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Effectiveness of the Extended Kalman Filter Through Difference Equations

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Abstract: The extended Kalman filter is extensively used in the nonlinear state estimation systems. As long as the system characteristics are correctly known, the extended Kalman filter gives the best performance. However, when the system information is partially known or incorrect, the extended Kalman filter (EKF) may diverge or give the biased estimates. To overcome this problem we introduced the new Riccati difference equation (RDE) which is used to study and examine the performance analysis of extended Kalman filter. We consider the special case of tracking a target with cluster, but with a probability arrival of small value. Finally the convergence analysis and stabilizing solution of Riccati difference equations arising from the standard extended Kalman filter is studied. Simulations results for convergence of EKF for the class of nonlinear filters are done through MATLAB.

Keywords: *convergence; extended Kalman filter; Riccati difference equations; feasibility and stabilizing solution.*

Mathematics Subject Classification (2010): 39A10, 39A30, 39A60, 39B82, 39B99.

1 Introduction

Several recent papers have been devoted to a study of nonlinear Riccati difference equations. The family of Kalman filters have been applied for state as well as parameter estimation for numerous linear as well as nonlinear systems. Though the standard Kalman filter is considered in an optimal estimator (in case of linear systems) with Gaussian noise characters, its nonlinear (extended Kalman filter) suboptimal counterpart is known to diverge under the influences of severe nonlinearities and uncertainties [4,7]. As a solution to this problem robust form of the EKF have been formulated for a wide class of uncertainities [13] in the form of new RDE.

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The paper is organized as follows. In Section 2, we introduced the new Riccati difference equation and algebraic Riccati equation, which are used to arrive the feasible solutions. Also we introduced some lemmas and assumptions which are useful for arriving the convergence analysis. Section 3 provides the conditions needed to ensure the convergence analysis and stabilizing the solutions of the new RDE with the initial conditions. Section 4 provides the simulation results for convergence of the EKF for the class of nonlinear systems through MATLAB [12]. Conclusions are made in Section 5.

2 Preliminaries

Consider the following linear discrete-time system [5, 10]

$$u_{k+1} = Ax_k + Bw_k \quad k \in N, \tag{1}$$

$$v_k = Cx_k + Du_k \quad k \in N, \tag{2}$$

$$z_k = Lx_k, \tag{3}$$

with the initial condition x_0 and $k = 0, 1, 2, \dots, N$, where $x_k \in R^n$ is the system state, $w_k \in R^q$ is the noise, $v_k \in R^m$ is the output measurements, $u_k \in R^m$ is the input measurements, $z_k \in R^p$ is a linear combination of the state variable to be estimated. A, B, C, D and L are known real constant matrices with appropriate dimensions. Time step k is defined as $Z_k = \{z_1, z_2, z_3, ..., z_k\}$, often this is referred to as the measurement.

It is worth noting that an estimator z_k is called an a priori filter if $\hat{z}_k$ is obtained with the output measurements [15] $\{v_0, v_1, \dots, v_{k-1}\}$, while $\hat{z}_k$ is referred to as a posteriori filter. This $\hat{z}_k$ is obtained by the measurements $\{v_0, v_1, \dots, v_k\}$.

Now we introduce the following new Riccati difference equation (RDE)

$$P_{k+1} = AP_kA^T - \left(AP_kC^T + BD^T\right)\left(CP_kC^T + R\right)^{-1}\left(CP_kA^T + DB^T\right), \tag{4}$$

and the Algebraic Riccati Equation (ARE) [14],

$$P = APA^T - \left(APC^T + BD^T\right)\left(CPC^T + R\right)^{-1}\left(CPA^T + DB^T\right). \tag{5}$$

It is clear that the existence of filter is related to the RDE (4) or ARE (5), and the fulfillment of a suitable matrix inequality (feasibility condition) [1], [3]. Now, we adopt the definition of feasible solution [6]. The feasiblility and convergence analysis problem studied in this paper is stated as follows: Given an arbitrarily large N, find the suitable conditions on the initial state P_0 such that the solution P_k is feasible at every step $k \in [0, N]$ and converges to the stabilizing solution P_s as $N \to \infty$ [8],[9]. We end this section by giving two preliminary results which play an important role in deriving the main results of this paper. The first is an extension of a comparison result of new RDE [16].

Lemma 2.1 *Consider the following Riccati difference equation*

$$P_{k+1} = AP_kA^T - \left(AP_kC^T + BD^T\right)\left(CP_kC^T + R\right)^{-1}\left(CP_kA^T + DB^T\right) + BB^T.$$

Let P_k^1 and P_k^2 be solutions of (4) with different initial conditions $P_0^1 = \bar{P}_0^1 \geq 0$ and $P_0^2 = \bar{P}_0^2 \geq 0$, respectively. Then the difference between the two solutions $\tilde{P}_k = P_k^2 - P_k^1$ satisfies the following equation

$$\tilde{P}_{k+1} = \tilde{A}_k\tilde{P}_k\tilde{A}_k^T - \tilde{A}_k\tilde{P}_kC^T\left(C\tilde{P}_kC^T + \tilde{R}_k\right)^{-1}C\tilde{P}_k\tilde{A}_k^T,$$

where $\tilde{A}_k = A - \left(AP_K^1C^T + BD^T\right)\left(CP_k^1C^T + R\right)^{-1}C$ *and* $\tilde{R}_k = CP_k^1C^T + R$.

In order to extend the above lemma, we need the following assumption.

Assumption 2.1 The matrix $\bar{A} = A - BD^T\left(DD^T\right)^{-1}C$ is invertible.

Lemma 2.2 *Consider Riccati difference equation* (4)*. Let* P_k^1 *and* P_k^2 *be the two solutions of* (4) *with different initial conditions* $P_0^2 > P_0^1 > 0$*. Then, under Assumption* 2.1*, when* P_k^2 *is feasible, it results that* $P_k^2 > P_k^1 > 0$ *and* P_k^1 *is feasible too. Furthermore, if* $P_0^2 > P_0^1$*, then* $P_k^2 > P_k^1$*.*

3 Convergence Analysis of Riccati Difference Equation

It is well known from filtering and control theory that the Kalman recursions lead to a recursive formula for the covariance matrix analysis [2]. This result is obtained by eliminating the Kalman gain from the recursion formula. This recursion formula is referred to as the Riccati difference equation [8]. The issue of the speed of convergence is an important one. So we introduced the following Lyapunov equation

$$\tilde{A}^TY\tilde{A} - Y = -M_-, \tag{6}$$

where $\tilde{A} = A - (AP_sC^T + BD^T)(CP_sC^T + R)^{-1}C$. Now we can formulate Kalman-like recursions for a general system as

$$M_k = \tilde{A}^{-T}\left(G + C^T\tilde{R}^{-1}C\right)\tilde{A}^{-1} - G_K, \tag{7}$$

$$G_k = -P_s^{-1} - P_s^{-1}\left(L^TL - P_s^{-1}\right)^{-1}P_s^{-1}, \tag{8}$$

$$R_k = CP_sC^T + R, \tag{9}$$

where k is the Kalman gain [5]. The following theorem establishes the relationship between the initial state P_0 and feasibile solution to RDE (4).

Theorem 3.1 *Consider the Riccati difference equation* (4)*. Let Assumption 2.1 hold, and let* Y *be the solution to the Lyapunov equation* (6)*. Then the solution* P_k *of RDE* (4) *is feasible over* $[0 \quad \infty)$ *if for some sufficiently small* $\epsilon > 0$*, the initial condition satisfies*

$$0 < P_0 < (G_k - Y + M_k + I)^{-1} + P_s. \tag{10}$$

Proof. The procedure of the proof is classified into three cases.

Case (i) $P_0 < P_s$. P_s is a constant feasible solution of (4), then the feasibility of P_k follows from Lemma 2.2 directly.

Case (ii) $P_0 > P_s$. Let's define $X_k = P_k - P_s$. Then, applying Lemma 2.1 to (4) and (5), immediately we obtain that X_k satisfies the following

$$\begin{aligned} X_{k+1} &= \hat{A}X_k\hat{A}^T - \hat{A}X_kC\left(CX_kC^T + \hat{R}\right)^{-1}CX_k\hat{A}^T \\ &= \hat{A}\left(X_k^{-1} + C^T\hat{R}^{-1}C\right)^{-1}\hat{A}^T, \end{aligned} \tag{11}$$

where $X_0 = p_0 - P_s$, $\hat{A} = A - \left(AP_sC^T + BD^T\right)\left(CP_sC^T + R\right)^{-1}C$ and $\hat{R} = CP_sC^T + R$. Now let $Z_k = X_k^{-1} - G_k$, where G_k is defined by (8). It is worth noting that $G_k \geq 0$, since P_s is feasible. Note that $\hat{A}$ is invertible as $\bar{A}$ is invertible and P_s is feasible. Then by (11), we have $Z_{k+1} = \hat{A}^{-T} Z_k \hat{A}^{-1} + M_k$, where M_k is defined by (7) and $Z_0 = (P_0 - P_s)^{-1} - G_k$. Since P_s is feasible and $X_k > 0$, then according to Lemma 2.2, it is clear that the feasibility of P_k is equivalent to the positive definiteness of Z_k, which follows from $Z_k = P - s^{-1}\left[\left(P - s^{-1} - P_k^{-1}\right)^{-1} - \left(P_s^{-1} - L^T L\right)^{-1}\right] P_s^{-1}$.

Now consider the following Lyapunov equation [11]

$$\hat{Z}_{k+1} = \hat{A}^{-T} \hat{Z}_{k+1} \hat{A}^{-1} + M_- \tag{12}$$

with $\hat{Z}_0 = Z_0$. By definition $M_k \geq M_-$, so that $Z_k \geq \hat{Z}_k$. Then $\hat{Z}_k > 0$ is sufficient to guarantee the positivity of Z_k. Now we compute (12) as follows

$$\begin{aligned} Z_k \geq \hat{Z}_k &= \left(\hat{A}^{-k}\right)^T \left(Z_0 + \sum_{j=1}^{k} \left(\hat{A}^j\right)^T M_- \hat{A}^j\right) \hat{A}^{-k} \\ &\geq \left(\hat{A}^{-k}\right)^T \left(Z_0 + \sum_{j=1}^{\infty} \left(\hat{A}^j\right)^T M_- \hat{A}^j\right) \hat{A}^{-k}, \end{aligned} \tag{13}$$

from (6), we deduce the value of Y,

$$\begin{aligned} Y &= \sum_{j=0}^{\infty} \left(\hat{A}^j\right)^T M_- \hat{A}^j \\ &= M_- + \sum_{j=1}^{\infty} \left(\hat{A}^j\right)^T M_- \hat{A}^j. \end{aligned} \tag{14}$$

Now comparing (13) and (14), we have

$$Z_k \geq \hat{Z}_k \geq \left(\hat{A}^{-k}\right)^T \left(Z_0 + Y - M_-\right) \hat{A}^{-k}. \tag{15}$$

So, if $Z_0 + Y - M_- > 0$, then $\hat{Z}_k > 0$ and in turn $Z_k > 0$. Here $Z_0 + Y - M_- > 0$. This is rewritten as

$$(P_0 - P_s)^{-1} - G_k + Y - M_- > 0. \tag{16}$$

Since $-Y + M_- \geq 0$ and $G_k \geq 0$, then (10) implies (16). Thus the proof of feasibility for the case of $P_0 > P_s$ is completed.

Case (iii). $P_0 - P_s$ is not a definite matrix. Initially we need to study the convergence of the solution of the RDE (4). It is easy to know that (4) satisfies the following matrix recursions

$$\begin{aligned} P_{k+1} &= \bar{A} S_k^{-1} \bar{A}^T + B\left[I - D^T \left(DD^T\right)^{-1} D\right] B^T, \\ S_k &= P_k^{-1} + C^T R^{-1} C, \end{aligned} \tag{17}$$

so S_k satisfies the following RDE

$$S_k = \left\{\bar{A} S_k^{-1} \bar{A}^T + B\left[I - D^T \left(DD^T\right)^{-1} D\right] B^T\right\}^{-1} + C^T R^{-1} C, \tag{18}$$

and the associated ARE is

$$S=\left\{\bar{A}S^{-1}\bar{A}^{T}+B\left[I-D^{T}\left(DD^{T}\right)^{-1}D\right]B^{T}\right\}^{-1}+C^{T}R^{-1}C. \tag{19}$$

Under Assumptions 2.1 and 19, we concluded that both the stabilizing solution S_s and antistabilizing solution S_a provides $S_s - S_a > 0$. This implies that there exists a $\bar{P}_0$ satisfying (10) and such that $\bar{P}_0 > P_0$ and $\bar{P}_0 > P_s$. Hence $P_0 - P_s$ is not a definite matrix.

The following theorem provides a sufficient condition for ensuring convergence as well as feasibility of the solution of the RDE (4) over $[0, \infty)$.

Theorem 3.2 *Consider the Riccati difference equation* (4)*. Let Assumption* 2.1 *hold, then the solution P_k of RDE* (4) *is feasible over* $[0 \quad \infty)$ *and converges to the stabilizing solution P_s of* (5) *as $k \to \infty$ if P_s is feasible and for some sufficiently small $\epsilon > 0$, then the initial condition satisfies*

$$0 < P_0 < \left(G_k - Y + M_- + \epsilon I\right)^{-1} + P_s, \tag{20}$$

where G, Y, and M are defined as in Theorem 3.1.

Proof. Initially, it is noted that P_k is feasible over $[0 \quad \infty)$ from Theorem 3.1. Consider (4), (5), (18) and (19), and the study of convergence of P_k is equivalent to the study of the convergence of S_k to S_s. So we focus on the convergence of S_k as follows, let $U = \{S_a - S_a\}^{-1}$, then from (19), we have

$$U=\tilde{A}^{T}U\tilde{A}+S_s^{-1}-P_s\hat{A}^{T}P_s^{-1}\hat{A}P_s. \tag{21}$$

Next, let $W = P_s\left[G_k - Y + M_- + P_s^{-1}\right]P_s$, then from (6), we have

$$W=\tilde{A}^{T}U\tilde{A}+S_s^{-1}-P_s\hat{A}^{T}P_s^{-1}\hat{A}P_s+N, \tag{22}$$

where

$$N=P_sC^{T}\hat{R}^{-1}CP_s+P_s\left(G_k-\hat{A}^{T}G_k\hat{A}\right)P_s-P_s\hat{A}^{T}M_-\hat{A}P_s=P_s\hat{A}^{T}M_+\hat{A}P_s\geq 0.$$

Comparing (21) and (22), we have $W \geq U$. Now consider (17) and (20), and we obtain

$$\begin{aligned}
S_0 &= P_0^{-1}+C^{T}R^{-1}C\\
&> \left[\left(G_k-Y+M_-+\epsilon I\right)^{-1}+P_s\right]^{-1}+C^{T}R^{-1}C\\
&= S_s-P_s^{-1}\left[G_k-Y+M_-+\epsilon I+P_s^{-1}\right]^{-1}P_s^{-1}\\
&\geq S-s-W^{-1}\\
&\geq S_s-U^{-1}=S_a.
\end{aligned} \tag{23}$$

From (23), we have $S_0 > S_a$. This implies that $\lim\limits_{k\to\infty} S_k = S_s$. It shows that P_k converges to P, and remains feasible at every step. Hence the proof.

4 Simulation Results

Example 4.1

Matrix States	Initial Estimations
Initial States	$\begin{bmatrix} 2 & 0 \\ 0 & 2.04 \end{bmatrix}$
Arbitrary Matrix P	$\begin{bmatrix} 0.6 & 1 \\ 1 & 0.4 \end{bmatrix}$
Arbitrary Matrix R	$\begin{bmatrix} 0.9 & 0 \\ 0 & 1.2 \end{bmatrix}$

Table 1: Initial values for Figure 1.

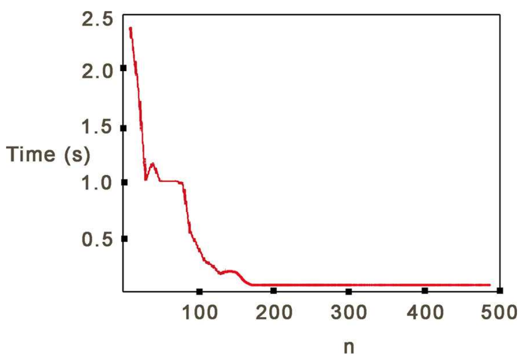

Figure 1: Convergence analysis for Table 1.

Example 4.2

Matrix States	Initial Estimations
Initial States	$\begin{bmatrix} 1.7 & 0 \\ 0 & 1.03 \end{bmatrix}$
Arbitrary Matrix P	$\begin{bmatrix} 0.2 & 1 \\ 1 & 0.7 \end{bmatrix}$
Arbitrary Matrix R	$\begin{bmatrix} 1.2 & 0 \\ 0 & 1.9 \end{bmatrix}$

Table 2: Initial Values for Figure 2.

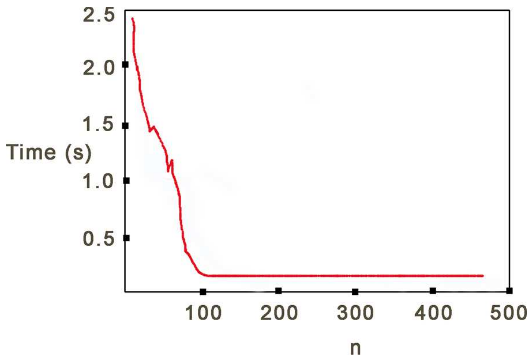

Figure 2: Convergence analysis for Table 2.

Example 4.3

Matrix States	Initial Estimations
Initial States	$\begin{bmatrix} 0.4 & 0 \\ 0 & 0.9 \end{bmatrix}$
Arbitrary Matrix P	$\begin{bmatrix} 1.7 & 1 \\ 1 & 2.4 \end{bmatrix}$
Arbitrary Matrix R	$\begin{bmatrix} 1.4 & 0 \\ 0 & 2.1 \end{bmatrix}$

Table 3: Initial Values for Figure 3.

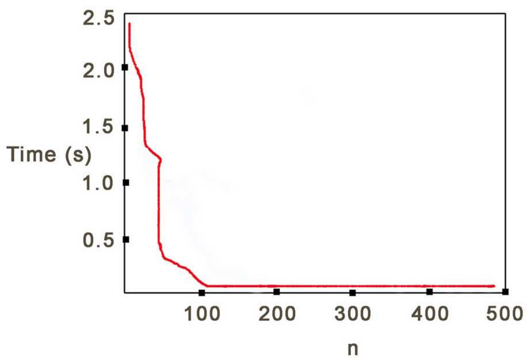

Figure 3: Convergence analysis for Table 3.

5 Conclusion

In this paper we classified the relationship between the initial state P_0 and the feasible solution through a new theorem. The estimation performance of the EKF is improved when we introduced the new RDE corresponding to ARE. Moreover, the convergence analysis is derived with the proposed RDE with good initial conditions alongwith a small ϵ. Furthermore, an additional theorem is formulated to ensure the convergence as well as feasible solutions of the new RDE. Simulation results show the performance of the proposed theorem even for the bad initializations.

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