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Existence of Even Homoclinic Solutions for a Class of Dynamical Systems

K. Khachnaoui *

Faculty of Sciences of Monastir Department of Mathematics,
5000 Monastir Tunisia

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Abstract: In this paper, we study the existence of even homoclinic solutions for a dynamical system

$$\ddot{x}(t) + A\dot{x}(t) + V'(t, x(t)) = 0,$$

where A is a skew-symmetric constant matrix, $t \in \mathbb{R}$, $x \in \mathbb{R}^N$ and $V \in C^1(\mathbb{R}\times\mathbb{R}^N, \mathbb{R})$, $V(t, x) = -K(t, x) + W(t, x)$. We assume that $W(t, x)$ does not satisfy the global Ambrosetti-Rabinowitz condition and that the norm of A is sufficiently small. For the proof we use the mountain pass theorem.

Keywords: *even homoclinic solution; dynamical system; mountain pass theorem; condition (C); critical point.*

Mathematics Subject Classification (2010): 34C37.

1 Introduction

The purpose of this work is to study the existence of even homoclinic solutions for the following system

$$\ddot{x}(t) + A\dot{x}(t) + V'(t, x(t)) = 0, \qquad (DS)$$

where A is a skew-symmetric constant matrix, $V \in C^1(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$, $V'(t, x) = \frac{\partial V}{\partial x}(t, x)$ and $x = (x_1, ..., x_N)$. We say that a solution $x(t)$ of dynamical system (DS) is homoclinic if $x(t) \to 0$ as $t \to \pm\infty$. In addition, x is called nontrivial if $x \not\equiv 0$. The theory of dynamical systems is a vast subject that can be studied from many different viewpoints. Particularly the existence of homoclinic solutions for DS is among the very important

* Corresponding author: mailto:k_khachnaoui@yahoo.com

problems which have been intensively studied. When $A = 0$, (DS) is just the following second order non-autonomous Hamiltonian system:

$$\ddot{x}(t) + V'(t, x(t)) = 0. \qquad (HS)$$

If the potential $V(t, x)$ is of type

$$V(t, x) = -\frac{1}{2}L(t)x.x + W(t, x), \tag{1}$$

where $L \in \mathcal{C}(\mathbb{R}, \mathbb{R}^{N^2})$ is a symmetric matrix depending continuously on t and $W \in C^1(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$, then the existence of homoclinic solutions of (HS) has been intensively studied by many mathematicians, see ([1], [6], [7], [11], [12], [14], [15], [22]) and the references therein. Assuming that $L(t)$ and $W(t, x)$ are T-periodic in t, $T > 0$, Rabinowitz [17] showed the existence of homoclinic solutions as a limit of $2kT$-periodic solutions of (HS). By the same method many authors have studied the existence of homoclinic solutions for the system (HS) via critical point theory and variational methods, see ([6], [9], [10], [11], [19]) and the references therein. In 2005, Izydorek and Janczewska [10] introduced a new type of potential $V(t, x)$ with which they studied the existence of homoclinic solutions for the system (HS), the potential $V(t, x)$ is T-periodic in t and of the form:

$$V(t, x) = -K(t, x) + W(t, x), \tag{2}$$

where K, $W \in C^1(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$, which has been extended in the recent paper [19]. They have proved the existence of homoclinic solutions as a limit of $2kT$-periodic solutions of (HS). If $K(t, x)$ and $W(t, x)$ are neither autonomous nor periodic in t, the problem of the existence of homoclinic solutions of (HS) is quite different from the ones just described, because of the lack of compactness of Sobolev embedding. In 2013, Benhassine and Timoumi [5] studied the existence of even homoclinic orbits of the system (HS) when the potential $V(t, x)$ is of the form (2) and satisfies a kind of new superquadratic conditions, in particular

(i) $W'(t, x).x > 2W(t, x) \geq 0$ for all $(t, x) \in \mathbb{R} \times (\mathbb{R}^N \setminus \{0\})$,
$\overline{W}(t, x) := \frac{1}{2}W'(t, x).x - W(t, x) \to +\infty$ as $|x| \to +\infty$ uniformly in t $\in \mathbb{R}$.
(ii) there exist constants $b_1 > 0$ such that

$$K(t, x) \geq b_1|x|^2.$$

When the potential $V(t, x)$ is of type (2), the existence of even homoclinic solutions of (DS) has not been studied. Motivated by the papers ([1], [3]- [11], [14]- [19], [21]), we prove the existence of even homoclinic solutions for (DS), as the limit of solutions of a sequence of boundary-value problems which are obtained by the minimax methods. Here and in the following $x.y$ denotes the inner product of $x, y \in \mathbb{R}^N$ and $|.|$ denotes the associated norm.

Our basic hypotheses on K and W are the following:
(H_1) For all $(t, x) \in \mathbb{R} \times \mathbb{R}^N$, $V'(t, x) \to 0$ *as* $|x| \to 0$ uniformly in $t \in \mathbb{R}$,
(H_2) There exists a constant $b_1 > 0$ such that

$$K(t, x) \geq b_1|x|^2, \quad K(t, x) \leq K'(t, x).x \leq 2K(t, x)$$

for all $(t, x) \in \mathbb{R} \times \mathbb{R}^N$,
(H_3) $W'(t, x) = o(|x|)$ as $|x| \to 0$ uniformly in $t \in \mathbb{R}$ and there exists some constant C_0

such that $\frac{|W'(t,x)|}{|x|} \leq C_0$ for all $(t,x) \in \mathbb{R} \times \mathbb{R}^N$,
(H_4) $W'(t,x).x > 2W(t,x) \geq 0$ for all $(t,x) \in \mathbb{R} \times (\mathbb{R}^N \setminus \{0\})$,
$\overline{W}(t,x) := \frac{1}{2}W'(t,x).x - W(t,x) \to +\infty$ as $|x| \to +\infty$ uniformly in t $\in \mathbb{R}$ and for any fixed $0 < r_1 < r_2$, $\inf_{t\in\mathbb{R}, r_1 \leq |x| \leq r_2} \dfrac{\overline{W}(t,x)}{|x|^2} \neq 0$,
(H_5) There exists constant $\xi_0 > 0$ such that

$$\liminf_{|x|\to+\infty} \frac{W(t,x)}{|x|^2} > \frac{2\pi^2 + \frac{\pi}{2}\bar{b}_1\xi_0}{\xi_0^2} + M_1$$

uniformly in $t \in [-\xi_0, \xi_0]$, where $M_1 = \sup_{t\in[-\xi_0,\xi_0],|x|=1} K(t,x)$, $\overline{b}_1 = \min\{1, 2b_1\}$ and b_1 is defined in (H_2).
(H_6) $\|A\| \leq \frac{1}{4}\bar{b}_1$.

Now we state our main results.

Theorem 1.1 *Assume that* (H_1)*–*(H_6) *hold, then the system (DS) has at least one even homoclinic solution* $x \in H^1(\mathbb{R}, \mathbb{R}^N)$ *such that* $\dot{x}(t) \to 0$ *as* $|t| \to +\infty$.

Remark 1.1 From (H_5), we see that there exist $a_1 > 0$ and $R > 0$ such that

$$\frac{W(t,x)}{|x|^2} \geq \frac{2\pi^2 + \frac{\pi}{2}\bar{b}_1\xi_0 + a_1}{\xi_0^2} + M_1,$$

for all $|x| > R$ and $t \in [-\xi_0, \xi_0]$. Let $M_3 = \max_{t\in[-\xi_0,\xi_0],|x|\leq R} W(t,x)$; we have

$$W(t,x) \geq (\frac{2\pi^2 + \frac{\pi}{2}\bar{b}_1\xi_0 + a_1}{\xi_0^2} + M_1)(|x|^2 - R^2) - M_3 \tag{3}$$

for all $x \in \mathbb{R}^N$ and $t \in [-\xi_0, \xi_0]$.

Moreover, $W'(t,x) = o(|x|)$ as $|x| \to 0$ uniformly in $t \in \mathbb{R}$, which implies that for any $\epsilon > 0$ there exists $\rho_0 > 0$ such that

$$|W'(t,x)| \leq \epsilon|x|,\ for\ (t,x) \in \mathbb{R} \times \mathbb{R}^N,\ |x| \leq \rho_0. \tag{4}$$

Now let us consider the following assumption:
(H_7) There exist $x_0 \in \mathbb{R}^N$ and $\xi_0 > 0$ such that

$$\int_{-\xi_0}^{\xi_0} (K(t,x_0) - W(t,x_0))dt < 0.$$

Our second result deals with the case of periodicity.

Theorem 1.2 *Assume that* V *is T-periodic in t,* $T > 0$ *and* (H_1)*-*(H_4)*,* (H_6) *and* (H_7) *hold, then the system (DS) has at least one even homoclinic solution* $x \in H^1(\mathbb{R}, \mathbb{R}^N)$ *such that* $\dot{x}(t) \to 0$ *as* $|t| \to +\infty$.

Example 1.1 Consider the functions

$$K(t,x) = |x|^2 + |x|^{\frac{3}{2}}, \quad W(t,x) = (e^{-t^2} + 2\pi)|x|^2\left(1 - \frac{1}{\ln(e + |x|)}\right).$$

A straightforward computation shows that W and K satisfy the assumptions of Theorem 1.1, but W does not satisfy the global Ambrosetti-Rabinowitz condition, and K cannot be written in the form $\frac{1}{2}(L(t)x, x)$ and does not satisfy the corresponding results in ([1], [3], [6]- [10], [12], [14], [17], [19], [21], [22]).

2 Proof of the Main Results.

By the idea of [11], we approximate an even homoclinic solution of (DS) by a solution of the following problem:

$$\begin{cases} \ddot{x}(t) + A\dot{x}(t) - K'(t,x(t)) + W'(t,x(t)) = 0 \ for \ t \ \in]-\xi,\xi[, \\ x(-t) = x(t) \ for \ t \ \in]-\xi,\xi[, x(-\xi) = x(\xi) = 0, \end{cases} \tag{5}$$

where ξ is a positive constant. The set

$$H_0^1([-\xi,\xi]) = \left\{ \begin{array}{l} x : [-\xi,\xi] \to \mathbb{R}^N / x \ is \ absolutely \ continuous, \\ x(-\xi) = x(\xi) = 0, \dot{x} \in L^2([-\xi,\xi],\mathbb{R}^N) \end{array} \right\}$$

is a Hilbert space with the norm

$$\|x\| = \left(\int_{-\xi}^{\xi} (|x(t)|^2 + |\dot{x}(t)|^2) dt \right)^{\frac{1}{2}}$$

and the associated inner product

$$\langle x, y \rangle = \int_{-\xi}^{\xi} (x(t).y(t) + \dot{x}(t).\dot{y}(t)) dt.$$

Consider the functional $I_\xi : H_0^1([-\xi,\xi]) \to \mathbb{R}$ defined by

$$I_\xi(x) = \int_{-\xi}^{\xi} \left[\frac{1}{2}|\dot{x}(t)|^2 + \frac{1}{2}(Ax(t).\dot{x}(t)) + K(t,x(t)) - W(t,x(t)) \right] dt.$$

It is easy to check that $I_\xi \in C^1(H_0^1([-\xi,\xi]),\mathbb{R})$ and by using the skew-symmetry of A, we have

$$I'_\xi(x)y = \int_{-\xi}^{\xi} [(\dot{x}(t).\dot{y}(t) - (A\dot{x}(t).y(t)) + K'(t,x(t)).y(t) - W'(t,x(t)).y(t)] \, dt. \tag{6}$$

Moreover, the critical points of I_ξ in $H_0^1([-\xi,\xi])$ are the classical solutions of (DS) in $[-\xi,\xi]$ satisfying $x(\xi) = x(-\xi) = 0$. We will obtain a critical point of I_ξ by using the Mountain Pass Theorem:

Lemma 2.1 *([16]) Let H be a real Banach space and $I \in C^1(H,\mathbb{R})$ satisfying the Palais-Smale condition. If I satisfies the following conditions:*

(i) $I(0) = 0$,

(ii) there exist constants ρ, $\alpha > 0$ such that $I_{|\partial B_\rho(0)} \geq \alpha$,

(iii) there exists $e \in H \backslash \overline{B}_\rho(0)$ such that $I(e) \leq 0$.

Then I possesses a critical value $c \geq \alpha$ given by

$$c = \inf_{g \in \Gamma} \max_{s \in [0,1]} I(g(s)),$$

where $B_\rho(0)$ is the open ball in H centered in 0, with radius ρ, $\partial B_\rho(0)$ as its boundary and

$$\Gamma = \{g \in C([0,1],H) : g(0) = 0, g(1) = e\}.$$

For a fixed $\xi > 0$, consider the subspace E_ξ of $H_0^1([-\xi,\xi])$ defined by

$$E_\xi = \left\{x \in H_0^1([-\xi,\xi]) | x(-t) = x(t),\ a.e.\ t \in]-\xi,\xi[\right\}.$$

We will proceed by successive lemmas.

Lemma 2.2 *The critical points of Φ_ξ on E_ξ are exactly the solutions of problem (5), where Φ_ξ is the restriction of I_ξ on E_ξ.*

Proof. Let

$$F_\xi = \left\{x \in H_0^1([-\xi,\xi]) / x(-t) = -x(t),\ a.e.\ t \in]-\xi,\xi[\right\}.$$

For every $x \in H_0^1([-\xi,\xi])$, set

$$y(t) = \frac{1}{2}\left(x(t) + x(-t)\right), \quad z(t) = \frac{1}{2}\left(x(t) - x(-t)\right),$$

then $y \in E_\xi$, $z \in F_\xi$ and $x = y + z$. So $H_0^1([-\xi,\xi]) = E_\xi + F_\xi$. Furthermore, for all $y \in E_\xi$, $z \in F_\xi$ we have

$$\langle y, z\rangle = \int_{-\xi}^{\xi} (y(t).z(t) + \dot{y}(t).\dot{z}(t))dt = \int_{\xi}^{-\xi} (y(-t).z(-t) + \dot{y}(-t).\dot{z}(-t))d(-t)$$

$$= \int_{-\xi}^{\xi} (y(t).(-z(t)) + (-\dot{y}(t)).\dot{z}(t))dt = -\langle y, z\rangle,$$

which implies that $\langle y, z\rangle = 0$ and then $E_\xi \perp F_\xi$. Hence $H_0^1([-\xi,\xi]) = E_\xi \oplus F_\xi$. If x is a critical point of Φ_ξ, for every $z \in E_\xi \subset C^0([-\xi,\xi],\mathbb{R}^N)$ (The space of continuous functions z on $[-\xi,\xi]$ such that $z(t) \to 0$ as $|t| \to +\infty$), then by (6) we have

$$\begin{aligned} \int_{-\xi}^{\xi} [\dot{x}(t).\dot{z}(t) - A\dot{x}(t).z(t)]dt &= \int_{-\xi}^{\xi} (\dot{x}(t) + Ax(t)).\dot{z}(t)dt \\ &= -\int_{\xi}^{\xi} (K'(t,x(t)) - W'(t,x(t))).z(t))dt \end{aligned}$$

which implies that $K'(t,x(t)) - W'(t,x(t))$ is the weak derivative of $\dot{x}(t) + Ax(t)$. Since $K, W \in C^1(\mathbb{R}\times\mathbb{R}^N,\mathbb{R})$ and $E_\xi \subset C^0([-\xi,\xi],\mathbb{R}^N)$, we see that $\dot{x}(t) + Ax(t)$ is continuous, which yields that $\dot{x}(t)$ is continuous and $x(t) \in C^2(\mathbb{R},\mathbb{R}^N)$; i.e $x \in E_\xi$ is a classical solutions of (5) if and only if it is a critical point of Φ_ξ on $H_0^1([-\xi,\xi])$. The proof of Lemma 2.2 is complete.

Lemma 2.3 *Assume that (H_2) holds. Then, for every $t \in [-\xi_0,\xi_0]$ and $x \in \mathbb{R}^N$, the following inequality holds:*

$$K(t,x) \le M_1|x|^2 + M_2, \tag{7}$$

where M_1 is defined in (H_5) and $M_2 = \sup_{t\in[-\xi_0,\xi_0],|x|\le 1} K(t,x)$.

Proof. To prove this lemma it suffices to show that for every $x \in \mathbb{R}^N$ and $t \in [-\xi_0, \xi_0]$ the function $(0, +\infty) \to \mathbb{R},\ s \mapsto K(t, s^{-1}x)s^2$ is nondecreasing; which is an immediate consequence of (H_2). The proof of Lemma 2.3 is complete.
By Sobolev's embedding theorem, $H^1(\mathbb{R}, \mathbb{R}^N)$ is continuously embedded into $L^p(\mathbb{R}, \mathbb{R}^N)$ for $p \in [2, +\infty]$. Thus there exists $\gamma_p > 0$ such that

$$\|x\|_{L^p(\mathbb{R},\mathbb{R}^N)} \leq \gamma_p \|x\|_{H^1(\mathbb{R},\mathbb{R}^N)}, \quad \forall\, p \in\ [2, +\infty], \forall\, x \in H^1(\mathbb{R}, \mathbb{R}^N).$$

Since $x \in H^1([-\xi, \xi])$ can be regarded as belonging to $H^1(\mathbb{R}, \mathbb{R}^N)$ if one extends it by zero in $\mathbb{R}\backslash[-\xi, \xi]$, then we have

$$\|x\|_{L^p([-\xi,\xi],\mathbb{R}^N)} \leq \gamma_p \|x\|, \quad \forall\, p \in\ [2, +\infty], \forall\, x \in H_0^1([-\xi, \xi]), \tag{8}$$

where γ_p is independent of $\xi > 0$.

Proposition 2.1 *Suppose that the conditions (H_1) - (H_6) or (H_1) - (H_4), (H_6) and (H_7) are satisfied, then for all $\xi \geq \xi_0$, the problem (5) possesses a nontrivial solution.*

Proof. **Step 1.** It is clear that $\Phi_\xi(0) = 0$. As shown in [2], a deformation lemma can be proved with condition (C) replacing the usual (PS) condition, and it turns out that the Mountain Pass Theorem in [16] holds true under condition (C), i.e., for every sequence $(y_j) \subset E_\xi$, (y_j) has a convergent subsequence if $\Phi_\xi(y_j)$ is bounded and $(1 + \|y_j\|) \left\|\Phi'_\xi(y_j)\right\|_{E^*_\xi} \to 0$ as $j \to +\infty$, where E^* is the dual space of E. Let $(y_j) \subset E_\xi$ be such that $\Phi_\xi(y_j)$ is bounded and $(1 + \|y_j\|) \left\|\Phi'_\xi(y_j)\right\|_{E^*_\xi} \to 0$ as $j \to +\infty$. Observe that for j large, it follows from (H_2) and (H_4) that there exists a constant M such that

$$M \geq \Phi_\xi(y_j) - \frac{1}{2}\Phi'_\xi(y_j)y_j =$$

$$\int_{-\xi}^{\xi} (\frac{1}{2}W'(t, y_j).y_j - W(t, y_j))dt + \int_{-\xi}^{\xi} (K(t, y_j) - \frac{1}{2}K'(t, y_j).y_j)dt$$

$$\geq \int_{-\xi}^{\xi} \overline{W}(t, y_j(t))dt. \tag{9}$$

By negation, if (y_j) is not bounded, passing to a subsequence if necessary we may assume that $\|y_j\| \to +\infty$ as $j \to +\infty$. Set $z_j = \frac{y_j}{\|y_j\|}$, then $\|z_j\| = 1$ and by (8) one has

$$\|z_j\|_{L^p([-\xi,\xi],\mathbb{R}^N)} \leq \gamma_p \|z_j\| = \gamma_p,\ \forall p \in\ [2, +\infty]. \tag{10}$$

By (H_2), (H_4) and (H_6) we have

$$\begin{aligned} 2M \geq 2\Phi_\xi(y_j) \quad &= \quad \int_{-\xi}^{\xi} |\dot{y_j}(t)|^2 dt - \int_{-\xi}^{\xi} (A\dot{y}_j(t).y_j(t))dt + 2\int_{-\xi}^{\xi} K(t, y_j(t))dt \\ &- \quad 2\int_{-\xi}^{\xi} W(t, y_j(t))dt \geq \overline{b}_1\|y_j\|^2 - \|A\|\|y_j\|^2 - \int_{-\xi}^{\xi} W'(t, y_j(t)).y_j(t)dt \\ &\geq \quad \|y_j\|^2 \left(\overline{b}_1 - \frac{\overline{b}_1}{4} - \int_{-\xi}^{\xi} \frac{W'(t, y_j(t)).y_j(t)}{\|y_j\|^2}dt\right), \end{aligned}$$

where $\overline{b}_1 = \min\{1, 2b_1\} > 0$. Thus implies that

$$\lim_{j\to+\infty}\int_{-\xi}^{\xi}\frac{W'(t,y_j(t)).y_j(t)}{\|y_j\|^2}dt \geq \frac{3}{4}\overline{b}_1. \tag{11}$$

Set

$$f(r) := \inf\left\{\overline{W}(t,x)|\ t\in[-\xi,\xi]\ and\ x\in\mathbb{R}^N\ with\ |x|\geq r\right\}$$

for $r \geq 0$. By (H_4) one has

$$f(r)\to+\infty\ as\ r\to+\infty.$$

For $0 \leq a \leq b$ let

$$\Omega_j(a,b) = \{t\in[-\xi,\xi]\mid a < y_j(t)\leq b\}$$

and

$$C_b^a = \inf\left\{\frac{\overline{W}(t,x)}{|x|^2}, t\in[-\xi,\xi]\ and\ a<|x|\leq b\right\}.$$

Obviously, we have

$$\overline{W}(t,y_j(t)) \geq C_b^a|y_j(t)|^2,\ for\ all\ t\in\ \Omega_j(a,b). \tag{12}$$

By (9) and (12) it follows

$$M \geq \int_{-\xi}^{\xi}\overline{W}(t,y_j)dt = \int_{\Omega_j(0,a)}\overline{W}(t,y_j)dt + \int_{\Omega_j(a,b)}\overline{W}(t,y_j)dt + \int_{\Omega_j(b,\infty)}\overline{W}(t,y_j(t))dt$$

$$\geq \int_{\Omega_j(0,a)}\overline{W}(t,y_j)dt + C_b^a\int_{\Omega_j(a,b)}|y_j|^2dt + f(b)meas(\Omega_j(b,\infty)), \tag{13}$$

which implies that

$$meas(\Omega_j(b,\infty)) \leq \frac{M}{f(b)} \to 0\ \ as\ b\to+\infty\ uniformly\ in\ j. \tag{14}$$

For any fixed $0 < a < b$ and by (8), (10) and (14) we have

$$\int_{\Omega_j(b,\infty)}|z_j|^2dt \leq \|z_j\|^2_{L^\infty([-\xi,\xi])}meas(\Omega_j(b,\infty)) \tag{15}$$

$$\leq \gamma_\infty^2 meas(\Omega_j(b,\infty)) \to 0$$

as $b \to +\infty$ uniformly in j. Moreover, by (13) we obtain

$$\int_{\Omega_j(a,b)}|z_j|^2dt = \frac{1}{\|y_j\|^2}\int_{\Omega_j(a,b)}|y_j|^2dt \leq \frac{M}{C_b^a\|y_j\|^2} \to 0 \tag{16}$$

as $j \to +\infty$. Let $0 < \varepsilon < \frac{\overline{b}_1}{4}$, by (H_3) there exist $a_\varepsilon > 0$ such that

$$|W'(t,x)| \leq \frac{\varepsilon}{\gamma_2^2}|x|\ \ for\ all\ |x|\leq a_\varepsilon.$$

Consequently,

$$\int_{\Omega_j(0,a_\varepsilon)} \frac{|W'(t,y_j)||z_j|^2}{|y_j|} dt \le \frac{\varepsilon}{\gamma_2^2} \int_{\Omega_j(0,a_\varepsilon)} |z_j|^2 dt \le \varepsilon. \tag{17}$$

By (15) we can take b_ε large such that

$$\int_{\Omega_j(b_\varepsilon,\infty)} |z_j|^2 dt \le \frac{\varepsilon}{C_0}.$$

Hence, by (H_3) we obtain

$$\int_{\Omega_j(b_\varepsilon,\infty)} \frac{|W'(t,y_j)||z_j|^2}{|y_j|} dt \le C_0 \int_{\Omega_j(b_\varepsilon,\infty)} |z_j|^2 dt \le \varepsilon. \tag{18}$$

By (16) there is j_0 such that

$$\int_{\Omega_j(a_\varepsilon,b_\varepsilon)} \frac{|W'(t,y_j)||z_j|^2}{|y_j|} dt \le C_0 \int_{\Omega_j(a_\varepsilon,b_\varepsilon)} |z_j|^2 dt \le \varepsilon, \tag{19}$$

for all $j \ge j_0$. Therefore, combining (17)-(19) we have

$$\int_{-\xi}^{\xi} \frac{W'(t,y_j).y_j}{\|y_j\|^2} dt \le \int_{[-\xi,\xi]\setminus\{t\in[-\xi,\xi]/|y_j(t)|=0\}} \frac{|W'(t,y_j)||z_j|^2}{|y_j|} dt \le 3\varepsilon < \frac{3}{4}\overline{b}_1,$$

which contradicts (11). Hence, (y_j) is bounded in E_ξ. Going if necessary to a subsequence, we can assume that there exists $y \in E_\xi$ such that $y_j \rightharpoonup y$ as $j \to +\infty$ in E_ξ, which implies that $y_j \to y$ as $j \to +\infty$ uniformly on $[-\xi,\xi]$. Hence $(\Phi'_\xi(y_j) - \Phi'_\xi(y))(y_j - y) \to 0$, $\|y_j - y\|_{L^2([-\xi,\xi],\mathbb{R}^N)} \to 0$ and $\int_{-\xi}^{\xi} (V'(t,y_j(t)) - V'(t,y(t))\,.(y_j(t) - y(t))dt \to 0$ and by the Hölder inequality, we have

$$\left| \int_{-\xi}^{\xi} (A\dot{y}_j(t) - A\dot{y}(t)).(y_j(t) - y(t))dt \right| \le \|A\|\|\dot{y}_j - \dot{y}\|_{L^2}\|y_j - y\|_{L^2} \to 0$$

as $j \to +\infty$. On the other hand, an easy computation shows that

$$\begin{aligned} &(\Phi'_\xi(y_j) - \Phi'_\xi(y))(y_j - y) \\ &= \|\dot{y}_j - \dot{y}\|^2_{L^2([-\xi,\xi],\mathbb{R}^N)} - \int_{-\xi}^{\xi} (A\dot{y}_j(t) - A\dot{y}(t).y_j(t) - y(t))dt \\ &\quad - \int_{-\xi}^{\xi} (V'(t,y_j(t))) - V'(t,y(t))).(y_j(t) - y(t))dt. \end{aligned}$$

and so $\|\dot{y}_j - \dot{y}\|_{L^2([-\xi,\xi],\mathbb{R}^N)} \to 0$. Consequently, $\|y_j - y\| \to 0$ as $j \to +\infty$. Hence, Φ_ξ satisfies condition (C).

Step 2. Now, let us show that Φ_ξ satisfies assumption (ii) of Lemma 2.1. By (H_3) there exists a constant $\rho_0 > 0$ such that

$$|W'(t,x)| \le \frac{\overline{b}_1}{2\gamma_2^2}|x|, \ \forall\, t \in \mathbb{R}, \ \forall\, |x| \le \rho_0.$$

It follows that

$$|W(t,x)| = \left|\int_0^1 W'(t,sx).xds\right| \leq \int_0^1 |W'(t,sx).x|ds$$

$$\leq \frac{\overline{b}_1}{2\gamma_2^2}\int_0^1 |x|^2 sds = \frac{\overline{b}_1}{4\gamma_2^2}|x|^2, \forall\, t\in\mathbb{R},\ \forall\, |x|\leq \rho_0. \tag{20}$$

Let $\rho = \frac{\rho_0}{\gamma_\infty}$ and $S = \{x \in E_\xi/\|x\| = \rho\}$. By (8), we have $\|x\|_{L^\infty([-\xi,\xi],\mathbb{R}^N)} \leq \rho_0$, for all $x \in S$, which together with (20), (H_2) and (H_6) implies that

$$\begin{aligned}
\Phi_\xi(x) &= \frac{1}{2}\int_{-\xi}^{\xi}|\dot{x}(t)|^2dt - \frac{1}{2}\int_{-\xi}^{\xi}(A\dot{x}(t).x(t))dt + \int_{-\xi}^{\xi}K(t,x(t))dt - \int_{-\xi}^{\xi}W(t,x(t))dt\\
&\geq \left(\frac{\overline{b}_1}{2} - \frac{\overline{b}_1}{8} - \frac{\overline{b}_1}{4}\right)\|x\|^2 = \frac{\overline{b}_1}{8}\rho^2 := \alpha,\ \forall\, x\in S.
\end{aligned}$$

Step 3. It remains to prove that Φ_ξ satisfies assumption(iii) of Lemma 2.1. If (H_5) holds, let

$$e(t) = \begin{cases} m|\sin(\omega t)|e_1, if\ t\in[-\xi_0,\xi_0],\\ 0, if\ t\in[-\xi,\xi]\backslash[-\xi_0,\xi_0],\end{cases}$$

where $\omega = \frac{2\pi}{\xi_0}$, $e_1 = (1,0,...,0)$ and $m \in \mathbb{R}\setminus\{0\}$. By the Hölder inequality, (H_6), Remark 1.1 and Lemma 2.3 we have

$$\begin{aligned}
\Phi_\xi(e) &= \frac{1}{2}\int_{-\xi}^{\xi}|\dot{e}(t)|^2dt + \frac{1}{2}\int_{-\xi}^{\xi}(Ae(t).\dot{e}(t))dt + \int_{-\xi}^{\xi}K(t,e(t))dt - \int_{-\xi}^{\xi}W(t,e(t))dt\\
&= \frac{1}{2}m^2\omega^2\int_{-\xi_0}^{\xi_0}|\cos(\omega t)|^2dt + \frac{1}{2}m^2\omega\int_{-\xi_0}^{\xi_0}(A|\sin(\omega t)|e_1.|\cos(\omega t)|e_1)dt\\
&+ \int_{-\xi_0}^{\xi_0}K(t,m|\sin(\omega t)|e_1)dt - \int_{-\xi_0}^{\xi_0}W(t,m|\sin(\omega t)|e_1)dt\\
&\leq \frac{1}{2}m^2\omega^2\int_{-\xi_0}^{\xi_0}|\cos(\omega t)|^2dt + m^2\omega\ \|A\|\xi_0 + M_1m^2\int_{-\xi_0}^{\xi_0}|\sin(\omega t)|^2dt + 2\xi_0M_2\\
&- \left(\frac{2\pi^2 + \frac{\pi}{2}\overline{b}_1\xi_0 + a_1}{\xi_0^2} + M_1\right)m^2\int_{-\xi_0}^{\xi_0}|\sin(\omega t)|^2dt\\
&+ 2\xi_0\left(R^2\left(\frac{2\pi^2 + \frac{\pi}{2}\overline{b}_1\xi_0 + a_1}{\xi_0^2} + M_1\right) + M_3\right)\\
&\leq m^2\left(-\frac{\pi\overline{b}_1}{2} - \frac{2a_1}{\xi_0}\right) + 2\xi_0\left(M_2 + R^2\left(\frac{2\pi^2 + \frac{\pi}{2}\overline{b}_1\xi_0 + a_1}{\xi_0^2} + M_1\right) + M_3\right) \to -\infty
\end{aligned}$$

as $m\to\infty$. If (H_7) holds, set $g(s) = s^{-2}W(t,sx_0)$ for $s>0$. Then it follows from (H_4) that

$$g'(s) = s^{-3}[-2W(t,sx_0) + W'(t,sx_0).sx_0] > 0,\ \ for\ t\in\ \mathbb{R},\ s>0.$$

Integrating the above from 1 to $\lambda > 1$, we obtain

$$W(t,\lambda x_0) \geq \lambda^2 W(t,x_0), for\ t\in\mathbb{R},\ \lambda>1. \tag{21}$$

By (H_2), it is easy to show that

$$K(t, \lambda x_0) \leq \lambda^2 K(t, x_0), for\ t \in \mathbb{R},\ \lambda > 1. \tag{22}$$

From (21) and (22) we have

$$\Phi_\xi(\lambda x_0) = \int_{-\xi}^{\xi} [K(t, \lambda x_0) - W(t, \lambda x_0)]dt$$

$$\leq \lambda^2 \left(\int_{-\xi}^{\xi} K(t, x_0)dt - \int_{-\xi}^{\xi} W(t, x_0)dt \right). \tag{23}$$

Choose $\sigma > 1$ such that $|\sigma x_0|\sqrt{2\xi_0} > \rho$ and let

$$e(t) = \begin{cases} \sigma x_0, if\ t \in [-\xi_0, \xi_0], \\ 0, if\ t \in [-\xi, \xi] \backslash [-\xi_0, \xi_0]. \end{cases}$$

By (23) and (H_7) we have

$$\begin{aligned} \Phi_\xi(e) &= \int_{-\xi}^{\xi} (K(t, e(t)) - W(t, e(t)))dt \\ &= \int_{-\xi_0}^{\xi_0} (K(t, \sigma x_0) - W(t, \sigma x_0))dt \\ &\leq \sigma^2 \int_{-\xi_0}^{\xi_0} (K(t, x_0) - W(t, x_0))dt < 0. \end{aligned}$$

All the assumptions of Lemma 2.1 are satisfied, so for all $\xi \geq \xi_0$, Φ_ξ possesses a critical value $c_\xi \geq \alpha > 0$ defined by

$$c_\xi \equiv \inf_{g \in \Gamma_\xi} \max_{s \in [0,1]} \Phi_\xi(g(s)),$$

where

$$\Gamma_\xi = \{g(t) \in C([0, 1], E_\xi)/g(0) = 0, g(1) = e\}.$$

Hence, for every $\xi > 0$, there exists $x_\xi \in E_\xi$ such that

$$\Phi_\xi(x_\xi) = c_\xi,\ \ \Phi'_\xi(x_\xi) = 0.$$

Since $c_\xi > 0$, x_ξ is nontrivial. The proof of Proposition 2.1 is complete.

Take a sequence $(\xi_n)_{n\in\mathbb{N}}$ with $\xi_0 \leq \xi_1 \leq \xi_2 \leq ... \to \infty$ and consider problem (5) on E_{ξ_n}, i.e.

$$\begin{cases} \ddot{x}(t) + A\dot{x}(t) - K'(t, x(t)) + W'(t, x(t)) = 0, for\ \ t \in] - \xi_n, \xi_n[, \\ x(-t) = x(t), for\ \ t \in] - \xi_n, \xi_n[, x(-\xi_n) = x(\xi_n) = 0. \end{cases} \tag{24}$$

Then by Proposition 2.1, for each $n \in \mathbb{N}$, (24) possesses a nontrivial solution x_n. Let $C^p_{loc}(\mathbb{R}, \mathbb{R}^N)$ $(p \in \mathbb{N})$ denote the space of C^p functions under the topology of almost uniformly convergence of functions and all derivatives up to order p. We have the following result.

Lemma 2.4 *The sequence* (x_n) *possesses a subsequence also denoted by* (x_n) *which converges to a* C^2 *function* x *in* $C^2_{loc}(\mathbb{R}, \mathbb{R}^N)$.

Proof. Let $q > k$, as any function in E_{ξ_k} can be regarded as belonging to E_{ξ_q} if one extends it by zero in $[-\xi_q, \xi_q]\backslash[-\xi_k, \xi_k]$, we have $\Gamma_{\xi_k} \subset \Gamma_{\xi_q}$ which implies $c_{\xi_q} \le c_{\xi_k}$. Thus $c_{\xi_n} \le c_{\xi_0}$ for any $n \in \mathbb{N}$.

As $\Phi_{\xi_n}(x_n) \le c_{\xi_0}$ and $(1 + \|x_n\|) \left\| \Phi'_{\xi_n}(x_n) \right\| = 0$, just as in the proof of condition (C) in Proposition 2.1, it is easy to prove that (x_n) is bounded uniformly in n. Therefore, there is a constant $C_1 > 0$ such that:

$$\|x_n\| \le C_1, \forall n \in \mathbb{N}. \tag{25}$$

Arguing as in Theorem 2.1 in [11], we conclude from the fact

$$|x_n(t_2) - x_n(t_1)| \le \int_{t_1}^{t_2} |\dot{x}(t)| dt \le (t_2 - t_1)^{1/2} \left(\int_{t_1}^{t_2} |\dot{x}(t)|^2 dt \right)^{1/2}$$

that the sequence (x_n) is equicontinuous on every interval $[-\xi_n, \xi_n]$. By (25) and Arzela-Ascoli theorem, the sequence (x_n) has a uniformly convergent subsequence on each $[-\xi_n, \xi_n]$.

Let $(x^1_{n_k})$ be a subsequence of (x_n) that converges on $[-\xi_1, \xi_1]$. Then $(x^1_{n_k})$ is equicontinuous and uniformly bounded on $[-\xi_2, \xi_2]$. So we can choose a subsequence $(x^2_{n_k})$ of $(x^1_{n_k})$ that converges uniformly on $[-\xi_2, \xi_2]$. Repeat this procedure for all n and take the diagonal sequence $(x^k_{n_k})$. It is obvious that $(x^k_{n_k})_k$ is a subsequence of $(x^i_{n_k})$ for any $1 \le i \le k$. Hence, it converges uniformly to a function $x(t)$ on any bounded interval. In the following, for simplicity, we denote the subsequence $(x^k_{n_k})$ also by (x_n). As (x_n) satisfies

$$\ddot{x}_n(t) + A\dot{x}_n(t) + V'(t, x_n(t)) = 0, \tag{26}$$

we conclude that the sequence $(\ddot{x}_n)$ and then also $(\dot{x}_n)$ converge uniformly on any bounded intervals. It is easy to see that

$$x_n(t) = \int_{-\xi_n}^{t} (t - s)\ddot{x}_n(s) ds,$$

then $x \in C^2(\mathbb{R}, \mathbb{R}^N)$ and $\ddot{x}_n \to \ddot{x}$ uniformly on any bounded intervals. Hence, by passing to the limit in (26) we conclude that x solves (DS). As x_n is even, the same is true for their limit x. The proof of Lemma 2.4 is complete.

Proof of Theorem 1.1. We have shown that x satisfies (DS). It remains to prove that x is nontrivial and homoclinic to 0.

Step 1. Let us show that x is nontrivial. Consider the function Ψ defined by $\Psi(0) = 0$ and for $s > 0$

$$\Psi(s) = \max_{t \in \mathbb{R}, 0 < |x| \le s} \frac{W'(t, x).x}{|x|^2}.$$

Then Ψ is a continuous, nondecreasing function and $\Psi(s) \ge 0$ for $s \ge 0$. The definition of Ψ implies that

$$\int_{-\xi_n}^{\xi_n} W'(t, x_n(t)).x_n(t) dt \le \Psi(\|x_n\|_{L^\infty([-\xi_n, \xi_n], \mathbb{R}^N)}) \|x_n\|^2, \tag{27}$$

for every $n \in \mathbb{N}$. Since $\Phi'_{\xi_n}(x_n).x_n = 0$, we have

$$\int_{-\xi_n}^{\xi_n} W'(t, x_n(t)).x_n(t)dt =$$

$$\int_{-\xi_n}^{\xi_n} |\dot{x}_n(t)|^2 dt - \int_{-\xi_n}^{\xi_n} (A\dot{x}_n(t).x_n(t))dt + \int_{-\xi_n}^{\xi_n} K'(t, x_n(t)).x_n(t)dt. \tag{28}$$

From (27), (28), (H_2) and (H_6), we obtain

$$\begin{aligned} \Psi(\|x_n\|_{L^\infty([-\xi_n,\xi_n],\mathbb{R}^N)})\|x_n\|^2 &\geq \int_{-\xi_n}^{\xi_n} |\dot{x}_n(t)|^2 dt - \int_{-\xi_n}^{\xi_n} (A\dot{x}_n(t).x_n(t))dt \\ &+ \int_{-\xi_n}^{\xi_n} K'(t, x_n(t)).x_n(t)dt \\ &\geq \int_{-\xi_n}^{\xi_n} |\dot{x}_n(t)|^2 dt + b_1 \int_{-\xi_n}^{\xi_n} |x_n(t)|^2 dt - \|A\|\|x_n\|^2 \\ &\geq (\min\{1, b_1\} - \|A\|)\|x_n\|^2. \end{aligned}$$

Since $\|x_n\| > 0$, it follows that

$$\Psi(\|x_n\|_{L^\infty([-\xi_n,\xi_n],\mathbb{R}^N)}) \geq (\min\{1, b_1\} - \|A\|) > 0.$$

If $\|x_n\|_{L^\infty([-\xi_n,\xi_n],\mathbb{R}^N)} \to 0$ as $n \to \infty$, we would have $\Psi(0) \geq (\min\{1, b_1\} - \|A\|) > 0$, a contradiction. Passing to a subsequence of (x_n) if necessary, there is a constant $C_3 > 0$ such that

$$\|x_n\|_{L^\infty([-\xi_n,\xi_n],\mathbb{R}^N)} \geq C_3 \tag{29}$$

for every $n \in \mathbb{N}$. Now, suppose $x \equiv 0$ and let x_n be the function defined in Lemma 2.4, extended by 0 in $\mathbb{R} \setminus [-\xi_n, \xi_n]$. For $A > 0$ we have

$$\begin{aligned} \|x_n\|^2 &= \int_{-\xi_n}^{\xi_n} (|\dot{x}_n(t)|^2 + |x_n(t)|^2)dt \\ &= \int_{\mathbb{R}} (|\dot{x}_n(t)|^2 + |x_n(t)|^2)dt \\ &= \int_{-A}^{A} (|\dot{x}_n(t)|^2 + |x_n(t)|^2)dt + \int_{\mathbb{R}\setminus[-A,A]} (|\dot{x}_n(t)|^2 + |x_n(t)|^2)dt \to 0 \; as \; A, n \to \infty. \end{aligned}$$

which is in contradiction with (29). Hence x is nontrivial.

Step 2. We prove that $x(t) \to 0$ as $|t| \to +\infty$. By the argument of Lemma 2.4, for each $i \in \mathbb{N}$ there is $n_i \in \mathbb{N}$ such that for all $n \geq n_i$ we have

$$\int_{-\xi_i}^{\xi_i} (|x_n(t)|^2 + |\dot{x}_n(t)|^2)dt \leq \|x_n\|^2 \leq C_1^2.$$

Letting $n \to +\infty$, we obtain

$$\int_{-\xi_i}^{\xi_i} (|x(t)|^2 + |\dot{x}(t)|^2)dt \leq C_1^2.$$

As $i \to +\infty$, we have

$$\int_{-\infty}^{+\infty} (|x(t)|^2 + |\dot{x}(t)|^2)dt \leq C_1^2.$$

Hence, we get

$$\int_{|t|\geq r} (|x(t)|^2 + |\dot{x}(t)|^2)dt \to 0 \; as \; r \to +\infty. \tag{30}$$

By Corollary 2.2 in [19], we have

$$|x(t)|^2 \leq \int_{t-1}^{t+1} (|x(s)|^2 + |\dot{x}(s)|^2)ds \tag{31}$$

for every $t \in \mathbb{R}$. By (30) and (31) we conclude that

$$x(t) \to 0 \;\; as \; |t| \to \infty.$$

Step 3. We have to show that $\dot{x}(t) \to 0 \;\; as \; |t| \to \infty$. By Corollary 2.2 in [19] we have

$$|\dot{x}(t)|^2 \leq \int_{t-1}^{t+1} (|x(s)|^2 + |\dot{x}(s)|^2)ds + \int_{t-1}^{t+1} |\ddot{x}(s)|^2 ds,$$

for every $t \in \mathbb{R}$. Since $x \in H^1(\mathbb{R}, \mathbb{R}^N)$, we get

$$\int_{t-1}^{t+1} (|x(s)|^2 + |\dot{x}(s)|^2)ds \to 0 \; as \; |t| \to \infty.$$

Hence, it suffices to prove that

$$\int_{t-1}^{t+1} |\ddot{x}(s)|^2 ds \to 0 \; as \; |t| \to \infty. \tag{32}$$

By (DS), we have

$$\begin{aligned}
\int_{t-1}^{t+1} |\ddot{x}(s)|^2 ds &= \int_{t-1}^{t+1} |A\dot{x}(s) + V'(t, x(s))|^2 ds \\
&\leq \|A\|^2 \int_{t-1}^{t+1} |\dot{x}(s)|^2 ds + \int_{t-1}^{t+1} |V'(t, x(s))|^2 ds \\
&+ 2\|A\| \left(\int_{t-1}^{t+1} |\dot{x}(s)|^2 ds\right)^{\frac{1}{2}} \left(\int_{t-1}^{t+1} |V'(t, x(s))|^2 ds\right)^{\frac{1}{2}}.
\end{aligned}$$

Since $\displaystyle\int_{t-1}^{t+1} |\dot{x}(s)|^2 ds \to 0$ as $|t| \to \infty$, $x(t) \to 0$ as $|t| \to \infty$ and $V'(t, x) \to 0$ as $|x| \to 0$ uniformly in $t \in \mathbb{R}$, then (32) follows. The proof of Theorem 1.1 is complete.

Proof of Theorem 1.2. Let
$H^1_{nT}(\mathbb{R}, \mathbb{R}^N) = \big\{x : \mathbb{R} \to \mathbb{R}^N, \; 2nT - periodic, \; x, \; \dot{x} \in L^2([-nT, nT], \mathbb{R}^N) \; and$
$x(-nT) = x(nT) = 0\big\}$. Consider the family of functionals $(\Phi_n)_{n\geq 1}$ defined on E_{nT} by

$$\Phi_n(x) = \int_{-nT}^{nT} \left[\frac{1}{2}|\dot{x}(t)|^2 + \frac{1}{2}(Ax(t).\dot{x}(t)) + K(t, x(t)) - W(t, x(t))\right] dt, \tag{33}$$

where

$$E_{nT} = \left\{x \in H^1_{nT}(\mathbb{R},\mathbb{R}^N)/x(-t) = x(t),\ a.e.t\ \in \mathbb{R}\right\}.$$

Arguing as in the proof of Theorem 1.1, we prove that assumptions (H_1)-(H_4), (H_6) and (H_7) imply that for every positive integer n, the problem

$$\begin{cases} \ddot{x}(t) + A\dot{x}(t) - K'(t,x(t)) + W'(t,x(t)) = 0,\ for\ t \in]-nT,nT[, \\ x(-t) = x(t), for\ t \in]-nT,nT[, x(-nT) = x(nT) = 0, \end{cases} \tag{34}$$

possesses a solution x_n. Moreover, the sequence (x_n) converges uniformly on any bounded interval to a homoclinic solution $x \in H^1(\mathbb{R},\mathbb{R}^N)$ satisfying $\dot{x}(t) \to 0$ as $|t| \to +\infty$. It remains to prove that $x(t) \not\equiv 0$. In the same way as in the proof of Theorem 1.1 it is easy to prove that there is a constant $C_4 > 0$ such that

$$\|x_n\|_{L^\infty([-nT,nT],\mathbb{R}^N)} \geq C_4 \tag{35}$$

for every $n \in \mathbb{N}$. Moreover, for all $j \in \mathbb{N}$, $t \mapsto x^j_n(t) = x_n(t + jT)$ is also a 2nT-periodic solution of problem (34). Hence, if the maximum of $|x_n|$ occurs in $\theta_n \in [-nT,nT]$ then the maximum of $|x^j_n|$ occurs in $\tau^j_n = \theta_n - jT$. Then there exists a $j_n \in \mathbb{Z}$ such that $\tau^{j_n}_n \in [-T,T]$. Consequently,

$$\|x^{j_n}_n\|_{L^\infty([-nT,nT],\mathbb{R}^N)} = \max_{t\in[-T,T]} |x^{j_n}_n(t)|.$$

Suppose contrary to our claim, that $x \equiv 0$. Then

$$\|x^{j_n}_n\|_{L^\infty([-nT,nT],\mathbb{R}^N)} = \max_{t\in[-T,T]} |x^{j_n}_n(t)| \to 0,$$

which contradicts (35). Then the proof of Theorem 1.2 is complete.

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