_k} is the rate of distribution of information between x_i and y_{i_k}. The functions f_i, g_{i_k} and h_{i_k} are the neuronal output response functions and are more commonly known as the signal functions. I_i, J_{i_k} are exogenous inputs.

We may use the terms main component or task or group element for x_i and subcomponent or task or group element for y_{i_k} synonymous with neuron, owing to the application for which system (1) is utilized. For example, (1) may be viewed as a management information system in which x_i are in layer (say managerial or lead group) monitoring the activities of related subgroups of y_{i_k}. Thus, (1) represents both (i) hierarchical systems in which x_i can wait till y_{i_k} complete their task and return to x_i (serial processing) and (ii) coordinating systems where x_i also work along with y_{i_k} to complete their part (parallel processing).

Several modifications of (1) are suggested in [20] that take care of interactions among the neurons as well as time delays. These models are left as open problems for further research. Two types of delays are common in such systems. First one is the time delay in transferring information/completed task from y_{i_k} to x_i, called transmission or propagation delay and the second is the one that occurs while carrying out the job by y_{i_k} themselves, namely, processing delay. Introduction of these two types of time delays into the system modifies (1) as following

$$\begin{aligned} x_i' &= -a_i x_i + \sum_{j=1}^{n} b_{ij} f_j(x_j) + \sum_{k=1}^{r_i} c_{ii_k} g_{i_k}(x_i, y_{i_k}(t - \tau_{i_k})) + I_i, \\ y_{i_k}' &= -c_{i_k} y_{i_k} + \sum_{l=1}^{r_i} d_{i_l} h_{i_l}(y_{i_l}(t - \tau_{i_l})) + J_{i_k}. \end{aligned} \tag{2}$$

Here $i = 1, 2, ..., n$, $k = 1, 2, ..., r_i$ and $1 \leq r_i \leq n$.

In (2) $\tau_{i_k} \geq 0$ denote delays in transmission of data/material from sub-system y_{i_k} to the main system while $\tau_{i_l} \geq 0$ denote processing delays with subcomponents. The present paper studies the qualitative behaviour of the solutions of (2) under the influence of time delays. The present study is important in the context of established influence of time delays on neural network systems and any physical system. (2) is quite general in the sense that it includes several modifications of (1) suggested in [20]. In fact (2) combines the models III and IV of [20].

The paper is organized as follows. For the system (2) we establish the conditions of existence and uniqueness of solutions, equilibria in Section 2. Different Lyapunov functionals are utilized to establish stability of equilibria in Section 3. Examples are provided for an illustration of the results. Finally the paper is concluded with a discussion in Section 4.

2 Existence of Solutions and Equilibria

From the theory of delay differential equations, local Lipschitz condition on the response functions (f_i, g_{i_k} and h_{i_k}) which are at least continuous in their domains of definitions, guarantees the existence of solutions to (2) (see [4,19,20]). However it is useful for researchers to note that conditions weaker than Lipschitz condition on these response functions that guarantee the existence of unique solutions to such systems are also available in literature (e.g., [19]). Thus, we may choose f_i, g_{i_k} and h_{i_k} from a very general class of functions. With this background, we tacitly assume that the system (2) possesses unique solutions that are continuable in their maximal intervals of existence. However, we need the following Lipschitz conditions on these functions to establish the existence of equilibria and their stability in subsequent sections:

$$\begin{aligned} \| g_{i_k}(x_i, y_{i_k}) - g_{i_k}(\overline{x}_i, \overline{y}_{i_k}) \| &\leq M_{1i_k}|y_{i_k} - \overline{y}_{i_k}| + M_{2i_k}|x_i - \overline{x}_i|, && (3)\\ |f_j(x_j) - f_j(\overline{x}_j)| &\leq p_j|x_j - \overline{x}_j|, && (4)\\ |h_{i_k}(y_{i_k}) - h_{i_k}(\overline{y}_{i_k})| &\leq q_{i_k}|y_{i_k} - \overline{y}_{i_k}|, && (5)\end{aligned}$$

where M_{1i_k}, M_{2i_k}, p_j *and* q_{i_k} are positive constants.

Since time delays do not disturb the presence of equilibria, as in [20], we have

Theorem 2.1. Let a_i *and* c_{i_k} $(i = 1$ *to* $n,\ k = 1$ *to* $r_i)$ be positive numbers such that

$$\begin{aligned} \sum_{j=1}^{n} |b_{ij}|p_j + \sum_{k=1}^{r_i} |c_{ii_k}|M_{2i_k} &< a_i, i = 1, 2, ..., n,\\ \sum_{l=1}^{r_i} |d_{i_l}|q_{i_l} + \sum_{k=1}^{r_i} |c_{ii_k}|M_{1i_k} &< c_{i_k}, \quad k = 1, 2, ..., r_i, \quad 1 \leq r_i \leq n. && (6)\end{aligned}$$

Then the system (2) possesses a unique positive equilibrium for each i, k. If we denote this equilibrium solution of (2) by $(x_i^*, y_{i_k}^*)$, then we should have

$$\begin{aligned} a_i x_i^* &= \sum_{j=1}^{n} b_{ij} f_j(x_j^*) + \sum_{k=1}^{r_i} c_{ii_k} g_{i_k}(x_i^*, y_{i_k}^*) + I_i, i = 1, 2, ..., n.\\ c_{i_k} y_{i_k}^* &= \sum_{l=1}^{r_i} d_{i_l} h_{i_l}(y_{i_l}^*) + J_{i_k}, \quad k = 1, 2, ..., r_i, \quad 1 \leq r_i \leq n. && (7)\end{aligned}$$

We shall now proceed to the stability of this unique equilibrium whose existence is ensured by Theorem 2.1.

3 Global Stability Results

In this section we shall establish criteria for the global asymptotic stability of the equilibrium patterns of system (2). The conditions for global stability of (1) are presented in [20]. We shall see how the presence of time delays influences the stability here in the context that time delays have tendency of disturbing the stability by introducing oscillations into the system. We begin with

Case 1. No processing delays within sub components:

We start with a special case of (2) in which we assume that $\tau_{i_l} = 0$ for all i_l. This means that we are considering a state when the sub components finish their part of job without any delay as required by x_i. However the system is characterized by the delays (i.e., $\tau_{i_k} \geq 0$) in transmission of these outcomes to main system.

We need the following inequality for our first result.

For all real numbers u, v and $\eta > 0$ we have

$$uv \leq \frac{1}{4\eta}u^2 + \eta v^2. \tag{8}$$

Theorem 3.1. *Assume that conditions (3)-(5) hold. The equilibrium $(x_i^*, y_{i_k}^*)$ of (2) is globally asymptotically stable for any length of time delays $\tau_{i_k} \geq 0$, for $i = 1, 2, ..., n$ and $k = 1, 2, ..., r_i$, provided the parameters satisfy any of the following sets of inequalities:*

$$a).\ \sum_{j=1}^{n} |b_{ij}| p_j \frac{1}{4\eta_1} + \sum_{j=1}^{n} |b_{ji}| p_i \eta_1 + \sum_{k=1}^{r_i} |c_{ii_k}| (M_{2i_k} + M_{1i_k}\eta_2) \quad < \quad a_i,$$

$$|c_{ii_k}| M_{1i_k} \frac{1}{4\eta_2} + \eta_3 \sum_{k=1}^{r_i} |d_{i_k}| q_{i_k}| + \frac{1}{4\eta_3} \sum_{k=1}^{r_i} |d_{i_k}| q_{i_k}| \quad < \quad c_{i_k},$$

$$b).\ \sum_{j=1}^{n} |b_{ij}| p_j^2 \frac{1}{4\eta_1} + \sum_{j=1}^{n} |b_{ji}| \eta_1 + \sum_{k=1}^{r_i} |c_{ii_k}| (M_{2i_k} + M_{1i_k}\eta_2) \quad < \quad a_i,$$

$$|c_{ii_k}| M_{1i_k} \frac{1}{4\eta_2} + \eta_3 \sum_{k=1}^{r_i} |d_{i_k}| q_{i_k}| + \frac{1}{4\eta_3} \sum_{k=1}^{r_i} |d_{i_k}| q_{i_k}| \quad < \quad c_{i_k},$$

$$c).\ \sum_{j=1}^{n} |b_{ij}| \frac{1}{4\eta_1} + \sum_{j=1}^{n} |b_{ji}| p_i^2 \eta_1 + \sum_{k=1}^{r_i} |c_{ii_k}| (M_{2i_k} + M_{1i_k}\eta_2) \quad < \quad a_i,$$

$$|c_{ii_k}| M_{1i_k} \frac{1}{4\eta_2} + \eta_3 \sum_{k=1}^{r_i} |d_{i_k}| q_{i_k}| + \frac{1}{4\eta_3} \sum_{k=1}^{r_i} |d_{i_k}| q_{i_k}| \quad < \quad c_{i_k},$$

where η_1, η_2 and η_3 are positive parameters chosen appropriately.

Proof. We construct a Lyapunov functional suitable for our purpose here. We first consider

$$V_1(t) = \sum_{i=1}^{n} \left\{ \frac{(x_i(t) - x_i^*)^2}{2} \right\}.$$

The derivative of V_1 along the solutions of (2), using (7), is given by

$$\begin{aligned}
V_1'(t) &= \sum_{i=1}^{n}\Big\{(x_i(t)-x_i^*)(x_i'(t)-x_i^{*'})\Big\}\\
&= \sum_{i=1}^{n}\Big\{(x_i(t)-x_i^*)\Big\{-a_i(x_i(t)-x_i^*)+\sum_{j=1}^{n}b_{ij}(f_j(x_j)-f_j(x_j^*))\\
&\quad +\sum_{k=1}^{r_i}c_{ii_k}(g_{i_k}(x_i,y_{i_k}(t-\tau_{i_k}))-g_{i_k}(x_i^*,y_{i_k}^*))\Big\}\Big\}\\
&\le \sum_{i=1}^{n}\Big\{\Big\{-a_i(x_i(t)-x_i^*)^2+|x_i(t)-x_i^*|\sum_{j=1}^{n}|b_{ij}|p_j|x_j(t)-x_j^*|\\
&\quad +|x_i(t)-x_i^*|\sum_{k=1}^{r_i}|c_{ii_k}|\Big[M_{2i_k}|x_i-x_i^*|+M_{1i_k}|y_{i_k}(t-\tau_{i_k}))-y_{i_k}^*|\Big]\Big\}\Big\},
\end{aligned}$$

utilizing (4),(3) for the last two terms respectively.

We utilize the inequality (8) for $\eta=\eta_1$ and $\eta=\eta_2$ in the second and fourth terms of the above inequality to get

$$\begin{aligned}
p_j|x_i(t)-x_i^*||x_j-x_j^*| &\le p_j\Big[\frac{1}{4\eta_1}(x_i(t)-x_i^*)^2+\eta_1(x_j-x_j^*)^2\Big],\\
|y_{i_k}(t-\tau_{i_k}))-y_{i_k}^*||x_i-x_i^*| &\le \Big[\frac{1}{4\eta_2}(y_{i_k}(t-\tau_{i_k})-y_{i_k}^*)^2+\eta_2(x_i-x_i^*)^2\Big]. \qquad (9)
\end{aligned}$$

Then we have

$$\begin{aligned}
V_1'(t) &\le \sum_{i=1}^{n}\Big[-a_i(x_i(t)-x_i^*)^2+\sum_{j=1}^{n}|b_{ij}|p_j\Big[\frac{1}{4\eta_1}(x_i(t)-x_i^*)^2+\eta_1(x_j-x_j^*)^2\Big]\\
&\quad +\sum_{k=1}^{r_i}|c_{ii_k}|M_{2i_k}(x_i-x_i^*)^2\\
&\quad +\sum_{k=1}^{r_i}|c_{ii_k}|M_{1i_k}\Big[\frac{1}{4\eta_2}(y_{i_k}(t-\tau_{i_k})-y_{i_k}^*)^2+\eta_2(x_i-x_i^*)^2\Big]\Big]\\
&= -\sum_{i=1}^{n}\Big[a_i-\sum_{j=1}^{n}|b_{ij}|p_j\frac{1}{4\eta_1}-\sum_{j=1}^{n}|b_{ji}|p_i\eta_1-\sum_{k=1}^{r_i}|c_{ii_k}|M_{2i_k}\\
&\quad -\sum_{k=1}^{r_i}|c_{ii_k}|M_{1i_k}\eta_2\Big](x_i-x_i^*)^2+\sum_{i=1}^{n}\sum_{k=1}^{r_i}|c_{ii_k}|M_{1i_k}\frac{1}{4\eta_2}(y_{i_k}(t-\tau_{i_k})-y_{i_k}^*)^2. \qquad (10)
\end{aligned}$$

Now define

$$V_2(t)=\sum_{i=1}^{n}\sum_{k=1}^{r_i}\frac{(y_{i_k}(t)-y_{i_k}^*)^2}{2}.$$

Then along the solutions of (2) we have

$$\begin{aligned} V_2'(t) &= \sum_{i=1}^{n}\sum_{k=1}^{r_i}(y_{i_k}(t)-y_{i_k}^*)(y_{i_k}'(t)-y_{i_k}^{*'}) \\ &= \sum_{i=1}^{n}\sum_{k=1}^{r_i}(y_{i_k}(t)-y_{i_k}^*)\Big[-c_{i_k}(y_{i_k}(t)-y_{i_k}^*)+\sum_{l=1}^{r_i}d_{i_l}[h_{i_l}(y_{i_l})-h_{i_l}(y_{i_l}^*)]\Big] \\ &\le \sum_{i=1}^{n}\sum_{k=1}^{r_i}\Big[-c_{i_k}(y_{i_k}(t)-y_{i_k}^*)^2+|y_{i_k}(t)-y_{i_k}^*|\sum_{l=1}^{r_i}|d_{i_l}|q_{i_l}|y_{i_l}-y_{i_l}^*|\Big] \\ &\le -\sum_{i=1}^{n}\sum_{k=1}^{r_i}\Big[c_{i_k}-\frac{1}{4\eta_3}\sum_{k=1}^{r_i}|d_{i_k}|q_{i_k}-\eta_3\sum_{k=1}^{r_i}|d_{i_k}|q_{i_k}\Big](y_{i_k}-y_{i_k}^*)^2, \end{aligned} \tag{11}$$

again utilizing the inequality (8) for $\eta=\eta_3>0$. Now consider

$$V_3(t) = \sum_{i=1}^{n}\sum_{k=1}^{r_i}|c_{ii_k}|M_{1i_k}\frac{1}{4\eta_2}\int_{t-\tau_{i_k}}^{t}(y_{i_k}(z)-y_{i_k}^*)^2dz.$$

Then we have

$$V_3'(t)=\sum_{i=1}^{n}\sum_{k=1}^{r_i}|c_{ii_k}|M_{1i_k}\frac{1}{4\eta_2}(y_{i_k}(t)-y_{i_k}^*)^2-\sum_{i=1}^{n}\sum_{k=1}^{r_i}|c_{ii_k}|M_{1i_k}\frac{1}{4\eta_2}(y_{i_k}(t-\tau_{i_k})-y_{i_k}^*)^2. \tag{12}$$

We now define our Lyapunov functional by $V(t)=V_1(t)+V_2(t)+V_3(t)$. Then along the solutions of (2) utilizing (10),(11) and (12), we get

$$\begin{aligned} V'(t) &= V_1'(t)+V_2'(t)+V_3'(t) \\ &\le -\sum_{i=1}^{n}\Big[a_i-\sum_{j=1}^{n}|b_{ij}|p_j\frac{1}{4\eta_1}-\sum_{j=1}^{n}|b_{ji}|p_i\eta_1-\sum_{k=1}^{r_i}|c_{ii_k}|M_{2i_k} \\ &\quad -\sum_{k=1}^{r_i}|c_{ii_k}|M_{1i_k}\eta_2\Big](x_i-x_i^*)^2+\sum_{i=1}^{n}\sum_{k=1}^{r_i}|c_{ii_k}|M_{1i_k}\frac{1}{4\eta_2}(y_{i_k}(t-\tau_{i_k})-y_{i_k}^*)^2 \\ &\quad -\sum_{i=1}^{n}\sum_{k=1}^{r_i}\Big[c_{i_k}-\eta_3\sum_{k=1}^{r_i}|d_{i_k}|q_{i_k}-\frac{1}{4\eta_3}\sum_{k=1}^{r_i}|d_{i_k}|q_{i_k}\Big](y_{i_k}(t)-y_{i_k}^*)^2 \\ &\quad +\sum_{i=1}^{n}\sum_{k=1}^{r_i}|c_{ii_k}|M_{1i_k}\frac{1}{4\eta_2}(y_{i_k}(t)-y_{i_k}^*)^2 \\ &\quad -\sum_{i=1}^{n}\sum_{k=1}^{r_i}|c_{ii_k}|M_{1i_k}\frac{1}{4\eta_2}(y_{i_k}(t-\tau_{i_k})-y_{i_k}^*)^2 \\ &= -\sum_{i=1}^{n}\Big[[a_i-\sum_{j=1}^{n}|b_{ij}|p_j\frac{1}{4\eta_1}-\sum_{i=1}^{n}|b_{ji}|p_i\eta_1-\sum_{k=1}^{r_i}|c_{ii_k}|(M_{2i_k}+M_{1i_k}\eta_2)\Big](x_i-x_i^*)^2 \\ &\quad -\sum_{i=1}^{n}\sum_{k=1}^{r_i}\Big[c_{i_k}-\eta_3\sum_{k=1}^{r_i}|d_{i_k}|q_{i_k}-\frac{1}{4\eta_3}\sum_{k=1}^{r_i}|d_{i_k}|q_{i_k}-|c_{ii_k}|M_{1i_k}\frac{1}{4\eta_2}\Big](y_{i_k}(t)-y_{i_k}^*)^2 \end{aligned}$$

If we choose

$$\begin{aligned} a_i &> \sum_{j=1}^{n} |b_{ij}|p_j \frac{1}{4\eta_1} + \sum_{j=1}^{n} |b_{ji}|p_i\eta_1 + \sum_{k=1}^{r_i} |c_{ii_k}|(M_{2i_k} + M_{1i_k}\eta_2), \\ c_{i_k} &> |c_{ii_k}|M_{1i_k}\frac{1}{4\eta_2} + \eta_3 \sum_{k=1}^{r_i} |d_{i_k}|q_{i_k} + \frac{1}{4\eta_3}\sum_{k=1}^{r_i} |d_{i_k}|q_{i_k}, \end{aligned}$$

as in assumption (a), then we have

$$V'(t) < 0.$$

Clearly V has all the properties of a Lyapunov functional to serve our purpose here. Rest of argument may be followed as in [4] or [19]. Hence $(x_i(t),\ y_{i_k}(t))$ converges to $(x_i^*,\ y_{i_k}^*)$ as $t \to \infty$.

The other two cases (b) and (c) may be proved on similar lines using the inequalities

$$\begin{aligned} p_j|x_i(t) - x_i^*||x_j - x_j^*| &\leq \Big[\frac{p_j^2}{4\eta_1}(x_i(t) - x_i^*)^2 + \eta_1(x_j(t) - x_j^*)^2\Big], \\ p_j|x_i(t) - x_i^*||x_j - x_j^*| &\leq \Big[\frac{1}{4\eta_1}(x_i(t) - x_i^*)^2 + \eta_1 p_j^2(x_j(t) - x_j^*)^2\Big], \end{aligned}$$

respectively in place of (9). The proof is complete.

The two-delay system:

We shall now consider the general case of (2) in which we assume delays both in transmission of information from and processing of information within subcomponents. The following result establishes sufficient conditions for the global asymptotic stability of equilibrium solution for this case.

Theorem 3.2. *Assume that the parameters of the system (2) satisfy the following conditions:*

$$a_i > \sum_{j=1}^{n} |b_{ji}|p_i + \sum_{k=1}^{r_i} |c_{ii_k}|M_{2i_k}, \quad c_{i_k} > \sum_{k=1}^{r_i} |d_{i_k}|q_{i_k} + \sum_{k=1}^{r_i} |c_{ii_k}|M_{1i_k},$$

for all $k = 1, 2, ..., r_i,\quad 1 \leq r_i \leq n,\ i = 1, 2, ..., n$. *Then the equilibrium* $(x_i^*, y_{i_k}^*)$ *is globally asymptotically stable independent of delays in the sense that all solutions of (2) satisfy the convergence requirement*

$$\lim_{t\to\infty} y_{i_k} \to y_{i_k}^*, \quad \lim_{t\to\infty} x_i \to x_i^*.$$

Proof. Utilizing (7) in (2), we rewrite (2) as

$$\begin{aligned} (x_i - x_i^*)' &= -a_i(x_i - x_i^*) + \sum_{j=1}^{n} b_{ij}[f_j(x_j) - f_j(x_j^*)] \\ &\quad + \sum_{k=1}^{r_i} c_{ii_k}[g_{i_k}(x_i, y_{i_k}(t - \tau_{i_k})) - g_{i_k}(x_i^*, y_{i_k}^*)], \\ (y_{i_k} - y_{i_k}^*)' &= -c_{i_k}(y_{i_k} - y_{i_k}^*) + \sum_{l=1}^{r_i} d_{i_l}[h_{i_l}(y_{i_l}(t - \tau_{i_l})) - h_{i_l}(y_{i_l}^*)]. \end{aligned}$$

We employ the functional

$$\begin{aligned} V(t) &= \sum_{i=1}^{n}\Big[|x_i - x_i^*| + |y_{i_k} - y_{i_k}^*| + \sum_{k=1}^{r_i}|c_{ii_k}|M_{1i_k}\int_{t-\tau_{i_k}}^{t}|y_{i_k}(s) - y_{i_k}^*|ds \\ &\quad + \sum_{l=1}^{r_i}|d_{i_l}|\int_{t-\tau_{i_l}}^{t}|h_{i_l}(y_{i_l}(s) - h_{i_l}(y^*{}_{i_l})|ds\Big], \end{aligned} \tag{13}$$

$$\begin{aligned} D^+V(t) &\le \sum_{i=1}^{n}\Big[-a_i|x_i - x_i^*| + \sum_{j=1}^{n}|b_{ij}||f_j(x_j) - f_j(x_j^*)| \\ &\quad + \sum_{k=1}^{r_i}|c_{ii_k}||g_{i_k}(x_i, y_{i_k}(t-\tau_{i_k})) - g_{i_k}(x_i, y_{i_k}^*) + g_{i_k}(x_i, y_{i_k}^*) - g_{i_k}(x_i^*, y_{i_k}^*)| \\ &\quad + \Big[-c_{i_k}|y_{i_k} - y_{i_k}^*| + \sum_{l=1}^{r_i}|d_{i_l}|h_{i_l}(y_{i_l}(t-\tau_{i_l})) - h_{i_l}(y_{i_l}^*)|\Big] \\ &\quad + \sum_{k=1}^{r_i}|c_{ii_k}|M_{1i_k}|y_{i_k} - y_{i_k}^*| - \sum_{k=1}^{r_i}|c_{ii_k}|M_{1i_k}|y_{i_k}(t-\tau_{i_k}) - y_{i_k}^*| \\ &\quad + \sum_{l=1}^{r_i}|d_{i_l}|h_{i_l}(y_{i_l}(t)) - h_{i_l}(y_{i_l}^*)| - \sum_{l=1}^{r_i}|d_{i_l}|h_{i_l}(y_{i_l}(t-\tau_{i_l})) - h_{i_l}(y_{i_l}^*)|\Big], \end{aligned}$$

$$\begin{aligned} D^+V(t) &\le \sum_{i=1}^{n}\Big[-a_i|x_i - x_i^*| + \sum_{j=1}^{n}|b_{ij}|p_j|x_j - x_j^*| \\ &\quad + \sum_{k=1}^{r_i}|c_{ii_k}||g_{i_k}(x_i, y_{i_k}(t-\tau_{i_k})) - g_{i_k}(x_i, y_{i_k}^*)| \\ &\quad + \sum_{k=1}^{r_i}|c_{ii_k}||g_{i_k}(x_i, y_{i_k}^*) - g_{i_k}(x_i^*, y_{i_k}^*)| \\ &\quad - c_{i_k}|y_{i_k} - y_{i_k}^*| + \sum_{l=1}^{r_i}|d_{i_l}|q_{i_l}|y_{i_l} - y_{i_l}^*| \\ &\quad + \sum_{k=1}^{r_i}|c_{ii_k}|M_{1i_k}|y_{i_k} - y_{i_k}^*| - \sum_{k=1}^{r_i}|c_{ii_k}|M_{1i_k}|y_{i_k}(t-\tau_{i_k}) - y_{i_k}^*|\Big], \end{aligned}$$

$$\begin{aligned} D^+V(t) &\le \sum_{i=1}^{n}\Big[-a_i|x_i - x_i^*| + \sum_{j=1}^{n}|b_{ij}|p_j|x_j - x_j^*| \\ &\quad + \sum_{k=1}^{r_i}|c_{ii_k}|M_{1i_k}|y_{i_k}(t-\tau_{i_k}) - y_{i_k}^*)| + \sum_{k=1}^{r_i}|c_{ii_k}|M_{2i_k}|x_i - x_i^*| \\ &\quad - c_{i_k}|y_{i_k} - y_{i_k}^*| + \sum_{l=1}^{r_i}|d_{i_l}|q_{i_l}|y_{i_l} - y_{i_l}^*| \\ &\quad + \sum_{k=1}^{r_i}|c_{ii_k}|M_{1i_k}|y_{i_k} - y_{i_k}^*| - \sum_{k=1}^{r_i}|c_{ii_k}|M_{1i_k}|y_{i_k}(t-\tau_{i_k}) - y_{i_k}^*|\Big], \end{aligned}$$

$$
\begin{aligned}
D^{+}V(t) &\leq -\sum_{i=1}^{n}\Big[[a_i-\sum_{j=1}^{n}|b_{ji}|p_i-\sum_{k=1}^{r_i}|c_{ii_k}|M_{2i_k}]|x_i-x_i^*| \\
&\quad +[c_{i_k}-\sum_{k=1}^{r_i}|d_{i_k}|q_{i_k}-\sum_{k=1}^{r_i}|c_{ii_k}|M_{1i_k}]|y_{i_k}-y_{i_k}^*|\Big] \\
&\leq -\widetilde{A}\sum_{i=1}^{n}\Big[|x_i-x_i^*|+|y_{i_k}-y_{i_k}^*|\Big] \\
&< 0,
\end{aligned}
$$

where $\widetilde{A}=\min\big\{\overline{A},\,\overline{B}\big\}>0$, in which

$$
\overline{A}=\min_{1\leq i\leq n}\Big\{a_i-\sum_{j=1}^{n}|b_{ji}|p_i-\sum_{k=1}^{r_i}|c_{ii_k}|M_{2i_k}\Big\}>0,
$$

$$
\overline{B}=min_{1\leq i\leq n}\Big\{c_{i_k}-\sum_{k=1}^{r_i}|d_{i_k}|q_{i_k}-\sum_{k=1}^{r_i}|c_{ii_k}|M_{1i_k}\Big\}>0.
$$

It is clear that V is the required Lyapunov functional and rest of the proof may be completed employing standard arguments (see e.g., [4,20]).

Remark 3.3. Stability of system (2) may be studied in two ways. Firstly, the subsystem $\{y_{i_k}\}$ may converge first and x_i converge then. In this case, the x_i wait for information for any length of time from their subsystems and finish the task only after y_{i_k} come up with their contribution. In the second case, x_i work along with subsystem y_{i_k} simultaneously to finish the job. That means, x_i and y_{i_k} converge together. The first approach was taken in [20] well. The present study is along second approach and Theorems 3.1 and 3.2 are in this direction.

For a delay free system (1) conditions for stability of equilibrium when x_i wait for y_{i_k} to converge first are given by (Theorem 4.1, [20])

$$
a_i > \sum_{j=1}^{n}|b_{ji}|p_i+\sum_{k=1}^{r_i}|c_{ii_k}|M_{2i_k},\ i=1,2,...,n; \tag{14}
$$

$$
c_{i_k} > \sum_{k=1}^{r_i}|d_{i_k}|q_{i_k},\quad 1\leq r_i\leq n. \tag{15}
$$

A straightforward comparison of parametric conditions of Theorems 3.1 and 3.2 of this paper with those of (14) and (15), shows that parameters are more strained here. However, this is tolerable when the system can not wait a long time for convergence of subsystems and have to compete the task all at a time, working in parallel with subsystem. This distinguishes the study here from earlier work ([20]). Further, since Theorems 3.1 and 3.2 are valid for $\tau_{i_k}=0=\tau_{i_l}$ also, these two results provide independent sets of sufficient conditions for global asymptotic stability of equilibrium solution of (1) also.

A close look at the parametric conditions of Theorems 3.1 and 3.2 for the choice of $\eta_1=\eta_2=\eta_3=\frac{1}{2}$ reveals that a part of strain on parameters c_{i_k} represented by $\eta_2\sum_{k=1}^{r_i}|c_{ii_k}|M_{1i_k}$ is taken by a_i. Thus, we remark that the x_i are actually sharing the burden of monitoring y_{i_k} and simultaneously converge with them. □

A more general case:

One may notice that primary units x_i are supported by y_{i_k}. But there is no information (input) nor instructions from x_i directly to y_{i_k}. Nor there is any check or supervision by x_i as far as dynamics in second equation of (2) are considered. What ever information provided by subsystem is taken up by x_i. That is, flow of information is uni-directional. This raises a doubt on the relevance of information/contrbution from y_{i_k}. To overcome this lapse in model, it was proposed in [20] that the inputs to y_{i_k} are from x_i but not mere constants, J_{i_k}. This may be more realistic in the sense that, y_{i_k} are chosen to aid x_i and hence, are motivated by x_i rather than some other inputs. Moreover x_i are also variables and thus, this choice reflects the presence of variable input which always influences the dynamics of y_{i_k}. To realize this, it was assumed that $J_{i_k} = J_{i_k}(x_i)$ for each i_k. In the present paper, to further enhance the quality of performance of y_{i_k}, we assume that the present task of y_{i_k} depends on some previous information/instructions from x_i. To be more specific, we admit time delays in these inputs also. That is, we consider, $J_{i_k} = J_{i_k}(x_i(t-\tau_i))$. This allows us to modify (2) as

$$\begin{aligned} x_i' &= -a_i x_i + \sum_{j=1}^{n} b_{ij} f_j(x_j) + \sum_{k=1}^{r_i} c_{ii_k} g_{i_k}(x_i, y_{i_k}(t-\tau_{i_k})) + I_i, i = 1,2,...,n; \\ y_{i_k}' &= -c_{i_k} y_{i_k} + \sum_{l=1}^{r_i} d_{i_l} h_{i_l}(y_{i_l}(t-\tau_{i_l})) + \sum_{k=1}^{r_i} J_{i_k}(x_i(t-\tau_i)), \quad 1 \le r_i \le n. \end{aligned} \tag{16}$$

An equilibrium solution, say $(x_i^*,\ y_{i_k}^*)$, for this system should satisfy the equations

$$\begin{aligned} a_i x_i^* &= \sum_{j=1}^{n} b_{ij} f_j(x_i^*) + \sum_{k=1}^{r_i} c_{ii_k} g_{i_k}(x_i^*, y_{i_k}^*) + I_i, i = 1,2,...,n; \\ c_{i_k} y_{i_k}^* &= \sum_{l=1}^{r_i} d_{i_l} h_{i_l}(y_{i_l}^*) + \sum_{k=1}^{r_i} J_{i_k}(x_i^*). \end{aligned} \tag{17}$$

We assume that the function J_{i_k} satisfies $|J_{i_k}(x_i(t)) - J_{i_k}(x_i^*)| \le \alpha_{i_k}|x_i - x_i^*|$, where $\alpha_{i_k} > 0$.

Assuming that the algebraic system (17) yields a unique solution (i.e., system (16) has a unique equilibrium pattern), we directly proceed to the global asymptotic stability of the equilibrium pattern of system (16). Using (17) in (16), we get

$$\begin{aligned} (x_i - x_i^*)' &= -a_i(x_i - x_i^*) + \sum_{j=1}^{n} b_{ij}[f_j(x_j) - f_j(x_j^*)] \\ &\quad + \sum_{k=1}^{r_i} c_{ii_k}[g_{i_k}(x_i, y_{i_k}(t-\tau_{i_k})) - g_{i_k}(x_i^*, y_{i_k}^*)], \\ (y_{i_k} - y_{i_k}^{*'})' &= -c_{i_k}(y_{i_k} - y_{i_k}^*) + \sum_{l=1}^{r_i} d_{i_l}[h_{i_l}(y_{i_l}(t-\tau_{i_l})) - h_{i_l}(y_{i_l}^*)] \\ &\quad + \sum_{k=1}^{r_i}[J_{i_k}(x_i(t-\tau_i)) - J(x_i^*)]. \end{aligned} \tag{18}$$

We employ the following Lyapunov functional for our purpose here

$$\begin{aligned} V(t) &= \sum_{i=1}^{n}\Big[|x_i - x_i^*| + |y_{i_k} - y_{i_k}^*| + \sum_{k=1}^{r_i}|c_{ii_k}|M_{1i_k}\int_{t-\tau_{i_k}}^{t}|y_{i_k}(s) - y_{i_k}^*|ds \\ &\quad + \sum_{l=1}^{r_i}|d_{i_l}|\int_{t-\tau_{i_l}}^{t}|h_{i_l}(y_{i_l}(s)) - h_{i_l}(y_{il}^*)|ds \\ &\quad + \sum_{k=1}^{r_i}\int_{t-\tau_i}^{t}|J_{i_k}(x_i(s) - J_{i_k}(x_i^*)|ds\Big]. \end{aligned} \tag{19}$$

The upper right derivative of V along the solutions of (16) employing (18) may be given by

$$\begin{aligned} D^+V(t) &\leq \sum_{i=1}^{n}\Big[-a_i|x_i - x_i^*| + \sum_{j=1}^{n}|b_{ij}||f_j(x_j) - f_j(x_j^*)| \\ &\quad + \sum_{k=1}^{r_i}|c_{ii_k}||g_{i_k}(x_i, y_{i_k}(t-\tau_{i_k})) - g_{i_k}(x_i^*, y_{i_k}^*)| \\ &\quad - c_{i_k}|y_{i_k} - y_{i_k}^*| + \sum_{l=1}^{r_i}|d_{i_l}||h_{i_l}(y_{i_l}(t-\tau_{i_l})) - h_{i_l}(y_{il}^*)| \\ &\quad + \sum_{k=1}^{r_i}|J_{i_k}(x_i(t-\tau_i)) - J_{i_k}(x_i^*)| \\ &\quad + \sum_{k=1}^{r_i}|c_{ii_k}|M_{1i_k}|y_{i_k} - y_{i_k}^*| - \sum_{k=1}^{r_i}|c_{ii_k}|M_{1i_k}|y_{i_k}(t-\tau_{i_k})) - y_{i_k}^*| \\ &\quad + \sum_{l=1}^{r_i}|d_{i_l}||h_{i_l}(y_{i_l}(t)) - h_{i_l}(y_{il}^*)| - \sum_{l=1}^{r_i}|d_{i_l}||h_{i_l}(y_{i_l}(t-\tau_{i_l})) - h_{i_l}(y_{il}^*)| \\ &\quad + \sum_{k=1}^{r_i}|J_{i_k}(x_i(t) - J_{i_k}(x_i^*)| - \sum_{k=1}^{r_i}|J_{i_k}(x_i(t-\tau_i)) - J_{i_k}(x_i^*)|\Big]. \end{aligned}$$

This, on further simplification, as done in earlier results, gives

$$\begin{aligned} D^+V(t) &\leq -\sum_{i=1}^{n}\Big[[a_i - \sum_{j=1}^{n}|b_{ji}|p_i - \sum_{k=1}^{r_i}|c_{ii_k}|M_{2i_k} - \sum_{k=1}^{r_i}\alpha_{i_k}]|x_i - x_i^*| \\ &\quad + \sum_{k=1}^{r_i}[c_{i_k} - \sum_{k=1}^{r_i}|d_{i_k}|q_{i_k} - \sum_{k=1}^{r_i}|c_{ii_k}|M_{1i_k}]|y_{i_k} - y_{i_k}^*|\Big] \\ &\leq -\widetilde{A}\Big[\sum_{i=1}^{n}\Big\{|x_i - x_i^*| + \sum_{k=1}^{r_i}|y_{i_k} - y_{i_k}^*|\Big\}\Big] \\ &< 0, \end{aligned} \tag{20}$$

provided

$$\widetilde{A} \equiv \min_{1\leq i\leq n}\Big\{a_i - \sum_{j=1}^{n}|b_{ij}|p_j - \sum_{k=1}^{r_i}|c_{ii_k}|M_{2i_k} - \sum_{k=1}^{r_i}\alpha_{i_k},\ c_{i_k} - \sum_{k=1}^{r_i}|d_{i_k}|q_{i_k} - \sum_{k=1}^{r_i}|c_{ii_k}|M_{1i_k}\Big\} > 0,$$

$k = 1, 2, ..., r_i$, holds. We are now in a position to state

Theorem 3.4. *The equilibrium solution of (12) is globally asymptotically stable provided the parameters satisfy*

$$a_i - \sum_{j=1}^{n} |b_{ij}| p_j - \sum_{k=1}^{r_i} |c_{ii_k}| M_{2i_k} - \sum_{k=1}^{r_i} \alpha_{i_k} > 0,$$

$$c_{i_k} - \sum_{k=1}^{r_i} |d_{i_k}| q_{i_k} - \sum_{k=1}^{r_i} |c_{ii_k}| M_{1i_k} > 0,$$

for all $i = 1, 2, ..., n$ *and* $\alpha_{i_k} > 0$ *is such that* $|J_{i_k}(x_i(t)) - J_{i_k}(x_i^*)| \leq \alpha_{i_k} |x_i - x_i^*|$.

Proof. The proof is obvious from standard arguments noticing that $V(t)$ defined by (19) and (20) is the required Lyapunov functional.

We shall illustrate the above results by means of numerical examples.

Example 3.5. Consider the following system having two neurons in X supported by two neurons in Y involving time delays as given by

$$\begin{aligned} \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} &= -\begin{pmatrix} 6x_1 \\ 8x_2 \end{pmatrix} + \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} f_1(x_1) \\ f_2(x_2) \end{pmatrix} \\ &+ \begin{pmatrix} -1 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} g_{1_1}(x_1, y_{1_1}(t - \tau_{1_1}) & g_{2_1}(x_2, y_{2_1}(t - \tau_{1_2})) \\ g_{1_2}(x_1, y_{1_2}(t - \tau_{2_1})) & g_{2_2}(x_2, y_{2_2}(t - \tau_{2_2})) \end{pmatrix} + \begin{pmatrix} I_1 \\ I_2 \end{pmatrix}, \end{aligned}$$

$$\begin{pmatrix} y_{1_1}' \\ y_{1_2}' \end{pmatrix} = -\begin{pmatrix} 4.5y_{1_1} \\ 8y_{1_2} \end{pmatrix} + \begin{pmatrix} \sqrt{2} & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} h_{1_1}(y_{1_1}(t - \tau_{1_1})) \\ h_{1_2}(y_{1_2}(t - \tau_{1_2})) \end{pmatrix} + \begin{pmatrix} J_{1_1} \\ J_{1_2} \end{pmatrix},$$

$$\begin{pmatrix} y_{2_1}' \\ y_{2_2}' \end{pmatrix} = -\begin{pmatrix} 6.5y_{2_1} \\ 7.5y_{2_2} \end{pmatrix} + \begin{pmatrix} 2 & -2 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} h_{2_1}(y_{2_1}(t - \tau_{2_1})) \\ h_{2_2}(y_{2_2}(t - \tau_{2_2})) \end{pmatrix} + \begin{pmatrix} J_{2_1} \\ J_{2_2} \end{pmatrix}.$$

Choose $f_i(x_i) = tanh(x_i), h_{i_k} = tanh(y_{i_k})$ and $g_{i_k}(x_i, y_{i_k}) = x_i + y_{i_k}$. Then $p_j = q_{i_k} = M_{1i_k} = M_{2i_k} = 1, i = 1, 2, k = 1, 2.$. Let us choose $\eta_1 = \frac{1}{2}$, $\eta_3 = \frac{1}{2}$. It is easy to see that for the above parametric values of the system, all the conditions of Theorem 3.1 are satisfied for the range of values of $\frac{3}{8} < \eta_2 < \frac{3}{7}$. Hence the equilibrium of the above system is globally asymptotically stable by virtue of Theorem 3.1 for $\tau_{i_l} = 0$.

Example 3.6. Consider the system

$$\begin{aligned} \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} &= -\begin{pmatrix} 6.1x_1 \\ 6.9x_2 \end{pmatrix} + \begin{pmatrix} 0.8 & 1 \\ -1 & 0.75 \end{pmatrix} \begin{pmatrix} f_1(x_1) \\ f_2(x_2) \end{pmatrix} \\ &+ \begin{pmatrix} 1.1 & 0.5 \\ -1.1 & 1 \end{pmatrix} \begin{pmatrix} g_{1_1}(x_1, y_{1_1}(t - \tau_{1_1}) & g_{2_1}(x_2, y_{2_1}(t - \tau_{1_2})) \\ g_{1_2}(x_1, y_{1_2}(t - \tau_{2_1})) & g_{2_2}(x_2, y_{2_2}(t - \tau_{2_2})) \end{pmatrix} + \begin{pmatrix} I_1 \\ I_2 \end{pmatrix}, \end{aligned}$$

$$\begin{pmatrix} y_{1_1}' \\ y_{1_2}' \end{pmatrix} = -\begin{pmatrix} 3.7y_{1_1} \\ 5.2y_{1_2} \end{pmatrix} + \begin{pmatrix} 1 & 0.5 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} h_{1_1}(y_{1_1}(t - \tau_{1_1})) \\ h_{1_2}(y_{1_2}(t - \tau_{1_2})) \end{pmatrix} + \begin{pmatrix} J_{1_1}(t - \tau_{1_1}) \\ J_{1_2}(t - \tau_{1_2}) \end{pmatrix},$$

$$\begin{pmatrix} y_{2_1}' \\ y_{2_2}' \end{pmatrix} = -\begin{pmatrix} 6.2y_{1_1} \\ 6.4y_{1_2} \end{pmatrix} + \begin{pmatrix} 0.25 & 0.5 \\ 1 & -0.5 \end{pmatrix} \begin{pmatrix} h_{2_1}(y_{2_1}(t - \tau_{2_1})) \\ h_{2_2}(y_{2_2}(t - \tau_{2_2})) \end{pmatrix} + \begin{pmatrix} J_{2_1}(t - \tau_{2_1}) \\ J_{2_2}(t - \tau_{2_2}) \end{pmatrix}$$

with all response functions as in above example. Then $p_j = q_{i_k} = M_{1i_k} = M_{2i_k} = 1, i = 1,2, k = 1,2$.

(i) Now choosing $\eta_1 = \frac{1}{4}$, $\eta_2 = \frac{1}{4}$, one may notice that all the conditions of Theorem 3.1 are satisfied for the range of values of $0.217 < \eta_3 < 1.189$ (approximately). Hence, the equilibrium of the above system is globally asymptotically stable by virtue of Theorem 3.1 for all $\tau_{i_l} = 0$ and $\tau_i = 0$.

(ii) Again all the parametric conditions of Theorem 3.2 are satisfied for all delays $\tau_{i_k} \geq 0$ $\tau_{i_l} \geq 0$ and $\tau_i = 0$, $i = 1,2$, and hence, the equilibrium solution is globally asymptotically stable by virtue of Theorem 3.2.

(iii) Choosing $J_{i_k}(x_i) = x_i$ for all $i, k = 1,2$, we get α_{i_k} $k = 1,2$. Then the parameters of the system satisfy all the conditions of Theorem 3.4 and hence, system tolerates all three types of delays involved.

4 Conclusion

In the present paper we have considered a cooperative and supportive neural network which is under influence of time delays both in processing of information with in the subgroup network and transmission of information from subgroup network to main network. Conditions on parameters are obtained so that the equilibrium is stable for any length of delays. Under these conditions the system behaves like delay independent system. However, it is also observed that the parameters are strained much for such stability. Hence conditions straining parameters less are welcome for more applicability of the network. Parametric conditions involving suitably restricted time delay parameters may be a better choice in this case. Our results in this direction will be reported soon. Another distinguishing feature of this paper is that the main components (x_i) of the system monitor the performance of the subcomponents (y_{i_k}) (work together attitude or parallel processing) unlike its earlier counter part. It is interesting to see how the system withstands if some of its subcomponents do not respond properly to the requirements of its main components. In other words, can the (x_i) converge even if some of the (y_{i_k}) do not converge or non cooperate? This will be a question of our future contention.

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Generalized Iterative Methods for Caputo Fractional Differential Equations via Coupled Lower and Upper Solutions with Superlinear Convergence

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Abstract: Existence of coupled lower and upper solutions for nonlinear differential equations guarantees the existence as well as interval of existence of the solution. In this work, a methodology has been developed to compute coupled lower and upper solutions using natural lower and upper solutions by iterative methods. Further, using the computed lower and upper solutions, sequences are developed which converge uniformly and monotonically to the unique solution. In addition, it has been shown that the convergence of these sequences is superlinear. Further the convergence of the sequences is accelerated by Gauss-Seidel method. Finally, some numerical examples are presented.

Keywords: *Caputo fractional differential equation; superlinear convergence.*

Mathematics Subject Classification (2010): 34A08, 34A12.

1 Introduction

It is well-known that qualitative and quantitative properties of fractional differential equations are very useful in applications. In addition, fractional differential equations in several situations have proved to be better and more economical models than their counterpart with integer derivatives. For details see [5, 9, 11] and the references therein. In the past thirty years there has been a rapid development in the qualitative study of fractional differential equation such as existence, uniqueness and stability results due to its applications. In particular, it has been very useful in biological sciences such

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as population models. However, most of the existence and uniqueness results for fractional differential equations are obtained by some type of fixed point theorem approach. See [1,2,17]. Unfortunately, these methods do not provide the interval of existence of the solution as well as a methodology to compute solutions. The method of lower and upper solutions and the method of coupled lower and upper solutions which guarantees the interval of existence, and is well-known for ordinary differential equations have now been extended to Riemann-Liouville and Caputo fractional differential equations in [4, 13]. Monotone method combined with lower and upper solutions provides both theoretical and constructive method of existence of the minimal and maximal solution or the unique solution if the uniqueness conditions are satisfied. See [6] for details. Monotone method works only when the nonlinear function is either increasing or could be made increasing by adding a linear term. Monotone method yields alternating sequences when the nonlinear function is decreasing with an additional assumption. In [18] and the references therein they have developed generalized monotone method for scalar first order ordinary differential equations. Generalized monotone method uses coupled lower and upper solutions and the method is very convenient to use when the nonlinear function is the sum of an increasing and decreasing functions. Furthermore, we do not need an additional assumption which we need when the nonlinear function is decreasing when we use an appropriate type of coupled lower and upper solutions, namely of type I. Generalized monotone method has been extended to scalar and system of Caputo fractional differential equations in [10, 16]. Generalized monotone method with coupled lower and upper solutions has an added advantage for fractional differential equations, since it avoids the computation of Mittag-Leffler function. The disadvantage of the generalized monotone method is the computation of coupled lower and upper solutions of type I on the interval of existence. The computation of coupled lower and upper solution is not a trivial matter. Using the generalized monotone method as a tool, both the theoretical and the numerical results for computing coupled lower and upper solutions for scalar and system of ordinary differential equations can be found in [15]. Computation of coupled lower and upper solution to any desired interval using generalized monotone method as a tool and the corresponding numerical results for scalar and system of Caputo fractional differential equations are developed in [11] and [14] respectively. However, the rate of convergence of the sequences is linear. In [13] generalized quasilinearization method was developed using coupled lower and upper solutions when the nonlinear function is the sum of a convex and a concave function. The method of generalized quasilinearization yields sequences which converge uniformly to the unique solution and the rate of convergence is quadratic. The complexity of this method is that the sequences are solutions of two systems of coupled linear equations. The solutions of these two systems are difficult even with constant coefficients for fractional differential equations. To overcome this difficulty, in this work we have taken the nonlinear function as the sum of a convex function and a non-increasing function. We have combined the method of generalized quasilinearization for the convex function and generalized monotone method for the non-increasing function. We compute the sequences as two systems of Caputo fractional differential equations which are decoupled. The method yields superlinear convergence. See [13] for details. In this work, we provide a methodology to compute coupled lower and upper solutions of type I, to any desired interval by using the mixed generalized quasilinearization method and generalized monotone method. The convergence is superlinear. Further we can accelerate the convergence by using Gauss-Seidel accelerated convergence. We have applied our theoretical results to the logistic equation. The first set of iterates is in terms of the Mittag-Leffler function. Computation of further iterates has led to interesting open problems, since it requires the exponential formula related to Mittag-Leffler function. The exponential properties of the Mittag-Leffler function are yet to be established. This has been addressed in our conclusion.

2 Preliminary and Auxiliary Results

In this section, we recall known results, some definitions which are needed for our main results.

Definition 2.1 Caputo fractional derivative of order q is given by:

$$^{c}D^{q}u(t) = \frac{1}{\Gamma(1-q)} \int_0^t (t-s)^{-q} u'(s)ds,$$

where $0 < q < 1$ and $\Gamma(q)$ is the Gamma function.

Although in this work, we study Caputo fractional differential equations, our comparison results follow from the relation between Riemann-Liouville derivative and Caputo fractional derivative. Hence the next definition is for the Riemann-Liouville derivative.

Definition 2.2 Riemann-Liouville fractional derivative of order q with respect to t is defined by:

$$D^{q}u(t) = \frac{1}{\Gamma(m-q)} \frac{d^m}{dt^m} \int_0^t (t-s)^{m-q-1} f(s)ds,$$

where $m-1 < q < m$.

In particular, if $0 < q < 1$, then

$$D^{q}u(t) = \frac{1}{\Gamma(1-q)} \frac{d}{dt} \int_0^t (t-s)^{-q} f(s)ds.$$

Here, and throughout this work, we will consider fractional differential equations of order q, where, $0 < q < 1$.

Consider the nonlinear Caputo fractional differential equation with initial condition of the form:

$$^{c}D^{q}u(t) = f(t, u(t)), \quad u(0) = u_0, \tag{1}$$

where $f \in C[J \times \mathbb{R}, \mathbb{R}]$ and $J = [0, T]$. The integral representation of (1) is given by:

$$u(t) = u_0 + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, u(s))ds. \tag{2}$$

The sequences we develop are always solutions of linear Caputo fractional differential equation. In order to compute the solution of the linear fractional differential equation with constant coefficients we need Mittag-Leffler function.

Definition 2.3 Mittag-Leffler function of two parameters q, r is given by

$$E_{q,r}(\lambda(t-t_0)^q) = \sum_{k=0}^{\infty} \frac{(\lambda(t-t_0)^q)^k}{\Gamma(qk+r)},$$

where $q, r > 0$. Also, for $t_0 = 0$ and $r = 1$, we get

$$E_{q,1}(\lambda t^q) = \sum_{k=0}^{\infty} \frac{(\lambda t^q)^k}{\Gamma(qk+1)},$$

where $q > 0$.

Also, consider linear Caputo fractional differential equation

$$^cD^q u(t) = \lambda u(t) + f(t), \quad u(0) = u_0, \ \ on \ \ J, \tag{3}$$

where $J = [0, T]$, λ is a constant and $f(t) \in C[J, \mathbb{R}]$. The solution of (3) exists and is unique. The explicit solution of (3) is given by:

$$u(t) = u_0 E_{q,1}(\lambda t^q) + \int_0^t (t-s)^{q-1} E_{q,q}(\lambda (t-s)^q) f(s) ds. \tag{4}$$

See [7] for details. In particular, if $\lambda = 0$, the solution $u(t)$ is given by:

$$u(t) = u_0 + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s) ds. \tag{5}$$

Also we recall known results related to scalar Caputo nonlinear fractional differential equations of the following form

$$^cD^q u(t) = f(t, u) + g(t, u), \quad u(0) = u_0 \ \ on \ \ J = [0, T], \tag{6}$$

where $0 < q < 1$. Results when $q = 1$ is proved in [18]. Here $f, g \in C(J \times \mathbb{R}, \mathbb{R})$, $f(t, u)$ is non-decreasing in u on J and $g(t, u)$ is non-increasing in u on J.

In order to prove the comparison result relative to coupled lower and upper solutions of (6) we recall a basic lemma relative to the Riemann-Liouville fractional derivative.

Lemma 2.1 *Let $m(t) \in C_p[J, \mathbb{R}]$ (where $J = [0, T]$) be such that for some $t_1 \in (0, T]$, $m(t_1) = 0$ and $m(t) \le 0$, on $(0, T]$. Then $D^q m(t_1) \ge 0$.*

Proof. See [4,7] for details. Note that the above result has been proved in [4] without using the Hölder continuity assumption of $m(t)$. □

The above lemma is true for Caputo derivative also, using the relation $^cD^q x(t) = D^q(x(t) - x(0))$ between the Caputo derivative and the Riemann-Liouville derivative. This is the version we will be using to prove our comparison results.

We recall the following known definitions which are needed for our main results.

Definition 2.4 The functions $\alpha_0, \beta_0 \in C^1(J, \mathbb{R})$ are called natural lower and upper solutions of (6) if :

$$\begin{cases} ^cD^q \alpha_0(t) \le f(t, \alpha_0) + g(t, \alpha_0), & \alpha_0(0) \le u_0, \\ ^cD^q \beta_0(t) \ge f(t, \beta_0) + g(t, \beta_0), & \beta_0(0) \ge u_0. \end{cases}$$

Definition 2.5 The functions $\alpha_0, \beta_0 \in C^1(J, \mathbb{R})$ are called coupled lower and upper solutions of (6) of type I if :

$$\begin{cases} ^cD^q \alpha_0(t) \le f(t, \alpha_0) + g(t, \beta_0), & \alpha_0(0) \le u_0, \\ ^cD^q \beta_0(t) \ge f(t, \beta_0) + g(t, \alpha_0), & \beta_0(0) \ge u_0. \end{cases}$$

See [10] for other types of coupled lower and upper solutions relative to (6).

Denoting $F(t,u) = f(t,u) + g(t,u)$, we state the next comparison result.

Theorem 2.1 *Let α, β be natural lower and upper solutions of (6), respectively. Suppose that $F(t,\beta) - F(t,\alpha) \leq L(\beta - \alpha)$ whenever $\beta \geq \alpha$, where L is a constant such that $L > 0$, then $\alpha(0) \leq \beta(0)$ implies that $\alpha(t) \leq \beta(t)$, $t \in J$.*

Proof. See [7] for details. □

Also, see [10, 16] for comparison result for coupled lower and upper solution of type I. Next, we recall a corollary of Theorem 2.1, which is useful in our main result.

Corollary 2.1 *Let $p \in C^1[J, \mathbb{R}]$. ${}^cD^q p(t) \leq Lp(t)$, where $L \geq 0$ and $p(0) \leq 0$. Then $p(t) \leq 0$ on J.*

We define the following sector Ω for convenience. That is,
$\Omega = [(t,u) : \alpha(t) \leq u(t) \leq \beta(t),\ t \in J]$.

Theorem 2.2 *Suppose $\alpha, \beta \in C^1[J, \mathbb{R}]$ are coupled lower and upper solutions of type I of (6) such that $\alpha(t) \leq \beta(t)$ on J and $F \in C(\Omega, \mathbb{R})$. Further, if $g(t,u)$ is decreasing in u, on J, then there exists a solution $u(t)$ of (6) such that $\alpha(t) \leq u(t) \leq \beta(t)$ on J, provided $\alpha(0) \leq u_0 \leq \beta(0)$.*

Proof. The proof follows from the scalar version of the result of [13]. □

Note that from the hypotheses of the above theorem, it follows that coupled lower and upper solution of type I are also natural lower and upper solutions.

The next results give the uniqueness theorem.

Theorem 2.3 *Let $\alpha, \beta \in C^1[J, \mathbb{R}]$, where α, β are coupled lower and upper solutions of (6) of type I, with $\alpha(t) \leq \beta(t)$ on J. If $f(t,u)$ is convex in u and $g(t,u)$ is decreasing in u, the hypotheses of Theorem 2.1 are satisfied. Then,(6) has a unique solution.*

The next result is useful in proving the equicontinuity of the sequences we develop in the next two theorems.

Theorem 2.4 *Let $\alpha_n(t)$ be a family of continuous functions on $[0,T]$, for each $n > 0$, where ${}^cD^q\alpha_n(t) = f(t, \alpha_n(t))$, $\alpha_n(0) = u_0$ and $|f(t, \alpha_n(t))| \leq M$ for $0 \leq t \leq T$. Then, the family $\{\alpha_n(t)\}$ is equicontinuous on $[0,T]$.*

Proof. See [7, 13] for details. □

Next, we provide two results relative to (6) where in the first result we assume f is convex in u and g is concave in u, and in the second result we assume f is convex in u and g is non-increasing in u. The first result we provide is related to the generalized quasilinearization method of (6) using coupled lower and upper solutions of type I.

Theorem 2.5 *Assume that*

(i) $\alpha_0, \beta_0 \in C^1[J, \mathbb{R}]$ are coupled lower solutions of type I, for (6) with $\alpha_0 \leq \beta_0$ on J,

(ii) $f, g \in C[\Omega, \mathbb{R}], f_u, g_u, f_{uu}$, and g_{uu} exist, are continuous and satisfy $f_{uu}(t,u) \geq 0, g_{uu}(t,u) \leq 0$ for $(t,u) \in \Omega$,

(iii) $g_u(t,u) \leq 0$ on Ω.

Then there exit monotone sequences $\{\alpha_n(t\}, \{\beta_n(t)\}$ which converge uniformly and monotonically to the unique solution of (6) and the convergence is quadratic.

Proof. See [13] for details. □

The next theorem is proved under the weaker assumption on $g(t,u)$. Also, this result mixes generalized quasilinearization method relative to the convex function $f(t,u)$ and generalized monotone method relative to the nonincreasing function $g(t,u)$ for $t \in J$.

Theorem 2.6 *Assume that*

(i) $\alpha_0, \beta_0 \in C^1[J,\mathbb{R}]$ *are coupled lower and upper solutions of type I, for (6) with* $\alpha_0 \leq \beta_0$ *on* J,

(ii) $f, g \in C[\Omega,\mathbb{R}], f_u, g_u,$ *and* f_{uu} *exist, are continuous and satisfy* $f_{uu}(t,u) \geq 0$ *for* $(t,u) \in \Omega$,

(iii) $g_u(t,u) \leq 0$ *on* Ω.

Then there exit monotone sequences $\{\alpha_n(t)\}, \{\beta_n(t)\}$ *which converge uniformly to the unique solution of (6) and the convergence is superlinear.*

Proof. See [13] for details. □

3 Main Results

In this section we will provide a method to compute coupled lower and upper solutions on any desired interval when we have the natural lower and upper solutions. Natural lower and upper solutions are relatively easy to compute. For example, equilibrium solutions are natural solutions. In the next result we use the superlinear convergence scheme as in Theorem 2.6, using natural lower and upper solutions. However, when we use natural lower and upper solutions, the results of Theorem 2.6 are true only when $\alpha_0 \leq \alpha_1$ and $\beta_0 \geq \beta_1$. This, in general, will not be true on the interval J, namely, the interval of existence of the solution. In the next result, monotone sequences constructed will converge to coupled minimal and maximal solutions as well as they are coupled lower and upper solutions on the interval of existence J.

Theorem 3.1 *Assume that*

(i) $\alpha_0, \beta_0 \in C^1[J,\mathbb{R}]$, α_0 *and* β_0 *are natural lower and upper solutions of (6) on* J *with* $\alpha_0 \leq \beta_0$ *on* J,

(ii) $f, g \in C[\Omega,\mathbb{R}], f_u, g_u, and f_{uu}$ *exist, are continuous and satisfy* $f_{uu}(t,u) \geq 0$ *for* $(t,u) \in \Omega$,

(iii) $g_u(t,u) \leq 0$ *on* Ω.

Then there exit monotone sequences $\{\alpha_n(t)\}, \{\beta_n(t)\}$ *which converge uniformly to the coupled lower and upper solution of (6). Here the sequences* $\{\alpha_n(t)\}$ *and* $\{\beta_n(t)\}$ *are computed using the following iterative scheme*

$$^cD^q\alpha_{n+1} = f(t,\alpha_n) + f_u(t,\alpha_n)(\alpha_{n+1} - \alpha_n) + g(t,\beta_n), \alpha_{n+1}(0) = u_0, \tag{7}$$

$$^cD^q\beta_{n+1} = f(t,\beta_n) + f_u(t,\alpha_n)(\beta_{n+1} - \beta_n) + g(t,\alpha_n), \beta_{n+1}(0) = u_0. \tag{8}$$

Proof. From the first iteration we will have $\alpha_0(t) \leq \alpha_1(t)$ on $[0,t_1]$ and $\beta_1(t) \leq \beta_0(t)$ on $[0,\bar{t}_1\ ]$. If $t_1 \geq T$, and $\bar{t}_1 \geq T$ there is nothing to prove, since one can use Theorem 2.6 to compute coupled minimal and maximal solutions. If not, certainly $t_1 < T$ and $\bar{t}_1 < T$. Also $\alpha_1(t_1) = \alpha_0(t_1)$. and $\beta_1(\bar{t}_1) = \beta_0(\bar{t}_1)$. We will now redefine $\alpha_1(t)$, and $\beta_1(t)$ on $[0,T]$ as follows:

$$^cD^q\alpha_1(t) = f(t,\alpha_0) + f_u(t,\alpha_0)(\alpha_1 - \alpha_0) + g(t,\beta_0), \ \ \alpha_1(0) = u_0 \ \ on \ \ [0,t_1],$$

$$^cD^q\beta_1(t) = f(t,\beta_0) + f_u(t,\alpha_0)(\beta_1 - \beta_0) + g(t,\alpha_0), \ \ \beta_1(0) = u_0 \ \ on \ \ [0, \bar{t}_1 \ ],$$

and

$$\alpha_1(t) = \alpha_0(t) \ \ on \ \ [t_1, T],$$
$$\beta_1(t) = \beta_0(t) \ \ on \ \ [\ \bar{t}_1, T].$$

Proceeding in this manner, we will have $\alpha_n(t_n) = \alpha_0(t_n)$, and $\beta_n(\bar{t}_n) = \beta_0(\bar{t}_n)$. Now we can redefine α_n, β_n as follows.

$$^cD^q\alpha_n(t) = f(t,\alpha_{n-1}) + f_u(t,\alpha_{n-1})(\alpha_n - \alpha_{n-1}) + g(t,\beta_{n-1}), \quad v_n(0) = u_0 \ \ on \ \ [0, t_n],$$

$$\alpha_n(t) = \alpha_0(t) \quad on \quad [t_n, T].$$

Similarly,

$$^cD^q\beta_n(t) = f(t,\beta_{n-1}) + f_u(t,\alpha_{n-1})(\beta_n - \beta_{n-1}) + g(t,\alpha_{n-1}), \quad \beta_n(0) = u_0 \ \ on \ \ [0, \bar{t}_n],$$

$$\beta_n(t) = \beta_0(t) \quad on \quad [\ \bar{t}_n, T],$$

where α_n, β_n intersect α_0 and β_0 at t_n, $\bar{t}_n$ respectively. If $t_n \geq T$, and $\bar{t}_n \geq T$ we can stop the process. Certainly $\alpha_n \leq \beta_n$ and α_n and β_n are coupled minimum and maximum solutions of (6) respectively.

Now we can show that the sequences $\{\alpha_n(t)\}$ and $\{\beta_n(t)\}$ constructed above are equicontinuous and uniformly bounded on J. Hence by Arzelá-Ascoli theorem, a subsequence converges uniformly and monotonically. Since the sequences are monotone, the entire sequence converges uniformly and monotonically to α and β respectively.

It is easy to observe that

$$^cD^q\alpha_n(t) = f(t,\alpha_{n-1}) + f_u(t,\alpha_{n-1})(\alpha_n - \alpha_{n-1}) + g(t,\beta_{n-1}), \ \ \alpha_n(0) = u_0 \ \ on \quad [0, t_n],$$

$$\alpha_n(t) = \alpha_0(t) \quad on \quad [t_{n-1}, T], \ \ such \ \ that \ \ \alpha_n(t_{n-1}) = \alpha_0(t_n),$$

and

$$^cD^q\beta_n(t) = f(t,\beta_{n-1}) + f_u(t,\alpha_{n-1})(\beta_n - \beta_{n-1}) + g(t,\alpha_{n-1}), \quad \beta_n(0) = u_0 \ \ on \ \ [0, \bar{t}_n],$$

$$\beta_n(t) = \beta_0(t) \quad on \quad [\ \bar{t}_{n-1}, T], \ \ such \ \ that \ \ \beta_n(\ \bar{t}_n) = \beta_0(\ \bar{t}_{n-1}),$$

for all $n \geq 1$.

As $n \to \infty$, $t_n, \bar{t}_n \to T$, $\alpha_n(t) \to \alpha(t)$, and $\beta_n(t) \to \beta(t)$, uniformly and monotonically.

Further,

$$^cD^q\alpha(t) = f(t,\alpha) + g(t,\beta), \quad \alpha(0) = u_0 \quad on \quad J,$$

and

$$^cD^q\beta(t) = f(t,\beta) + g(t,\alpha), \quad \beta(0) = u_0 \quad on \quad J.$$

Hence α, β are coupled lower and upper solutions of (6) such that $\alpha \leq \beta$ on J. This concludes the proof. □

Theorem 3.2 *Assume that*

(i) $\alpha_0, \beta_0 \in C^1[J, \mathbb{R}]$, α_0 *and* β_0 *are natural lower and upper solutions of (6) on* J *with* $\alpha_0 \leq \beta_0$ *on* J,

(ii) $f, g \in C[\Omega, \mathbb{R}], f_u, g_u, f_{uu}$ *exist, are continuous and satisfy* $f_{uu}(t,u) \geq 0$ *for* $(t,u) \in \Omega$,

(iii) $g_u(t,u) \leq 0$ *on* Ω.

Then there exit monotone sequences $\{\alpha_n(t)\}, \{\beta_n(t)\}$ *which converge uniformly to the unique solution of (6) and the convergence is superlinear.*

Proof. Theorem 3.1 proves that, there exit monotone sequences $\{\alpha_n(t)\}, \{\beta_n(t)\}$ such that $\{\alpha_n(t)\} \longrightarrow \alpha(t)$ and $\{\beta_n(t)\} \longrightarrow \beta(t)$ uniformly and monotonically and (α, β) are coupled lower and upper solutions of type I for (6) respectively on J. However, it is easy to observe that each pair of $\alpha_n(t), \beta_n(t)$ computed are also coupled lower and upper solutions of (6) on the common interval of $[0, t_n]$ and $[0, \bar{t}_n]$. Suppose that for some $n = k$ both t_k and $\bar{t}_k$ are $\geq T$, then the computation of $\alpha_{k+1}(t), \beta_{k+1}(t)$ will no longer need $\alpha_0(t), \beta_0(t)$. Then it is easy to observe that $\alpha_{k+1}(t), \beta_{k+1}(t)$ will be coupled lower and upper solutions of type I for (6) respectively on J. Also this sequence will converge uniformly and monotonically to α, β using Theorem 3.1. This implies that $\alpha \leq \beta$ on J. By hypotheses and using Theorem 2.3, it can be shown that $\alpha \equiv \beta \equiv u$, where u is the unique solution of (6) on J. In order to prove superlinear convergence we let $p_n(t) = u(t) - \alpha_n(t)$ and $q_n(t) = \beta_n(t) - u(t)$. It is easy to see that $p_n(0) = 0$, $q_n(0) = 0$. Using Gronwall type of Lemma and the estimate on f_{uu} and g_u on J, we can prove that $\max_J|p_n + q_n| \leq \max_J(|(p_{n-1} + q_{n-1})|^2 + |(p_{n-1} + q_{n-1})|)$ which proves superlinear convergence. See [13]for details. □

Note that if $g(t, u)$ is non-increasing in u on J, then α, β constructed above are also natural lower and upper solutions. By the existence theorem, there exists a solution of (6) on J such that $\alpha \leq u \leq \beta$ provided, $\alpha(0) \leq u_0 \leq \beta(0)$.

Remark 3.1 Note that Theorem 3.1 provides coupled lower and upper solutions of (6) on J. Now we can develop sequences $\{\alpha_n\}$ and $\{\beta_n\}$ using Theorem 2.6. These sequences converge uniformly and monotonically to coupled minimal and maximal solutions. Further if uniqueness condition is satisfied, the sequences converge to the unique solution of (6). Further we can apply Gauss-Seidel method such that the sequences converge faster. This is what we have proved in the next result.

Theorem 3.3 *Let all the hypotheses of Theorem 2.6 hold with the iterative scheme given by*

$$^cD^q\alpha^*_{n+1} = f(t, \alpha^*_n) + f_u(t, \alpha^*_n)(\alpha^*_{n+1} - \alpha^*_n) + g(t, \beta^*_n), \alpha^*_{n+1}(0) = u_0, \tag{9}$$

$$^cD^q\beta^*_{n+1} = f(t, \beta^*_n) + f_u(t, \alpha^*_{n+1})(\beta^*_{n+1} - \beta^*_n) + g(t, \alpha^*_n), \beta^*_{n+1}(0) = u_0. \tag{10}$$

starting with $\alpha^*_0 = \alpha_1$ on J. Then there exist monotone sequences $\{\alpha_n\}$ and $\{\beta_n\}$, which converge uniformly to the unique solution of (6) and the convergence is faster than superlinear.

Proof. We provide a brief proof. Initially compute α_1 using $^cD^q\alpha_1 = f(t, \alpha_0) + f_u(t, \alpha_0)(\alpha_1 - \alpha_0) + g(t, \beta_0), \alpha_1(0) = u_0$. Relabel $\alpha_1 = \alpha^*_0$. Now compute β_1 using $\beta_0$and α^*_0. That is $^cD^q\beta_1 = f(t, \beta_0) + f_u(t, \alpha^*_0)(\beta_1 - \beta_0) + g(t, \alpha^*_0), \beta_1(0) = u_0$. One can easily see that $\alpha_0(t) \leq \alpha_1(t)$ on J. Now it is enough if we prove that $\beta^*_0 \leq \beta_1$.

$$\begin{gathered}
\text{Let } p(t) = \beta^*_0 - \beta_1, \quad p(0) = 0. \\
^cD^qp(t) = {^cD^q\beta_0}^* - {^cD^q\beta_1} \\
= f(t, \beta_0) + f_u(t, \alpha_1)(\beta_1 - \beta_0) + g(t, \alpha_1) - (f(t, \beta_0) + f_u(t, \alpha_0)(\beta_1 - \beta_0) + g(t, \alpha_0)) \\
= (f_u(t, \alpha_1) - f_u(t, \alpha_0))(\beta_1 - \beta_0) + g(t, \alpha_1) - g(t, \alpha_0) \\
\leq 0, \quad \text{since } \alpha_1(t) \geq \alpha_0(t) \text{ on } J.
\end{gathered}$$

This implies $p(t) \leq 0$ on J, using Corollary 2.1. That is $\beta^*_0 \leq \beta_1$ on J. Continuing the process, we can show that that the sequences $\{\alpha^*_n\}$ and $\{\beta^*_n\}$ converge faster than the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ which are computed using Theorem 3.1.

4 Numerical Results

In this section, we provide a numerical example as an application of our main results. We take a simple logistic equation and apply Theorem 3.1. In order to apply Theorem 3.1, we assume that α_1 and β_1 should satisfy $\alpha_0 \leq \alpha_1$, $\beta_1 \leq \beta_0$ on $[0, T]$. If $q = 1$, the solution of the logistic equation can be computed explicitly. However, if $0 < q < 1$, we cannot compute the solution explicitly. Method of lower and upper solution guarantees the interval of existence. The equilibrium solutions play the role of lower and upper solutions.

Consider the example

$$ {}^cD^q u(t) = u - u^2, \quad u(0) = \frac{1}{2}, \quad t \in [0, T], \quad T \geq 1. \tag{11} $$

It is easy to observe that $\alpha_0(t) = 0$ and $\beta_0(t) = 1$ are natural lower and upper solutions respectively of (11) such that $\alpha_0 \leq \beta_0$ on $[0, T]$. Here $f(t, u) = u$ and $g(t, u) = -u^2$.

Using the iterative schemes as in Theorem 3.1 we obtain

$$ {}^cD^q \alpha_1(t) = \alpha_1 - \beta_1^2 \quad and \quad {}^cD^q \beta_1(t) = \beta_1 - \alpha_1^2. $$

Solving for α_1 and β_1, we arrive at

$$ \alpha_1 = 1 - \tfrac{1}{2} E_{q,1}(t^q) \text{ and } \beta_1 = \tfrac{1}{2} E_{q,1}(t^q) $$

Similarly, the next iteration gives rise to

$$ {}^cD^q \alpha_2(t) = \alpha_2 - \beta_1^2 \text{ and } {}^cD^q \beta_2(t) = \beta_2 - \alpha_1^2 $$
$$ {}^cD^q \alpha_2(t) = \alpha_2 - (\tfrac{1}{2} E_{q,1}(t^q))^2 \text{ and } {}^cD^q \beta_2(t) = \beta_2 - (1 - \tfrac{1}{2} E_{q,1}(t^q))^2. $$

In order to compute α_2 and β_2, we use (3) with $\lambda = 1$, and $f(t)$ as $-(\frac{1}{2} E_{q,1}(t^q))^2$ and $-(1 - \frac{1}{2} E_{q,1}(t^q))^2$ respectively. Here, we have computed $(\frac{1}{2} E_{q,1}(t^q))^2$ and $(1 - \frac{1}{2} E_{q,1}(t^q))^2$ using the product formula. The product formula is given by

$$ E_{q,1}(\lambda(t - t_0)^q) * E_{q,1}(\mu(t - t_0)^q) = \sum_{k=0}^{\infty} \frac{(t - t_0)^{qk}}{\Gamma(qk + 1)} (\lambda + \mu)_{q,1}^k, $$

where

$$ (\lambda + \mu)_{q,1}^k = \sum_{l=0}^{k} \frac{\lambda^l \mu^{k-l} \Gamma(qk + 1)}{\Gamma(ql + 1)\Gamma(q(k - l) + 1)}, $$

which is the generalized binomial formula. Further we need to multiply this by $E_{q,q}((t - s)^q)$ as in formula (4) to compute α_2 and β_2. Computing α_2 and β_2, we arrive at

$$ \alpha_2 = \tfrac{1}{2} E_{q,1}(t^q) - \tfrac{1}{4} s_1 \text{ and } \beta_2 = 1 - \tfrac{1}{2} E_{q,1}(t^q) - \tfrac{1}{4} s_1 + s_2, $$

where

$$ s_1 = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{l=0}^{k} \frac{t^{q+jq+kq} \Gamma(1 + kq)}{\Gamma(lq + 1)\Gamma(kq - lq + 1)\Gamma(q + jq + kq + 1)}, $$

$$ s_2 = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{t^{q+jq+kq}}{\Gamma(q + jq + kq + 1)}. $$

The graphs of α_1, β_1 and α_2, β_2 have been drawn in Figure 1 where $q = \frac{1}{2}, t_0 = 0$.

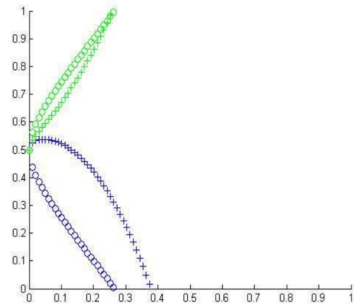

Figure 1: Coupled Lower and Upper Solutions of (11) with $q = 1/2$ using Theorem 3.1.

5 Conclusion

In this work we have mixed generalized quasilinearization method and generalized monotone method to compute the coupled lower and upper solution of type I on the desired interval. In addition, the method also provides the unique solution of the nonlinear problem. This mixed method yields superlinear convergence. Computation of the solution of the coupled lower and upper solutions numerically involves the generalized Mittag-Leffler function which involves the generalized binomial coefficients. In Figure 1, we can see that $\overline{t_2} \not> \overline{t_1}$, since the evaluation of β_2 is not accurate. This is due to the lack of knowledge of product of Mittag-Leffler function and its accurate computation. We plan to develop the necessary properties of the Mittag-Leffler function in our future work and obtain better estimates for the sequences $\{\alpha_n\}$ and $\{\beta_n\}$.

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Peakons and Soliton Solutions of Newly Developed Benjamin-Bona-Mahony-Like Equations

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Abstract: This paper establishes a set of Benjamin-Bona-Mahony-like equations (BBM-like) equations. By means of an advection dispersion equation, we can develop several BBM-like equations. We show that these established equations share some of the solitary wave solutions of the BBM equation. We also show that these developed equations give paekon solutions, for specific values of the parameter included in these equations, although these equations are not of the Camassa-Holm type of equations. We also derive a variety of solitonic solutions.

Keywords: *BBM-like equation; peakons; solitons.*

Mathematics Subject Classification (2010): 74D10, 74D30, 37G20, 34A45.

1 Introduction

Nonlinear equations have been a subject of intensive study for decades in several areas of mathematics, physics, engineering and other sciences. The study of these nonlinear equations has been the topic of major research projects in nonlinear sciences. Another interesting class of excitations consists of establishing nonlinear equations with significant physical features [1–10].

The KdV equation reads

$$u_t + uu_x + u_{xxx} = 0. \tag{1}$$

This equation models a variety of nonlinear wave phenomena such as shallow water waves, acoustic waves in a harmonic crystal, internal gravity waves in oceans, blood pressure pulses, and ion-acoustic waves in plasmas [1–7]. The KdV equation is completely integrable and admits multiple-soliton solutions and exhibits an infinite number of conserved

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quantities. The Korteweg-de Vries (KdV) equation was derived to describe shallow water waves of long wavelength and small amplitude. The KdV equation admits soliton solutions which have been the subject of intense study for the last few decades. Researchers remain intrigued by the physical properties of the KdV equation, in particular the complete integrability and the possess of an infinite number of conserved quantities.

While the KdV equation has remarkable properties [3], some other aspects of this equation are less favorable. This includes, e.g., an unbounded dispersion relation, that is obviously non-physical [3]. Several noticeable attempts to improve the KdV model were taken over the years. Benjamin-Bona-Mahony introduced the regularized long-wave equation, or the BBM equation that reads

$$u_t + u_x + uu_x - u_{xxt} = 0, \tag{2}$$

replaces the third-order derivative in the KdV equation (1) by a mixed derivative $-uxxt$, which, in turn, results in a bounded dispersion relation [3]. The BBM equation (2) can be used to describe the behavior of an undular bore, in water, which comprises a smooth wavefront followed by a train of solitary waves [6,7]. An undular bore can be interpreted as the dispersive analog of a shock wave in classical non-dispersive, dissipative hydrodynamics [7, 11-20].

Studies on nonlinear evolution equations are growing rapidly because these equations describe real features in science, technology, and engineering fields. In the past decades, a vast variety of powerful methods has been established to determine the exact solutions for these equations and to study the scientific features of these solutions from many points of view. Examples of these methods are the Hirota bilinear method [4], the simplified Hirota's method [6], the Bäcklund transformation method, Darboux transformation, Pfaffian technique, the inverse scattering method [4], the Painlevé analysis, the generalized symmetry method, the subsidiary ordinary differential equation method, and many other methods that can be found in [13–20].

The BBM equation is not integrable and admits one soliton solution given by

$$u(x,t) = -\frac{12k^2}{4k^2-1}\operatorname{sech}^2\left(kx + \frac{k}{4k^2-1}\,t\right), \tag{3}$$

where $k \neq \pm\frac{1}{2}$.

Moreover, the BBM equation has the singular solution

$$u(x,t) = \frac{12k^2}{4k^2-1}\operatorname{csch}^2\left(kx + \frac{k}{4k^2-1}\,t\right). \tag{4}$$

The present paper is aimed at the derivation of entirely new Benjamin-Bona-Mahony-like (BBM-like) equations that will give peakon solutions, i.e peak-shaped soliton solutions, in addition to other travelling wave solutions, although these equations are not of the Camassa-Holm type of equations. The derivation process, as will be seen later, leads to an infinite number of such equations. We will also show that these new forms share the solutions (3)–(4) with the BBM equation (2). To achieve our goals we will use several tools that will be applied in order to extract exact solutions.

2 Formulations of the BBM-Like Equations

In this section, we will establish a class of BBM-like equations with distinct structures. In a manner parallel to that used in [5], we introduce a generalized form of an advection dispersion equation as

$$u_t + Vu_x + \delta u_{xxt} = 0, \tag{5}$$

where δ is an arbitrary dimensionless parameter and $V(u, u_x, u_{xx}, \cdots)$ is an arbitrary function. We assume that the travelling wave

$$u(x,t) = f(x - ct) = f(\xi), \tag{6}$$

solves the BBM equation (2) and also solves the advection dispersion equation (5) for the same speed c. Using $\xi = x - ct$ transforms (2) and (5) to

$$-cf' + (1 + f)f' - u_{xxt} = 0, \tag{7}$$

and

$$-cf' + Vf' + \delta u_{xxt} = 0, \tag{8}$$

respectively. Eliminating u_{xxt} from these two equations, and by noting that $f' \neq 0$, we obtain

$$V = (\delta + 1)c - \delta(1 + f) = (\delta + 1)c - \delta(1 + u). \tag{9}$$

The advection dispersion equations, or the BBM-like equations can be obtained by using a variety of values of the speed c, that can be obtained by integrating or differentiating (7) as many times as we want and if possible.

We first solve (7) for c where we find

$$c = 1 + u - \frac{u_{xxt}}{u_x}. \tag{10}$$

Substituting (10) into (9) gives

$$V = (\delta + 1)(1 + u - \frac{u_{xxt}}{u_x}) - \delta(1 + u). \tag{11}$$

Substituting (11) into the generalized advection dispersion equation (5) gives

$$u_t + \left\{(\delta + 1)(1 + u - \frac{u_{xxt}}{u_x}) - \delta(1 + u)\right\} u_x + \delta u_{xxt} = 0, \tag{12}$$

which gives the standard BBM equation (2) for any value of δ.

We now turn for the derivation of the BBM-like equations. Integrating (7) and solving for c we find

$$c = 1 + \frac{1}{2}u - \frac{u_{xt}}{u}. \tag{13}$$

Substituting for c from (13) into (9) gives

$$V_1 = (\delta + 1)\left(1 + \frac{1}{2}u - \frac{u_{xt}}{u}\right) - \delta(1 + u). \tag{14}$$

Inserting this result into the advection dispersion equation (5) gives

$$u_t + \left\{(\delta + 1)\left(1 + \frac{1}{2}u - \frac{u_{xt}}{u}\right) - \delta(1 + u)\right\} u_x + \delta u_{xxt} = 0, \tag{15}$$

that will be termed the first BBM-like equation.

To determine more values for the speed c, we can differentiate (7) as many times as we want. For example, differentiating (7) once and solving for c we find

$$c = 1 + u + \frac{u_x^2 - u_{xxxt}}{u_{xx}}, \tag{16}$$

and by differentiating (7) again and solving for c we obtain

$$c = 1 + u + \frac{3u_x u_{xx} - u_{xxxxt}}{u_{xxx}}. \tag{17}$$

Other values for c can be determined by differentiating (7) as many times as we want. Substituting (16) and (17) into (9) and simplifying one finds

$$V_2 = (\delta + 1)\left(1 + u + \frac{u_x^2 - u_{xxxt}}{u_{xx}}\right) - \delta(1 + u), \tag{18}$$

and

$$V_3 = (\delta + 1)\left(1 + u + \frac{3u_x u_{xx} - u_{xxxxt}}{u_{xxx}}\right) - \delta(1 + u). \tag{19}$$

Notice that V_2 and V_3 involve higher order derivatives than the dispersive term u_{xxt} of the BBM equation. Substituting V_2 and V_3 into (5) gives the following BBM-like equations

$$u_t + \left\{1 + u + (\delta + 1)\left(\frac{u_x^2 - u_{xxxt}}{u_{xx}}\right)\right\} u_x + \delta u_{xxt} = 0, \tag{20}$$

and

$$u_t + \left\{1 + u + (\delta + 1)\left(\frac{3u_x u_{xx} - u_{xxxxt}}{u_{xxx}}\right)\right\} u_x + \delta u_{xxt} = 0, \tag{21}$$

that will be termed the second and the third BBM-like equations respectively.

The first conclusion that we can make here is that the three derived BBM-like equations (15), (20) and (21) share the same soliton and singular solutions (3) and (4) that we derived earlier for the standard BBM equation (2).

Because our main concern of this work is to establish peakon solutions for the derived BBM-like equations, which are not of the CH or DP type, in addition to other solutions, we found that peakon solutions exist only for specific value of δ for each equation. Using selected values of δ for the equations (15), (20) and (21), we obtain the following specific BBM-like equations

$$u_t + \left\{1 - 2\frac{u_{xt}}{u}\right\} u_x + u_{xxt} = 0, \delta = 1, \tag{22}$$

$$u_t + \left\{1 + u - \left(\frac{u_x^2 - u_{xxxt}}{u_{xx}}\right)\right\} u_x - 2u_{xxt} = 0, \delta = -2, \tag{23}$$

and

$$u_t + \left\{1 + u - \frac{1}{3}\left(\frac{3u_x u_{xx} - u_{xxxxt}}{u_{xxx}}\right)\right\} u_x - \frac{4}{3}u_{xxt} = 0, \delta = -\frac{4}{3}. \tag{24}$$

In what follows we will employ distinct tools to derive exact solutions for each of the aforementioned forms (15), (20), and (21), that will be referred to as Form I, Form II, and Form III respectively. Recall that peakon solutions exist only for specific values of the parameter δ, whereas other solutions will be obtained for any selective value of δ.

3 The Nonlinear BBM-Like Equation: Form I

In this section we will study form I of the nonlinear BBM-like equation

$$u_t + \left\{ (\delta+1)\left(1+\frac{1}{2}u - \frac{u_{xt}}{u}\right) - \delta(1+u) \right\} u_x + \delta u_{xxt} = 0, \tag{25}$$

where we will derive peakon solutions and other travelling wave solutions.

3.1 Peakon solution

As stated before, we found that peakon solution exists for (25) only for $\delta = 1$, where (25) becomes

$$u_t + \left\{ 1 - 2\frac{u_{xt}}{u} \right\} u_x + u_{xxt} = 0. \tag{26}$$

To determine a peakon solution to (26), we assume the peakon solution is of the form

$$u(x,t) = Re^{-|kx-ct|}. \tag{27}$$

Substituting this assumption into (26) we solve the resulting equation to find that

$$c = -\frac{k}{k^2-1},\ k \neq \pm 1, \tag{28}$$

and R can be any selective real number such as c. Consequently, the peakon solution is given by

$$u(x,t) = Re^{-|kx+\frac{k}{k^2-1}t|}. \tag{29}$$

Recall that the standard BBM equation gives soliton solutions but not peakon solutions.

Figure 1 below shows the peakon solution (29).

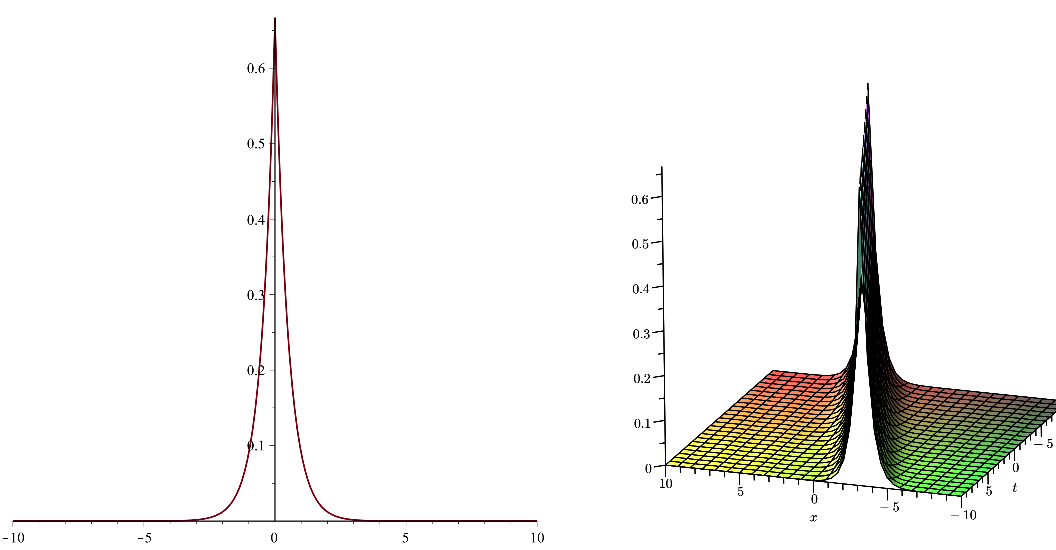

Figure 1: (a) Peakon solution with $R = \frac{2}{3}, k = 2,\ -10 \leq x \leq 10$,
(b) Peakon solution with $R = \frac{2}{3}, k = 2,\ -10 \leq x, t \leq 10$.

3.2 Soliton solutions

In this section, we will derive soliton solutions that satisfy the generalized BBM-like equation (25) for specific values of the parameter δ. For this reason, we assume that the solution for (25) has the form

$$u(x,t) = R + \operatorname{sech}^2(kx - ct). \tag{30}$$

Substituting this assumption into the nonlinear BBM-like equation (25), and solving the resulting equation for R and c, we find two sets of solutions given by

$$\begin{aligned} c &= \tfrac{k}{4k^2+1}, \\ R &= -\tfrac{2}{3}, \end{aligned} \tag{31}$$

valid for $\delta = 1$, and

$$\begin{aligned} c &= \tfrac{1}{12k}, \\ R &= \tfrac{1-16k^2}{12k^2}, \end{aligned} \tag{32}$$

valid for $\delta = -1$.

This in turn gives the soliton solutions

$$u(x,t) = -\frac{2}{3} + \operatorname{sech}^2(kx - \frac{k}{4k^2+1}t), \delta = 1, \tag{33}$$

and

$$u(x,t) = \frac{1-16k^2}{12k^2} + \operatorname{sech}^2(kx - \frac{1}{12k}t), \delta = -1. \tag{34}$$

We point out that the first solution justifies also the BBM equation, whereas the second solution satisfies only the BBM-like equation (25).

In a similar manner, we can derive the singular soliton solutions

$$u(x,t) = \frac{2}{3} + \operatorname{csch}^2(kx - \frac{k}{4k^2+1}t), \delta = 1. \tag{35}$$

and

$$u(x,t) = -\frac{1+8k^2}{12k^2} + \operatorname{csch}^2(kx + \frac{1}{12k}t), \delta = -1. \tag{36}$$

Unlike the previous results of the soliton solutions, the first singular soliton solution (35) satisfies the BBM-like equation (25), whereas the second solution (36) satisfies the BBM and the BBM-like equations.

3.3 Travelling waves solutions

In this section, we will derive more exact solutions that satisfy the generalized BBM-like equation (25), for specific values of the parameter δ. In what follows, we will present the approaches that will be used to derive these new solutions.

3.3.1 Solutions in the $\sec^2$ or $\csc^2$ form

We assume that the solution for (25) has the form

$$u(x,t) = R + \sec^2(kx - ct). \tag{37}$$

Substituting this assumption into the nonlinear BBM-like equation (25), and solving the resulting equation for R and c, we find two sets of solutions given by

$$\begin{aligned} c &= -\tfrac{k}{4k^2-1}, \\ R &= -\tfrac{2}{3}. \end{aligned} \tag{38}$$

valid for $\delta = 1$, and

$$\begin{aligned} c &= -\tfrac{1}{12k}, \\ R &= -\tfrac{1+16k^2}{12k^2}. \end{aligned} \tag{39}$$

valid for $\delta = -1$. This gives the exact solutions

$$u(x,t) = -\frac{2}{3} + \sec^2(kx + \frac{k}{4k^2-1}t), \delta = 1, \tag{40}$$

and

$$u(x,t) = -\frac{1+16k^2}{12k^2} + \sec^2(kx + \frac{1}{12k}t), \delta = -1, \tag{41}$$

3.3.2 Solutions in the $\sin^2$ or $\cos^2$ form

We assume that the solution for (25) has the form

$$u(x,t) = R + \sin^2(kx - ct). \tag{42}$$

Substituting this assumption into the nonlinear BBM-like equation (25), and solving the resulting equation for R and c, we find only one set of solutions given by

$$\begin{aligned} c &= -\tfrac{k}{4k^2+1}, \\ R &= -\tfrac{1}{2}. \end{aligned} \tag{43}$$

valid for $\delta = 1$ This gives the exact solution

$$u(x,t) = -\frac{1}{2} + \sec^2(kx + \frac{k}{4k^2+1}t), \delta = 1. \tag{44}$$

In a similar manner, we can derive the solution

$$u(x,t) = -\frac{1}{2} + \cos^2(kx + \frac{k}{4k^2+1}t), \delta = 1. \tag{45}$$

4 The Nonlinear BBM-Like Equation: Form II

In this section we will study form II of the nonlinear BBM-like equation

$$u_t + \left\{1 + u + (\delta+1)\frac{u_x^2 - u_{xxxt}}{u_{xx}}\right\} u_x + \delta u_{xxt} = 0, \tag{46}$$

where we will derive peakon solutions and other travelling wave solutions.

4.1 Peakon solution

As stated before, we found that peakon solution exists for (46) only for $\delta = -2$, where (46) becomes

$$u_t + \left\{1 + u - \frac{u_x^2 - u_{xxxt}}{u_{xx}}\right\} u_x - 2u_{xxt} = 0, \delta = -2, \tag{47}$$

To determine a peakon solution to (47), we assume the peakon solution is of the form

$$u(x,t) = Re^{-|kx-ct|}. \tag{48}$$

Substituting this assumption into (47) we solve the resulting equation to find that

$$c = -\frac{k}{k^2-1}, \; k \neq \pm 1, \tag{49}$$

and R can be any selective real number such as c. Consequently, the peakon solution is given by

$$u(x,t) = Re^{-|kx+\frac{k}{k^2-1}t|}. \tag{50}$$

Recall that the standard BBM equation gives soliton solutions but not peakon solutions. Moreover, the obtained peakon solution (50) is identical to the peakon solution obtained earlier for the first form.

4.2 Soliton solutions

In this section, we will derive soliton solutions that satisfy the generalized BBM-like equation (46). For this reason, we assume that the solution for (46) has the form

$$u(x,t) = R + \operatorname{sech}^2(kx - ct). \tag{51}$$

Substituting this assumption into the nonlinear BBM-like equation (46), and solving the resulting equation for R and c, we find one set of solutions given by

$$\begin{aligned} c &= \tfrac{1}{12k}, \\ R &= \tfrac{1-16k^2}{12k^2}, \end{aligned} \tag{52}$$

valid for any real value of δ.

This in turn gives the soliton solutions

$$u(x,t) = \frac{1-16k^2}{12k^2} + \operatorname{sech}^2(kx - \frac{1}{12k}t), \tag{53}$$

which also satisfies the BBM equation.

In a similar manner, we can derive the singular soliton solutions

$$u(x,t) = -\frac{1+8k^2}{12k^2} + \operatorname{csch}^2(kx + \frac{1}{12k}t), \tag{54}$$

which also satisfies the BBM equation.

4.3 Travelling waves solutions

In this section, we will derive more exact solutions that satisfy the generalized BBM-like equation (46). In what follows, we will present the approaches that will be used to derive these new solutions.

4.3.1 Solutions in the $\sec^2$ or $\csc^2$ form

We assume that the solution for (46) has the form

$$u(x,t) = R + \sec^2(kx - ct). \tag{55}$$

Substituting this assumption into the nonlinear BBM-like equation (46), and solving the resulting equation for R and c, we find one set of solutions given by

$$\begin{array}{rcl} c &=& -\frac{1}{12k}, \\ R &=& -\frac{1+16k^2}{12k^2}. \end{array} \tag{56}$$

valid for any real value of δ. This gives the exact solutions

$$u(x,t) = -\frac{1+16k^2}{12k^2} + \sec^2(kx + \frac{1}{12k}t). \tag{57}$$

In a like manner, we can derive another exact solution of the form

$$u(x,t) = -\frac{1+16k^2}{12k^2} + \csc^2(kx + \frac{1}{12k}t). \tag{58}$$

5 The Nonlinear BBM-Like Equation: Form III

In this section we will study form III of the nonlinear BBM-like equation

$$u_t + \left\{1 + u + (\delta+1)\frac{3u_x u_{xx} - u_{xxxxt}}{u_{xxx}}\right\} u_x + \delta u_{xxt} = 0, \tag{59}$$

where we will derive peakon solutions and other travelling wave solutions.

5.1 Peakon solution

As stated before, we found that peakon solution exists for (59) only for $\delta = -\frac{4}{3}$, where (59) becomes

$$u_t + \left\{1 + u - \frac{1}{3}(\frac{3u_x u_{xx} - u_{xxxxt}}{u_{xxx}})\right\} u_x - \frac{4}{3}u_{xxt} = 0. \tag{60}$$

To determine a peakon solution to (60), we assume the peakon solution is of the form

$$u(x,t) = Re^{-|kx-ct|}. \tag{61}$$

Substituting this assumption into (60) we solve the resulting equation to find that

$$c = -\frac{k}{k^2-1},\ k \neq \pm 1, \tag{62}$$

and R can be any selective real number such as c. Consequently, the peakon solution is given by

$$u(x,t) = Re^{-|kx+\frac{k}{k^2-1}t|}. \qquad (63)$$

It is obvious that although the three forms of the BBM-like equations differ in its structures, but all three models gave the same peakon solution.

5.2 Soliton solutions

In this section, we will derive soliton solutions that satisfy the generalized BBM-like equation (59) for specific values of the parameter δ. For this reason, we assume that the solution for (59) has the form

$$u(x,t) = R + \operatorname{sech}^2(kx - ct). \qquad (64)$$

Substituting this assumption into the nonlinear BBM-like equation (59), and solving the resulting equation for R and c, we find two sets of solutions given by

$$\begin{aligned} c &= \tfrac{1}{12k}, \\ R &= \tfrac{1-16k^2}{12k^2}, \end{aligned} \qquad (65)$$

valid for any real value of δ.

This in turn gives the soliton solutions

$$u(x,t) = \frac{1-16k^2}{12k^2} + \operatorname{sech}^2(kx - \frac{1}{12k}t), \delta = -1. \qquad (66)$$

In a similar manner, we can derive the singular soliton solutions

$$u(x,t) = -\frac{1+8k^2}{12k^2} + \operatorname{csch}^2(kx + \frac{1}{12k}t), \delta = -1. \qquad (67)$$

5.3 Travelling waves solutions

In this section, we will derive more exact solutions that satisfy the generalized BBM-like equation (59), for specific values of the parameter δ. In what follows, we will present the approaches that will be used to derive these new solutions.

5.3.1 Solutions in the $\sec^2$ or $\csc^2$ form

We assume that the solution for (59) has the form

$$u(x,t) = R + \sec^2(kx - ct). \qquad (68)$$

Substituting this assumption into the nonlinear BBM-like equation (59), gives the same solution obtained before for form II, namely

$$u(x,t) = -\frac{1+16k^2}{12k^2} + \sec^2(kx + \frac{1}{12k}t), \qquad (69)$$

and

$$u(x,t) = -\frac{1+16k^2}{12k^2} + \csc^2(kx + \frac{1}{12k}t). \qquad (70)$$

5.3.2 Solutions in the $\sin^2$ or $\cos^2$ form

We assume that the solution for (59) has the form

$$u(x,t) = R + \sin^2(kx - ct). \tag{71}$$

Substituting this assumption into the nonlinear BBM-like equation (59), and solving the resulting equation for R and c, we find only one set of solutions given by

$$\begin{aligned} \delta &= -\tfrac{4}{3}, \\ c &= \tfrac{k(3+2R}{2(4k^2+1)}, \end{aligned} \tag{72}$$

where R is left as a free parameter. This gives the exact solution

$$u(x,t) = R + \sin^2(kx + \frac{k}{4k^2+1}t), \tag{73}$$

In a similar manner, we can derive the solution

$$u(x,t) = R + \cos^2(kx + \frac{k}{4k^2+1}t). \tag{74}$$

6 Conclusion

In this work we established three (BBM-like) equations that share some of the solitary wave solutions with the standard BBM equation. We showed that these forms, although are not of the same type as the CH or DP list of equations, but give peakon solutions provided that each form has specific value of the parameter δ included in the equation. This shows that the developed BBM-like equations can model solitary wave solutions and peaked solitary waves solutions. In addition, the developed equations contain terms with higher derivatives than the third-order term u_{xxt}.

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