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CONTENTS

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CONTENTS

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The Problem of Stability by Nonlinear Approximation

to the 85th Birthday of Professor V.I. Zubov

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Abstract: In the present paper, Vladimir Zubov's results on the problem of stability by nonlinear approximation are surveyed together with their recent developments and extensions.

Keywords: *stability; homogeneous system; nonlinear approximation; Lyapunov function; perturbations; estimates of solutions.*

Mathematics Subject Classification (2010): 34A34, 34D20.

1 Introduction

The outstanding Russian mathematician and mechanical engineer Vladimir Ivanovich Zubov (1930-2000) made an invaluable contribution to the development of Stability Theory and Control Theory.

V. I. Zubov was born on April 14, 1930 in Kashira town, Moscow region, Russia. In 1945 he finished a middle school. At the age of 14, Vladimir was wounded by a hand grenade explosed accidently and soon failed eyesight. In 1949 he finished the Leningrad special school for blind and visually impaired children and entered the Mathematical and Mechanical Faculty of the Leningrad State University. In 1953, after graduating with honors, he joined the University faculty and since then his career was inseparably associated with the Leningrad (now, Saint Petersburg) State University.

In 1955, V. I. Zubov defended his PhD thesis "Boundaries of the Asymptotic Stability Domain" in which he proved the theorem on the asymptotic stability domain. This result is now known as *Zubov's theorem.*

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Further Zubov's activities involved both pure fundamental investigations and solution of applied real-life problems in several fields — from spacecraft to ship control.

In 1969, the Faculty of Applied Mathematics and Control Processes was founded at the Leningrad State University with Vladimir Zubov's appointment as its first dean. Two years later, a Research Institute of Computational Mathematics and Control Processes was set up by the USSR Government. Zubov became its brains-and-heart. In particular, he headed the projects on the design, development and operation of systems of self-guided winged missiles, and tactical schemes construction for the USSR Navy to oppose aircraft carriers of the potential enemy.

Zubov's scientific activities was surveyed in the paper [8] dedicated to his 80th Birthday. In the present review, we would like to focus on Zubov's works on the problem of stability by nonlinear approximation together with their ramifications in the last decade publications.

2 Stability Analysis by Nonlinear Approximation

The basic tool for the stability analysis of motions of differential equation systems is the Lyapunov direct method (or the Lyapunov functions method). However, it should be recalled that until now, a general algorithm has not been yet constructed for the Lyapunov function generation for an arbitrary nonlinear system. The most common approach to the problem consists in, firstly, reduction of an original system to a simpler one, secondly, stability investigation of the reduced system via the Lyapunov function construction, and, thirdly, subsequent testing of this function as a potential candidate for the Lyapunov function of the original system.

A. M. Lyapunov has determined conditions under which the conclusion on the stability of the zero solution for a nonlinear system can be obtained via the analysis of the corresponding system of linear approximation [22]. However, it is worth mentioning that in numerous applications it is required to study differential equation systems for which the expansions of the right-hand sides in powers of the phase variables do not contain linear terms at all. Thus, there arises a problem of stability by nonlinear approximation.

The first theorems on stability by nonlinear approximation were proved by I. G. Malkin, N. N. Krasovskii and V. I. Zubov [21, 23, 35, 36]. In these papers, systems with homogeneous right-hand sides were considered as the first approximation.

Definition 2.1 Let $\mathbb{R}$ be the field of real numbers, $\mathbb{R}^n$ denote the n-dimensional Euclidean space. A function $f(\mathbf{x}) : \mathbb{R}^n \to \mathbb{R}$ is called homogeneous of the order μ, where μ is a positive rational with the odd denominator, if

$$f(\lambda \mathbf{x}) = \lambda^{\mu} f(\mathbf{x}) \tag{1}$$

for $\lambda \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^n$. In the case when μ is a positive real number, and equality (1) holds for $\lambda \geq 0$ and $\mathbf{x} \in \mathbb{R}^n$, the function $f(\mathbf{x})$ is called positive homogeneous of the order μ.

Consider the system of differential equations

$$\dot{\mathbf{x}}(t) = \mathbf{F}(\mathbf{x}(t)) \tag{2}$$

and the corresponding perturbed system

$$\dot{\mathbf{x}}(t) = \mathbf{F}(\mathbf{x}(t)) + \mathbf{G}(t, \mathbf{x}(t)). \tag{3}$$

Here $\mathbf{x}(t) \in \mathbb{R}^n$ is the state vector; components of the vector $\mathbf{F}(\mathbf{x})$ are homogeneous functions of the order $\mu > 0$ which are continuous for all $\mathbf{x} \in \mathbb{R}^n$; vector function $\mathbf{G}(t, \mathbf{x})$ is continuous for $t \geq 0$, $\|\mathbf{x}\| < H$ and satisfies the inequality $\|\mathbf{G}(t, \mathbf{x})\| \leq c\|\mathbf{x}\|^\sigma$, where c and σ are positive constants, $0 < H \leq +\infty$, and $\| \cdot \|$ denotes the Euclidean norm of a vector. Thus, systems (2) and (3) admit the zero solution.

It is required to determine conditions under which the asymptotic stability of the zero solution of (2) implies the same type of stability for the zero solution of the perturbed system (3).

In [21, 23], the case has been studied when components of the vector $\mathbf{F}(\mathbf{x})$ are homogeneous forms of an integer order $\mu > 1$. It was proved that if the inequality $\sigma > \mu$ holds, then the perturbations do not disturb the asymptotic stability of the zero solution.

It is worth mentioning that Malkin's proof was based on a geometric approach [23]. A family of closed surfaces surrounding the origin were constructed, and angles between these surfaces and trajectories of system (3) were estimated. To prove the theorem on the stability by nonlinear approximation, Krasovskii has used the Lyapunov direct method, see [21]. He has determined conditions under which for system (3) there exists a Lyapunov function solving the stability problem and satisfying estimates of a special form.

V. I. Zubov has extended the results of [21, 23] to wider classes of systems, see [34–36]. Unlike [21, 23], in [34–36] it was assumed that the components of the vector $\mathbf{F}(\mathbf{x})$ are, in general, not forms, but homogeneous functions of the order $\mu > 0$. Zubov has established the following properties of solutions of homogeneous systems:

(i) if $\mathbf{x}(t, \mathbf{x}_0)$ is a solution of (2) starting from the point $\mathbf{x}_0$ at $t = 0$, then, for any $c \in \mathbb{R}$, the function $c\mathbf{x}(c^{\mu-1}t, \mathbf{x}_0)$ is the solution of (2) as well;

(ii) the zero solution of (2) can be asymptotically stable only in the case when μ is a rational with the odd numerator and denominator;

(iii) if the zero solution of (2) is asymptotically stable, then it is globally asymptotically stable.

Zubov has investigated conditions under which for a homogeneous system there exists a homogeneous Lyapunov function satisfying the assumptions of the Lyapunov asymptotic stability theorem. He has obtained the following result, see [35, 36].

Theorem 2.1 *Let for solutions of* (2) *the inequality* $\|\mathbf{x}(t, \mathbf{x}_0)\| \leq bt^{-\alpha}$ *be valid for* $t \geq T$, $\|\mathbf{x}_0\| = 1$, *where* T, b, α *are positive constants. Then there exist functions* $V(\mathbf{x})$ *and* $W(\mathbf{x})$ *possessing the properties:*

(a) $V(\mathbf{x})$ *and* $W(\mathbf{x})$ *are continuous for* $\mathbf{x} \in \mathbb{R}^n$ *positive homogeneous functions of the orders* γ *and* $\gamma + \mu - 1$ *respectively, where* γ *is sufficiently large positive number;*

(b) functions $V(\mathbf{x})$ *and* $W(\mathbf{x})$ *are positive definite;*

(c) function $V(\mathbf{x})$ *is differentiable with respect to solutions of system* (2), *and the equality* $\dot{V}\big|_{(2)} = -W(\mathbf{x})$ *holds.*

Moreover, in the case when the right-hand sides of (2) *are* k *times continuously differentiable functions for* $\mathbf{x} \in \mathbb{R}^n$, *where* $k \geq 1$, *while constructing functions* $V(\mathbf{x})$ *and* $W(\mathbf{x})$, *one can choose* $V(\mathbf{x})$ *in the class of* k *times continuously differentiable functions.*

It was shown, see [36], that if the function $\mathbf{F}(\mathbf{x})$ is continuously differentiable for $\mathbf{x} \in \mathbb{R}^n$, then the functions $V(\mathbf{x})$ and $W(\mathbf{x})$ satisfy the system of partial differential equations

$$\left(\frac{\partial V(\mathbf{x})}{\partial \mathbf{x}}\right)^T \mathbf{F}(\mathbf{x}) = -W(\mathbf{x}), \qquad \left(\frac{\partial V(\mathbf{x})}{\partial \mathbf{x}}\right)^T \mathbf{x} = \gamma V(\mathbf{x}). \tag{4}$$

Zubov has studied the problem of solvability of this system [36]. In particular, in the case when $n = 2$, he has proposed a constructive approach for finding solutions of (4).

On the basis of Theorem 2.1, Zubov has determined the following stability and ultimate boundedness criteria for the perturbed system (3), see [35, 36].

Theorem 2.2 *Let the vector function* $\mathbf{F}(\mathbf{x})$ *be continuously differentiable for* $\mathbf{x} \in \mathbb{R}^n$, *and inequality* $\sigma > \mu$ *hold. Then from the asymptotic stability of the zero solution of* (2) *it follows that the zero solution of* (3) *is asymptotically stable as well.*

Theorem 2.3 *Let the vector function* $\mathbf{F}(\mathbf{x})$ *be continuously differentiable for* $\mathbf{x} \in \mathbb{R}^n$, *and the inequality* $\sigma < \mu$ *hold. Then from the asymptotic stability of the zero solution of* (2) *it follows that solutions of* (3) *are uniformly ultimately bounded.*

Moreover, new stability conditions were established in the critical case of several zero roots and in the critical case of several pairs of purely imaginary roots of characteristic equation, see [36, 37, 39, 40].

Zubov has also derived estimates for the convergence rate of solutions for asymptotically stable homogeneous system (2) and for the perturbed system (3), see [36]. He has proved that if $\mu > 1$, function $\mathbf{F}(\mathbf{x})$ is continuously differentiable for $\mathbf{x} \in \mathbb{R}^n$, and the zero solution of (2) is asymptotically stable, then, there exist positive constants c_1, c_2, c_3, c_4 such that for solutions of (2) the inequalities

$$\|\mathbf{x}_0\| \left(c_1 + c_2\|\mathbf{x}_0\|^{\mu-1}\, t\right)^{-\frac{1}{\mu-1}} \le \|\mathbf{x}(t, \mathbf{x}_0)\| \le \|\mathbf{x}_0\| \left(c_3 + c_4\|\mathbf{x}_0\|^{\mu-1}\, t\right)^{-\frac{1}{\mu-1}} \qquad (5)$$

hold for any $\mathbf{x}_0 \in \mathbb{R}^n$ and for $t \ge 0$. For the case when $0 < \mu < 1$, Zubov has obtained conditions under which every solution of system (2) gets to the origin in a finite time, and remains at this point thereafter. In his later works [41, 43], this property of homogeneous systems with homogeneity orders less than one was used for the design of feedback controls providing finite-time synchronization of dynamical systems motions.

Furthermore, Zubov has extended the above results to systems with generally homogeneous right-hand sides [38, 39].

Definition 2.2 A function $f(\mathbf{x}) : \mathbb{R}^n \to \mathbb{R}$ is called generally homogeneous of the order ν with respect to the dilation $(m_1, \dots, m_n)$, where $\nu, m_1, \dots, m_n$ are positive rationals with the odd denominators, if

$$f(\lambda^{m_1} x_1, \dots, \lambda^{m_n} x_n) = \lambda^{\nu} f(\mathbf{x}) \qquad (6)$$

for all $\lambda \in \mathbb{R}^n$ and $\mathbf{x} \in \mathbb{R}^n$. In the case when $\nu, m_1, \dots, m_n$ are positive real numbers, and equality (6) holds for $\lambda \ge 0$ and $\mathbf{x} \in \mathbb{R}^n$, the function $f(\mathbf{x})$ is called positive generally homogeneous of the order ν with respect to the dilation $(m_1, \dots, m_n)$.

Definition 2.3 A vector field $\mathbf{F}(\mathbf{x}) = (f_1(\mathbf{x}), \dots, f_n(\mathbf{x}))^T : \mathbb{R}^n \to \mathbb{R}^n$ is called generally homogeneous of the order μ with respect to the dilation $(m_1, \dots, m_n)$, where $\mu, m_1, \dots, m_n$ are rationals with the odd denominators, such that $m_i > 0$ and $\mu + m_i > 0$, $i = 1, \dots, n$, if $f_i(\lambda^{m_1} x_1, \dots, \lambda^{m_n} x_n) = \lambda^{\mu + m_i} f_i(x_1, \dots, x_n)$, $i = 1, \dots, n$, for all $\lambda \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^n$. System (2) is called generally homogeneous if its vector field $\mathbf{F}(\mathbf{x})$ is generally homogeneous.

In [38, 39], for generally homogeneous systems, conditions of the existence of generally homogeneous Lyapunov functions were obtained, and criteria of stability and ultimate boundedness by generally homogeneous approximation were found.

V. I. Zubov has also set up a problem of the stability by the first, in a broad sense, approximation. He has investigated the conditions for stability of the zero solution for arbitrary admissible functions included in the first approximation [39, 42].

In particular, systems of the form

$$\dot{x}_i(t) = \sum_{j=1}^{n} p_{ij}(t) f_j(x_j(t)) + g_i(t, \mathbf{x}(t)), \qquad i = 1, \ldots, n, \tag{7}$$

have been considered [42]. Here coefficients $p_{ij}(t)$ are continuous for $t \geq 0$; functions $f_j(x_j)$ are continuous for $|x_j| < H$ $(0 < H \leq +\infty)$ and belong to a sector-like constrained set defined as follows: $x_j f_j(x_j) > 0$ for $x_j \neq 0$, $j = 1, \ldots, n$; the perturbations $g_i(t, \mathbf{x})$ are given and continuous for $t \geq 0$, $\|\mathbf{x}\| < H$.

The following issues were investigated:

(i) under what conditions the zero solution of the unperturbed system ($g_i(t, \mathbf{x}) \equiv 0$, $i = 1, \ldots, n$) is asymptotically stable for any admissible functions $f_j(x_j)$?

(ii) under what conditions perturbations do not destroy the asymptotic stability of the zero solution?

On the basis of the obtained results, Zubov has developed new and effective approaches to the problem of stability analysis of nonlinear systems in the cases being critical in the Lyapunov sense.

3 Some Extensions of Zubov's Rezults

3.1 Existence of homogeneous Lyapunov functions

From Zubov's results it follows that for system (2) with homogeneous polynomial right-hand sides possessing the asymptotic stability property for its zero solution, it is always possible to choose a Lyapunov function in the class of homogeneous functions. In [31], the problem has been discussed whether it is possible to choose this function in the class of homogeneous polynomials (forms) or not. The answer proves to be negative. For any given positive integer γ, there exists a system from the family

$$\dot{x}_1(t) = (\alpha - \varepsilon) x_1^3(t) - x_2^3(t), \quad \dot{x}_2(t) = x_1^3(t) - \alpha x_2^3(t), \quad 0 < \varepsilon < \alpha < 1,$$

such that the zero solution of this system is asymptotically stable but the derivative of any form of the order γ with respect to this system is not sign-definite.

L. Rosier has proved that it is possible to guarantee the existence of continuously differentiable homogeneous functions for homogeneous systems under less conservative conditions than those imposed in Zubov's theorems, see [28].

Theorem 3.1 *Let vector function $\mathbf{F}(\mathbf{x})$ be continuous for $\mathbf{x} \in \mathbb{R}^n$ positive generally homogeneous of the order $\mu \in \mathbb{R}$ with respect to the dilation $(m_1, \ldots, m_n)$, where $m_i > 0$ and $\mu + m_i > 0$, $i = 1, \ldots, n$. If the zero solution of (2) is asymptotically stable, then, for any positive integer k, there exists a Lyapunov function $V(\mathbf{x})$ possessing the following properties:*

(a) the function $V(\mathbf{x})$ is k times continuously differentiable at the point $\mathbf{x} = \mathbf{0}$, and it is infinitely differentiable for $\mathbf{x} \neq \mathbf{0}$;

(b) function $V(\mathbf{x})$ is positive definite;

(c) function $V(\mathbf{x})$ is positive generally homogeneous of the order γ with respect to the dilation $(m_1, \dots, m_n)$, where γ is an arbitrary number greater than $k \max_{i=1,\dots,n} m_i$;

(d) the derivative of $V(\mathbf{x})$ with respect to system (2) is negative definite.

The application of Theorem 3.1 permits us to weaken the conditions of known criteria of the stability and the ultimate boundedness by nonlinear approximation.

For homogeneous systems, the problem of existence of homogeneous Lyapunov functions satisfying the assumptions of the first Lyapunov instability theorem was studied in [18].

3.2 Stability analysis of nonlinear systems via averaging

In [2, 3], nonlinear nonstationary systems whose right-hand sides are homogeneous with respect to phase variables have been studied. For such systems, an approach for Lyapunov functions constructing was proposed. Its application permits us to show that if the order of homogeneity of the right-hand sides of the time-varying system under consideration is greater than one, then the asymptotic stability of the zero solution of the corresponding averaged system implies the same property for the zero solution of the original system. These results have been further developed in [4, 5, 15, 26, 27, 30]. In particular, in [30], a modification of the approach for the Lyapunov functions construction was suggested. Other techniques for the determination of similar asymptotic stability conditions for time-varying homogeneous systems have been developed in [26, 27].

Compared with the known stability conditions obtained by the application of averaging technique, the principal novelty of the above results is that, to guarantee the asymptotic stability for a nonstationary homogeneous system, the right-hand sides of the system need not be fast time-varying. It is shown that in the averaging technique, instead of a small parameter providing the fast time-variation of a vector field, the orders of homogeneity can be used.

3.3 Stability of nonlinear complex and hybrid systems

In [24], a motion polystability problem for differential equation systems has been studied. In terms of matrix-valued Lyapunov functions, conditions of polystability for nonlinear systems with separable motions by nonlinear and psevdo-linear approximation were found.

Sufficient conditions of the asymptotic stability with respect to a part of variables for equilibrium positions of nolinear complex systems have been derived in [4, 29].

In [20], an approach for the stability analysis of multiconnected systems by nonlinear approximation was suggested. In [9, 10], the results of [20] were strengthened and extended to wider classes of systems. It is worth mentioning that the approach in [20] is based on the vector Lyapunov functions method, whereas in [9, 10] scalar Lyapunov functions were proposed.

The stability problem for hybrid homogeneous systems was studied in [7, 32]. Sufficient conditions were obtained under which a family of homogeneous subsystems admits a common Lyapunov function. The fulfilment of these conditions provides global asymptotic stability of the zero solution of the corresponding switched system for any admissible switching law. For the case when we can not guarantee the existence of such a function, in [7], the multiple Lyapunov function and the dwell-time approaches were used to determine the classes of switching signals for which the zero solution of the hybrid homogeneous system is locally or globally asymptotically stable. Stability conditions for some types of nonlinear multiconnected systems with a variable structure were found in [11, 33].

3.4 Preservation of stability under the digitization

In the papers [12, 14], the problem of preservation of stability under the digitization for certain classes of nonlinear differential equation systems was studied.

In [12], the homogeneous system (2) and the corresponding difference system

$$\mathbf{y}(k+1) = \mathbf{y}(k) + h\,\mathbf{F}(\mathbf{y}(k)) \tag{8}$$

have been considered. Here $\mathbf{y}(k) \in \mathbb{R}^n$; components of the vector $\mathbf{F}(\mathbf{x})$ are homogeneous functions of the order $\mu > 1$ which are are continuously differentiable for all $\mathbf{x} \in \mathbb{R}^n$; $h > 0$ is a digitisation step; $k = 0, 1, \ldots$.

The following theorem was proved.

Theorem 3.2 *If the zero solution of system* (2) *is asymptotically stable, then the zero solution of* (8) *is asymptotically stable for any value of* $h > 0$*.*

Thus, unlike the case of linear systems, for essentially nonlinear homogeneous systems, the preservation of stability while passing from differential systems to difference ones can be guaranteed for an arbitrary digitization step.

Furthermore, in [12, 14], theorems on the stability by nonlinear approximation were obtained for various classes of difference systems.

3.5 Stability analysis of nonlinear time-delay systems

In the papers [5, 6, 13, 15], certain classes of nonlinear time-delay systems have been studied. It was assumed that the trivial solution of a system is asymptotically stable when delay is equal to zero. The Lyapunov direct method and the Razumikhin theorem were used to show that if the system is essentially nonlinear, i.e., the right-hand sides of the system do not contain linear terms, then the asymptotic stability of the zero solution is preserved for an arbitrary positive value of the delay. On the basis of the proposed approach, new delay-independent stability conditions have been obtained for wide classes of nonlinear systems, see [5, 6, 13, 15].

In particular, in [6], homogeneous time-delay system of the form

$$\dot{\mathbf{x}}(t) = \mathbf{F}(\mathbf{x}(t), \mathbf{x}(t-\tau)) \tag{9}$$

has been considered. Here $\mathbf{x}(t) \in \mathbb{R}^n$; the components of the vector $\mathbf{F}(\mathbf{x}, \mathbf{y})$ are homogeneous functions of the order $\mu > 1$, defined for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, and continuous with respect to their variables, and continuously differentiable with respect to $\mathbf{y}$; τ is a constant positive delay. This means that system (9) admits the zero solution.

Theorem 3.3 *Let the zero solution of the corresponding delay free system* $\dot{\mathbf{x}}(t) = \mathbf{F}(\mathbf{x}(t), \mathbf{x}(t))$ *be asymptotically stable. Then the zero solution of* (9) *is asymptotically stable for any value of* $\tau > 0$*.*

3.6 Estimates of the convergence rate of solutions

Zubov's results on the finite-time stability and synchronization have been vigorously developed during the past decades, see [16, 17] and the references cited therein.

In [12], a discrete-time counterpart of estimates (5) was obtained for nonlinear homogeneous difference systems.

In [15], in terms of the Razumikhin approach, a procedure for the estimation of the convergence rate of solutions for essentially nonlinear time-delay systems was developed.

3.7 Stability by the first, in a broad sense, approximation

Some results on the stability by the first, in a broad sense, approximation have been obtained in [1, 5, 9, 10, 19, 20]. For instance, in [9, 10], a generalization of system (7) was studied. With the aid of the well-known Martynyuk–Obolenskij stability criteria for autonomous Wazewskij systems, see [25], an approach to the construction of Lyapunov functions for the system in question was proposed, and existence conditions for such functions were found. By the use of the Lyapunov functions constructed, new theorems on the stability and ultimate boundedness by nonlinear approximation have been proved [9, 10].

4 Conclusion

Vladimir Zubov was a prominent scholar, engineer and university lecturer. In the previous sections we have reviewed just only one area of scientific activity of his own and his successors.

Zubov is the author of about 200 publications including 31 monographs and text books. He was an advisor for 20 DSc and about 100 PhD dissertations. Under Zubov's supervision, a worldwide famous school in control theory was developed in St. Petersburg.

In 1968 V. I. Zubov became the USSR State Prize winner for his pioneer works in Control Theory. In 1981 he was elected a corresponding member of the Soviet Union Academy of Sciences, and in 1998 he was awarded the title of the Honor Scholar of the Russian Federation. In 1996, the Zubov scientific school "Processes of control and stability" was the winner of the competition for the State support of leading scientific schools of Russia. In 2001, the Research Institute of Computational Mathematics and Control Processes of St. Petersburg State University was named after him.

For outstanding merits to the world science, Zubov's name was perpetuated as a name of minor planet 'ZUBOV 10022'. This asteroid has a size of 6 km, a brightness of 13.8 magnitude, and the greatest orbit's semiaxis of 2.369 astronomical units.

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Nonlinear Dynamics and Systems Theory, **15** (3) (2015) 231–241

Mathematical Contributions to the Dynamics of the Josephson Junctions: State of the Art and Open Problems

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Abstract: Mathematical models related to some Josephson junctions are pointed out and attention is drawn to the solutions of certain initial boundary problems and to some of their estimates. In addition, results of rigorous analysis of the behaviour of these solutions when $t \to \infty$ and when the small parameter ε tends to zero are cited. These analyses lead us to mention some of the open problems.

Keywords: *third order parabolic operator; fundamental solution; superconductivity; Josephson junction.*

Mathematics Subject Classification (2010): 82D55, 74K30, 35K35, 35E05.

1 Introduction

Our purpose is to:

i) furnish a short review of the mathematical contributions to the dynamics of the Josephson junctions,

ii) introduce some possible open problems.

From the mathematical point of view, many descriptions of superconductivity phenomena have been developed and an important contribution has been given by Brian David Josephson. He predicted in 1962 the tunnelling of superconducting Cooper pairs through an insulating barrier to pass from one superconductor to another (Josephson effect). He also predicted the exact form of the current and voltage relations for the junction (Josephson junction) [1]. (Experimental work proved that his theory was right, and Josephson was awarded the 1973 Nobel Prize in Physics.)

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The flux-dynamics of a Josephson junction, i.e., two layers of superconductors separated by a very thin layer of insulating material, can be described by means of Sine–Gordon equation (SGE):

$$u_{xx} - u_{tt} = \sin u, \tag{1}$$

where x denotes the direction of propagation, t is time and the variable $u = u(x,t)$ represents the difference between the phases of the wave functions of the two superconductors.

However, in dealing with real junctions it seems necessary to take into account other effects such as losses and bias. Therefore, many authors prefer to consider the so-called perturbed Sine–Gordon equation (PSGE):

$$\varepsilon u_{xxt} + u_{xx} - u_{tt} - a u_t = \sin u - \gamma. \tag{2}$$

In this case, terms εu_{xxt} and $a u_t$ represent respectively the dissipative normal electron current flow along and across the junction, (longitudinal and shunt losses) while γ is the normalized current bias [2]. The value's range for a and ε depends on the real junction. Indeed, there are cases with $0 < a, \varepsilon < 1$ and, when the shunt resistance of the junction is low, the case a large with respect to 1 arises [2–4].

In some cases, extra terms must be considered. For example in a semiannular or in a S-shaped Josephson junction, when an applied magnetic field b parallel to the plane of the dielectric barrier is considered, the dynamic equation is:

$$\varepsilon u_{xxt} + u_{xx} - u_{tt} - a u_t = \sin u - \gamma - b\cos(kx), \tag{3}$$

where the last term evaluates a transient force on the trapped fluxons and locates these ones at the center of the junction [2,5,6]. Moreover, if an annular junction, also provided with a microshort, is considered, the vortex dynamics in a static magnetic field is modelled with the general perturbed sine–Gordon equation (see, f.i. [7]):

$$\varepsilon u_{xxt} + u_{xx} - u_{tt} - a u_t = [1 - \delta(x)\mu]\sin u - \gamma - b\cos(kx), \tag{4}$$

where μ is the current density associated with the microshort.

Nowadays, in addition to rectangular or annular junctions, many other geometries for Josephson junctions have been proposed. For instance, window Josephson junctions (WJJ) ([8] and reference therein) or exponentially shaped Josephson junctions (ESJJ) [9–12]. This type of junction is only a particular case of a structure covering a region

$$0 \le x \le L, \qquad g_2(x) \le y \le g_1(x). \tag{5}$$

Denoting by

$$0 < w(x) = g_1(x) - g_2(x) \ll 1, \tag{6}$$

the evolution of the phase inside the junction is given by:

$$\varepsilon u_{xxt} + u_{xx} - u_{tt} - a u_t = \sin u - \Gamma(x) - \frac{\dot{w}(x)}{w(x)}(u_x + \varepsilon u_{xt}) + \eta_y \frac{\dot{w}(x)}{w(x)}, \tag{7}$$

where $\Gamma(x) = \frac{\eta_x|_{g_2} - \eta_x|_{g_1}}{w(x)}$ and η_x η_y is the normalized magnetic field respectively in the x and y directions [10]. When one assumes $g_1(x) = -g_2(x) = w_o\, e^{-\lambda x}$, where λ is a constant that, generally, is less than one, an ESJJ is obtained. Moreover, assuming that there is no bias current so that $\Gamma(x) = 0$ and $\eta_y = 0$, the equation achieved is the following:

$$\varepsilon u_{xxt} + u_{xx} - u_{tt} - \varepsilon\lambda u_{xt} - \lambda u_x - a u_t = \sin u. \tag{8}$$

The current due to the tapering is represented by terms λu_x and $\lambda \varepsilon u_{xt}$. In particular λu_x characterizes the geometrical force driving the fluxons from the wide edge to the narrow edge. These junctions assure many advantages compared to rectangular ones, such as a voltage which is not chaotic anymore, but rather periodic excluding, in this way, some among the possible causes of large spectral width. It is also proved that the problem of trapped flux can be avoided (see f.i. [10]).

There exist numerous applications of Josephson junctions especially as superconducting quantum interference device (SQUID), which consists of a loop of superconductor with one or more Josephson junctions. These devices are one of the most important applications of superconductivity. They are basically extremely sensitive sensors of magnetic flux. This peculiarity allows to diagnose heart and/or blood circuit problems using magnetocardiograms and even to evaluate magnetic fields generated by electric currents in the brain using magnetoencephalography -MEG- [2]. SQUIDs are also used in nondestructive testing as a convenient alternative to ultra sound or x-ray methods (see [2] and reference therein). In geophysics, instead, they are used as gradiometers [3] or as gravitational wave detectors (see [4] and reference therein). SQUIDs play an important role in the study of the potential virtues of superconducting digital electronics, too [13].

2 Mathematical Models and Equivalences

All equations previously considered have something in common. More precisely, if one denotes by $\mathcal{L}$ the following linear third order parabolic operator:

$$\mathcal{L} = \varepsilon\partial_{xxt} - \partial_{tt} + \partial_{xx} - \alpha\partial_t, \tag{9}$$

(1)-(4) and (8) can be expressed by means of the unique equations:

$$\mathcal{L}u = f(x, t, u). \tag{10}$$

According to the meaning of f, numerous other examples of dissipative phenomena can be considered. For example, equation (10) arises in the motion of viscoelastic fluids or solids (see [14–17] and references therein) and in the study of viscoelastic plates with memory, when the relaxation function is given by an exponential function [18]. It can also be employed in the analysis of phase-change problems for an extended heat conduction model [19,20]. In addition, equation (10) arises also in heat conduction at low temperature [15, 21] and in the propagation of localized magnetohydrodinamic models in plasma physics [22]. Still, it is possible to find others in [23–26].

Then, an equivalence between the third order equation (10), typical of Josephson junctions, and biological phenomena has been pointed out in [27]. Indeed, let us consider the FitzHugh-Nagumo system (FHN) [28,29]:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \varepsilon \dfrac{\partial^2 u}{\partial x^2} - v - a u + u^2 (a + 1 - u) \qquad (0 < a < 1), \\ \dfrac{\partial v}{\partial t} = b u - \beta v, \end{cases} \tag{11}$$

where $u(x, t)$ represents a membrane potential of a nerve axon at distance x and time t, and $v(x, t)$ is a recovery variable that models the transmembrane current.

This reaction-diffusion model characterizes the theory of the propagation of nerve impulses, and the connection between a third order equation like (10) and the (FHN) system can be realized changing the first one into the second one under continuous parameter variations [27].

An equation that is able to model all these physical problems has been introduced in [30] and it is represented by the following parabolic integro-differential equation:

$$\mathcal{L}_R\, u \equiv u_t - \varepsilon u_{xx} + au + b \int_0^t e^{-\beta(t-\tau)}\, u(x,\tau)\, d\tau \;=\; F(x,t,u). \tag{12}$$

Indeed, it has been proved that (12) characterizes both reaction diffusion models like the FitzHugh-Nagumo system and superconductive models [30–34].

In particular, perturbed Sine-Gordon equation (2) can be obtained by (12) as soon as one assumes

$$a \;=\; \alpha - \frac{1}{\varepsilon}, \qquad b = -\,\frac{a}{\varepsilon}, \qquad \beta \;=\frac{1}{\varepsilon} \tag{13}$$

and F is such that

$$F(x,t,u) \;=\; -\int_0^t e^{-\,\frac{1}{\varepsilon}\,(t-\tau\,)}\;[\; sen\, u(x,\tau)\;-\gamma\;]\; d\tau. \tag{14}$$

Furthermore, the integro-differential equation (12) is able to describe the evolution inside an exponentially shaped Josephson junction, too. Indeed, as it has already been underlined in [12], assuming

$$\beta \;= \frac{1}{\varepsilon}, \qquad b = \beta^2\,(1-\alpha\,\varepsilon), \qquad a\,\beta = \;\frac{\lambda^2}{4}\;-\,b, \tag{15}$$

$$F \;=\; -\int_0^t \; e^{-\,\frac{1}{\varepsilon}(t-\tau)} f_1(x,\tau,u)\, d\tau,$$

with

$$f_1 \;=\; e^{-\frac{\lambda}{2}\,x}\,[\,\sin\,(e^{\,x\,\lambda/2}\;u)\;-\gamma], \tag{16}$$

from the integro-differential equation (12) it follows:

$$\varepsilon u_{xxt}\;-\;u_{tt}\;+\;u_{xx} - \,(\alpha\;+\varepsilon\,\frac{\lambda^2}{4})u_t\;\;-\;\frac{\lambda^2}{4}\,u = \;f_1. \tag{17}$$

Therefore, assuming $e^{\frac{\lambda}{2}\,x}\;u = \bar{u},$ (17) turns into equation (8).

Remark: In (12) the kernel $e^{-\beta(t-\tau)}\,u(x,\tau)$ can be modified as physical situations demand and in this way many other physical phenomena could be described (see, f.i. [35–38] and references therein). The particular choice made here is due to describe the superconductive and biological models considered.

3 Mathematical Results

There exist many significant analytic results concerning the qualitative analysis of equations related to Josephson junctions and many initial-boundary problems have been discussed in a lot of papers (see [15, 39–43] and references therein).

A first analysis, where the fundamental solution is determined, concerns operator $\mathcal{L}$ in case $\alpha = 0$ [14,44]. Later, in [45,46], the fundamental solution of the whole operator $\mathcal{L}$ of (9) is explicitly determined and various properties are analyzed. Estimates and properties of continuous dependence for the solution of initial value problem are determined, too.

Moreover, in [47], in order to deduce an exhaustive asymptotic analysis, the Green function of the linear operator $\mathcal{L}$ of (9) has been determined by Fourier series and by means of its properties, an exponential decrease of solution related to the Dirichlet problem is deduced. And still by means of Fourier series, existence and uniqueness for Dirichlet, Neumann and pseudoperiodic initial-boundary conditions are achieved, too [42,43].

The Dirichlet problem is still considered with respect to equation (8) and in [11] the problem is reduced to an integral equation with kernel G endowed with rapid convergence and exponentially vanishing as t tends to infinity. Indeed, let

$$\gamma_n = \frac{n\pi}{l}, \quad b_n = (\gamma_n^2 + \lambda^2/4), \quad g_n = \frac{1}{2}(\alpha + \varepsilon\, b_n), \quad \omega_n = \sqrt{g_n^2 - b_n} \tag{18}$$

and

$$G_n(t) = \frac{1}{\omega_n}\, e^{-g_n\, t}\, sinh(\omega_n t), \tag{19}$$

the Green function is given by

$$G(x,t,\xi) = \frac{2}{l}\, e^{\frac{\lambda}{2}\, x} \sum_{n=1}^{\infty} G_n(t) \quad sin\gamma_n\xi \quad sin\gamma_n x. \tag{20}$$

The initial boundary problem with Dirichlet conditions is analyzed and an appropriate analysis implies results on the existence and uniqueness of the solution.

That is, indicating by

$$\Omega_T \equiv \{ (x,t) : 0 \le x \le L \; ; \; 0 < t \le T \},$$

the following initial boundary problem

$$\begin{cases} (\partial_{xx} - \lambda\,\partial_x)\,(\varepsilon u_t + u) - \partial_t(u_t + \alpha\, u) = F(x,t,u), \quad (x,t) \in \Omega_T, \\ u(x,0) = h_0(x), \quad u_t(x,0) = h_1(x), \quad x \in [0,L], \\ u(0,t) = g_1(t), \quad u(l,t) = g_2(t), \quad 0 < t \le T. \end{cases} \tag{21}$$

for $g_1 = g_2 = 0$ admits the following integral equation:

$$u(x,t) = (\partial_t + \alpha + \varepsilon\,\lambda\partial_x - \varepsilon\partial_{xx}) \int_0^L h_0(\xi) e^{-\frac{\lambda\xi}{2}} G(x,\xi,t) d\xi \tag{22}$$

$$+ \int_0^L h_1(\xi) e^{-\frac{\lambda\xi}{2}} G(x,\xi,t) d\xi + \int_0^t d\tau \int_0^L G(x,\xi,t-\tau) e^{-\frac{\lambda\xi}{2}} F(\xi,\tau,u(\xi,\tau)) d\xi.$$

So, a priori estimates, continuous dependence and asymptotic behaviour of the solution, are deduced, too.

When boundary data are non null, in order to achieve explicit estimates of boundary contributions related to the Dirichlet problem, equivalence between the equation describing the evolution inside an (ESJJ) and the integro-differential equation (12) has been considered. Indeed, operator $\mathcal{L}_R$ of (12) has already been extensively examined in [30] and the fundamental solution K with many of its properties have been determined.

More in detail, if a, b, ε, β are positive constants, $r = |x|/\sqrt{\varepsilon}$ and $J_n(z)$ denotes the Bessel function of first kind and order n, let us consider the function

$$K(r,t) = \frac{e^{-\frac{r^2}{4t}}}{2\sqrt{\pi\varepsilon t}}e^{-at} - \frac{1}{2}\sqrt{\frac{b}{\pi\varepsilon}}\int_0^t \frac{e^{-\frac{r^2}{4y}-ay}}{\sqrt{t-y}}e^{-\beta(t-y)}J_1(2,\sqrt{by(t-y)})dy. \tag{23}$$

The following theorem has been proved:

Theorem 3.1 *The function K has the same basic properties of the fundamental solution of the heat equation, that is: $K(x,t) \in C^\infty$ for $t > 0$, $x \in \Re$.*

For fixed $t > 0$, K and its derivatives are exponentially vanishing as fast as $|x|$ tends to infinity.

For any fixed $\delta > 0$, uniformly for all $|x| \geq \delta$, it results:

$$\lim_{t\downarrow 0} K(x,t) = 0. \tag{24}$$

For $t > 0$, it is $\mathcal{L}_R K = 0$.

Moreover, it results

$$|K(x,t)| \leq \frac{e^{-\frac{x^2}{4\varepsilon t}}}{2\sqrt{\pi\varepsilon t}}\left[e^{-at} + bt\,\frac{e^{-at} - e^{-\beta t}}{\beta - a}\right]. \tag{25}$$

Previous estimates show, as well, that K exponentially decays to zero as t increases. These and other properties also allowed to prove in [12] numerous properties of the following function which is similar to *theta functions*:

$$\theta(x,t) = K(x,t) + \sum_{n=1}^{\infty}[K(x+2nL,t) + K(x-2nL,t)] = \sum_{n=-\infty}^{\infty} K(x+2nL,t). \tag{26}$$

So that, as for problem (21), denoting by

$$G(x,\xi,t) = \theta(|x-\xi|,t) - \theta(x+\xi,t)$$

and

$$F(x,t,u) = e^{-\frac{\lambda}{2}x}\left[\int_0^t e^{-\frac{1}{\varepsilon}(t-\tau)}[\sin(e^{x\lambda/2}u) - \gamma]\,d\tau - h_1(x)\,e^{-\frac{t}{\varepsilon}}\right],$$

it has been proved that the problem admits the following integral equation:

$$u(x,t) = \int_0^L G(x,\xi,t)e^{-\frac{\lambda}{2}x}h_0(\xi)d\xi + \int_0^t d\tau\int_0^L G(x,\xi,t)F(\xi,\tau,u(x,\tau))d\xi \tag{27}$$

$$-2\varepsilon\int_0^t \theta_x(x,\,t-\tau)g_1(\tau)d\tau + 2\varepsilon\int_0^t \theta_x(x-L,t-\tau)e^{-\frac{\lambda L}{2}}g_2(\tau)d\tau.$$

Besides, a priori estimates and asymptotic properties have proved that when t tends to infinity, the effect due to the initial disturbances (h_0, h_1) is vanishing, while the effect of the non linear source is bounded for all t. Furthermore, for large t, the effects due to boundary disturbances g_1, g_2 are null or at least everywhere bounded.

Indeed, if $h_0 = h_1 = 0$ and $F = 0$, the following theorem holds:

Theorem 3.2 *When t tends to infinity and data g_i $(i = 1, 2)$ are two continuous functions convergent for large t, one has:*

$$u = g_{1,\infty} \frac{\sinh \sigma_0 \, (L - x)}{\sinh \, \sigma_0 \, L} + g_{2,\infty} \frac{\sinh \sigma_0 \, x}{\sinh \, \sigma_0 \, L}, \tag{28}$$

where $\sigma_0 = \frac{\lambda}{2}$ and $g_{i,\infty} = \lim_{t \to \infty} g_i$, $(i = 1, 2)$. Otherwise, when $\dot{g}_i \in L_1[0, \infty]$ $(i = 1, 2)$ too, the effects determined by boundary disturbance vanish.

Another aspect frequently highlighted in many papers is that the linear third order operator $\mathcal{L}$ is an example of wave operator perturbed by higher order viscous terms. The behaviour of solution of (10) when $\alpha = 0$, has been analyzed in various applications of artificial viscosity method [48, 49]. Moreover, in [50], when ε is vanishing, the interaction between diffusion effects and pure waves has been evaluated by means of slow time εt and fast times t/ε. These aspects are also analyzed in [16] referring to the strip problem for equation (10) with a linear source term f, while in the non-linear case, the Neumann boundary problem has been discussed in [51].

Also equation (8) can be considered as a semilinear hyperbolic equation perturbed by viscous terms described by higher-order derivatives with small diffusion coefficients ε. In [52], the influence of the dissipative terms has been estimated proving that they are both bounded when ε tends to zero and when time tends to infinity, giving a mathematical proof of what has been observed in [9].

As for explicit solutions, an extensive literature exists, and more recently, various classes of solutions for (SGE) have been determined (see, f.i., [53, 54]). Furthermore, when $\varepsilon = 0$, some travelling-wave solutions for (2) have been obtained both for $|\gamma|$ not larger than 1 and for $|\gamma| > 1$ [55, 56]. Still when $\varepsilon = 0$, some classes of explicit solutions have been determined for equation (8), too [52].

4 Open Problems

In light of what has been stated until now, many open problems can be highlighted.

It would be interesting, for example, to study equation (2) when interface conditions for the phase (and its normal gradient) are added, connecting, in this way, with the problems of window Josephson junctions (WJJ) when the influence of an external magnetic field must be considered [57]. Indeed, letting $\varepsilon = 0$, (2) exactly recalls one of the equations usually considered for (WJJ).

When, on the other hand, ε is not vanishing, a viscous term, represented by the third order term, appears. So that, it would be interesting to give an estimate of the diffusive effects due to the ε-term, too.

Moreover, according to the analogy between superconductor equations and reaction-diffusion models, the Robin boundary problem would be considered in order to achieve results for many biological phenomena, too [58, 59].

Besides, as for analysis on asymptotic effects due to the boundary perturbations related to equation (8), as it has been pointed out, the Dirichlet boundary problem has already been considered in [12]. So, the evaluation could be extended to other boundary problems, such as, for instance, Neumann and mixed ones.

Of course, in order to achieve estimates for other more significant physical problems, this analysis and many other estimates could be carried out for solution of equation (3) and for equations like (4) where the presence of a gap in the vacuum chamber is considered, too [41].

The analysis conducted so far required that in (12) constants a, b, ε, β were all positive. This can be valid if we look for an analogy with an (ESJJ), but excludes application of (12) to some other junctions. Therefore it would be interesting to extend the analysis of operator $\mathcal{L}_R$ for any value of a, b, ε, β.

Finally a qualitative analysis of operators should be made in case $\varepsilon, \alpha, \lambda$ were not constant.

5 Conclusion

The state of the art proves that many significant analytic results concerning the qualitative analysis of equations related to Josephson junctions have been obtained and many initial-boundary problems have been discussed. However other many important open problems may be considered and solved.

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Nonlinear Dynamics and Systems Theory, **15** (3) (2015) 242–271

An Inversion of a Fractional Differential Equation and Fixed Points

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Abstract: This is a study of the scalar fractional differential equation of Riemann-Liouville type

$$D^q x(t) = f(t, x(t)), \quad \lim_{t\to 0^+} t^{1-q}x(t) = x^0,$$

where $q \in (0,1)$ and $x^0 \neq 0$. This is first written as a Volterra integral equation

$$x(t) = x^0 t^{q-1} + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} f(s, x(s))\, ds.$$

After two existence results for a solution on a short interval $(0, T]$ are presented, it is then transformed in two steps into an integral equation

$$y(t) = F(t) + \int_0^t R(t-s)\left[y(s) + \frac{f(s+T, y(s))}{J}\right] ds,$$

where $y(t) = x(t+T)$. The function R is completely monotone on $(0,\infty)$ and $\int_0^\infty R(t)\, dt = 1$. When f is bounded and continuous for y bounded and continuous on $[0,\infty)$, then the integral maps sets of bounded continuous functions into sets of bounded equicontinuous functions. Moreover, F is uniformly continuous on $[0,\infty)$, $F(t) \to 0$, and $F \in L^1[0,\infty)$, while J is an arbitrary positive constant. A growth condition on f is used to show that all of these equations share solutions.

The point of the work is that an integral equation with two singularities and a kernel having infinite integral is transformed into an equation with a mildly singular kernel and finite integral. That final form is very suitable for a variety of fixed point theorems yielding qualitative properties of solutions of each of the stated equations.

Keywords: *fixed points; fractional differential equations; integral equations; Riemann-Liouville operators; singular kernels.*

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1 Introduction

A myriad of real-world problems can be modeled by the fractional differential equation of Riemann-Liouville type

$$D^q x(t) = f(t, x(t)), \quad \lim_{t \to 0^+} t^{1-q} x(t) = x^0 \quad (0 < q < 1). \tag{1.1}$$

Substantial treatments are found in Diethelm [10], Kilbas et al. [12], Lakshmikantham et al. [14], and Podlubny [19]. An annotated bibliography is found in Oldham and Spanier [17].

Under certain conditions it is known that this initial value problem and the Volterra equation

$$x(t) = x^0 t^{q-1} + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, x(s))\, ds \tag{1.2}$$

share solutions. Equation (1.2) is far more familiar to most analysts than is (1.1), so there is good reason to pursue a study of (1.2) and its relation to (1.1). It can be argued that this last equation has essentially three singularities and a kernel which does not belong to $L^1(0, \infty)$. The singular forcing function immediately feeds back into the function f producing a singularity which can cause us to restrict the values of q for which a solution will exist. These properties offer a strong challenge. Our goal is to transform it into a far more tractable equation.

The conditions with (1.1), and subsequently transferred to (1.2), are of critical importance. Both the literature and the results which we will obtain here dictate very precise properties for solutions contained in this definition.

Definition 1.1 For a given $q \in (0, 1)$, a function $\phi\colon (0, T] \to \Re$ is said to be a *solution* of (1.2) if ϕ is continuous, if ϕ satisfies (1.2) on $(0, T]$, and if

$$t^{1-q}\phi(t) \text{ is continuous on } [0, T] \text{ with } \lim_{t \to 0^+} t^{1-q}\phi(t) = x^0.$$

The first task is to obtain some general existence theorems for solutions on a short interval $(0, T]$ which will get us past the singularities and facilitate the transformation. The continuing work does not rely on these particular existence results, but asks only local existence.

Next, we improve the kernel by transforming the Volterra equation into an intermediate equation

$$x(t) = z(t) + \int_0^t R(t-s) \left[x(s) + \frac{f(s, x(s))}{J} \right] ds \tag{1.3}$$

in which J is an arbitrary positive constant, while R is a completely monotone kernel residing in $L^1(0, \infty)$, while

$$z(t) = x^0 t^{q-1} - \int_0^t R(t-s) x^0 s^{q-1} ds \tag{1.4}$$

still contains the singularity in the forcing function. Thus, we make one more transformation mapping that last equation into

$$y(t) = F(t) + \int_0^t R(t-s) \left[y(s) + \frac{f(s+T, y(s))}{J} \right] ds, \tag{1.5}$$

where $y(t) = x(t+T)$. Not only is the kernel nice but now our function F is uniformly continuous on $[0,\infty)$, $F(t) \to 0$ as $t \to \infty$, and $F \in L^1[0,\infty)$.

With this we have achieved our goal. We have transformed the fractional equation into a very standard Volterra equation with a mildly singular kernel. From this the investigator can now move out and apply classical techniques to obtain qualitative properties of solutions of the original fractional differential equation. There is a more complete summary and guide for further work located in the first part of Section 4.

While our goal is the transformation, in the process there emerges a property which seems entirely new. In order to obtain our existence theorem for a solution on $(0,T]$, we ask a growth condition

$$|f(t,x)| \le |f(t,0)| + Kt^{r_1}|x|^{r_2}, \quad \int_0^T |f(t,0)|\,dt < \infty \tag{1.6}$$

for $0 < t \le T$ with $r_1 > -1$ and other technical conditions including

$$r_1 + r_2(q-1) + 1 > 0. \tag{1.7}$$

The local existence follows from these, a contraction mapping, and a nonlinear Lipschitz condition. A similar growth condition is also used (in work to be offered elsewhere because of its length), together with Schauder's theorem, to obtain existence without a Lipschitz condition.

1.1 Two central issues

There are two properties which will play central roles in this paper and we want to alert the reader to them early. The first issue is that existence theory must place restrictions on the values of q in $(0,1)$. As r_1 and r_2 are constants inherently part of $f(t,x)$, (1.7) restricts the values of q to an interval $q_0 < q < 1$ for some $q_0 \ge 0$, a restriction not seen in the aforementioned references. However, Example 2.3 shows that general existence theorems must contain such restrictions. A study of the references reveals that such restrictions were missed since the investigators ask for either a Lipschitz condition, a severe bound on f, or both. See Section 2.5 of [14], [19, p. 127], or [10, p. 77] for example. This brings in the property which to a large extent ties this paper together. Every existence result which we have encountered either in the literature cited just now or in our own work presented here and in preparation has a condition subsumed by

$$|f(t,x)| \le u(t) + K_2 t^{r_1}|x|^{r_2} \tag{1.6*}$$

with mild conditions on $u(t)$ and technical relations between q, r_1, r_2. It is common to find q restricted to an interval smaller than $(0,1)$ [13, p. 1 and Lemma 1] for reasons other than existence theory.

The second issue is encountered almost immediately and continues to be foremost in the considerations. A main sufficient condition to transfer from (1.1) to (1.2) and again to transfer from (1.2) to (1.3) is that a solution on a short interval $(0,T]$ must satisfy

$$\int_0^T [|x(s)| + |f(s,x(s))|]\,ds < \infty. \tag{1.8}$$

Now, the two issues are brought together using Lemma 2.1 and Theorem 2.6. It is shown that if there is a solution and if f does satisfy (1.6) then it will also satisfy (1.8).

Thus, every existence theorem we encounter asks (1.6) and, hence, has as a corollary (1.8). And (1.8) is a main sufficient condition to pass from (1.1) to (1.2) and is also a main sufficient condition to pass from (1.2) to (1.3). The passage from (1.3) to (1.5) is just a translation. Hence, our entire stated problem of passing from (1.1) to (1.5) rests in an essential way on (1.6), and consequently on (1.8), whether we use one of our own existence results or one of the cited works.

2 Existence and Uniqueness

We are concerned with the fractional differential equation and initial condition

$$D^q x(t) = f(t, x(t)), \quad \lim_{t\to 0^+} t^{1-q} x(t) = x^0 \quad (0 < q < 1), \tag{2.1}$$

where $x^0 \in \Re$, $x^0 \neq 0$, $f\colon (0,T] \times \Re \to \Re$ is continuous for some $T > 0$. The symbol D^q denotes the *Riemann-Liouville fractional differential operator of order* q, which for $0 < q < 1$ is defined by

$$D^q x(t) := \frac{1}{\Gamma(1-q)} \frac{d}{dt} \int_0^t (t-s)^{-q} x(s)\, ds,$$

where $\Gamma\colon (0,\infty) \to \Re$ is Euler's Gamma function:

$$\Gamma(x) := \int_0^\infty t^{x-1} e^{-t}\, dt.$$

Our study will focus on the integral equation

$$x(t) = x^0 t^{q-1} + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, x(s))\, ds, \tag{2.2}$$

where $q \in (0,1)$ and $x^0 \in \Re$. However, we exclude $x^0 = 0$ from consideration since this particular value would remove the singularity at $t = 0$, thereby changing (2.2) to a different type of equation.

Notice that this equation contains essentially three singularities. The singular forcing function and kernel are clear. But there is instantaneous feedback of the forcing function into the function f resulting in a complicated singularity in the integrand. This will become more clear as we study existence problems and examine growth properties of f.

The following result given in [4] establishes mild conditions under which (2.1) and (2.2) are equivalent in the sense that they share solutions.

Theorem 2.1 *Let $q \in (0,1)$ and $x^0 \neq 0$. Let $f(t,x)$ be a function that is continuous on the set*

$$\mathcal{B} := \{(t,x) \in \Re^2 : 0 < t \leq T, x \in I\},$$

where $I \subset \Re$ denotes an unbounded interval. Suppose a function $x\colon (0,T] \to I$ is continuous and that both $x(t)$ and $f(t,x(t))$ are absolutely integrable on $(0,T]$. Then $x(t)$ satisfies the initial value problem (2.1) on the interval $(0,T]$ if and only if it satisfies the Volterra integral equation (2.2) on this same interval.

It can give the reader pause to be confronted with the need to show $x(t)$ and $f(t, x(t))$ absolutely integrable. But according to the discussion in the subsection of the introduction, a sufficient condition is that f satisfy (1.6) and that requires no knowledge of the solution.

In order to get past the singularity in the forcing function, t^{q-1}, we will first present two existence results for a short interval $(0, T]$. In the first existence result (cf. Theorem 2.5), we assume there is a positive constant K_2 so that $f\colon [0, T] \times \Re \to \Re$ is continuous and satisfies the Lipschitz condition

$$|f(t, x) - f(t, y)| \leq K_2|x - y| \tag{2.3}$$

for $0 \leq t \leq T$ and all $x, y \in \Re$. Then, because of the continuity of f, there is also a positive constant K_1 such that

$$|f(t, x)| \leq |f(t, 0)| + K_2|x| \leq K_1 + K_2|x| \tag{2.4}$$

for $0 \leq t \leq T$ and all $x \in \Re$. In the second existence result (cf. Theorem 2.7), f is allowed to have a singularity at $t = 0$ and the Lipschitz condition (2.3) is replaced with a more general condition.

All of our work on existence will be done in a certain weighted space $(X, |\cdot|_g)$, which we define next. The term *g-norm* is what we call $|\cdot|_g$.

Definition 2.1 For a fixed $T > 0$ and for $g(t) := t^{q-1}$, let $(X, |\cdot|_g)$, or simply X, denote the space of continuous functions $\phi\colon (0, T] \to \Re$ for which

$$|\phi|_g := \sup_{0<t\leq T} \frac{|\phi(t)|}{g(t)}$$

is finite.

Theorem 2.2 *The space $(X, |\cdot|_g)$ is a Banach space.*

Proof. It is a straightforward exercise to show that X is a subspace of the vector space of all continuous functions on $(0, T]$ and to verify that $|\cdot|_g$ is a norm. Thus, $(X, |\cdot|_g)$ is a normed vector space. To show that it is also complete, let $\{x_n\} \subset X$ be a Cauchy sequence. This translates into $\{t^{1-q}x_n(t)\}$ being a uniformly Cauchy sequence of continuous functions on $(0, T]$. By the Cauchy criterion, it converges uniformly on $(0, T]$ to a limit function φ, which is also continuous on $(0, T]$. Finally, $\varphi \in (X, |\cdot|_g)$. In order to see this, choose N large enough so that

$$\left| \frac{\varphi(t)}{t^{q-1}} - \frac{x_N(t)}{t^{q-1}} \right| < 1$$

for all $t \in (0, T]$. Then we have

$$\frac{|\varphi(t)|}{t^{q-1}} \leq \left| \frac{\varphi(t)}{t^{q-1}} - \frac{x_N(t)}{t^{q-1}} \right| + \left| \frac{x_N(t)}{t^{q-1}} \right| < 1 + \frac{|x_N(t)|}{t^{q-1}}.$$

Hence

$$\sup_{0<t\leq T} \frac{|\varphi(t)|}{t^{q-1}} \leq 1 + \sup_{0<t\leq T} \frac{|x_N(t)|}{t^{q-1}} < \infty.$$

We now define a mapping P by $\phi \in X$ implies that

$$(P\phi)(t) := x^0 t^{q-1} + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, \phi(s))\, ds \tag{2.5}$$

and show that $P\colon X \to X$. The next theorem involves the integral

$$\mathcal{H}(t) := \int_0^t (t-s)^{n-1} \phi(s)\, ds. \tag{2.6}$$

It follows from classical theorems for Lebesgue integrals depending on a parameter (e.g. [2, Thm. 10.38]) that if the function ϕ is continuous on a closed interval $[0, T]$, then so is $\mathcal{H}$. Part of the proof of this result depends on ϕ being bounded on $[0, T]$. However, even if $\phi(s)$ has a singularity at $s = 0$, we still have the following lemma.

Lemma 2.1 *Let $n \in \Re^+$. If a function ϕ is continuous and absolutely integrable on $(0, T]$, then the integral $\mathcal{H}$ given by (2.6) defines a function that is also continuous and absolutely integrable on $(0, T]$.*

A proof of this lemma can be found in [4, Lemma 4.6]. It will be used twice in the proof of the following theorem. The transformation of Section 3 will rest heavily on it.

Theorem 2.3 *Let P be the mapping defined by (2.5).*

(i) *If $\phi \in X$, then $\int_0^t (t-s)^{q-1} \phi(s)\, ds \in X$.*

(ii) *If for each $\phi \in X$ a function $\psi_\phi \in X$ exists with*

$$|f(t, \phi(t))| \le \psi_\phi(t) \tag{2.7}$$

for all $0 < t \le T$, then $\int_0^t (t-s)^{q-1} f(s, \phi(s))\, ds \in X$.

(iii) *If (ii) holds and if $\phi \in X$, then $P\phi \in X$.*

Proof. According to the definition of the weighted space, we must show that the integral function in (i) is continuous and that

$$\sup_{0<t\le T} \frac{1}{g(t)} \left| \int_0^t (t-s)^{q-1} \phi(s)\, ds \right| < \infty,$$

where $g(t) = t^{q-1}$.

As for continuity, first notice that as $\phi \in X$ then

$$|\phi(t)| \le |\phi|_g\, t^{q-1}$$

for $0 < t \le T$. Hence, ϕ is absolutely integrable on $(0, T]$ since

$$\int_0^T |\phi(t)|\, dt \le |\phi|_g \int_0^T t^{q-1}\, dt = |\phi|_g \frac{T^q}{q} < \infty.$$

It then follows that $\int_0^t (t-s)^{q-1} \phi(s)\, ds$ is continuous on $(0, T]$ by Lemma 2.1. As for the second part of the proof of (i), we have

$$\frac{1}{g(t)}\left|\int_0^t (t-s)^{q-1}\phi(s)\,ds\right| \le \frac{1}{t^{q-1}}\int_0^t (t-s)^{q-1}|\phi(s)|\,ds$$
$$\le \frac{1}{t^{q-1}}\int_0^t (t-s)^{q-1}|\phi|_g\, s^{q-1}\,ds = t^{1-q}|\phi|_g\int_0^t (t-s)^{q-1}s^{q-1}\,ds$$

for $0 < t \le T$. With the change of variable $s = tv$, the integral becomes

$$\int_0^t (t-s)^{q-1}s^{q-1}\,ds = t^{2q-1}\int_0^1 v^{q-1}(1-v)^{q-1}\,dv.$$

Now it can be expressed in terms of the *Beta function*, namely, the function $B(p,q)$ that is defined by

$$B(p,q) := \int_0^1 v^{p-1}(1-v)^{q-1}\,dv \tag{2.8}$$

and which converges if and only if both p and q are positive. Hence,

$$\int_0^t (t-s)^{q-1}s^{q-1}\,ds = t^{2q-1}B(q,q) < \infty$$

since $B(q,q)$ converges as $q > 0$. Since the Beta function is related to the Gamma function (cf. [11, p. 200] or [18, p. 521]) by the equation

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)},$$

we obtain

$$\int_0^t (t-s)^{q-1}s^{q-1}\,ds = t^{2q-1}\frac{\Gamma^2(q)}{\Gamma(2q)}.$$

As a result, we have

$$\frac{1}{g(t)}\left|\int_0^t (t-s)^{q-1}\phi(s)\,ds\right| \le t^{1-q}\,|\phi|_g\,t^{2q-1}\frac{\Gamma^2(q)}{\Gamma(2q)} \le \frac{T^q\Gamma^2(q)}{\Gamma(2q)}\,|\phi|_g < \infty$$

for all $t \in (0,T]$. This concludes the proof of (i).

Let $\phi \in X$. Then, as a function $\psi_\phi \in X$ exists satisfying (2.7), we have

$$\int_0^T |f(t,\phi(t)|\,dt \le \int_0^T \psi_\phi(t)\,dt < \infty.$$

This allows us to invoke Lemma 2.1 again to conclude that the integral function in (ii) is continuous on $(0,T]$. Also, as $\psi_\phi \in X$, it follows from (i) that

$$\frac{1}{g(t)}\left|\int_0^t (t-s)^{q-1}f(s,\phi(s))\,ds\right|$$
$$\le \frac{1}{g(t)}\int_0^t (t-s)^{q-1}|f(s,\phi(s))|\,ds \le \frac{1}{g(t)}\int_0^t (t-s)^{q-1}\psi_\phi(s)\,ds$$
$$\le \sup_{0<t\le T}\frac{1}{g(t)}\left|\int_0^t (t-s)^{q-1}\psi_\phi(s)\,ds\right| < \infty$$

for all $t \in (0,T]$, which completes the proof of (ii).

Finally, it follows from (ii) and $x^0t^{q-1} \in X$ that all terms of P belong to X. Since X is a vector space, $P\phi \in X$.

Theorem 2.4 *Let* $f\colon (0,T] \times \Re \to \Re$ *be continuous. Suppose that a function* $x\colon (0,T_0] \to \Re$ *is a solution of*

$$x(t) = x^0 t^{q-1} + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s,x(s))\, ds \tag{2.2}$$

on $(0,T_0]$ *where* $T_0 \leq T$. *Then, for each* $\epsilon \in (0,|x^0|)$, *there is a* $T^* \leq T_0$ *so that*

$$(|x^0| - \epsilon)t^{q-1} < |x(t)| < (|x^0| + \epsilon)t^{q-1} < 2|x^0|t^{q-1} \tag{2.9}$$

for $0 < t \leq T^*$.

Proof. We have

$$t^{1-q}x(t) = x^0 + t^{1-q}\frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s,x(s))\, ds$$

and $t^{1-q}x(t)$ is continuous on the closed interval $[0,T_0]$ (cf. Def. 1.1). It follows that

$$t^{1-q} \int_0^t (t-s)^{q-1} f(s,x(s))\, ds$$

is continuous on $[0,T_0]$. Now

$$0 = \lim_{t\to 0^+} |t^{1-q}x(t) - x^0| = \frac{1}{\Gamma(q)} \lim_{t\to 0^+} \left| t^{1-q} \int_0^t (t-s)^{q-1} f(s,x(s))\, ds \right|.$$

For a given $\epsilon \in (0,|x^0|)$, there is a $T^* \in (0,T_0]$ such that $0 \leq t \leq T^*$ implies that

$$\frac{1}{\Gamma(q)} \left| t^{1-q} \int_0^t (t-s)^{q-1} f(s,x(s))\, ds \right| < \epsilon.$$

So, for $0 < t \leq T^*$, we have

$$\begin{aligned} &\big||x(t)| - |x^0|t^{q-1}\big| \\ &\qquad \leq \big|x(t) - x^0t^{q-1}\big| = \frac{1}{\Gamma(q)} \left| \int_0^t (t-s)^{q-1} f(s,x(s))\, ds \right| < \epsilon t^{q-1}. \end{aligned}$$

Using the first and last terms, we obtain

$$-\epsilon t^{q-1} < |x(t)| - |x^0|t^{q-1} < \epsilon t^{q-1}$$

so that

$$\big(|x^0| - \epsilon\big)t^{q-1} < |x(t)| < \big(|x^0| + \epsilon\big)t^{q-1} < 2|x^0|t^{q-1},$$

as required.

Corollary 2.1 *For the* T^* *of Theorem 2.4, the solution* $x(t)$ *has the sign of* x^0. *Moreover,* $x(t)$ *is absolutely integrable on* $(0,T^*]$.

Proof. For the given $\epsilon \in (0, |x^0|)$ and T^* in the proof of Theorem 2.4, we see that

$$|t^{1-q}x(t) - x^0| < \epsilon$$

or

$$(x^0 - \epsilon)t^{q-1} < x(t) < (x^0 + \epsilon)t^{q-1}$$

for $0 < t \le T^*$. And so if $x^0 > 0$, then $\epsilon < x^0$ and

$$x(t) > \frac{x^0 - \epsilon}{t^{1-q}} > 0$$

for $0 < t \le T^*$. If $x^0 < 0$, then $\epsilon < -x^0$ and

$$x(t) < \frac{x^0 + \epsilon}{t^{1-q}} < 0$$

for $0 < t \le T^*$.

Finally, as $|x(t)| < 2|x^0|t^{q-1}$ for $0 < t \le T^*$,

$$\int_0^{T^*} |x(s)|\, ds \le 2|x^0| \int_0^{T^*} s^{q-1}\, ds = \frac{2|x^0|}{q}(T^*)^q < \infty.$$

Corollary 2.2 *Let $f\colon [0,T] \times \Re \to \Re$ be continuous and satisfy condition (2.4). If $x(t)$ is a solution of (2.2) on the interval $(0, T^*]$ as in Theorem 2.4, then both $x(t)$ and $f(t, x(t))$ are absolutely integrable on $(0, T^*]$.*

Proof. It follows from (2.4) that

$$|f(t, x(t))| \le K_1 + K_2|x(t)|$$

for $0 \le t \le T^*$. We have already shown in Corollary 2.1 that $x(t)$ is absolutely integrable on $(0, T^*]$. Thus,

$$\int_0^{T^*} |f(t, x(t))|\, dt \le K_1 T^* + K_2 \int_0^{T^*} |x(t)|\, dt < \infty.$$

Applications.

(a_1) Theorem 2.4 tells us precisely where to look for a function $x(t)$ satisfying the integral equation (2.2). For a sufficiently small $T^* \in (0, T]$, it will lie in the set

$$M := \{\phi \in C(0, T^*] \mid |\phi(t)| \le 2|x^0|t^{q-1}\},$$

where $C(0, T^*]$ denotes the set of all continuous functions on $(0, T^*]$; and it will be sandwiched between two constant multiples of $x^0 t^{q-1}$, as in (2.9).

(a_2) In this paper we mainly consider the growth of $f(t, x)$, not its sign. However in situations where the sign becomes important, then it will be critical to replace M with the following set M^+. Suppose $x^0 > 0$. Then we see from Corollary 2.1 with $\epsilon = x^0/2$ and $T^* \in (0, T]$ sufficiently small that the solution $x(t)$ will reside in the set

$$M^+ := \{\phi \in C(0, T^*] \mid \tfrac{1}{2}x^0 < t^{1-q}\phi(t) < \tfrac{3}{2}x^0\}.$$

There is a parallel statement for $x^0 < 0$.

(b) A suitable space of functions for a fixed point mapping would be the Banach space $(X, |\cdot|_g)$ described in Definition 2.1.

(c) To find a solution of (2.2) we would contrive to define a mapping $P\colon M \to X$ by $\phi \in M$ implies that

$$(P\phi)(t) = x^0 t^{q-1} + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, \phi(s))\, ds$$

and a fixed point of P in M would satisfy (2.2). We would then examine $f(t,x)$ in the light of the sandwich inequality (2.9) to determine the range of values of q for which the remainder of the definition of solution would hold. The sandwich inequality tells us that if there is a solution it will lie very near $x^0 t^{q-1}$. For reasonable functions f, such as polynomials, we will be able to use that sandwich inequality information to tell precisely which values of q will generate a solution.

(d) The absolute integrability of the solution will be used in Theorem 2.1 to show that a solution of (2.2) is a solution of (2.1).

We now prepare to obtain a solution. Let X be the Banach space of continuous functions $\phi\colon (0,T] \to \Re$ satisfying Definition 2.1. Note that because of (2.4) the conditions of part (ii) in Theorem 2.3 are satisfied. As a result, $\phi \in X$ implies

$$\int_0^t (t-s)^{q-1} f(s, \phi(s))\, ds \in X. \tag{2.10}$$

For the given $x^0 \neq 0$ and some $T_0 \in (0,T]$ to be determined, define the set M as before by

$$M := \{\phi \in X \colon |\phi|_g \leq 2|x^0|\}. \tag{2.11}$$

Then for each $\phi \in M$,

$$|\phi(t)| \leq 2|x^0| t^{q-1}$$

for $0 < t \leq T_0$. For the set X, define the natural mapping P by $\phi \in X$ implies that

$$(P\phi)(t) := x^0 t^{q-1} + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, \phi(s))\, ds \tag{2.12}$$

for $0 < t \leq T$.

The following theorem can be proved by showing that P is a *contraction* on the set M: that is to say, $P\colon M \to M$ and a constant $\alpha \in (0,1)$ exists such that

$$\rho(Px, Py) \leq \alpha \rho(x,y) \tag{2.13}$$

for all $x, y \in M$, where $\rho(x,y) := |x-y|_g$ is the metric provided by the norm $|\cdot|_g$. Then *Banach's contraction mapping principle* asserts that P has a unique fixed point in M, i.e., a unique $\phi \in M$ such that $P\phi = \phi$. Since this theorem will turn out to be a special case of Theorem 2.7, the proof is omitted.

Theorem 2.5 *Let $f\colon [0,T] \times \Re \to \Re$ be continuous and satisfy the Lipschitz condition (2.3). Then, for each $q \in (0,1)$, there is a $T_0 \in (0,T]$ such that (2.2) has a unique continuous solution ϕ on $(0,T_0]$ with*

$$\lim_{t\to 0^+} t^{1-q} \int_0^t (t-s)^{q-1} f(s,\phi(s))\, ds = 0, \qquad \lim_{t \to 0^+} t^{1-q}\phi(t) = x^0. \tag{2.14}$$

Finally, both $\phi(t)$ and $f(t,\phi(t))$ are absolutely integrable.

Earlier we pointed out that the Volterra equation (2.2) has a singularity at $t = 0$ due to the forcing function, a singularity at the upper limit of integration t due to the kernel, and whatever singularity might arise from f. In the following example, f has an obvious singularity at $t = \pi^2/4$.

Example 2.1 The Volterra equation

$$x(t) = \frac{1}{\sqrt{t}} - \frac{1}{2}\int_0^t \frac{1}{\sqrt{t-s}} \cdot \frac{\mathfrak{J}_1(\sqrt{s})}{\cos(\sqrt{s}\,)}\, x(s)\, ds,$$

where $\mathfrak{J}_1(t)$ denotes the Bessel function of the first kind of order 1, has a unique continuous solution $\phi(t)$ on an interval $(0, T_0]$ for some value of $T_0 \in (0, \pi^2/4)$. It satisfies (2.14), where

$$f(t,x) = -\frac{\sqrt{\pi}\,\mathfrak{J}_1(\sqrt{t})}{2\cos(\sqrt{t}\,)}\, x,$$

and both $\phi(t)$ and $f(t, \phi(t))$ are absolutely integrable on $(0, T_0]$. Furthermore, $\phi(t)$ is also the unique continuous solution of the initial value problem

$$D^{1/2}x(t) = -\frac{\sqrt{\pi}\,\mathfrak{J}_1(\sqrt{t})}{2\cos(\sqrt{t}\,)}\, x(t), \qquad \lim_{t\to 0^+} \sqrt{t}\, x(t) = 1$$

on the interval $(0, T_0]$.

Proof. Comparing the Volterra equation with (2.2), we see that $x^0 = 1$, $q = 1/2$, and the function f is as given above. Since $\mathfrak{J}_1(z)$ is an entire function of z in the complex plane, $\mathfrak{J}_1(\sqrt{t})$ is continuous for all $t \geq 0$. Thus, for any fixed $T \in (0, \pi^2/4)$, the part of f depending only on t is continuous on the closed interval $[0, T]$. This implies there are positive constants K_1, K_2 such that (2.3) and (2.4) hold for $0 \leq t \leq T$ and all $x, y \in \Re$. As a result, all of the conclusions stated in the example, except for the very last one, follow from Theorem 2.5. The last one follows from Theorem 2.1.

Remark 2.1 In fact, the function

$$\phi(t) := \frac{\cos(\sqrt{t})}{\sqrt{t}}$$

is the unique continuous solution of the Volterra equation on all of $(0, \pi^2/4)$. To verify this, use the change of variable $\sqrt{s} = \sqrt{t}\sin\theta$. Then

$$\begin{aligned}\int_0^t \frac{1}{\sqrt{t-s}} \cdot \frac{\mathfrak{J}_1(\sqrt{s})}{\cos(\sqrt{s}\,)}\, \phi(s)\, ds &= \int_0^t \frac{1}{\sqrt{t-s}} \cdot \frac{\mathfrak{J}_1(\sqrt{s})}{\sqrt{s}}\, ds \\ &= 2\int_0^{\pi/2} \mathfrak{J}_1(\sqrt{t}\sin\theta)\, d\theta.\end{aligned}$$

From an integration formula in [20, p. 374], we see that

$$\sqrt{\frac{2z}{\pi}} \int_0^{\pi/2} \mathfrak{J}_1(z\sin\theta)\, d\theta = \mathbb{H}_{1/2}(z),$$

where $\mathbb{H}_{1/2}$ denotes Struve's function of order $\frac{1}{2}$. From [1, (12.1.16)], we have

$$\mathbb{H}_{1/2}(z) = \sqrt{\frac{2}{\pi z}}\,(1-\cos z).$$

Thus,

$$\int_0^{\pi/2} \mathfrak{J}_1(z\sin\theta)\,d\theta = \frac{1-\cos z}{z}.$$

Therefore, letting $x(t)=\phi(t)$, we obtain

$$\begin{aligned}\frac{1}{2}\int_0^t \frac{1}{\sqrt{t-s}}\cdot\frac{\mathfrak{J}_1(\sqrt{s})}{\cos(\sqrt{s}\,)}\,\phi(s)\,ds &= \int_0^{\pi/2} \mathfrak{J}_1(\sqrt{t}\sin\theta)\,d\theta \\ &= \frac{1-\cos\sqrt{t}}{\sqrt{t}} = \frac{1}{\sqrt{t}} - \phi(t)\end{aligned}$$

for $0 < t < \pi^2/4$.

In [4] we also verify directly that the function $\phi(t)$ is a solution of the fractional differential equation and its accompanying initial condition in Example 2.1.

The proof of Theorem 2.5 rests on the Lipschitz condition (2.3). Let us generalize this theorem by replacing the Lipschitz condition with a more general condition (cf. item (iii)) below. In addition, consider the modifications listed below in items (i)–(ii).

(i) For some $T > 0$ let $f\colon (0,T]\times\Re \to \Re$ be continuous.

(ii) Let $r_1 > -1$. Let $r_2 = m/n$, where m, n are positive integers with no common factors and n is odd, and $r_2 \geq 1$. (Note then that $x^{r_2} \in \Re$ for all $x \in \Re$.) Furthermore, let r_1, r_2 satisfy the inequality

$$\mu := 1 + r_1 + (q-1)r_2 > 0. \tag{2.15}$$

(iii) Let the function f satisfy the additional condition that a constant $K > 0$ exists such that

$$|f(t,x) - f(t,y)| \leq K t^{r_1} |x^{r_2} - y^{r_2}| \tag{2.16}$$

for $t \in (0,T]$ and for all $x, y \in \Re$.

Example 2.2 The function $f(t,x) = \sqrt[3]{tx^4}$ satisfies conditions (i), (ii), and (iii) with $K=1$, $r_1 = 1/3$, $r_2 = 4/3$ for any fixed $q \in (0,1)$. The function $f(t,x) = t^{-1/2}x^{4/3}$ satisfies conditions (i) and (iii) with $K=1$, $r_1 = -1/2$, $r_2 = 4/3$. As for (ii), $\mu > 0$ if $q \in (5/8, 1)$.

Before we generalize Theorem 2.5, we present a theorem that will aid in its proof and will be crucial for other results in Sections 3 and 4. The function G in part b) of the theorem is defined by

$$G(t,x) := -\left[x + \frac{f(t,x)}{J}\right],$$

where J is a positive constant.

Theorem 2.6 *Suppose that $f\colon (0,T]\times\Re\to\Re$ is continuous where*

$$|f(t,x)| \le |f(t,0)| + Kt^{r_1}|x|^{r_2}$$

with r_1, r_2 satisfying item (ii) containing (2.15), that $f(t,0)$ is absolutely integrable on $(0,T]$, and that $x(t)$ is a continuous solution of (2.2) on an interval $(0,T_0] \subset (0,T]$ satisfying $|x(t)| \le 2|x^0|t^{q-1}$. Then:

a) *There is a constant $\kappa > 0$ with*

$$\int_0^{T_0} [|x(s)| + |f(s,x(s))|]\,ds = \kappa.$$

b) *For each $t \in (0,T_0]$, there is a nonnegative $\mathfrak{D}(t) \in \Re$ with*

$$\int_0^t \int_0^s (t-s)^{q-1}(s-u)^{q-1}|G(u,x(u))|\,du\,ds = \mathfrak{D}(t).$$

Proof. We have

$$\begin{aligned}
&\int_0^{T_0} [|x(s)| + |f(s,x(s))|]\,ds \\
&\qquad \le \int_0^{T_0} [2|x^0|s^{q-1} + |f(s,0)| + Ks^{r_1}(2|x^0|s^{q-1})^{r_2}]\,ds \\
&\qquad = 2|x^0|\frac{{T_0}^q}{q} + \int_0^{T_0} |f(s,0)|\,ds + K(2|x^0|)^{r_2}\int_0^{T_0} s^{r_1}s^{r_2(q-1)}\,ds
\end{aligned}$$

which is finite because $r_1 + r_2(q-1) + 1 > 0$, completing the proof of part a).

The continuity of f and x implies that

$$\phi(s) := |G(s,x(s))|$$

is continuous on $(0,T_0]$ while part a) implies that it is absolutely integrable on this interval. Now, apply Lemma 2.1 to see that

$$\psi(s) := \int_0^s (s-u)^{q-1}\phi(u)\,du$$

is continuous and absolutely integrable. Hence

$$\int_0^t (t-s)^{q-1}\psi(s)\,ds$$

is continuous and absolutely integrable on $(0,T_0]$.

Note. If $f(t,x) = -Jx$ for a given $J > 0$, then $G \equiv 0$ and $\mathfrak{D}(t) = 0$ for $t \in (0,T_0]$. Then (2.2) simplifies to

$$x(t) = x^0t^{q-1} - \frac{J}{\Gamma(q)}\int_0^t (t-s)^{q-1}x(s)\,ds.$$

It is well-established that this linear equation has a unique continuous solution on the entire interval $(0,\infty)$, which can be expressed in terms of the resolvent function. For more details, see (3.3) and (3.4) in Section 3. For nonlinear equations, the focus of this paper, the function G is not identically zero and so $\mathfrak{D}(t)$ is positive.

Theorem 2.7 *Suppose conditions* (i)–(iii) *listed before Example 2.2 hold and that*

$$\lim_{t\to 0^+} t^{1-q}\int_0^t (t-s)^{q-1}|f(s,0)|\,ds = 0. \tag{2.17}$$

Then, for each $q \in (0,1)$ satisfying (2.15), a $T_0 \in (0,T]$ exists such that the integral equation (2.2) has a unique continuous solution ϕ on $(0,T_0]$. Furthermore, both $\phi(t)$ and $f(t,\phi(t))$ are absolutely integrable on $(0,T_0]$. Also, ϕ satisfies (2.14) and is the unique continuous solution of the initial value problem (2.1) on $(0,T_0]$.

Proof. We first show that (2.17) implies that $f(t,0)$ is absolutely integrable on $(0,T]$. Let $\epsilon = 1$. Then there exists a $\delta \in (0,T]$ such that

$$t^{1-q}\int_0^t (t-s)^{q-1}|f(s,0)|\,ds < 1$$

for $t \in (0,\delta)$. And so for $t \in (0,\delta)$, we have

$$\begin{aligned}0 \le \int_0^t |f(s,0)|ds &= \int_0^t (t-s)^{1-q}(t-s)^{q-1}|f(s,0)|\,ds\\ &\le \int_0^t t^{1-q}(t-s)^{q-1}|f(s,0)|\,ds \le 1.\end{aligned}$$

It follows that $f(s,0)$ is absolutely integrable on $(0,\delta)$. Thus $f(s,0)$ is absolutely integrable on $(0,T]$ because of the continuity of f.

Next consider the set M and the mapping P defined by (2.11) and (2.12), respectively. If $T_0 \in (0,T]$ is sufficiently small, we will show that the "generalized Lipschitz condition" (2.16) implies $P\colon M \to M$. First observe from (2.16) that

$$|f(t,x)| \le |f(t,0)| + Kt^{r_1}|x|^{r_2}$$

for $0 < t \le T$. It follows from this, the integrability of $f(t,0)$, and the proof of Theorem 2.6 a) that for every $\phi \in M$, $f(t,\phi(t))$ is absolutely integrable on $(0,T_0]$. Of course, ϕ being in M is continuous and absolutely integrable on this interval. The absolute integrability and continuity of $f(t,\phi(t))$ imply the integral term of $P\phi$ is continuous on $(0,T_0]$ by Lemma 2.1. Thus $P\phi$ itself is continuous on $(0,T_0]$.

Now we show $P\colon M \to M$. For any $\phi \in M$,

$$\begin{aligned}|(P\phi)(t)| &\le |x^0|t^{q-1} + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}|f(s,\phi(s))|\,ds\\ &\le |x^0|t^{q-1} + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\big[Ks^{r_1}|\phi(s)|^{r_2} + |f(s,0)|\big]\,ds\\ &\le |x^0|t^{q-1} + \frac{K}{\Gamma(q)}\int_0^t (t-s)^{q-1}s^{r_1}\big(2|x^0|s^{q-1}\big)^{r_2}\,ds\\ &\qquad + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}|f(s,0)|\,ds.\end{aligned}$$

Using the assumption that $r_1 + (q-1)r_2 = \mu - 1$ and the Beta function, we have

$$\begin{aligned}|(P\phi)(t)| &\leq |x^0|t^{q-1} + \frac{K\big(2|x^0|\big)^{r_2}}{\Gamma(q)} \int_0^t s^{\mu-1}(t-s)^{q-1}\,ds \\ &\qquad + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1}|f(s,0)|\,ds \\ &= |x^0|t^{q-1} + \frac{K\big(2|x^0|\big)^{r_2}\Gamma(\mu)}{\Gamma(\mu+q)}\, t^{\mu+q-1} \\ &\qquad + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1}|f(s,0)|\,ds.\end{aligned}$$

And so for $\phi \in M$,

$$\begin{aligned}|(P\phi)(t)| \leq \bigg\{|x^0| &+ \frac{K\Gamma(\mu)\big(2|x^0|\big)^{r_2}}{\Gamma(\mu+q)}\, t^{\mu} \\ &+ \frac{t^{1-q}}{\Gamma(q)} \int_0^t (t-s)^{q-1}|f(s,0)|\,ds\bigg\} t^{q-1}.\end{aligned}$$

Thus, as $\mu > 0$, $1-q > 0$, and because of (2.17),

$$|(P\phi)(t)| \leq 2|x^0|t^{q-1}$$

for $0 < t \leq T_0$, if T_0 is sufficiently small. For such a T_0, $PM \subset M$.

To prepare the way for showing that P is a contraction mapping in the weighted norm $|\cdot|_g$, first consider the difference $x^{r_2} - y^{r_2}$ for a given pair $x, y \in \Re$ and a given rational number $r_2 > 1$ satisfying the conditions listed in item (ii). It follows from the Mean Value Theorem that there exists a number ξ between x and y such that

$$|x^{r_2} - y^{r_2}| = r_2|\xi|^{r_2-1}|x-y|.$$

Since $r_2 > 1$, the function z^{r_2-1} is increasing on $[0,\infty)$. Consequently, as $|\xi| \leq \max\{|x|,|y|\}$,

$$|\xi|^{r_2-1} \leq \big(\max\{|x|,|y|\}\big)^{r_2-1}.$$

Thus, for $\phi, \psi \in M$,

$$\begin{aligned}\big|(\phi(t))^{r_2} - (\psi(t))^{r_2}\big| &\leq r_2\big(\max\{|\phi(t)|,|\psi(t)|\}\big)^{r_2-1}|\phi(t)-\psi(t)| \\ &\leq r_2\left(2|x^0|t^{q-1}\right)^{r_2-1}|\phi(t)-\psi(t)|\end{aligned}$$

for $0 < t \leq T_0$.

It follows from the previous inequality and (2.16) that

$$\begin{aligned}\frac{|(P\phi)(t)-(P\psi)(t)|}{t^{q-1}} &\leq \frac{t^{1-q}}{\Gamma(q)} \int_0^t (t-s)^{q-1}|f(s,\phi(s)) - f(s,\psi(s))|\,ds \\ &\leq \frac{t^{1-q}}{\Gamma(q)} \int_0^t (t-s)^{q-1}Ks^{r_1}\big|(\phi(s))^{r_2} - (\psi(s))^{r_2}\big|\,ds \\ &\leq \frac{Kt^{1-q}}{\Gamma(q)} \int_0^t (t-s)^{q-1}s^{r_1}r_2\left(2|x^0|s^{q-1}\right)^{r_2-1}|\phi(s)-\psi(s)|\,ds.\end{aligned}$$

Hence, because of the definition of the g-norm and (2.15),

$$\begin{aligned}
&\frac{|(P\phi)(t)-(P\psi)(t)|}{t^{q-1}} \\
&\le \frac{r_2Kt^{1-q}}{\Gamma(q)}\big(2|x^0|\big)^{r_2-1}\int_0^t (t-s)^{q-1}s^{r_1+(q-1)(r_2-1)}|\phi(s)-\psi(s)|\,ds \\
&\le \frac{r_2Kt^{1-q}}{\Gamma(q)}\big(2|x^0|\big)^{r_2-1}|\phi-\psi|_g\int_0^t (t-s)^{q-1}s^{r_1+(q-1)r_2}\,ds \\
&\le \frac{r_2Kt^{1-q}}{\Gamma(q)}\big(2|x^0|\big)^{r_2-1}|\phi-\psi|_g\int_0^t s^{\mu-1}(t-s)^{q-1}\,ds.
\end{aligned}$$

Evaluating the integral with the Beta function, we obtain

$$\begin{aligned}
\frac{|(P\phi)(t)-(P\psi)(t)|}{t^{q-1}} &\le \frac{r_2Kt^{1-q}}{\Gamma(q)}\big(2|x^0|\big)^{r_2-1}|\phi-\psi|_g\cdot t^{\mu+q-1}\frac{\Gamma(\mu)\Gamma(q)}{\Gamma(\mu+q)} \\
&= \left[\frac{r_2K\Gamma(\mu)}{\Gamma(\mu+q)}\big(2|x^0|\big)^{r_2-1}t^{\mu}\right]|\phi-\psi|_g.
\end{aligned}$$

Although this was derived for $r_2>1$, it is also true for $r_2=1$. Since $\mu>0$, the bracketed quantity is less than 1 for $t\in(0,T_0]$ if T_0 is small enough. We conclude a $T_0\in(0,T]$ exists such that $P\colon M\to M$ and P is a contraction on M. Therefore, by Banach's contraction mapping principle, there is a unique $\phi\in M$ such that $P\phi=\phi$.

Both the fixed point $\phi(t)$ and the function $f(t,\phi(t))$ are absolutely integrable on $(0,T_0]$ since this is true of all functions in M, as we saw earlier in the proof.

It follows from the bound we obtained for $P\phi$ that

$$\begin{aligned}
&\frac{t^{1-q}}{\Gamma(q)}\int_0^t (t-s)^{q-1}|f(s,\phi(s))|\,ds \\
&\qquad\le \frac{K\big(2|x^0|\big)^{r_2}\Gamma(\mu)}{\Gamma(\mu+q)}\,t^{\mu}+\frac{t^{1-q}}{\Gamma(q)}\int_0^t (t-s)^{q-1}|f(s,0)|\,ds,
\end{aligned}$$

where $\mu>0$. This along with (2.17) implies the first limit in (2.14). This in turn implies the second limit in (2.14) as

$$\lim_{t\to0^+} t^{1-q}\phi(t)=\lim_{t\to0^+} t^{1-q}(P\phi)(t)=x^0.$$

Finally, the fixed point ϕ fulfills all of the conditions of Theorem 2.1. Therefore it is also the unique continuous solution of (2.1) on $(0,T_0]$.

Remark 2.2 Two items worth noticing are:

(i) Consider the function $f(t,x)$ when $x=0$. The continuity of f in Theorem 2.5 implies that $f(t,0)$ is bounded on the closed interval $[0,T]$. Contrast this with Theorem 2.7 which no longer requires it to be defined at $t=0$ nor bounded as long as it satisfies (2.17).

(ii) Theorem 2.7 generalizes Theorem 2.5.

Item (ii) follows from observing that if condition (2.3) holds on a closed interval $[0, T]$, then condition (2.16) certainly holds on the half-open interval $(0, T]$ with $r_1 = 0$ and $r_2 = 1$. Also, condition (2.15) holds as

$$\mu = 1 + r_1 + (q-1)r_2 = 1 + 0 + (q-1)(1) = q > 0.$$

Furthermore, as $f(t, 0)$ is bounded on $[0, T]$, condition (2.17) holds.

Example 2.3 Part 1: the range of q. We now show that existence must take q into account. We will examine an assumed solution of (2.2) taking $f(t, x) = x^{2n+1}$ with n a positive integer and $x^0 > 0$. The work takes place in the context of Theorem 2.4 and (a_2) in the applications located just after Corollary 2.2. Thus, any solution $x \colon (0, T] \to \Re$ will be continuous, while $t^{1-q}x(t)$ will be continuous on the closed interval $[0, T]$ and for T small enough will satisfy

$$(1/2)x^0 t^{q-1} \le x(t) \le (3/2)x^0 t^{q-1}.$$

Here is the important part and it can be used to test many functions in the same way to determine permissible values of q. The function $f(t, x)$ is increasing in $x > 0$ so it preserves inequalities: for $x^0 > 0$ and for s small we have

$$\left[(1/2)x^0 s^{q-1}\right]^{2n+1} \le \left[x(s)\right]^{2n+1} \le \left[(3/2)x^0 s^{q-1}\right]^{2n+1}$$

which we write for convenience as

$$A(s) \le B(s) \le C(s).$$

Moreover,

$$Q(t) := t^{1-q} \int_0^t (t-s)^{q-1}(x(s))^{2n+1} ds$$

must be continuous on $[0, T]$ for some sufficiently small positive T; in particular, the limit as $t \downarrow 0$ of $Q(t)$ must exist. But notice that

$$t^{1-q} \int_0^t (t-s)^{q-1} A(s)\, ds \le t^{1-q} \int_0^t (t-s)^{q-1} B(s)\, ds \le t^{1-q} \int_0^t (t-s)^{q-1} C(s)\, ds.$$

However the end terms differ only by a multiplicative constant so if we can prove that the end terms both have limit zero as $t \downarrow 0$, then the middle term will have the same limit of zero. We will see that the end terms have a limit if and only if

$$q > \frac{2n}{2n+1} \tag{2.18}$$

and that limit is zero. If that fails to hold, then both of the end terms are unbounded. This means that if (2.18) fails then the middle term can not have a limit, while if (2.18) holds then the middle term has the same limit of zero.

Using the Beta function to compute the integral, we obtain

$$t^{1-q} \int_0^t (t-s)^{q-1} s^{q(2n+1)-2n-1}\, ds = K t^{q(2n+1)-2n}$$

for some $K > 0$. This gives the required convergence if and only if (2.18) holds.

Now let us return to Theorem 2.7 and condition (2.15). We have

$$\mu = 1 + 0 + (q-1)(2n+1) > 0$$

which is the same as (2.18). Note that $f(t,x) = x^{2n+1}$ trivially satisfies all of the other conditions of Theorem 2.7. We conclude a continuous solution of (2.2) exists on some interval $(0,T]$ if and only if

$$\frac{2n}{2n+1} < q < 1.$$

Moreover, one of the statements of Theorem 2.7 tells us that the solution $x(t)$ as well as $f(t,x(t))$ are absolutely integrable on $(0,T]$. As a result, we also conclude from Theorem 2.1 that $x(t)$ is also a continuous solution of (2.1) on $(0,T]$.

Part 2: the third singularity.

We readily see that (2.2) has a singularity in the forcing function and one in the kernel. But both of them are mild in a technical sense. However they coalesce as $t \downarrow 0$. From (2.9) we see that as $\epsilon \downarrow 0$ then $x(s)$ in the integrand of (2.2) gets as close to $x^0 s^{q-1}$ as we please on a sufficiently short interval $(0,T]$. For instance, with $f(t,x) = x^{2n+1}$, (2.2) is approximated arbitrarily well for very small t by

$$x(t) = x^0 t^{q-1} + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} (x^0 s^{q-1})^{2n+1}\, ds.$$

As $t \downarrow 0$ both terms in the integrand tend to infinity producing a product of singularities which is no longer mild in any technical sense.

3 A Transformation

Equation (2.2) and its solution on some short interval $[0,T)$ that is ensured by Theorem 2.7 hold many challenges if we wish to continue that solution beyond T. The forcing function is singular at $t = 0$ so the solution is singular there too and that introduces another singularity in the integrand besides the one already at $s = t$. In Example 2.3, Part 2 we discussed how this added singularity will coalesce with the singularity in the kernel producing a singularity of a radically different type than either that in the forcing function or in the kernel. And this added singularity cannot be avoided because it occurs as $t \downarrow 0$. This is also the situation in Example 2.1. Note however its integrand has even more singularities: those located at the zeroes of $\cos(\sqrt{t})$, which were avoided in that example by simply restricting the interval under consideration.

To make matters worse, consider the integral in the mapping P, calling it H for now, and suppose for the moment that f satisfies a global Lipschitz condition with constant $\alpha < 1$. If the functions $\phi_i : [0,\infty) \to \Re$ $(i = 1,2)$ are bounded and continuous with the supremum norm $\|\cdot\|$, then

$$H(t,x) := \int_0^t (t-s)^{q-1} f(s,x(s))\, ds$$

satisfies

$$
\begin{aligned}
|H(t,\phi_1(t)) - H(t,\phi_2(t)| &\le \int_0^t (t-s)^{q-1}\alpha|\phi_1(s)-\phi_2(s)|\,ds \\
&\le \alpha\|\phi_1-\phi_2\| \int_0^t s^{q-1}\,ds \\
&= \alpha\|\phi_1-\phi_2\| t^q/q.
\end{aligned}
$$

So for this situation H, as well as P, is not a contraction for $t \ge (q/\alpha)^{1/q}$. Moreover, if $f(t,x)$ contains a bounded additive function $u(t)$, then it transforms into $\int_0^t (t-s)^{q-1}u(s)\,ds$ passing from a bounded u to a possibly unbounded integral.

There is a simple way out of all these difficulties. In [5] we introduced a transformation for a fractional differential equation of Caputo type which has turned out to be very useful in the construction of fixed point mappings. The first part of it will now be given. It will take a second step to make it work for fractional differential equations of Riemann-Liouville type because of the singular forcing function.

Let J be an arbitrary positive constant and write (2.2) as

$$
\begin{aligned}
x(t) &= x^0 t^{q-1} + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}[-Jx(s) + Jx(s) + f(s,x(s))]\,ds \\
&= x^0 t^{q-1} - \frac{J}{\Gamma(q)}\int_0^t (t-s)^{q-1}x(s)\,ds \\
&\quad + \frac{J}{\Gamma(q)}\int_0^t (t-s)^{q-1}\left[x(s) + \frac{f(s,x(s))}{J}\right] ds,
\end{aligned}
$$

which we then rewrite as

$$
x(t) = x^0 t^{q-1} - \int_0^t C(t-s)[x(s) + G(s,x(s))]\,ds, \tag{3.1}
$$

where

$$
C(t) := \frac{Jt^{q-1}}{\Gamma(q)} \tag{3.2a}
$$

and

$$
G(s,x(s)) := -\left[x(s) + \frac{f(s,x(s))}{J}\right]. \tag{3.2b}
$$

Now view the linear equation

$$
z(t) = x^0 t^{q-1} - \int_0^t C(t-s)z(s)\,ds \tag{3.3}
$$

as the linear part of the nonlinear equation (3.1). Closely allied to both (3.1) and (3.3) is the *resolvent equation*

$$
R(t) = C(t) - \int_0^t C(t-s)R(s)\,ds. \tag{3.4}
$$

It is well-established that (3.4) has a unique continuous solution on $(0,\infty)$, which is known as the *resolvent* (cf. [3, Thm. 4.2]). Because of this uniqueness, it follows from multiplying both sides of (3.4) by $x^0\Gamma(q)/J$ that (3.3) also has a unique continuous solution on $(0,\infty)$, namely

$$z(t) = \frac{x^0\Gamma(q)}{J}R(t).$$

Substituting this for $z(s)$ in the integrand of (3.3) and using (3.2a), we obtain

$$\begin{aligned} z(t) &= x^0 t^{q-1} - \int_0^t C(t-s)\frac{x^0\Gamma(q)}{J}R(s)\,ds \\ &= x^0 t^{q-1} - \int_0^t \frac{J}{\Gamma(q)}(t-s)^{q-1}\frac{x^0\Gamma(q)}{J}R(s)\,ds \\ &= x^0 t^{q-1} - \int_0^t (t-s)^{q-1}x^0 R(s)\,ds. \end{aligned}$$

With an obvious change of variable, we can also write this as

$$z(t) = x^0 t^{q-1} - \int_0^t R(t-s)x^0 s^{q-1}\,ds. \tag{3.5}$$

Important properties of $R(t)$ (cf. [15, p. 212 f.]) that we rely on are

$$0 < R(t) \le C(t), \qquad \int_0^\infty R(s)\,ds = 1 \tag{3.6}$$

and the fact that $R(t)$ is completely monotone on $(0,\infty)$ ([15, p. 224]).

Suppose the conditions of an existence theorem, such as Theorem 2.5 or 2.7, are satisfied so that a solution $x(t)$ of (2.2), equivalently of (3.1), is known to exist on an interval $(0, T_0]$. In that case, a variation of parameters formula found in Miller [15, (1.4), p. 192] states that $x(t)$ will also satisfy the equation

$$x(t) = z(t) - \int_0^t R(t-s)G(s,x(s))\,ds \tag{3.7}$$

provided

$$\begin{aligned} &\int_0^t \int_0^s R(t-s)C(s-u)G(u,x(u))\,du\,ds \\ &\qquad = \int_0^t \int_u^t R(t-s)C(s-u)G(u,x(u))\,ds\,du \end{aligned} \tag{3.8}$$

for $0 < t \le T_0$. Note that this interchange in the order of integration is valid if the conditions of either Theorem 2.5 or Theorem 2.7 are satisfied since those conditions imply part b) of Theorem 2.6, which in turn implies (3.8) by the Hobson-Tonelli test (cf. [16, p. 93]). We further note a function satisfying (3.7) and (3.8) will also satisfy (2.2) since, as Miller [15, p. 192] points out, the steps from (3.1) to (3.7) are reversible.

The solution z of (3.5) will play a major role in the subsequent work and the following result offers its properties.

Lemma 3.1 *For each $\epsilon > 0$, the function z defined by (3.5) is bounded on $[\epsilon,\infty)$ and tends to zero as $t \to \infty$. Furthermore*

$$|z(t)| \le |x^0| t^{q-1}\left[1 - \int_0^t R(s)\,ds\right] \tag{3.9}$$

for all $t > 0$. The bounds on both z and $\int_0^t R(t-s)s^{q-1}\,ds$ are independent of the positive constant J. Moreover,

$$z(t) = \frac{x^0\Gamma(q)}{J}R(t).$$

Proof. From (3.4) we see that

$$\begin{aligned}
R(t) &= \frac{J}{\Gamma(q)}t^{q-1} - \frac{J}{\Gamma(q)}\int_0^t (t-s)^{q-1}R(s)\,ds \\
&= \frac{J}{\Gamma(q)}\left[t^{q-1} - \int_0^t R(t-s)s^{q-1}\,ds\right] \\
&= \frac{J}{x^0\Gamma(q)}z(t).
\end{aligned}$$

Consequently, for $t > 0$,

$$0 \le \int_0^t R(t-s)s^{q-1}\,ds \le t^{q-1} \tag{3.10}$$

as $R(t) > 0$ for $t > 0$ [15, p. 222]. Note this is independent of J. Hence (3.5) has the following limit:

$$\lim_{t\to\infty} z(t) = x^0\left[\lim_{t\to\infty} t^{q-1} - \lim_{t\to\infty}\int_0^t R(t-s)s^{q-1}ds\right] = 0.$$

This limit, along with the continuity of $z(t)$ on $(0,\infty)$, implies that $z(t)$ is bounded on $[\epsilon,\infty)$ for each $\epsilon > 0$.

By the previous inequality,

$$|z(t)| = |x^0|\left|t^{q-1} - \int_0^t R(t-s)s^{q-1}\,ds\right| = |x^0|\left(t^{q-1} - \int_0^t R(t-s)s^{q-1}\,ds\right).$$

As t^{q-1} is decreasing,

$$\int_0^t R(t-s)s^{q-1}\,ds \ge t^{q-1}\int_0^t R(t-s)\,ds.$$

Therefore,

$$|z(t)| \le |x^0|\left(t^{q-1} - t^{q-1}\int_0^t R(t-s)\,ds\right) = |x^0|t^{q-1}\left[1 - \int_0^t R(u)\,du\right]. \tag{3.11}$$

4 A Translation

We have one more step to take and it is a large one. Fix x^0 and let x be a solution of (2.2). Redefine the interval of definition and say that it is a solution on the interval $(0, 2T]$ satisfying Definition 1.1 as well as (3.7) and (3.8). In particular, we have

$$|x(t)| \le 2|x^0|t^{q-1}$$

on that interval. However, we must keep in mind that z still has a singularity. This section is devoted to showing that a translation with $y(t) = x(t+T)$ will transform (3.7) into

$$y(t) = F(t) + \int_0^t R(t-s)\left[y(s) + \frac{f(s+T, y(s))}{J}\right] ds,$$

where F is bounded, continuous, in $L^1[0,\infty)$, and converges to zero as $t \to \infty$. But most of all we want to remember (3.6).

The value of x^0 determines $z(t)$ and we know from Lemma 3.1 that $z(t)$ is bounded and continuous for $t \geq T$ and that $z(t) \to 0$ as $t \to \infty$. Now translate (3.7) as follows:

$$\begin{aligned} x(t+T) &= z(t+T) - \int_0^{t+T} R(t+T-s)G(s, x(s))\, ds \\ &= z(t+T) + \int_0^T R(t+T-s)\left[x(s) + \frac{f(s, x(s))}{J}\right] ds \\ &\quad + \int_T^{t+T} R(t+T-s)\left[x(s) + \frac{f(s, x(s))}{J}\right] ds \\ &= F(t) + \int_0^t R(t-s)\left[x(s+T) + \frac{f(s+T, x(s+T))}{J}\right] ds, \end{aligned} \tag{4.1}$$

where

$$F(t) := z(t+T) + \int_0^T R(t+T-s)\left[x(s) + \frac{f(s, x(s))}{J}\right] ds. \tag{4.2}$$

Next, let

$$y(t) := x(t+T)$$

and rewrite the translated equation (4.1) as

$$y(t) = F(t) + \int_0^t R(t-s)\left[y(s) + \frac{f(s+T, y(s))}{J}\right] ds, \tag{4.3}$$

where $y(0) = x(T)$. From this we see how to define an appropriate mapping for establishing solutions in the Banach space of bounded continuous functions on $[0,\infty)$ with the sup norm, which we denote by $(BC, \|\cdot\|)$. For a specified subset Q of this space, define the mapping $P\colon Q \to BC$ by $\phi \in Q$ implies

$$(P\phi)(t) := F(t) + \int_0^t R(t-s)\left[\phi(s) + \frac{f(s+T, \phi(s))}{J}\right] ds. \tag{4.4}$$

The last line of the following theorem need not be disquieting. If we ask that f satisfy (1.6), then we invoke Theorem 2.6 and find that $x(s)$ and $f(s, x(s))$ are absolutely integrable on $(0, T]$ so that (3.8) is assured by the Hobson-Tonelli theorem.

Theorem 4.1 *Let $q \in (0,1)$, $f\colon (0,\infty) \times \Re \to \Re$ be continuous, and $x^0 \in \Re$ with $x^0 \neq 0$. Let $x(t)$ be a solution of*

$$x(t) = x^0 t^{q-1} + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, x(s))\, ds \tag{2.2}$$

on an interval $(0, T]$. *For a given constant* $J > 0$, *let* $z(t)$ *denote the unique continuous solution of (3.3) on* $(0, \infty)$ *and let a function* $F\colon [0, \infty) \to \Re$ *be defined by (4.2). Lastly, let* $y(t)$ *be a solution of (4.3) on an interval* $[0, \tau]$ *for some* $\tau > 0$.

If the piecewise-defined function

$$x_c(t) := \begin{cases} x(t), & \text{if} \quad 0 < t \le T, \\ y(t-T), & \text{if} \quad T < t \le T + \tau, \end{cases} \tag{4.5}$$

satisfies (3.8) at each $t \in (0, T+\tau]$, *then it is a solution of (2.2) on* $(0, T+\tau]$.

Proof. Suppose $x_c(t)$ satisfies (3.8) at each $t \in (0, T+\tau]$. Then the solution $x(t)$ must satisfy (3.8) at each $t \in (0, T]$. Hence, by the variation of parameters formula $x(t)$ is also a solution of (3.7). Thus,

$$x_c(t) = z(t) - \int_0^t R(t-s)G(s, x_c(s))\, ds \qquad (0 < t \le T), \tag{4.6}$$

where from (3.2b)

$$G(s, x_c(s)) = -\left[x_c(s) + \frac{f(s, x_c(s))}{J}\right].$$

Since $y(t)$ is a solution of (4.3) for $0 \le t \le \tau$, $y(0) = F(0)$. Setting $t = 0$ in (4.2) and replacing $x(t)$ with $x_c(t)$, we get

$$y(0) = z(T) - \int_0^T R(T-s)G(s, x_c(s))\, ds.$$

Comparing this with (4.6) when $t = T$, we see $y(0) = x_c(T)$. And so

$$y(0) = x(T).$$

Thus, as x and y are continuous functions on their respective domains, the piecewise-defined function x_c is continuous on the interval $(0, T+\tau]$.

Since the function $y(t)$ satisfies (4.3) on the interval $[0, \tau]$, we have

$$\begin{aligned} y(t) &= F(t) - \int_0^t R(t-s)G(s+T, y(s))\, ds \\ &= z(t+T) - \int_0^T R(t+T-s)G(s, x(s))\, ds \\ &\qquad - \int_0^t R(t-s)G(s+T, y(s))\, ds. \end{aligned}$$

With the change of variable $u = s + T$, this becomes

$$\begin{aligned} y(t) = z(t+T) &- \int_0^T R(t+T-s)G(s, x(s))\, ds \\ &- \int_T^{t+T} R(t+T-u)G(u, y(u-T))\, du. \end{aligned}$$

Rewriting the right-hand side in terms of the function x_c, we have

$$y(t) = z(t+T) - \int_0^T R(t+T-s)G(s, x_c(s))\, ds$$
$$- \int_T^{t+T} R(t+T-u)G(u, x_c(u))\, du.$$

And so

$$y(t) = z(t+T) - \int_0^{t+T} R(t+T-s)G(s, x_c(s))\, ds$$

for $0 \le t \le \tau$. Or,

$$y(t-T) = z(t) - \int_0^{t} R(t-s)G(s, x_c(s))\, ds$$

for $T \le t \le T+\tau$. That is,

$$x_c(t) = z(t) - \int_0^{t} R(t-s)G(s, x_c(s))\, ds. \qquad (T \le t \le T+\tau)$$

This and (4.6) implies that the function $x_c(t)$ is a solution of the intermediate equation (3.7) on the interval $(0, T+\tau]$.

Finally since $x_c(t)$ satisfies (3.8) for $0 < t \le T+\tau$, we can invoke the variation of parameters result to conclude $x_c(t)$ is also a solution of the integral equation (2.2) on $(0, T+\tau]$.

Summary

Our stated goal was to transform (2.2) into a standard Volterra integral equation with a singularity only in the kernel which was to be completely monotone and have integral equal to one. That final equation is (4.3). In the next subsection we will develop the properties of the function F because F did not appear in (2.2). The solution of (2.2) will be that original solution on the short interval $(0, T]$ and then continued with the solution y of (4.3). Here are details which should guide the investigator.

Suppose that some existence theorem yields a solution of (2.2) on an interval $(0, T]$. That solution resides in

$$\text{A)} \quad M = \{\phi \in X : |\phi(t)| \le 2|x^0|t^{q-1}\}$$

provided that T is sufficiently small.

Regardless of which existence theorem we might use, suppose that we have assumed $f : (0, \infty) \times \Re \to \Re$ is continuous and satisfies

$$\text{B)} \quad |f(t,x)| \le |f(t,0)| + Kt^{r_1}|x|^{r_2}$$

and with the assumption that $f(t,0)$ is absolutely integrable on $(0, T]$. We need only note from A) and B) above that

$$\int_0^T s^{r_1}(s^{q-1})^{r_2}\, ds = \int_0^T s^{r_1 + r_2(q-1)}\, ds = kt^{r_1+r_2(q-1)+1}$$

for some $k > 0$ and our basic requirement for integrability is

$$\text{C)} \quad r_1 + r_2(q-1) + 1 > 0.$$

By Theorem 2.6 if A), B), and C) hold then

D) $|x| + |f(t,x)|$ is integrable on $(0,T]$

so Theorem 2.1 holds, as does (3.8) making (2.2), (3.7), and (4.3) equivalent in the sense of Theorem 4.1 as long as the solution extending $x(t)$ from $(0,T]$ to $(0,T+\tau]$ is continuous.

In conclusion, after verifying B), C), and any existence result, the investigator may go directly to (4.3) and begin the task of extracting properties of continuous solutions. Those properties are inherited by both (2.1) and (2.2).

4.1 Properties of the forcing function

Properties of the function F defined by (4.2) that will govern solutions of (4.3) are stated in the next theorem. We noted earlier that a solution of (2.2) lies in the set M defined in (2.11) so it is absolutely integrable. We gave conditions in Theorem 2.6 to ensure that a solution x will have $|x| + |f(t,x)|$ integrable. Moreover, when that holds so does conclusion b) of that theorem which is a sufficient condition for (3.8) to hold and, indeed, to assure us by Theorem 2.1 that the solution satisfies (2.1). That, in turn, was used together with a solution of (2.2) on a short interval to pass from (2.2) to (3.7) and then on to our final equation (4.3). This paragraph then is describing the fundamental position of item (iii) in the next theorem. Items (i) and (ii) are reminding us of Definition 1.1.

Theorem 4.2 *Let $f\colon (0,T_1] \times \Re \to \Re$ be continuous. Suppose there exists a $T \in (0, T_1/2]$ and a continuous function $x\colon (0,2T] \to \Re$ that is absolutely integrable and satisfies the equation*

$$\begin{aligned} x(t) &= z(t) - \int_0^t R(t-s)G(s,x(s))\,ds \qquad (3.7)\\ &= z(t) + \int_0^t R(t-s)\left[x(s) + \frac{f(s,x(s))}{J}\right] ds \end{aligned}$$

on $(0,2T]$. Suppose further that

(i) $t^{1-q}x(t)$ is continuous on $[0,2T]$,

(ii) $\lim_{t\to 0^+} t^{1-q}x(t) = x^0$,

(iii) $f(t,x(t))$ is absolutely integrable on $(0,2T]$.

Then the function $F\colon [0,\infty) \to \Re$ defined by (4.2), where J denotes an arbitrary positive constant, is uniformly continuous on $[0,\infty)$, tends to zero as $t \to \infty$, and is in $L^1[0,\infty)$. Moreover, a bound for F exists that is independent of the value of J.

The proof of this theorem is a consequence of the following three lemmas, namely Lemmas 4.1–4.3.

Lemma 4.1 *Under the conditions of Theorem 4.2, for any given constant $J > 0$, the function*

$$\begin{aligned} \mathsf{G}(t) &:= -\int_0^T R(t+T-s)G(s,x(s))\,ds \qquad (4.7)\\ &= \int_0^T R(t+T-s)\left[x(s) + \frac{f(s,x(s))}{J}\right] ds \end{aligned}$$

is uniformly continuous on $[\eta, \infty)$ *for any* $\eta > 0$.

Proof. Fix an arbitrary $J > 0$. Let $\epsilon > 0$. We see from (3.6) that $R(t)$ is continuous on $[\eta, \infty)$ and $R(t) \to 0$ as $t \to \infty$, which implies it is uniformly continuous on this interval. For a given $\lambda > 0$, there is a $\gamma > 0$ such that distinct $t_1, t_2 \in [\eta, \infty)$ and

$$|t_1 + T - s - t_2 - T + s| = |t_1 - t_2| < \gamma$$

and $T - s \geq 0$ imply that

$$|R(t_1 + T - s) - R(t_2 + T - s)| < \lambda.$$

Hence for these t_i, we have

$$\begin{aligned} |\mathsf{G}(t_1) - \mathsf{G}(t_2)| &\leq \int_0^T \big|R(t_1 + T - s) - R(t_2 + T - s)\big|\big|G(s, x(s))\big|\, ds \\ &\leq \lambda \int_0^T \big|G(s, x(s))\big|\, ds \\ &\leq \lambda \int_0^T \left[|x(s)| + \frac{|f(s, x(s))|}{J}\right] ds =: \lambda H_J, \end{aligned}$$

where H_J denotes the constant defined by the last integral. So choose $\lambda < \epsilon/H_J$. Therefore, for the given $\epsilon > 0$, there is a $\gamma > 0$ such that $|t_1 - t_2| < \gamma$ implies $|G(t_1) - G(t_2)| < \epsilon$.

The function F in the following two lemmas refers to the function defined by (4.2).

Lemma 4.2 *Under the conditions of Theorem 4.2, for any given constant* $J > 0$*, the function* $F(t)$ *is right-continuous at* $t = 0$*. Furthermore, it is uniformly continuous on* $[0, \infty)$.

Proof. By hypothesis, a continuous function $x(t)$ exists satisfying (3.7) on $(0, 2T]$. Recall earlier we defined y by

$$y(t) := x(t + T) \tag{4.8}$$

with the purpose of continuing the solution of (3.7) beyond $2T$. This then yielded equation (4.3), which we find convenient here to write as

$$y(t) = F(t) + L(t), \tag{4.9}$$

where

$$L(t) := \int_0^t R(t - s)\left[y(s) + \frac{f(s + T, y(s))}{J}\right] ds. \tag{4.10}$$

This suggests defining $L(0) = 0$. In so doing, we have from (4.9), (4.2), and (3.7) that

$$y(0) = F(0) = z(T) - \int_0^T R(T - s)G(s, x(s))\, ds = x(T).$$

Note this is consistent with (4.8). Thus let $L(0) := 0$.

Since x is continuous on $(0, 2T]$, we see from (4.8) that y is continuous on $(-T, T]$. So it follows from (4.9) that if we can show that $L(t)$ is right-continuous at $t = 0$, then the same will be true of F. It follows from the hypothesis that the function f is continuous on $(0, 2T] \times \Re$. Thus, as y is continuous on $[0, T]$, there is a constant C_J such that

$$\left| y(s) + \frac{f(s+T, y(s))}{J} \right| \le C_J$$

on $[0, T]$. This along with (3.6) implies

$$|L(t)| \le C_J \int_0^t R(t-s)\, ds \le \frac{C_J J}{\Gamma(q)} \int_0^t s^{q-1}\, ds \le \frac{C_J J t^q}{q\Gamma(q)}. \tag{4.11}$$

Hence $L(t) \to L(0)$ as $t \to 0^+$. As a result, F is right-continuous at $t = 0$.

As for uniform continuity, first observe from (4.2) and (4.7) that

$$F(t) = z(t+T) + \mathsf{G}(t). \tag{4.12}$$

By default, G is right-continuous at $t = 0$ since $z(t+T)$ is continuous on $[0, \infty)$. Because of this and Lemma 4.1, we see that G is continuous on $[0, \infty)$. This, together with the uniform continuity of G on $[\eta, \infty)$ for every $\eta > 0$, implies that G is uniformly continuous on $[0, \infty)$. It follows from Lemma 3.1 that $z(t+T) \to 0$ as $t \to \infty$. This and the continuity of $z(t+T)$ on $[0, \infty)$ imply that $z(t+T)$ is also uniformly continuous on $[0, \infty)$. Since the sum of uniformly continuous functions is uniformly continuous, we conclude F is uniformly continuous on $[0, \infty)$.

Finally note that the foregoing argument is valid for any given $J > 0$. This concludes the proof.

Lemma 4.3 *Under the conditions of Theorem 4.2, $F \in L^1[0, \infty)$, $F(t) \to 0$ as $t \to \infty$, and F is bounded on $[0, \infty)$. Moreover, there is a bound for F on $[0, \infty)$ that is independent of J.*

Proof. Let us start with the first term of $F(t)$, namely $z(t+T)$. We have already determined that $z(t+T) \to 0$ as $t \to \infty$. Now consider $\mathsf{G}(t)$, the other term of $F(t)$. Recall that R is completely monotone, so it is decreasing on $(0, \infty)$. Consequently,

$$\begin{aligned} |\mathsf{G}(\mathsf{t})| &\le \int_0^T R(t+T-s) \big| G(s, x(s)) \big|\, ds \\ &\le R(t) \int_0^T \left| x(s) + \frac{f(s, x(s))}{J} \right| ds \\ &\le R(t) \int_0^T \left[|x(s)| + \frac{|f(s, x(s))|}{J} \right] ds = K R(t), \end{aligned} \tag{4.13}$$

where K denotes the last integral, which has a finite value since both $x(s)$ and $f(s, x(s))$ are absolutely integrable on $(0, T]$. As $t \to \infty$, $\mathsf{G}(t) \to 0$ since $R(t) \to 0$. Because both terms of $F(t)$ tend to zero, so does $F(t)$.

It also follows from (4.13) that $\mathsf{G} \in L^1[0, \infty)$ because $R \in L^1[0, \infty)$. Moreover, $z(t+T) \in L^1[0, \infty)$ since $z(t)$ is proportional to $R(t)$ (cf. Lemma 3.1). Hence, $F \in L^1[0, \infty)$.

Recall from Lemma 4.2 that F is uniformly continuous on $[0, \infty)$. This together with $F(t) \to 0$ as $t \to \infty$ implies that F is bounded on $[0, \infty)$.

We now come to the final part of the proof, which is to show that a bound for F exists independent of J. Consider $z(t+T)$, the first term of $F(t)$. From (3.9) we see that

$$|z(t+T)| \leq |x^0|(t+T)^{q-1} \leq |x^0|T^{q-1} \tag{4.14}$$

for $t \geq 0$, yielding a bound for $z(t+T)$ on $[0,\infty)$ that depends on T but not on J. So what remains is to prove that the integral term $\mathsf{G}(t)$ in (4.12) also has a bound on $[0,\infty)$ independent of J.

Condition (i) of Theorem 4.2 implies the existence of a constant k such that $|x(t)| \leq kt^{q-1}$ for all $t \in (0, 2T]$. Then in view of (3.6), (3.2a), and the monotonicity of R, we have

$$\begin{aligned}
|\mathsf{G(t)}| &\leq \int_0^T R(t+T-s)\left[|x(s)| + \frac{|f(s,x(s))|}{J}\right] ds \\
&\leq \int_0^T R(t+T-s)ks^{q-1}\, ds \\
&\qquad + \int_0^T \frac{J}{\Gamma(q)}(t+T-s)^{q-1}\frac{|f(s,x(s))|}{J}\, ds \\
&\leq k\int_0^T R(T-s)s^{q-1}\, ds + \frac{1}{\Gamma(q)}\int_0^T (t+T-s)^{q-1}|f(s,x(s))|\, ds.
\end{aligned}$$

Applying (3.10), we obtain the bound

$$|\mathsf{G(t)}| \leq kT^{q-1} + \frac{1}{\Gamma(q)}\int_0^T (T-s)^{q-1}|f(s,x(s))|\, ds. \tag{4.15}$$

By hypothesis, $f(s,x(s))$ is absolutely integrable on $(0,T]$. So the integral in (4.15) converges according to Lemma 2.1. Because of this, (4.12), and (4.14), we have

$$|F(t)| \leq \big(|x^0| + k\big)T^{q-1} + \frac{1}{\Gamma(q)}\int_0^T (T-s)^{q-1}|f(s,x(s))|\, ds$$

for $t \geq 0$. Thus the right-hand side serves as a bound for F. Since it does not depend on J, the proof is complete.

The completion of the proof of this last lemma also completes the proof of Theorem 4.2.

5 Future Work

The objective was to reduce the fractional differential equation to a very common Volterra integral equation. Equation (4.3) now has three properties which make it ideal for fixed point theory.

First, the kernel $R(t-s)$ has two properties used extensively in fixed point theory. If Q is a convex set in the Banach space of bounded continuous functions with the supremum norm and if $f(s+T, y(s))$ is bounded for $y \in Q$, then

$$\int_0^t R(t-s)\left[y(s) + \frac{f(s+T,y(s))}{J}\right] ds$$

maps Q into an equicontinuous set [6, Thm. 5.1] all ready for numerous fixed point theorems of the Schauder type. Further work will actually give compactness of the mapping provided that Q is a ball in the Banach space [9]. On the other hand, if the function in large brackets defines a contraction, it will be preserved by that same integral.

Next, we have said nothing of J, but it serves a prime function, together with that extra $y(s)$ in the integrand. These work together to secure a self mapping set parallel to that seen in [6] concerning Caputo problems.

There are many directions we can take from here. Our next project involves offering an existence theorem based on the growth condition given here, but without any kind of contraction assumption. We then pick up (4.3) and obtain results on bounded solutions, solutions in $L^1[T,\infty)$, and solutions which are asymptotically periodic. While such results are of interest in themselves, they put us in a position to compare and contrast the behavior of solutions of Caputo equations having a Volterra representation parallel to (2.2) of the form

$$x(t) = x^0 + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, x(s))\, ds.$$

The difference in initial conditions is clear and that is a prime reason for considering them. The present work leading up to (4.3) reveals new differences which an investigator would like to take into account. For example, the function $z(t)$ discussed in Lemma 3.1 is in $L^1[T,\infty)$, but it is seen in [7] the corresponding $z(t)$ for the Caputo equation is in $L^p[0,\infty)$ if and only if $p > 1/q$. Other differences appear in the study of asymptotically periodic solutions in (4.3) compared to those for the Caputo equation as shown in [8].

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Mild Solution for Impulsive Neutral Integro-Differential Equation of Sobolev Type with Infinite Delay

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Abstract: In this work, we consider an impulsive neutral integro-differential equation of Sobolev type with infinite delay in an arbitrary Banach space X. The existence of mild solution is obtained by using resolvent operator and Hausdorff measure of noncompactness. We give an example based on the theory and provide the conclusion at the end of the paper.

Keywords: *resolvent operator; impulsive differential equation; neutral integro-differential equation; measure of noncompactness.*

Mathematics Subject Classification (2010): 34K37, 34K30, 35R11, 47N20.

1 Introduction

In our recent work [19], we have studied the impulsive neutral integro-differential equation with infinite delay in a Banach space $(X, \|\cdot\|)$,

$$\begin{aligned} \frac{d}{dt}[u(t) - F(t, u_t)] &= A[u(t) + \int_0^t f(t-s)u(s)ds] + G(t, u_t, \int_0^t \mathcal{E}(t, s, u_s)ds), \\ & \quad t \in J = [0, T_0],\ t \neq t_k,\ k = 1, 2, \cdots, m, \qquad (1) \\ u_0 &= \phi \in \mathfrak{B}, \qquad (2) \\ \Delta u(t_i) &= I_i(u_{t_i}),\ i = 1, 2, \cdots, m, \qquad (3) \end{aligned}$$

where $0 < T_0 < \infty$, A is a closed linear operator defined on a Banach space $(X; \|\cdot\|)$ with dense domain $D(A) \subset X$; $f(t), t \in [0, T_0]$ is a bounded linear operator. The functions $F : [0, T_0] \times \mathfrak{B} \to X$, $G : [0, T_0] \times \mathfrak{B} \times X \to X$,

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$\mathcal{E} : [0,T_0] \times [0,T_0] \times \mathfrak{B} \to X,\ I_i : X \to X,\ i = 1,\cdots,m$ are appropriate functions and $0 < t_1 < t_2 < \cdots < t_m < t_{m+1} = T_0$ are pre-fixed numbers. The symbol $\Delta u(t) = u(t^+) - u(t^-)$ denotes the jump of the function u at t i.e., $u(t^-)$ and $u(t^+)$ denotes the end limits of the $u(t)$ at t. The history $u_t : (-\infty, 0] \to X$ is a continuous function defined as $u_t(s) = u(t+s), s \leq 0$ belongs to the abstract phase space $\mathfrak{B}$ and $\mathfrak{B}$ is the phase space defined axiomatically later in Section 2. We have established the existence results by using Hausdorff measure of noncompactness and Darbo fixed point theorem with the assumption that A generates an analytic resolvent operator and G satisfies the Caratheòdary condition.

In [20], the authors have discussed the regularity of solutions of the semilinear integro-differential equations of Sobolev type in Banach space which is illustrated as

$$\frac{d}{dt}[Ey(t)] = A[y(t) + \int_0^t f(t-s)y(s)ds] + F(t, y(t)), \tag{4}$$

$$y(0) = y_0, \qquad t \in [0, T_0],\ \ 0 < T_0 < \infty, \tag{5}$$

where E and A are considered as closed linear operators such that the domains contained in Banach space X and ranges contained in Banach space Y, $f(t), t \in [0, T_0]$ is a bounded linear operator such that Y is continuously and densely embedded in X. The nonlinear function $F : [0, T_0] \times X \to Y$ is a continuous function. The authors have obtained the results by using Banach fixed point theorem and resolvent operator.

As in the above mentioned work, our aim in this paper is to investigate the existence of mild solution of the following impulsive Sobolev type neutral integro-differential equation with infinite delay in a Banach space $(X, \|\cdot\|)$,

$$\frac{d}{dt}[Ey(t) + F(t, y_t, \int_0^t a(t,s,y_s)ds)] = A[y(t) + \int_0^t f(t-s)y(s)ds]$$
$$+G(t, u_t, \int_0^t \mathcal{E}(t,s,u_s)ds),\ t \in J = [0, T_0],\ t \neq t_i, \tag{6}$$

$$u_0 = \phi \in \mathfrak{B}, \tag{7}$$

$$\Delta u(t_i) = I_i(u_{t_i}),\ i = 1, 2, \cdots, m, \tag{8}$$

where E and A are the same operators as defined in equation (4). The functions $F : [0,T_0] \times \mathfrak{B} \times X \to Y,\ G : [0,T_0] \times \mathfrak{B} \times X \to Y,\ \mathcal{E} : [0,T_0] \times [0,T_0] \times \mathfrak{B} \to X,$ $I_i : X \to X,\ i = 1, \cdots, m$ are appropriate functions satisfying some suitable conditions to be mentioned in Section 3.

Recently, impulsive differential equations have been rising as an important area of study due to their wide applicability in sciences and engineering such as physics, control theory, biology, population dynamics, medical domain and many others, and hence they have earned considerable attention of researchers. The process or phenomena subject to short-term external influences can be modeled by the impulsive differential equations which allow for discontinuities in the evolution of the state. For more study of such differential equations and their applications, we refer to the monographs [12], [24] and papers. Moreover, Sobolev type semilinear integrodifferential equation can be used to describe the flow of fluid through fissured rocks [2], thermodynamics and shear in second order fluids and many others. For wide study of Sobolev type differential equation, we refer to papers [20] – [23]. A lot of natural phenomena emerging from numerous areas, for example, fluid dynamics, electronics and kinetics, can be modeled in the form of the integro-differential equation. Integro-differential equation of neutral type with delay describe the system of rigid heat conduction with finite wave spaces.

The organization of the paper is as follows: Section 2 provides some basic facts, lemmas and theorems which will be used for establishing the result. Section 3 focuses on the existence of a mild solution by means of Hausdorff measure of noncompactness and analytic semigroup. Section 4 provides an example based on the obtained abstract theory. The last section of the paper is devoted to providing conclusion.

2 Preliminaries and Assumptions

In this section, we provide some fundamental definition, lemmas and theorems which will be utilized all around this paper.

Let X be a Banach space. The symbol $C([a,b];X)$, $(a,b\in\mathbb{R})$ stands for the Banach space of all the continuous functions from $[a,b]$ into X equipped with the norm $\| z(t)\|_C = \sup_{t\in[a,b]}\| z(t)\|_X$ and $L^p((a,b);X)$ stands for Banach space of all Bochner-measurable functions from (a,b) to X with the norm

$$\| z\|_{L^p} = (\int_{(a,b)} \| z(s)\|_X^p ds)^{1/p}.$$

For the differential equation with infinite delay, Kato and Hale [9] have proposed the phase space $\mathfrak{B}$ satisfying certain fundamental axioms.

Definition 2.1 The linear space of all functions from $(-\infty,0]$ into Banach space X with a seminorm $\|\cdot\|_{\mathfrak{B}}$ is known as phase space $\mathfrak{B}$. The fundamental axioms on $\mathfrak{B}$ are the following:

(A) If $y:(-\infty,d+T_0]\to X$, $T_0>0$ is a continuous function on $[d,d+T_0]$ such that $y_d\in\mathfrak{B}$ and $y|_{[d,d+T_0]}\in\mathfrak{B}\in\mathcal{PC}([d,d+T_0];X)$, then for every $t\in[d,d+T_0)$, the following conditions hold:

(i) $y_t\in\mathfrak{B}$,

(ii) $H\| y_t\|_{\mathfrak{B}}\geq \| y(t)\|$,

(iii) $\| y_t\|_{\mathfrak{B}}\leq N(t+d)\| y_d\|_{\mathfrak{B}}+K(t-d)\sup\{\| y(s)\| : d\leq s\leq t\}$,

where H is a positive constant; $N,\ K:[0,\infty)\to[1,\infty)$, N is locally bounded, K is continuous and $K,\ H,\ N$ are independent of $y(\cdot)$.

(A1) For the function y in $(A1)$, y_t is a $\mathfrak{B}$-valued continuous function for $t\in[d,d+T_0]$.

(B) The space $\mathfrak{B}$ is complete.

Consider the following integro-differential equation

$$\frac{d}{dt}[Ey(t)] \quad = \quad A[y(t)+\int_0^t f(t-s)y(s)ds]. \tag{9}$$

To prove the result, we impose the following data on operators A and E. The following conditions are fulfilled by operators $A:D(A)\subset X\to Y$ and $E:D(E)\subset X\to Y$:
$(E1)$ A and E are closed linear operators,

$(E2)$ $D(E) \subset D(A)$ and E is bijective,
$(E3)$ $E^{-1} : Y \to D(E)$ is continuous operator and $E^{-1}B = BE^{-1}$,
$(E4)$ $AE^{-1} : Y \to Y$ is the infinitesimal generator of uniformly continuous semigroup of bounded linear operators in X.

To set the structure for our primary existence results, we have to introduce the following definitions.

Definition 2.2 A family $\{R(t)\}_{t\in[0,T_0]}$ of bounded linear operators is said to be a resolvent operator for equation (9) if the following conditions are satisfied

(i) $R(0) = I$, where I is the identity operator on X.

(ii) $R(t)$ is strongly continuous for $t \in [0, T_0]$.

(iii) $R(t) \in B(Z),\ t \in [0, T_0]$. For $z \in Z$ and $R(\cdot)z \in C([0, T_0]; Z) \cap C^1([0, T_0]; Z)$, we have

$$\frac{d}{dt}R(t)z = AE^{-1}[R(t)z + \int_0^t f(t-s)R(s)zds], \tag{10}$$

$$= R(t)AE^{-1}z + \int_0^t R(t-s)AE^{-1}f(s)zds, \quad t \in [0, T_0]. \tag{11}$$

Here $B(Z)$ denotes the space of bounded linear operators defined on Z and Z is a Banach space formed from $D(A)$ with the graph norm.

Throughout the work, the resolvent operator $\{R(t)\}_{t\geq 0}$ is assumed to be analytic in Banach space X and there exist positive constants N_1 and N_2 such that $\| R(t)\| \leq N_1$ and $\|f(t)\| \leq N_2$ for each $t \in [0, T_0]$.

To consider the mild solution for the impulsive problem, we propose the set $\mathcal{PC}([0, T_0]; X) = \{y : [0, T_0] \to X : y$ is continuous at $t \neq t_i$ and left continuous at $t = t_i$ and $y(t_i^+)$ exists, for all $i = 1, \cdots, m\}$. Clearly, $\mathcal{PC}([0, T_0]; X)$ is a Banach space endowed with the norm $\| u\|_{\mathcal{PC}} = \sup_{t\in[0,T_0]} \| u(s)\|$. For a function $y \in \mathcal{PC}([0, T_0]; X)$ and $i \in \{0, 1, \cdots, m\}$, we define the function $\widetilde{y_i} \in C([t_i, t_{i+1}], X)$ such that

$$\widetilde{y_i}(t) = \begin{cases} y(t), & \text{for } t \in (t_i, t_{i+1}], \\ y(t_i^+), & \text{for } t = t_i. \end{cases} \tag{12}$$

For $W \subset \mathcal{PC}([0, T_0]; X)$ and $i \in \{0, 1, \cdots, m\}$, we have $\widetilde{W_i} = \{\widetilde{y_i} : y \in W\}$ and the following Accoli-Arzelà type criteria.

Lemma 2.1 *[7]. A set $W \subset \mathcal{PC}([0, T_0]; X)$ is relatively compact if and only if each set $\widetilde{W_i} \subset C([t_i, t_{i+1}], X)$ $(i = 0, 1 \cdots, m)$ is relatively compact.*

Now, we discuss some basic definition of measure of noncompactness (MNC).

Definition 2.3 [10] The Hausdorff's measure of noncompactness (H'MNC) χ_Y is defined as

$$\chi_Y(U) = \inf\{\varepsilon > 0 : U \text{ can be covered by a finite number of balls with radius } \varepsilon\}, \tag{13}$$

for the bounded set $U \subset Y$, where Y is a Banach space.

Lemma 2.2 *[10] For any bounded set U, $V \subset Y$, where Y is a Banach space. Then the following conditions are fulfilled:*

(i) $\chi_Y(U) = 0$ *if and only if U is pre-compact;*

(ii) $\chi_Y(U) = \chi_Y(conv\ U) = \chi_Y(\overline{U})$, *where conv U and $\overline{U}$ denote the convex hull and closure of U respectively;*

(iii) $\chi_Y(U) \subset \chi_Y(V)$, *when* $U \subset V$;

(iv) $\chi_Y(U + V) \leq \chi_Y(U) + \chi_Y(V)$, *where* $U + V = \{u + v : u \in U,\ v \in V\}$;

(v) $\chi_Y(U \cup V) \leq \max\{\chi_Y(U), \chi_Y(V)\}$;

(vi) $\chi_Y(\lambda U) = \lambda \cdot \chi_Y(U)$, *for any* $\lambda \in \mathbb{R}$;

(vii) If the map $P : D(P) \subset Y \to \mathcal{Z}$ is continuous and satisfy the Lipschitsz condition with constant κ, then we have that $\chi_{\mathcal{Z}}(PU) \leq \kappa\chi_Y(U)$ for any bounded subset $U \subset D(P)$, where Y and $\mathcal{Z}$ are Banach spaces.

Definition 2.4 [10] A bounded and continuous map $P : \mathcal{D} \subset Z \to Z$ is a χ_Z-contraction if there exists a constant $0 < \kappa < 1$ such that $\chi_Z(P(U)) \leq \kappa\chi_Z(U)$, for any bounded closed subset $U \subset \mathcal{D}$, where Z is a Banach space.

Lemma 2.3 *[16] Let $\mathcal{D} \subset Z$ be closed, convex with $0 \in \mathcal{D}$ and the continuous map $P : \mathcal{D} \to \mathcal{D}$ be a χ_Z-contraction. If the set $\{u \in \mathcal{D} : u = \lambda Pu,\ for\ \ 0 < \lambda < 1\}$ is bounded, then the map P has a fixed point in $\mathcal{D}$.*

Lemma 2.4 *(Darbo-Sadovskii) [10]. Let $\mathcal{D} \subset Z$ be bounded, closed and convex. If the continuous map $P : \mathcal{D} \to \mathcal{D}$ is a χ_Z-contraction, then the map P has a fixed point in $\mathcal{D}$.*

In this paper, we consider that χ denotes the Hausdorff's measure of noncompactness (H'MNC)in X, χ_C denotes the Hausdorff's measure of noncompactness in $C([0, T_0]; X)$ and $\chi_{\mathcal{PC}}$ denotes the Hausdorff's measure of noncompactness in $\mathcal{PC}([0, T_0]; X)$.

Lemma 2.5 *([10]. If U is bounded subset of $C([0, T_0]; X)$, then we have that $\chi(U(t)) \leq \chi_C(U)$, $\forall\ \ t \in [0, T_0]$, where $U(t) = \{u(t); u \in U\} \subseteq X$. Furthermore, if U is equicontinuous on $[0, T_0]$, then $\chi(U(t))$ is continuous on the interval $[0, T_0]$ and*

$$\chi_C(U) = \sup_{t \in [0,T_0]} \{\chi(U(t))\}. \tag{14}$$

Lemma 2.6 *[10] If $U \subset C([0, T_0]; X)$ is bounded and equicontinuous, then $\chi(U(t))$ is continuous and*

$$\chi(\int_0^t U(s)ds) \leq \int_0^t \chi(U(s))ds,\ \forall\ \ t \in [0, T_0], \tag{15}$$

where $\int_0^t U(s)ds = \{\int_0^t u(s)ds, u \in U\}$.

Lemma 2.7 *[14]*

(1) If $U \subset \mathcal{PC}([0,T_0];X)$ is bounded, then $\chi(U(t)) \leq \chi_{\mathcal{PC}}(U)$, $\forall\ \ t \in [0,T_0]$, where $U(t) = \{u(t) : u \in U\} \subset X$;

(2) If U is piecewise equicontinuous on $[0,T_0]$, then $\chi(U(t))$ is piecewise continuous for $t \in [0,T_0]$ and

$$\chi_{\mathcal{PC}}(U) = \sup\{\chi(U(t)) : t \in [0,T_0]\}; \tag{16}$$

(3) If $U \subset \mathcal{PC}([0,T_0];X)$ is bounded and equicontinuous, then $\chi(U(t))$ is piecewise continuous for $t \in [0,T_0]$ and

$$\chi(\int_0^t U(s)ds) \leq \int_0^t \chi(U(s))ds, \forall\ \ t \in [0,T_0], \tag{17}$$

where $\int_0^t U(s)ds = \{\int_0^t u(s)ds : u \in U\}$.

Now, we present the definition of mild solution for the system (6)-(8).

Definition 2.5 A piecewise continuous function $y : [-\infty, T_0]$ is said to be a mild solution for the system (6)-(8) if $y_0 = \phi$, $y(\cdot)|_{[0,T_0]} \in \mathcal{PC}$ and the following integral equation

$$\begin{aligned} y(t) \quad = \quad & E^{-1}R(t)E\phi(0) + E^{-1}R(t)F(0,\phi,0) - E^{-1}F(t,y_t,\int_0^t a(t,s,y_s)ds) \\ & -E^{-1}\int_0^t R(t-s)AE^{-1}F(s,y_s,\int_0^s a(s,\tau,y_\tau)d\tau)ds \\ & -E^{-1}\int_0^t R(t-s)AE^{-1}\int_0^s f(s-\tau)F(\tau,y_\tau,\int_0^\tau a(\tau,\xi,y_\xi)d\xi)d\tau ds \\ & +E^{-1}\int_0^t R(t-s)G(s,y_s,\int_0^s \mathcal{E}(s,\tau,y_\tau)d\tau)ds \\ & + \sum_{0<t_i<t} E^{-1}R(t-t_i)I_i(y_{t_i}), \quad t \in [0,T_0], \end{aligned} \tag{18}$$

is verified.

3 Main Results

We assume the following conditions which will be required to establish the result.

(E5) The function $F : [0,T_0] \times \mathfrak{B} \times X \to X$ is a continuous function and there exist positive constants L_{F_1} and L_{F_2} such that

$$\begin{aligned} \|F(t_1,w_1,z_1) - F(t_2,w_2,z_2)\| &\leq L_{F_1}[|t_1-t_2| + \|w_1-w_2\|_{\mathfrak{B}} + \|z_1-z_2\|_X], \\ \|AF(t,w_1,z_1) - AF(t,w_2,z_2)\| &\leq L_{F_2}[\|w_1-w_2\|_{\mathfrak{B}} + \|z_1-z_2\|_X], \end{aligned} \tag{19}$$

for all $t_1, t_2, t \in [0,T_0]$, $w_1, w_2 \in \mathfrak{B}$ and $z_1, z_2 \in X$ with $L_1 = \sup_{t\in[0,T_0]} \|F(t,0,0)\|$, $L_2 = \sup_{t\in[0,T_0]} \|AF(t,0,0)\|$.

(E6) (1). The function $a(t,s,\cdot) : \mathfrak{B} \to X$ is continuous for each $(t,s) \in [0,T_0] \times [0,T_0]$ and $a(\cdot,\cdot,w)$, $\mathcal{E}(\cdot,\cdot,w) : [0,T_0] \times [0,T_0] \to X$ are strongly measurable for all $w \in \mathfrak{B}$.

The function $a : J \times J \times \mathfrak{B} \to X$ is a continuous function and there exists constant $a_1 > 0$ such that

$$\| \int_0^t [a(t,s,w) - a(t,s,z)]ds\| \quad \leq \quad a_1 \|w - z\|_{\mathfrak{B}}, \tag{20}$$

for each $(t, s) \in J \times J$ and $z, w \in \mathfrak{B}$.
(2). There exist functions $m_a, m_{\mathcal{E}} : [0, T_0] \times [0, T_0] \to [0, +\infty)$ such that $m_a, m_{\mathcal{E}}$ are differentiable, a.e., with respect to the first variable and $\int_0^t m_a(t,s)ds$, $\int_0^t m_{\mathcal{E}}(t,s)ds$, $\int_0^t \frac{\partial, m_a(t,s)[\text{or } m_{\mathcal{E}}(t,s)]}{\partial t}ds$ are bounded on $[0, T_0]$ and $\frac{\partial m_{\mathcal{E}}}{\partial t} \geq 0$, for a.e., $0 \leq s < t \leq T_0$ such that

$$\begin{aligned} \|a(t,s,w)\| &\leq m_a(t,s)W_a(\|w\|_{\mathfrak{B}}), \\ \|\mathcal{E}(t,s,w)\| &\leq m_{\mathcal{E}}(t,s)W_{\mathcal{E}}(\|w\|_{\mathfrak{B}}), \end{aligned} \tag{21}$$

for each $0 \leq s < t \leq T_0,\ w \in \mathfrak{B}$ and $W_a, W_{\mathcal{E}} : [0, \infty) \to (0, \infty)$ are continuous nondecreasing functions.

(E7) $G : [0, T_0] \times \mathfrak{B} \times X \to X$ is a nonlinear function such that
(1) For each $y : (-\infty, T_0] \to X,\ y_0 = \phi \in \mathfrak{B},\ G(t, \cdot, \cdot)$ is continuous a.e. for $t \in [0, T_0]$ and function $t \mapsto G(t, y_t, \int_0^t \mathcal{E}(t, s, y_s)ds)$ is strongly measurable for $y \in \mathcal{PC}([0, T_0]; X)$.
(2) There are integrable functions $\alpha, \beta : J \to [0, \infty)$ and continuously differentiable increasing functions $\Omega,\ \mathfrak{W} : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\| G(\tau, w, z)\| \leq \alpha(\tau)\Omega(\| w\|_{\mathfrak{B}}) + \beta(\tau)\mathfrak{W}(\| z\|),\ \tau \in [0, T_0],\ (w, z) \in \mathfrak{B} \times X. \tag{22}$$

(3) There is an integrable function $\xi : J \to [0, \infty)$ such that for any bounded subsets $H_1 \subset \mathcal{PC}((-\infty, 0]; X)$, $H_2 \subset X$, we have that

$$\chi(R(\tau)G(\tau, H_1, H_2)) \leq \xi(\tau)\{\sup_{-\infty \leq \theta \leq 0} \chi(H_1(\theta)) + \chi(H_2)\}, \tag{23}$$

a.e. for $t \in [0, T_0]$. Where $H_1(\theta) = \{u(\theta) : u \in H_1\}$.

(E8) (1) The functions $I_i : \mathfrak{B} \to X,\ i = 1, 2, \cdots, m$ are continuous and there are constants $L_i > 0$ $(i = 1, 2, \cdots, m)$ such that

$$\| I_i(x) - I_i(y)\| \leq L_i\| x - y\|_{\mathfrak{B}},\ \forall\, x, y \in \mathfrak{B}. \tag{24}$$

(2) There exist positive constants K_i^1 and K_i^2,$(i = 1, \cdots, m)$ such that

$$\| I_i(x)\| = K_i^1\| x\|_{\mathfrak{B}} + K_i^2,\ x \in \mathfrak{B}. \tag{25}$$

(E9)

$$\int_0^{T_0} b(s)ds \leq \int_e^{+\infty} [W_a(\vartheta) + \Omega(\vartheta) + \frac{W_{\mathcal{E}}(\vartheta)}{\Omega'(\vartheta)}\mathfrak{W}'(LW_{\mathcal{E}}(\vartheta))]^{-1}ds, \tag{26}$$

where

$$
\begin{aligned}
b_1(t) &= \frac{1}{1-\mathcal{C}_2}[(N_{T_0}\Lambda L_{F_1} + \Lambda^2 N_1 T_0 L_{F_2} + \Lambda^2 N_2 N_1 T_0^2 L_{F_2})(m_a(t,t) \\
&\quad + \int_0^t \frac{\partial m_a(t,s)}{\partial t} ds)], \\
b_2(t) &= \frac{N_{T_0}\Lambda N_1 p(t)}{1-\mathcal{C}_2}, \ \ b_3(t) = m_{\mathcal{E}}(t,t) + \int_0^t \|\frac{\partial m_{\mathcal{E}}(t,s)}{\partial t}\| ds, \\
p(t) &= \max\{\alpha(t),\beta(t)\} \ \ b(t) = \max\{b_1(t), b_2(t), b_3(t)\} \ d = \frac{\mathcal{C}_1}{1-\mathcal{C}_2}, \\
\mathcal{C}_1 &= N_{T_0}[\Lambda N_1(L_{F_1}T_0 + L_1) + \Lambda L_1 + \Lambda^2 N_1 T_0 L_2(1 + N_2 T_0) + N_1 + \sum_{0<t_i<t} K_i^2] \\
&\quad + [N_1 L_{F_1} N_{T_0} + (N_{T_0}\Lambda\Lambda' N_1 H + K_{T_0})]\|\phi\|_{\mathfrak{B}}, \\
\mathcal{C}_2 &= N_{T_0}[\Lambda L_{F_1} + \Lambda^2 N_1 T_0 L_{F_2} + \Lambda^2 N_2 N_1 T_0^2 L_{F_2} + \Lambda N_1 \sum_{0<t_i<t} K_i^1] < 1, \\
e &= \Omega^{-1}(\Omega(d) + \mathfrak{W}(d)), \quad \int_0^t m_{\mathcal{E}}(t,s)ds < L_0, \\
&\quad \Omega_1 \ \text{ is arbitrary positive constant.}
\end{aligned}
$$

We consider the function $z : (-\infty, T_0] \to X$ defined by $z_0 = \phi$ and $z(t) = E^{-1}R(t)E\phi(0)$ on $[0, T_0]$. It is easy to see that $\|z_t\| \leq [N_{T_0}\Lambda\Lambda' N_1 H + K_{T_0}]\|\phi\|_{\mathfrak{B}}$, where $N_{T_0} = \sup_{t\in[0,T_0]} N(t)$, $K_{T_0} = \sup_{t\in[0,T_0]} K(t)$ and $\Lambda = \|E^{-1}\|$, $\Lambda' = \|E\|$.

Theorem 3.1 *If the assumptions* $(E1) - (E9)$ *are fulfilled and*

$$
\begin{aligned}
N_{T_0}[\Lambda(1+a_1)(L_{F_1} + \Lambda N_1 T_0 L_{F_2} + \Lambda N_1 N_2 T_0^2 L_{F_2}) + \Lambda N_1 \sum_{0<t_i<t} L_i] \\
+ \Lambda(1 + L_0\Omega_1)\int_0^t \xi(s)ds < 1. \qquad (27)
\end{aligned}
$$

Then, there exists at least one solution for the system (6)-(8).

Proof. Let $\mathcal{S}(T_0) = \{y : (-\infty, T_0] \to X : y_0 = \phi, \ y|_{[0,T_0]} \in \mathcal{PC}\}$ with the supremum norm $(\|\cdot\|_{T_0})$ be the space. Now, we consider the operator $\Pi : \mathcal{S}(T_0) \to \mathcal{S}(T_0)$ defined by

$$
\Pi y(t) = \begin{cases}
0, & t \in (-\infty, 0], \\
E^{-1}R(t)F(0,\phi,0) - E^{-1}F(t, y_t + z_t, \int_0^t a(t,s,y_s+z_s)ds) \\
-E^{-1}\int_0^t R(t-s)AE^{-1}F(s, y_s+z_s, \int_0^s a(s,\tau,y_\tau+z_\tau)d\tau)ds \\
-E^{-1}\int_0^t R(t-s)AE^{-1}\int_0^s f(s-\tau)F(\tau, y_\tau+z_\tau, \int_0^\tau a(\tau,\xi,y_\xi+z_\xi)d\xi)d\tau ds \\
+E^{-1}\int_0^t R(t-s)G(s, y_s+z_s, \int_0^s \mathcal{E}(s,\tau,y_\tau+z_\tau)d\tau)ds \\
+\sum_{0<t_i<t} E^{-1}R(t-t_i)I_i(y_{t_i}+z_{t_i}), & t \in [0, T_0].
\end{cases} \qquad (28)
$$

Clearly, we have $\|y_t + z_t\|_{\mathfrak{B}} \leq [N_{T_0}\Lambda\Lambda' N_1 H + K_{T_0}]\|\phi\|_{\mathfrak{B}} + N_{T_0}\|y\|_t$, where $\|y\|_t = \sup_{s\in[0,t]}\|y(s)\|$. From the axioms A, our assumptions and the strong continuity of $R(t)$, we can see that $\Pi y \in \mathcal{PC}$. For $y \in S(T_0)$, we get

$$\|R(t-s)AE^{-1}F(s,y_s+z_s,\int_0^s a(s,\tau,y_\tau+z_\tau)d\tau)\| \leq \Lambda N_1[L_{F_2}(\|y_s+z_s\|_{\mathfrak{B}} + \int_0^t m_a(t,s)W_a(\|y_s+z_s\|_{\mathfrak{B}}))+L_2],$$

and

$$\|f(s-\tau)AE^{-1}F(\tau,y_\tau+z_\tau,\int_0^\tau a(\tau,\xi,y_\xi+z_\xi)d\xi)d\tau\| \leq N_2\Lambda[L_{F_2}(\|y_s+z_s\|_{\mathfrak{B}} + \int_0^t m_a(t,s)W_a(\|y_s+z_s\|_{\mathfrak{B}}))+L_2].$$

Thus, from the Bocher theorem it takes after that $AR(t-s)F(s,y_s+z_s,\int_0^s a(s,\tau,y_\tau+z_\tau)d\tau)$ is integrable. So, we deduce that Π is well defined on $\mathcal{S}(T_0)$. Next, we give the demonstration of Theorem 3.1 in numerous steps.

Step 1. The set $\{y \in \mathcal{PC}([0,T_0],X): \ y(t) = \lambda \Pi y(t), \text{ for } 0<\lambda<1\}$ is bounded. For $\lambda \in (0,1)$, let y_λ be a solution for $y = \lambda \Pi y$. We obtain

$$\| y_{\lambda_t} + z_t\| \leq [N_{T_0}\Lambda\Lambda' N_1 H + K_{T_0}]\|\phi\|_{\mathfrak{B}} + N_{T_0}\|y_\lambda\|_t. \tag{29}$$

Let $u_\lambda(t) = [N_{T_0}\Lambda\Lambda' N_1 H + K_{T_0}]\|\phi\|_{\mathfrak{B}} + N_{T_0}\|y_\lambda\|_t$ for each $t \in [0,T_0]$ and $\lambda \in (0,1)$. $\|y_\lambda(t)\| = \|\lambda \Pi y_\lambda(t)\| \leq \|\Pi y_\lambda(t)\|$

$$\begin{aligned}
\leq\ & \|E^{-1}R(t)F(0,\phi,0)\| + \|E^{-1}F(t,y_{\lambda_t}+z_t,\int_0^t a(t,s,y_{\lambda_s}+z_s)ds)\| \\
&+\|E^{-1}\int_0^t R(t-s)AE^{-1}F(s,y_{\lambda_s}+z_s,\int_0^s a(s,\tau,y_{\lambda_\tau}+z_\tau)d\tau)ds\| \\
&+\|E^{-1}\int_0^t R(t-s)AE^{-1}\int_0^s f(s-\tau)F(\tau,y_{\lambda_\tau}+z_\tau,\int_0^\tau a(\tau,\xi,y_{\lambda_\xi}+z_\xi)d\xi)d\tau ds\| \\
&+\|\int_0^t R(t-s)E^{-1}G(s,y_{\lambda_s}+z_s,\int_0^s \mathcal{E}(s,\tau,y_{\lambda_\tau}+z_\tau)d\tau)ds\| \\
&+\sum_{0<t_i<t}\|E^{-1}R(t-t_i)I_i(y_{\lambda_{t_i}}+z_{t_i})\|, \\
\leq\ & \Lambda N_1(L_{F_1}(T_0+\|\phi\|_{\mathfrak{B}})+L_1) + \Lambda[L_{F_1}(u_\lambda(t)+\int_0^t m_a(t,s)W_a(u_\lambda(s))ds)+L_1] \\
&+\Lambda^2 N_1 T_0[L_{F_2}(u_\lambda(t)+\int_0^t m_a(t,s)W_a(u_\lambda(s))ds)+L_2] \\
&+\Lambda^2 N_2 N_1 T_0^2[L_{F_2}(u_\lambda(s)+\int_0^t m_a(t,s)W_a(u_\lambda(s))ds)+L_2] \\
&+\Lambda N_1\int_0^t \alpha(s)\Omega(u_\lambda(s))+\beta(s)\mathfrak{W}(\int_0^s m_{\mathcal{E}}(s,\tau)W_{\mathcal{E}}(u_\lambda(\tau))d\tau)ds \\
&+\Lambda N_1\sum_{0<t_i<t}(K_i^1 u_\lambda(t)+K_i^2),
\end{aligned}$$

which gives that
$\|y_\lambda(t)\|$

$$\begin{aligned} &\leq \Lambda N_1(L_{F_1}T_0+L_1)+\Lambda L_1+\Lambda^2N_1T_0L_2(1+N_2T_0)+N_1\sum_{0<t_i<t}K_i^2+N_1L_{F_1}\|\phi\|_{\mathfrak{B}} \\ &+[\Lambda L_{F_1}+\Lambda^2N_1T_0L_{F_2}+\Lambda^2N_2N_1T_0^2L_{F_2}+\Lambda N_1\sum_{0<t_i<t}K_i^1]u_\lambda(t) \\ &+[\Lambda L_{F_1}+\Lambda^2N_1T_0L_{F_2}+\Lambda^2N_2N_1T_0^2L_{F_2}]\int_0^t m_a(t,s)W_a(u_\lambda(s))ds \\ &+\Lambda N_1\int_0^t \alpha(s)\Omega(u_\lambda(s))+\beta(s)\mathfrak{W}(\int_0^s m_{\mathcal{E}}(s,\tau)W_{\mathcal{E}}(u_\lambda(\tau))d\tau)ds. \end{aligned}$$

Thus, we estimate

$$\begin{aligned} u_\lambda(t) &\leq \frac{\mathcal{C}_1}{1-\mathcal{C}_2}+\frac{N_{T_0}}{1-\mathcal{C}_2}[\Lambda L_{F_1}+\Lambda^2N_1T_0L_{F_2} \\ &+\Lambda^2N_2N_1T_0^2L_{F_2}]\int_0^t m_a(t,s)W_a(u_\lambda(s))ds \\ &+\frac{N_{T_0}\Lambda N_1}{1-\mathcal{C}_2}\int_0^t \alpha(s)\Omega(u_\lambda(s))+\beta(s)\mathfrak{W}(\int_0^s m_{\mathcal{E}}(s,\tau)W_{\mathcal{E}}(u_\lambda(\tau))d\tau)ds. \end{aligned}$$

Take $d=\frac{\mathcal{C}_1}{1-\mathcal{C}_2}$ and get

$$\begin{aligned} u_\lambda(t) &\leq d+\frac{N_{T_0}}{1-\mathcal{C}_2}[\Lambda L_{F_1}+\Lambda^2N_1T_0L_{F_2}+\Lambda^2N_2N_1T_0^2L_{F_2}]\int_0^t m_a(t,s)W_a(u_\lambda(s))ds \\ &+\frac{N_{T_0}\Lambda N_1}{1-\mathcal{C}_2}\int_0^t \alpha(s)\Omega(u_\lambda(s))+\beta(s)\mathfrak{W}(\int_0^s m_{\mathcal{E}}(s,\tau)W_{\mathcal{E}}(u_\lambda(\tau))d\tau)ds. \end{aligned} \tag{30}$$

Let

$$\begin{aligned} \mu_\lambda(t) &= d+\frac{N_{T_0}}{1-\mathcal{C}_2}[\Lambda L_{F_1}+\Lambda^2N_1T_0L_{F_2}+\Lambda^2N_2N_1T_0^2L_{F_2}]\int_0^t m_a(t,s)W_a(u_\lambda(s))ds \\ &+\frac{N_{T_0}\Lambda N_1}{1-\mathcal{C}_2}\int_0^t \alpha(s)\Omega(u_\lambda(s))+\beta(s)\mathfrak{W}(\int_0^s m_{\mathcal{E}}(s,\tau)W_{\mathcal{E}}(u_\lambda(\tau))d\tau)ds, \end{aligned} \tag{31}$$

then, we get $\mu_\lambda(0)=d$ and $u_\lambda(t)\leq\mu_\lambda$ for each $t\in[0,T_0]$. Thus, we get

$$\begin{aligned} \mu_\lambda^{'}(t) &\leq \frac{N_{T_0}}{1-\mathcal{C}_2}[\Lambda L_{F_1}+\Lambda^2N_1T_0L_{F_2}+\Lambda^2N_2N_1T_0^2L_{F_2}] \\ &\times(a_0(t,t)W_a(u_\lambda(t))+\int_0^t\frac{\partial m_a(t,s)}{\partial t}W_a(u_\lambda(t))ds) \\ &+\frac{N_{T_0}\Lambda N_1}{1-\mathcal{C}_2}[\alpha(t)\Omega(u_\lambda(t))+\beta(t)\mathfrak{W}(\int_0^t m_{\mathcal{E}}(t,s)W_{\mathcal{E}}(u_\lambda(s))ds)]. \end{aligned}$$

Let $\vartheta(t)$ be such that

$$\Omega(\vartheta)=\Omega(\mu_\lambda)+\mathfrak{W}(\int_0^t m_{\mathcal{E}}(t,s)W_{\mathcal{E}}(\mu_\lambda)ds). \tag{32}$$

We also have $\vartheta \geq \mu_\lambda$. We differentiate the above equation and get

$$\begin{aligned}
\Omega^{'}(\vartheta)\vartheta^{'} &= \Omega^{'}(\mu_\lambda)\mu_\lambda^{'} + \mathfrak{W}^{'}(\int_0^t m_{\mathcal{E}}(t,s)W_{\mathcal{E}}(\mu_\lambda)ds) \\
&\quad \times[\int_0^t \frac{\partial m_{\mathcal{E}}}{\partial t}(t,s)W_{\mathcal{E}}(\mu_\lambda)ds + m_{\mathcal{E}}(t,t)W_{\mathcal{E}}(\mu_\lambda)], \\
\Omega^{'}(\vartheta)\vartheta^{'} &\leq \Omega^{'}(\vartheta)[\frac{N_{T_0}}{1-\mathcal{C}_2}(\Lambda L_{F_1} + \Lambda^2 N_1 T_0 L_{F_2} + \Lambda^2 N_2 N_1 T_0^2 L_{F_2}) \\
&\quad \times W_a(\vartheta)(a_0(t,t) + \int_0^t \frac{\partial m_a(t,s)}{\partial t}ds) \\
&\quad + \frac{N_{T_0}\Lambda N_1}{1-\mathcal{C}_2}p(t)\Omega(\vartheta)] + \mathfrak{W}^{'}(W_{\mathcal{E}}(\vartheta)\int_0^t m_{\mathcal{E}}(t,s)ds) \\
&\quad \times W_{\mathcal{E}}(\vartheta)[\int_0^t \|\frac{\partial m_{\mathcal{E}}}{\partial t}(t,s)\|ds + m_{\mathcal{E}}(t,t)] \qquad (33)
\end{aligned}$$

Furthermore, from the hypotheses on Ω, we get

$$\Omega^{'}(\vartheta) \geq \Omega^{'}(\mu_\lambda) \geq \Omega(\mu_\lambda(0)) \geq \Omega^{'}(\Lambda\Lambda N_1\|\phi\|_{\mathfrak{B}}) > 0.$$

Thus, we get

$$\begin{aligned}
\vartheta^{'} &\leq \frac{1}{1-\mathcal{C}_2}[(N_{T_0}\Lambda L_{F_1} + \Lambda^2 N_1 T_0 L_{F_2} + \Lambda^2 N_2 N_1 T_0^2 L_{F_2}) \times W_a(\vartheta)(a_0(t,t) \\
&\quad + \int_0^t \frac{\partial m_a(t,s)}{\partial t}ds) + N_{T_0}\Lambda N_1 p(t)\Omega(\vartheta)] + \frac{W_{\mathcal{E}}(\vartheta)}{\Omega'(\vartheta)}\mathfrak{W}^{'}(W_{\mathcal{E}}(\vartheta)\int_0^t m_{\mathcal{E}}(t,s)ds) \\
&\quad \times[\int_0^t \|\frac{\partial m_{\mathcal{E}}}{\partial t}(t,s)\|ds + m_{\mathcal{E}}(t,t)]. \qquad (34)
\end{aligned}$$

By the assumption $(E9)$, we estimate

$$\begin{aligned}
\vartheta^{'} &\leq [b_1 W_a(\vartheta) + b_2\Omega(\vartheta) + \frac{b_3 W_E(\vartheta)}{\Omega'(\vartheta)}\mathfrak{W}^{'}(LW_{\mathcal{E}}(\vartheta))], \\
&\leq b(t)(W_a(\vartheta) + \Omega(\vartheta) + \frac{W_{\mathcal{E}}(\vartheta)}{\Omega'(\vartheta)}\mathfrak{W}^{'}(LW_{\mathcal{E}}(\vartheta))). \qquad (35)
\end{aligned}$$

Thus, for $t \in [0, T_0]$

$$\begin{aligned}
&\int_{\vartheta(0)}^{\vartheta(t)} [W_a(\vartheta) + \Omega(\vartheta) + \frac{W_{\mathcal{E}}(\vartheta)}{\Omega'(\vartheta)}\mathfrak{W}^{'}(LW_{\mathcal{E}}(\vartheta))]^{-1}ds \\
&\quad \leq \int_0^{T_0} b(s)ds, \\
&\quad \leq \int_e^{+\infty} [W_a(\vartheta) + \Omega(\vartheta) + \frac{W_{\mathcal{E}}(\vartheta)}{\Omega'(\vartheta)}\mathfrak{W}^{'}(LW_{\mathcal{E}}(\vartheta))]^{-1}ds, \qquad (36)
\end{aligned}$$

it implies that the function $\vartheta(t)$ is bounded function on $[0, T_0]$. Thus, we obtain that the function $u_\lambda(t)$ is bounded on $[0, T_0]$. Hence, $y_\lambda(\cdot)$ is bounded on $[0, T_0]$.

Step 2. Π is a χ-contraction.
Now, we introduce the decomposition of $\Pi = \Pi_1 + \Pi_2$ defined by

$$\begin{aligned}
\Pi_1 y(t) &= E^{-1}R(t)F(0,\phi,0) - E^{-1}F(t, y_t + z_t, \int_0^t a(t,s,y_s+z_s)ds) \\
&\quad -E^{-1}\int_0^t R(t-s)AE^{-1}F(s, y_s+z_s, \int_0^s a(s,\tau,y_\tau+z_\tau)d\tau)ds \\
&\quad -E^{-1}\int_0^t R(t-s)AE^{-1}\int_0^s f(s-\tau)F(\tau, y_\tau+z_\tau, \int_0^\tau a(\tau,\xi,y_\xi+z_\xi)d\xi)d\tau ds \\
&\quad + \sum_{0<t_i<t} E^{-1}R(t-t_i)I_i(y_{t_i}+z_{t_i}), \qquad (37) \\
\Pi_2 y(t) &= E^{-1}\int_0^t R(t-s)G(s, y_s+z_s, \int_0^s E(s,\tau,y_\tau+z_\tau)d\tau)ds. \qquad (38)
\end{aligned}$$

Now, we firstly show that Π is Lipschitz continuous with Lipschitz constant $\mathcal{K}_1$. Let $y_1,\ y_2 \in \mathcal{S}(T_0)$. Then, we obtain

$$\begin{aligned}
&\|\Pi_1 y_1(t) - \Pi_1 y_2(t)\| \leq \\
&\quad \|E^{-1}F(t, {y_1}_t + z_t, \int_0^t a(t,s,{y_1}_s+z_s)ds) - E^{-1}F(t, {y_2}_t + z_t, \int_0^t a(t,s,{y_2}_s+z_s)ds)\| \\
&\quad +\|E^{-1}\| \int_0^t \|R(t-s)AE^{-1}[F(s, {y_1}_s+z_s, \int_0^s a(s,\tau,{y_1}_\tau+z_\tau)d\tau) \\
&\quad -F(s, {y_2}_s+z_s, \int_0^s a(s,\tau,{y_2}_\tau+z_\tau)d\tau)]\|ds \\
&\quad +\|E^{-1}\| \int_0^t \|R(t-s)AE^{-1}\int_0^s f(s-\tau)F(\tau, {y_1}_\tau+z_\tau, \int_0^\tau a(\tau,\xi,{y_1}_\xi+z_\xi)d\xi) \\
&\quad -F(\tau, {y_2}_\tau+z_\tau, \int_0^\tau a(\tau,\xi,{y_2}_\xi+z_\xi)d\xi)]d\tau\|ds \\
&\quad + \sum_{0<t_i<t} \|E^{-1}R(t-t_i)\| \cdot \|I_i({y_1}_{t_i}+z_{t_i}) - I_i({y_2}_{t_i}+z_{t_i})\|, \\
&\leq \Lambda L_{F_1}(1+a_1)\|{y_1}_t - {y_2}_t\|_{\mathfrak{B}} + \Lambda^2 N_1 T_0 L_{F_2}(1+a_1)\|{y_1}_t - {y_2}_t\|_{\mathfrak{B}} \\
&\quad +\Lambda^2 N_1 N_2 T_0^2 L_{F_2}(1+a_1)\|{y_1}_t - {y_2}_t\|_{\mathfrak{B}} + \Lambda N_1 \sum_{0<t_i<t} L_i \|{y_1}_t - {y_2}_t\|_{\mathfrak{B}}, \\
&\leq N_{T_0}[\Lambda(1+a_1)(L_{F_1} + \Lambda N_1 T_0 L_{F_2} + \Lambda N_1 N_2 T_0^2 L_{F_2}) + \Lambda N_1 \sum_{0<t_i<t} L_i] \\
&\quad \times \|y_1 - y_2\|_{T_0}, \qquad (39)
\end{aligned}$$

which implies that Π_1 is Lipschitz continuous with Lipschitz constant $\mathcal{K}_1 = N_{T_0}[\Lambda(1+a_1)(L_{F_1} + \Lambda N_1 T_0 L_{F_2} + \Lambda N_1 N_2 T_0^2 L_{F_2}) + \Lambda N_1 \sum_{0<t_i<t} L_i] < 1$.

Let B be an arbitrary subset of $\mathcal{S}(T_0)$. Besides, $R(t)$ is equicontinuous resolvent operator. Therefore, from the assumption (HG) and the strong continuity of $R(t)$, we have that $R(t-s)G(s, x_s+y_s, \int_0^s \mathcal{E}(s,\tau,x_\tau+y_\tau)d\tau)$ is piecewise equicontinuous. Then, by Lemma 2.6 we have

$$
\begin{aligned}
&\chi(\Pi_2(B(t)))\\
&\leq \chi(E^{-1}\int_0^t R(t-s)G(s,B_s+z_s,\int_0^s \mathcal{E}(s,\tau,B_\tau+z_\tau)d\tau)ds),\\
&\leq \Lambda\int_0^t \xi(s)\cdot(\sup_{-\infty<\theta\leq 0}\chi(B(s+\theta)+z(s+\theta))+\chi(\int_0^s E(s,\tau,B_\tau+z_\tau)d\tau))ds,\\
&\leq \Lambda\int_0^t \xi(s)\sup_{-\infty<\theta\leq 0}[\chi(B(s+\theta)+z(s+\theta))+L_0\chi(W_{\mathcal{E}}(B(s+\theta)+z(s+\theta)))]ds,\\
&\leq \Lambda\int_0^t \xi(s)\sup_{0\leq\tau\leq s}(\chi(B(\tau))+L_0\chi(W_{\mathcal{E}}(B(\tau))))ds,\\
&\leq \Lambda\ \chi_{\mathcal{PC}}(B)[1+\Omega_1 L_0]\int_0^t \xi(s)ds,\ [\because\ \chi(W_{\mathcal{E}}(B(\tau)))\leq \Omega_1\chi(B(\tau))], \qquad (40)
\end{aligned}
$$

for every bounded set $B\subset\mathcal{PC}$. Here Ω_1 is constant and $\int_0^t m_{\mathcal{E}}(t,s)ds\leq L_0$.

Now we can see that for any bounded subset $B\in\mathcal{PC}$

$$
\begin{aligned}
\chi_{\mathcal{PC}}(\Pi(B)) &= \chi_{\mathcal{PC}}(\Pi_1 B+\Pi_2 B),\\
&\leq \chi_{\mathcal{PC}}(\Pi_1 B)+\chi_{\mathcal{PC}}(\Pi_2 B),\\
&\leq (\mathcal{K}_1+\Lambda(1+L_0\Omega_1)\int_0^t \xi(s)ds)\chi_{\mathcal{PC}}(B), \qquad (41)
\end{aligned}
$$

from the above inequality we obtain that Π is χ-contraction. Hence Π has at least one fixed point in B by Darbo fixed point theorem. Let y be the fixed point of the map Π on $S(T_0)$. Thus $u=y+z$ is a mild solution for the problem (6)-(8). Therefore, this completes the proof of the theorem.

Theorem 3.2 *Let us assume that the hypotheses* $(E1)$-$(E4)$ *and* $(E5)$-$(E9)$ *are satisfied and*

$$
\begin{aligned}
&N_{T_0}[\Lambda L_{F_1}+\Lambda^2 N_1 T_0 L_{F_2}+\Lambda^2 N_1 N_2 T_0^2 L_{F_2}+N_1\Lambda\sum_{0<t_i<t}K_i^1]\\
&+(\Lambda L_{F_1}+\Lambda^2 N_1 T_0 L_{F_2}+\Lambda^2 N_1 N_2 T_0^2 L_{F_2})\times\int_0^{T_0} m_a(T_0,s)\lim_{\tau\to\infty}\sup\frac{W_a(\tau)}{\tau}ds\\
&+\Lambda N_1\int_0^{T_0}[\alpha(s)\lim_{\tau\to\infty}\sup\frac{\Omega(\tau)}{\tau}+\beta(s)\lim_{\tau\to\infty}\sup\frac{\mathfrak{W}(\tau)}{\tau}]ds<1. \qquad (42)
\end{aligned}
$$

Then, there exists at least one mild solution for Sobolev type equation (6)-(8).

Proof. The proof of the theorem is similar to the proof of the previous Theorem 3.1. We consider the operator Π defined by the equation (28). Next, we show that there exist a positive constant k such that $\Pi(B_k)\subset B_k$, here B_k denotes the closed and convex ball with center at the origin and radius k i.e.,$B_k=\{y\in\mathcal{S}(T_0):\|y\|_{T_0}\leq k\}$. To show the claim, we assume that for any $k>0$, there exists $y_k\in B_k$ and $t_k\in[0,T_0]$ such that $k<\|\ \Pi y_k(t_k)\|$. For $y_k\in B_k$ and $t_k\in[0,T_0]$, we get

$$
\begin{aligned}
k < {} & \|\Pi y_k(t_k)\| \\
\leq {} & \Lambda N_1(L_{F_1}T_0 + L_1)\|\phi\|_{\mathfrak{B}} + \Lambda[L_{F_1}(\|y_{k t_k} + z_{t_k}\|_{\mathfrak{B}} \\
& + \int_0^{t_k} m_a(t_k, s)W_a(\|y_{k t_k} + z_{t_k}\|_{\mathfrak{B}})ds) + L_1] \\
& + \Lambda^2 N_1 T_0[L_{F_2}(\|y_{k t_k} + z_{t_k}\|_{\mathfrak{B}} + \int_0^{t_k} m_a(t_k, \tau)W_a(\|y_{k\tau} + z_\tau\|_{\mathfrak{B}})d\tau) + L_2] \\
& + \Lambda^2 N_1 N_2 T_0^2[L_{F_2}(\|y_{k s} + z_s\|_{\mathfrak{B}} + \int_0^{t_k} m_a(t_k, \tau)W_a(\|y_{k\tau} + z_\tau\|_{\mathfrak{B}})d\tau) + L_2] \\
& + \Lambda N_1 \int_0^{t_k} \alpha(s)\Omega(\|y_{k s} + z_s\|_{\mathfrak{B}}) + \beta(s)\mathfrak{W}(\int_0^s m_{\mathcal{E}}(s, \tau)W_{\mathcal{E}}(\|y_{k\tau} + z_\tau\|_{\mathfrak{B}})d\tau)ds \\
& + N_1\Lambda \sum_{0<t_i<t}(K_i^1\|y_{k t_k} + z_{t_k}\|_{\mathfrak{B}} + K_i^2), \\
\leq {} & N_1(L_{F_1}T_0 + L_1)\|\phi\|_{\mathfrak{B}} + \Lambda L_1 + \Lambda^2 N_1 T_0 L_2 + \Lambda^2 N_1 N_2 T_0^2 L_2 + N_1\Lambda \sum_{0<t_i<t} K_i^2 \\
& + [\Lambda L_{F_1} + \Lambda^2 N_1 T_0 L_{F_2} + \Lambda^2 N_1 N_2 T_0^2 L_{F_2} + N_1\Lambda \sum_{0<t_i<t} K_i^1] \times \|y_{k t_k} + z_{t_k}\|_{\mathfrak{B}} \\
& + (\Lambda L_{F_1} + \Lambda^2 N_1 T_0 L_{F_2} + \Lambda^2 N_1 N_2 T_0^2 L_{F_2}) \int_0^{t_k} m_a(t_k, s)W_a(\|y_{k t_k} + z_{t_k}\|_{\mathfrak{B}})ds \\
& + \Lambda N_1 \int_0^{t_k} [\alpha(s)\Omega(\|y_{k s} + z_s\|_{\mathfrak{B}}) + \beta(s)\mathfrak{W}(\int_0^s m_{\mathcal{E}}(s, \tau)W_{\mathcal{E}}(\|y_{k\tau} + z_\tau\|_{\mathfrak{B}})d\tau)]ds, \\
\leq {} & N_1(L_{F_1}T_0 + L_1)\|\phi\|_{\mathfrak{B}} + \Lambda L_1 + \Lambda^2 N_1 T_0 L_2 + \Lambda^2 N_1 N_2 T_0^2 L_2 + N_1\Lambda \sum_{0<t_i<t} K_i^2 \\
& + [\Lambda L_{F_1} + \Lambda^2 N_1 T_0 L_{F_2} + \Lambda^2 N_1 N_2 T_0^2 L_{F_2} + N_1\Lambda \sum_{0<t_i<t} K_i^1] \\
& \times [(N_{T_0}\Lambda\Lambda' N_1 H + K_{T_0})\|\phi\|_{\mathfrak{B}} + N_{T_0}k] + (\Lambda L_{F_1} + \Lambda^2 N_1 T_0 L_{F_2} + \Lambda^2 N_1 N_2 T_0^2 L_{F_2}) \\
& \times \int_0^{t_k} m_a(t_k, s)W_a((N_{T_0}\Lambda\Lambda' N_1 H + K_{T_0})\|\phi\|_{\mathfrak{B}} + N_{T_0}k)ds \\
& + \Lambda N_1 \int_0^{t_k} [\alpha(s)\Omega((N_{T_0}\Lambda\Lambda' N_1 H + K_{T_0})\|\phi\|_{\mathfrak{B}} + N_{T_0}k) \\
& + \beta(s)\mathfrak{W}(\int_0^s m_{\mathcal{E}}(s, \tau)W_{\mathcal{E}}(N_{T_0}\Lambda\Lambda' N_1 H + K_{T_0})\|\phi\|_{\mathfrak{B}} + N_{T_0}k)d\tau)]ds \qquad (43)
\end{aligned}
$$

Dividing the above inequality by k and taking $k \to \infty$, we conclude

$$
\begin{aligned}
1 < {} & N_{T_0}[\Lambda L_{F_1} + \Lambda^2 N_1 T_0 L_{F_2} + \Lambda^2 N_1 N_2 T_0^2 L_{F_2} + N_1\Lambda \sum_{0<t_i<t} K_i^1] \\
& + (\Lambda L_{F_1} + \Lambda^2 N_1 T_0 L_{F_2} + \Lambda^2 N_1 N_2 T_0^2 L_{F_2}) \\
& \times \int_0^{T_0} m_a(T_0, s) \lim_{k\to\infty}\sup \frac{W_a((N_{T_0}\Lambda\Lambda' N_1 H + K_{T_0})\|\phi\|_{\mathfrak{B}} + N_{T_0}k)}{k} ds \\
& + \Lambda N_1 \int_0^{T_0} [\alpha(s) \lim_{k\to\infty}\sup \frac{\Omega((N_{T_0}\Lambda\Lambda' N_1 H + K_{T_0})\|\phi\|_{\mathfrak{B}} + N_{T_0}k)}{k}
\end{aligned}
$$

$$
\begin{aligned}
&+\beta(s) \lim_{k\to\infty} \sup \frac{\mathfrak{W}(\int_0^{T_0} m_{\mathcal{E}}(T_0,\tau) W_{\mathcal{E}}(N_{T_0}\Lambda\Lambda' N_1 H + K_{T_0})\|\phi\|_{\mathfrak{B}} + N_{T_0}k)d\tau)}{k}]ds \\
\le\ & N_{T_0}[\Lambda L_{F_1} + \Lambda^2 N_1 T_0 L_{F_2} + \Lambda^2 N_1 N_2 T_0^2 L_{F_2} + N_1\Lambda \sum_{0<t_i<t} K_i^1] \\
&+(\Lambda L_{F_1} + \Lambda^2 N_1 T_0 L_{F_2} + \Lambda^2 N_1 N_2 T_0^2 L_{F_2}) \times \int_0^{T_0} m_a(T_0,s) \lim_{\tau\to\infty}\sup \frac{W_a(\tau)}{\tau} ds \\
&+\Lambda N_1 \int_0^{T_0} [\alpha(s) \lim_{\tau\to\infty}\sup \frac{\Omega(\tau)}{\tau} + \beta(s) \lim_{\tau\to\infty}\sup \frac{\mathfrak{W}(\tau)}{\tau}]ds \qquad (44)
\end{aligned}
$$

which gives a contradiction with the inequality (42). Hence, we obtain that $\Pi(B_k) \subset B_k$. As in the proof of Theorem 3.1, we conclude that there exists at least one mild solution for the system (6)-(8).

4 Application

Consider the following first order impulsive Sobolev type integro-differential equation with unbounded delay in a Banach space $(X, \| \cdot \|)$

$$
\begin{aligned}
&\frac{d}{dt}[x(t,u) + x_{uu}(t,u) - F(t, x(t-k,u), \int_0^t g_1(t,s,x(s-k,u))ds)] \\
&= \frac{\partial^2}{\partial u^2}[x(t,u) + \int_0^t f(t-s,u)x(s,u)ds] \\
&+ \int_0^t a(t,u,s-t)G(x(s,u), \int_0^s E(s,\tau,x_\tau)d\tau)ds, \quad t\in[0,T_0], \quad u\in[0,\pi], \qquad (45) \\
&x(t,0) = x(t,\pi) = 0, \quad t \in [0,T_0], \qquad (46) \\
&x(\tau,u) = \phi(\tau,u), \quad \tau \le 0,\ 0 \le u \le \pi, \qquad (47) \\
&\Delta x(t_i)(u) = \int_{-\infty}^t c_i(t_i - s)x(s,u)ds, \qquad (48)
\end{aligned}
$$

where $\phi \in C_0 \times L^2(h,X)$ ($\mathfrak{B}$-Phase space) and $0 < t_1 < t_2 < \cdots < t_m < b$ are fixed numbers.

The functions f, a, G, E, c_i, F satisfy the following conditions:

(A1) The operator $f(t), t \ge 0$ is bounded and $\| f(t,u)\| \le N_2$;

(A2) $a(t,u,\tau)$ is continuous function on $[0,T_0]\times[0,\pi]\times(-\infty,0]$ with $\int_{-\infty}^0 a(t,u,\tau)d\tau = n(t,u) < \infty$;

(A3) G is a continuous function, satisfying $G(x_1,x_2) \le \Omega_1(\| x_1\|) + \Omega_2(\| x_2\|)$, where $\Omega_1(\cdot)$ and $\Omega_2(\cdot)$ are continuous, increasing and positive functions on $[0,\infty)$;

(A4) The function $E(\cdot)$ is a continuous function, satisfying $0 \le E(t,s,u) \le m_E(t,s)\omega(\| u\|)$, where ω is a positive increasing continuous function on $[0,\infty)$ and m_E is differentiable a.e., with respect to the first variable with $\int_0^t m_E(t,s)ds$, $\int_0^t \frac{\partial m_E(t,s)}{\partial t} ds$ are bounded on $[0,T_0]$ and $\frac{\partial m_E(t,s)}{\partial t} \ge 0$;

(A5) The functions $c_i \in C([0,\infty);\mathbb{R})$ and $K_i^3 = (\int_{-\infty}^0 \frac{(c_i(s))^2}{h(s)} ds)^{1/2} < 0, \forall\, i = 1,\cdots,m$;

(A6) F is an appropriate Lipschitz continuous function satisfying assumption $(E5)$.

We define the operators $A : D(A) \subset X \to X$ and $E : D(E) \subset X \to X$ such that

$$Ax = x'', \qquad Ex = x + x'',$$

where $D(A)$ and $D(B)$ are defined by

$$\{x \in X : x,\ x_u \text{ are absolutely continuous},\ \ x_{uu} \in X,\ x(0) = x(\pi) = 0\}. \tag{49}$$

Then, we get

$$\begin{aligned} Ax &= \sum_{n=1}^{\infty} n^2 < x, x_n > x_n, \quad x \in D(A), \\ Ez &= \sum_{n=1}^{\infty} (1+n^2) < x, x_n > x_n, \quad x \in D(E), \end{aligned} \tag{50}$$

with $x_n(u) = \sqrt{2/\pi}\sin(nu)$, $\ n = 1, \cdots$, is the orthogonal set of vectors of A. Moreover, $x \in X$, we get

$$\begin{aligned} E^{-1}z &= \sum_{n=1}^{\infty} \frac{1}{1+n^2} < x_n, x > x_n, \\ AE^{-1} &= \sum_{n=1}^{\infty} \frac{n^2}{1+n^2} < x_n, x > x_n, \\ R(t)x &= \sum_{n=1}^{\infty} \exp\left(\frac{n^2 t}{1+n^2}\right) < x_n, x > x_n. \end{aligned} \tag{51}$$

Clearly, AE^{-1} is the infinitesimal generator of a strongly continuous resolvent operator $R(t)$ on Y. Applying Theorem 3.1, we conclude that there exists at least one mild solution for the system (45)-(48).

5 Conclusion

The existence of mild solution for an impulsive neutral integro-differential equation of Sobolev type was investigated. The sufficient condition for ensuring the existence of mild solution was provided by using Darbo-Sadovskii fixed point theorem, analytic semigroup and Hausdorff measure of noncompactness without assuming Lipschitz continuity of nonlinear part G and compactness of semigroup. An example was studied for explaining the feasibility of the discussed results.

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Effectiveness of the Extended Kalman Filter Through Difference Equations

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Abstract: The extended Kalman filter is extensively used in the nonlinear state estimation systems. As long as the system characteristics are correctly known, the extended Kalman filter gives the best performance. However, when the system information is partially known or incorrect, the extended Kalman filter (EKF) may diverge or give the biased estimates. To overcome this problem we introduced the new Riccati difference equation (RDE) which is used to study and examine the performance analysis of extended Kalman filter. We consider the special case of tracking a target with cluster, but with a probability arrival of small value. Finally the convergence analysis and stabilizing solution of Riccati difference equations arising from the standard extended Kalman filter is studied. Simulations results for convergence of EKF for the class of nonlinear filters are done through MATLAB.

Keywords: *convergence; extended Kalman filter; Riccati difference equations; feasibility and stabilizing solution.*

Mathematics Subject Classification (2010): 39A10, 39A30, 39A60, 39B82, 39B99.

1 Introduction

Several recent papers have been devoted to a study of nonlinear Riccati difference equations. The family of Kalman filters have been applied for state as well as parameter estimation for numerous linear as well as nonlinear systems. Though the standard Kalman filter is considered in an optimal estimator (in case of linear systems) with Gaussian noise characters, its nonlinear (extended Kalman filter) suboptimal counterpart is known to diverge under the influences of severe nonlinearities and uncertainties [4,7]. As a solution to this problem robust form of the EKF have been formulated for a wide class of uncertainities [13] in the form of new RDE.

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The paper is organized as follows. In Section 2, we introduced the new Riccati difference equation and algebraic Riccati equation, which are used to arrive the feasible solutions. Also we introduced some lemmas and assumptions which are useful for arriving the convergence analysis. Section 3 provides the conditions needed to ensure the convergence analysis and stabilizing the solutions of the new RDE with the initial conditions. Section 4 provides the simulation results for convergence of the EKF for the class of nonlinear systems through MATLAB [12]. Conclusions are made in Section 5.

2 Preliminaries

Consider the following linear discrete-time system [5, 10]

$$u_{k+1} = Ax_k + Bw_k \quad k \in N, \tag{1}$$

$$v_k = Cx_k + Du_k \quad k \in N, \tag{2}$$

$$z_k = Lx_k, \tag{3}$$

with the initial condition x_0 and $k = 0, 1, 2, \dots, N$, where $x_k \in R^n$ is the system state, $w_k \in R^q$ is the noise, $v_k \in R^m$ is the output measurements, $u_k \in R^m$ is the input measurements, $z_k \in R^p$ is a linear combination of the state variable to be estimated. A, B, C, D and L are known real constant matrices with appropriate dimensions. Time step k is defined as $Z_k = \{z_1, z_2, z_3, ..., z_k\}$, often this is referred to as the measurement.

It is worth noting that an estimator z_k is called an a priori filter if $\hat{z}_k$ is obtained with the output measurements [15] $\{v_0, v_1, \dots, v_{k-1}\}$, while $\hat{z}_k$ is referred to as a posteriori filter. This $\hat{z}_k$ is obtained by the measurements $\{v_0, v_1, \dots, v_k\}$.

Now we introduce the following new Riccati difference equation (RDE)

$$P_{k+1} = AP_kA^T - \left(AP_kC^T + BD^T\right)\left(CP_kC^T + R\right)^{-1}\left(CP_kA^T + DB^T\right), \tag{4}$$

and the Algebraic Riccati Equation (ARE) [14],

$$P = APA^T - \left(APC^T + BD^T\right)\left(CPC^T + R\right)^{-1}\left(CPA^T + DB^T\right). \tag{5}$$

It is clear that the existence of filter is related to the RDE (4) or ARE (5), and the fulfillment of a suitable matrix inequality (feasibility condition) [1], [3]. Now, we adopt the definition of feasible solution [6]. The feasiblility and convergence analysis problem studied in this paper is stated as follows: Given an arbitrarily large N, find the suitable conditions on the initial state P_0 such that the solution P_k is feasible at every step $k \in [0, N]$ and converges to the stabilizing solution P_s as $N \to \infty$ [8],[9]. We end this section by giving two preliminary results which play an important role in deriving the main results of this paper. The first is an extension of a comparison result of new RDE [16].

Lemma 2.1 *Consider the following Riccati difference equation*

$$P_{k+1} = AP_kA^T - \left(AP_kC^T + BD^T\right)\left(CP_kC^T + R\right)^{-1}\left(CP_kA^T + DB^T\right) + BB^T.$$

Let P_k^1 and P_k^2 be solutions of (4) with different initial conditions $P_0^1 = \bar{P}_0^1 \geq 0$ and $P_0^2 = \bar{P}_0^2 \geq 0$, respectively. Then the difference between the two solutions $\tilde{P}_k = P_k^2 - P_k^1$ satisfies the following equation

$$\tilde{P}_{k+1} = \tilde{A}_k\tilde{P}_k\tilde{A}_k^T - \tilde{A}_k\tilde{P}_kC^T\left(C\tilde{P}_kC^T + \tilde{R}_k\right)^{-1}C\tilde{P}_k\tilde{A}_k^T,$$

where $\tilde{A}_k = A - \left(AP_K^1 C^T + BD^T\right)\left(CP_k^1 C^T + R\right)^{-1} C$ *and* $\tilde{R}_k = CP_k^1 C^T + R$.

In order to extend the above lemma, we need the following assumption.

Assumption 2.1 The matrix $\bar{A} = A - BD^T \left(DD^T\right)^{-1} C$ is invertible.

Lemma 2.2 *Consider Riccati difference equation* (4). *Let* P_k^1 *and* P_k^2 *be the two solutions of* (4) *with different initial conditions* $P_0^2 > P_0^1 > 0$. *Then, under Assumption* 2.1, *when* P_k^2 *is feasible, it results that* $P_k^2 > P_k^1 > 0$ *and* P_k^1 *is feasible too. Furthermore, if* $P_0^2 > P_0^1$, *then* $P_k^2 > P_k^1$.

3 Convergence Analysis of Riccati Difference Equation

It is well known from filtering and control theory that the Kalman recursions lead to a recursive formula for the covariance matrix analysis [2]. This result is obtained by eliminating the Kalman gain from the recursion formula. This recursion formula is referred to as the Riccati difference equation [8]. The issue of the speed of convergence is an important one. So we introduced the following Lyapunov equation

$$\tilde{A}^T Y \tilde{A} - Y = -M_-, \tag{6}$$

where $\tilde{A} = A - (AP_s C^T + BD^T)(CP_s C^T + R)^{-1} C$. Now we can formulate Kalman-like recursions for a general system as

$$M_k = \tilde{A}^{-T}\left(G + C^T \tilde{R}^{-1} C\right)\tilde{A}^{-1} - G_K, \tag{7}$$

$$G_k = -P_s^{-1} - P_s^{-1}\left(L^T L - P_s^{-1}\right)^{-1} P_s^{-1}, \tag{8}$$

$$R_k = CP_s C^T + R, \tag{9}$$

where k is the Kalman gain [5]. The following theorem establishes the relationship between the initial state P_0 and feasibile solution to RDE (4).

Theorem 3.1 *Consider the Riccati difference equation* (4). *Let Assumption 2.1 hold, and let* Y *be the solution to the Lyapunov equation* (6). *Then the solution* P_k *of RDE* (4) *is feasible over* $[0 \quad \infty)$ *if for some sufficiently small* $\epsilon > 0$, *the initial condition satisfies*

$$0 < P_0 < (G_k - Y + M_k + I)^{-1} + P_s. \tag{10}$$

Proof. The procedure of the proof is classified into three cases.

Case (i) $P_0 < P_s$. P_s is a constant feasible solution of (4), then the feasibility of P_k follows from Lemma 2.2 directly.

Case (ii) $P_0 > P_s$. Let's define $X_k = P_k - P_s$. Then, applying Lemma 2.1 to (4) and (5), immediately we obtain that X_k satisfies the following

$$\begin{aligned} X_{k+1} &= \hat{A} X_k \hat{A}^T - \hat{A} X_k C\left(C X_k C^T + \hat{R}\right)^{-1} C X_k \hat{A}^T \\ &= \hat{A}\left(X_k^{-1} + C^T \hat{R}^{-1} C\right)^{-1} \hat{A}^T, \end{aligned} \tag{11}$$

where $X_0 = p_0 - P_s$, $\hat{A} = A - \left(AP_s C^T + BD^T\right)\left(CP_s C^T + R\right)^{-1} C$ and $\hat{R} = CP_s C^T + R$. Now let $Z_k = X_k^{-1} - G_k$, where G_k is defined by (8). It is worth noting that $G_k \geq 0$, since P_s is feasible. Note that $\hat{A}$ is invertible as $\bar{A}$ is invertible and P_s is feasible. Then by (11), we have $Z_{k+1} = \hat{A}^{-T} Z_k \hat{A}^{-1} + M_k$, where M_k is defined by (7) and $Z_0 = (P_0 - P_s)^{-1} - G_k$. Since P_s is feasible and $X_k > 0$, then according to Lemma 2.2, it is clear that the feasibility of P_k is equivalent to the positive definiteness of Z_k, which follows from $Z_k = P - s^{-1}\left[\left(P - s^{-1} - P_k^{-1}\right)^{-1} - \left(P_s^{-1} - L^T L\right)^{-1}\right] P_s^{-1}$.

Now consider the following Lyapunov equation [11]

$$\hat{Z}_{k+1} = \hat{A}^{-T} \hat{Z}_{k+1} \hat{A}^{-1} + M_- \tag{12}$$

with $\hat{Z}_0 = Z_0$. By definition $M_k \geq M_-$, so that $Z_k \geq \hat{Z}_k$. Then $\hat{Z}_k > 0$ is sufficient to guarantee the positivity of Z_k. Now we compute (12) as follows

$$\begin{aligned} Z_k \geq \hat{Z}_k &= \left(\hat{A}^{-k}\right)^T \left(Z_0 + \sum_{j=1}^{k} \left(\hat{A}^j\right)^T M_- \hat{A}^j\right) \hat{A}^{-k} \\ &\geq \left(\hat{A}^{-k}\right)^T \left(Z_0 + \sum_{j=1}^{\infty} \left(\hat{A}^j\right)^T M_- \hat{A}^j\right) \hat{A}^{-k}, \end{aligned} \tag{13}$$

from (6), we deduce the value of Y,

$$\begin{aligned} Y &= \sum_{j=0}^{\infty} \left(\hat{A}^j\right)^T M_- \hat{A}^j \\ &= M_- + \sum_{j=1}^{\infty} \left(\hat{A}^j\right)^T M_- \hat{A}^j. \end{aligned} \tag{14}$$

Now comparing (13) and (14), we have

$$Z_k \geq \hat{Z}_k \geq \left(\hat{A}^{-k}\right)^T \left(Z_0 + Y - M_-\right) \hat{A}^{-k}. \tag{15}$$

So, if $Z_0 + Y - M_- > 0$, then $\hat{Z}_k > 0$ and in turn $Z_k > 0$. Here $Z_0 + Y - M_- > 0$. This is rewritten as

$$\left(P_0 - P_s\right)^{-1} - G_k + Y - M_- > 0. \tag{16}$$

Since $-Y + M_- \geq 0$ and $G_k \geq 0$, then (10) implies (16). Thus the proof of feasibility for the case of $P_0 > P_s$ is completed.

Case (iii). $P_0 - P_s$ is not a definite matrix. Initially we need to study the convergence of the solution of the RDE (4). It is easy to know that (4) satisfies the following matrix recursions

$$\begin{aligned} P_{k+1} &= \bar{A} S_k^{-1} \bar{A}^T + B\left[I - D^T\left(DD^T\right)^{-1} D\right] B^T, \\ S_k &= P_k^{-1} + C^T R^{-1} C, \end{aligned} \tag{17}$$

so S_k satisfies the following RDE

$$S_k = \left\{\bar{A} S_k^{-1} \bar{A}^T + B\left[I - D^T\left(DD^T\right)^{-1} D\right] B^T\right\}^{-1} + C^T R^{-1} C, \tag{18}$$

and the associated ARE is

$$S = \left\{ \bar{A} S^{-1} \bar{A}^T + B \left[I - D^T \left(D D^T \right)^{-1} D \right] B^T \right\}^{-1} + C^T R^{-1} C. \tag{19}$$

Under Assumptions 2.1 and 19, we concluded that both the stabilizing solution S_s and antistabilizing solution S_a provides $S_s - S_a > 0$. This implies that there exists a $\bar{P}_0$ satisfying (10) and such that $\bar{P}_0 > P_0$ and $\bar{P}_0 > P_s$. Hence $P_0 - P_s$ is not a definite matrix.

The following theorem provides a sufficient condition for ensuring convergence as well as feasibility of the solution of the RDE (4) over $[0, \infty)$.

Theorem 3.2 *Consider the Riccati difference equation* (4)*. Let Assumption* 2.1 *hold, then the solution P_k of RDE* (4) *is feasible over $[0 \quad \infty)$ and converges to the stabilizing solution P_s of* (5) *as $k \to \infty$ if P_s is feasible and for some sufficiently small $\epsilon > 0$, then the initial condition satisfies*

$$0 < P_0 < \left(G_k - Y + M_- + \epsilon I \right)^{-1} + P_s, \tag{20}$$

where G, Y, and M are defined as in Theorem 3.1.

Proof. Initially, it is noted that P_k is feasible over $[0 \quad \infty)$ from Theorem 3.1. Consider (4), (5), (18) and (19), and the study of convergence of P_k is equivalent to the study of the convergence of S_k to S_s. So we focus on the convergence of S_k as follows, let $U = \{S_a - S_a\}^{-1}$, then from (19), we have

$$U = \tilde{A}^T U \tilde{A} + S_s^{-1} - P_s \hat{A}^T P_s^{-1} \hat{A} P_s. \tag{21}$$

Next, let $W = P_s \left[G_k - Y + M_- + P_s^{-1} \right] P_s$, then from (6), we have

$$W = \tilde{A}^T U \tilde{A} + S_s^{-1} - P_s \hat{A}^T P_s^{-1} \hat{A} P_s + N, \tag{22}$$

where

$$N = P_s C^T \hat{R}^{-1} C P_s + P_s \left(G_k - \hat{A}^T G_k \hat{A} \right) P_s - P_s \hat{A}^T M_- \hat{A} P_s = P_s \hat{A}^T M_+ \hat{A} P_s \geq 0.$$

Comparing (21) and (22), we have $W \geq U$. Now consider (17) and (20), and we obtain

$$\begin{aligned} S_0 &= P_0^{-1} + C^T R^{-1} C \\ &> \left[\left(G_k - Y + M_- + \epsilon I \right)^{-1} + P_s \right]^{-1} + C^T R^{-1} C \\ &= S_s - P_s^{-1} \left[G_k - Y + M_- + \epsilon I + P_s^{-1} \right]^{-1} P_s^{-1} \\ &\geq S - s - W^{-1} \\ &\geq S_s - U^{-1} = S_a. \end{aligned} \tag{23}$$

From (23), we have $S_0 > S_a$. This implies that $\lim\limits_{k \to \infty} S_k = S_s$. It shows that P_k converges to P, and remains feasible at every step. Hence the proof.

4 Simulation Results

Example 4.1

Matrix States	Initial Estimations
Initial States	$\begin{bmatrix} 2 & 0 \\ 0 & 2.04 \end{bmatrix}$
Arbitrary Matrix P	$\begin{bmatrix} 0.6 & 1 \\ 1 & 0.4 \end{bmatrix}$
Arbitrary Matrix R	$\begin{bmatrix} 0.9 & 0 \\ 0 & 1.2 \end{bmatrix}$

Table 1: Initial values for Figure 1.

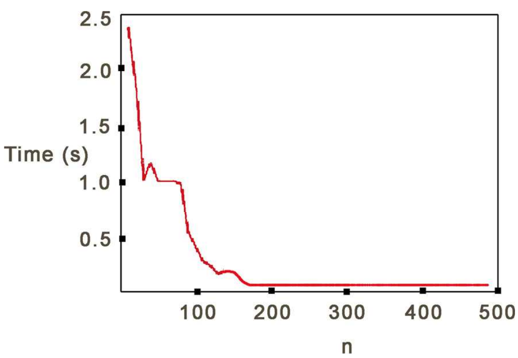

Figure 1: Convergence analysis for Table 1.

Example 4.2

Matrix States	Initial Estimations
Initial States	$\begin{bmatrix} 1.7 & 0 \\ 0 & 1.03 \end{bmatrix}$
Arbitrary Matrix P	$\begin{bmatrix} 0.2 & 1 \\ 1 & 0.7 \end{bmatrix}$
Arbitrary Matrix R	$\begin{bmatrix} 1.2 & 0 \\ 0 & 1.9 \end{bmatrix}$

Table 2: Initial Values for Figure 2.

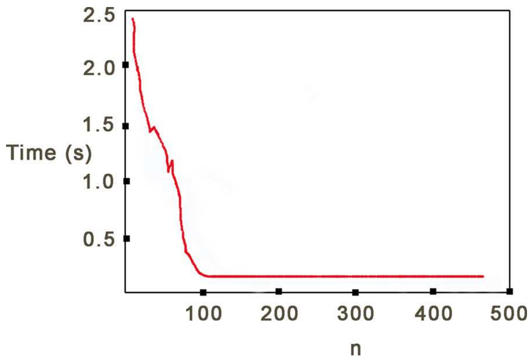

Figure 2: Convergence analysis for Table 2.

Example 4.3

Matrix States	Initial Estimations
Initial States	$\begin{bmatrix} 0.4 & 0 \\ 0 & 0.9 \end{bmatrix}$
Arbitrary Matrix P	$\begin{bmatrix} 1.7 & 1 \\ 1 & 2.4 \end{bmatrix}$
Arbitrary Matrix R	$\begin{bmatrix} 1.4 & 0 \\ 0 & 2.1 \end{bmatrix}$

Table 3: Initial Values for Figure 3.

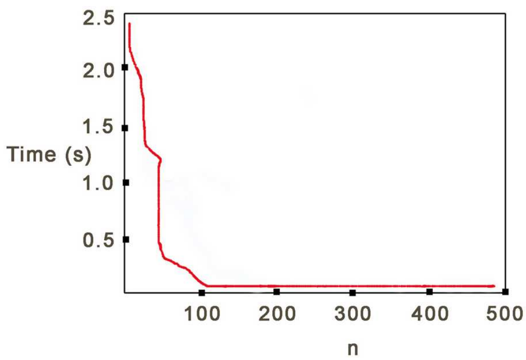

Figure 3: Convergence analysis for Table 3.

5 Conclusion

In this paper we classified the relationship between the initial state P_0 and the feasible solution through a new theorem. The estimation performance of the EKF is improved when we introduced the new RDE corresponding to ARE. Moreover, the convergence analysis is derived with the proposed RDE with good initial conditions alongwith a small ϵ. Furthermore, an additional theorem is formulated to ensure the convergence as well as feasible solutions of the new RDE. Simulation results show the performance of the proposed theorem even for the bad initializations.

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Existence of Even Homoclinic Solutions for a Class of Dynamical Systems

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Abstract: In this paper, we study the existence of even homoclinic solutions for a dynamical system

$$\ddot{x}(t) + A\dot{x}(t) + V'(t, x(t)) = 0,$$

where A is a skew-symmetric constant matrix, $t \in \mathbb{R}$, $x \in \mathbb{R}^N$ and $V \in C^1(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$, $V(t, x) = -K(t, x) + W(t, x)$. We assume that $W(t, x)$ does not satisfy the global Ambrosetti-Rabinowitz condition and that the norm of A is sufficiently small. For the proof we use the mountain pass theorem.

Keywords: *even homoclinic solution; dynamical system; mountain pass theorem; condition (C); critical point.*

Mathematics Subject Classification (2010): 34C37.

1 Introduction

The purpose of this work is to study the existence of even homoclinic solutions for the following system

$$\ddot{x}(t) + A\dot{x}(t) + V'(t, x(t)) = 0, \qquad (DS)$$

where A is a skew-symmetric constant matrix, $V \in C^1(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$, $V'(t, x) = \frac{\partial V}{\partial x}(t, x)$ and $x = (x_1, ..., x_N)$. We say that a solution $x(t)$ of dynamical system (DS) is homoclinic if $x(t) \to 0$ as $t \to \pm\infty$. In addition, x is called nontrivial if $x \not\equiv 0$. The theory of dynamical systems is a vast subject that can be studied from many different viewpoints. Particularly the existence of homoclinic solutions for DS is among the very important

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problems which have been intensively studied. When $A = 0$, (DS) is just the following second order non-autonomous Hamiltonian system:

$$\ddot{x}(t) + V'(t, x(t)) = 0. \qquad (HS)$$

If the potential $V(t,x)$ is of type

$$V(t,x) = -\frac{1}{2}L(t)x.x + W(t,x), \tag{1}$$

where $L \in \mathcal{C}(\mathbb{R}, \mathbb{R}^{N^2})$ is a symmetric matrix depending continuously on t and $W \in C^1(\mathbb{R}\times\mathbb{R}^N, \mathbb{R})$, then the existence of homoclinic solutions of (HS) has been intensively studied by many mathematicians, see ([1], [6], [7], [11], [12], [14], [15], [22]) and the references therein. Assuming that $L(t)$ and $W(t,x)$ are T-periodic in t, $T > 0$, Rabinowitz [17] showed the existence of homoclinic solutions as a limit of $2kT$-periodic solutions of (HS). By the same method many authors have studied the existence of homoclinic solutions for the system (HS) via critical point theory and variational methods, see ([6], [9], [10], [11], [19]) and the references therein. In 2005, Izydorek and Janczewska [10] introduced a new type of potential $V(t,x)$ with which they studied the existence of homoclinic solutions for the system (HS), the potential $V(t,x)$ is T-periodic in t and of the form:

$$V(t,x) = -K(t,x) + W(t,x), \tag{2}$$

where K, $W \in C^1(\mathbb{R}\times\mathbb{R}^N, \mathbb{R})$, which has been extended in the recent paper [19]. They have proved the existence of homoclinic solutions as a limit of $2kT$-periodic solutions of (HS). If $K(t,x)$ and $W(t,x)$ are neither autonomous nor periodic in t, the problem of the existence of homoclinic solutions of (HS) is quite different from the ones just described, because of the lack of compactness of Sobolev embedding. In 2013, Benhassine and Timoumi [5] studied the existence of even homoclinic orbits of the system (HS) when the potential $V(t,x)$ is of the form (2) and satisfies a kind of new superquadratic conditions, in particular

(i) $W'(t,x).x > 2W(t,x) \geq 0$ for all $(t,x) \in \mathbb{R}\times(\mathbb{R}^N \setminus \{0\})$,
$\overline{W}(t,x) := \frac{1}{2}W'(t,x).x - W(t,x) \to +\infty$ as $|x| \to +\infty$ uniformly in t $\in \mathbb{R}$.
(ii) there exist constants $b_1 > 0$ such that

$$K(t,x) \geq b_1|x|^2.$$

When the potential $V(t,x)$ is of type (2), the existence of even homoclinic solutions of (DS) has not been studied. Motivated by the papers ([1], [3]- [11], [14]- [19], [21]), we prove the existence of even homoclinic solutions for (DS), as the limit of solutions of a sequence of boundary-value problems which are obtained by the minimax methods. Here and in the following $x.y$ denotes the inner product of $x, y \in \mathbb{R}^N$ and $|.|$ denotes the associated norm.

Our basic hypotheses on K and W are the following:

(H_1) For all $(t,x) \in \mathbb{R}\times\mathbb{R}^N$, $V'(t,x) \to 0$ *as* $|x| \to 0$ uniformly in $t \in \mathbb{R}$,
(H_2) There exists a constant $b_1 > 0$ such that

$$K(t,x) \geq b_1|x|^2, \quad K(t,x) \leq K'(t,x).x \leq 2K(t,x)$$

for all $(t,x) \in \mathbb{R}\times\mathbb{R}^N$,
(H_3) $W'(t,x) = o(|x|)$ as $|x| \to 0$ uniformly in $t \in \mathbb{R}$ and there exists some constant C_0

such that $\frac{|W'(t,x)|}{|x|} \leq C_0$ for all $(t, x) \in \mathbb{R} \times \mathbb{R}^N$,
(H_4) $W'(t, x).x > 2W(t, x) \geq 0$ for all $(t, x) \in \mathbb{R} \times (\mathbb{R}^N \setminus \{0\})$,
$\overline{W}(t, x) := \frac{1}{2}W'(t, x).x - W(t, x) \to +\infty$ as $|x| \to +\infty$ uniformly in t $\in \mathbb{R}$ and for any fixed $0 < r_1 < r_2$, $\inf\limits_{t\in\mathbb{R}, r_1\leq|x|\leq r_2} \dfrac{\overline{W}(t, x)}{|x|^2} \neq 0$,
(H_5) There exists constant $\xi_0 > 0$ such that

$$\liminf_{|x|\to+\infty} \frac{W(t, x)}{|x|^2} > \frac{2\pi^2 + \frac{\pi}{2}\bar{b}_1\xi_0}{\xi_0^2} + M_1$$

uniformly in $t \in [-\xi_0, \xi_0]$, where $M_1 = \sup\limits_{t\in[-\xi_0,\xi_0],|x|=1} K(t, x)$, $\overline{b}_1 = \min\{1, 2b_1\}$ and b_1 is defined in (H_2).
(H_6) $\|A\| \leq \frac{1}{4}\bar{b}_1$.

Now we state our main results.

Theorem 1.1 *Assume that (H_1)–(H_6) hold, then the system (DS) has at least one even homoclinic solution $x \in H^1(\mathbb{R}, \mathbb{R}^N)$ such that $\dot{x}(t) \to 0$ as $|t| \to +\infty$.*

Remark 1.1 From (H_5), we see that there exist $a_1 > 0$ and $R > 0$ such that

$$\frac{W(t, x)}{|x|^2} \geq \frac{2\pi^2 + \frac{\pi}{2}\bar{b}_1\xi_0 + a_1}{\xi_0^2} + M_1,$$

for all $|x| > R$ and $t \in [-\xi_0, \xi_0]$. Let $M_3 = \max\limits_{t\in[-\xi_0,\xi_0],|x|\leq R} W(t, x)$; we have

$$W(t, x) \geq (\frac{2\pi^2 + \frac{\pi}{2}\bar{b}_1\xi_0 + a_1}{\xi_0^2} + M_1)(|x|^2 - R^2) - M_3 \tag{3}$$

for all $x \in \mathbb{R}^N$ and $t \in [-\xi_0, \xi_0]$.

Moreover, $W'(t, x) = o(|x|)$ as $|x| \to 0$ uniformly in $t \in \mathbb{R}$, which implies that for any $\epsilon > 0$ there exists $\rho_0 > 0$ such that

$$|W'(t, x)| \leq \epsilon|x|, \ for\ (t, x) \in \mathbb{R} \times \mathbb{R}^N, \ |x| \leq \rho_0. \tag{4}$$

Now let us consider the following assumption:
(H_7) There exist $x_0 \in \mathbb{R}^N$ and $\xi_0 > 0$ such that

$$\int_{-\xi_0}^{\xi_0} (K(t, x_0) - W(t, x_0))dt < 0.$$

Our second result deals with the case of periodicity.

Theorem 1.2 *Assume that V is T-periodic in t, $T > 0$ and (H_1)-(H_4), (H_6) and (H_7) hold, then the system (DS) has at least one even homoclinic solution $x \in H^1(\mathbb{R}, \mathbb{R}^N)$ such that $\dot{x}(t) \to 0$ as $|t| \to +\infty$.*

Example 1.1 Consider the functions

$$K(t, x) = |x|^2 + |x|^{\frac{3}{2}}, \quad W(t, x) = (e^{-t^2} + 2\pi)|x|^2 \left(1 - \frac{1}{\ln(e + |x|)}\right).$$

A straightforward computation shows that W and K satisfy the assumptions of Theorem 1.1, but W does not satisfy the global Ambrosetti-Rabinowitz condition, and K cannot be written in the form $\frac{1}{2}(L(t)x, x)$ and does not satisfy the corresponding results in ([1], [3], [6]- [10], [12], [14], [17], [19], [21], [22]).

2 Proof of the Main Results.

By the idea of [11], we approximate an even homoclinic solution of (DS) by a solution of the following problem:

$$\begin{cases} \ddot{x}(t) + A\dot{x}(t) - K'(t,x(t)) + W'(t,x(t)) = 0 \; for \; t \; \in]-\xi,\xi[, \\ x(-t) = x(t) \; for \; t \; \in]-\xi,\xi[, x(-\xi) = x(\xi) = 0, \end{cases} \tag{5}$$

where ξ is a positive constant. The set

$$H_0^1([-\xi,\xi]) = \left\{ \begin{array}{l} x : [-\xi,\xi] \to \mathbb{R}^N / x \; is \; absolutely \; continuous, \\ x(-\xi) = x(\xi) = 0, \dot{x} \in L^2([-\xi,\xi],\mathbb{R}^N) \end{array} \right\}$$

is a Hilbert space with the norm

$$\|x\| = \left(\int_{-\xi}^{\xi} (|x(t)|^2 + |\dot{x}(t)|^2) dt \right)^{\frac{1}{2}}$$

and the associated inner product

$$\langle x, y \rangle = \int_{-\xi}^{\xi} (x(t).y(t) + \dot{x}(t).\dot{y}(t)) dt.$$

Consider the functional $I_\xi : H_0^1([-\xi,\xi]) \to \mathbb{R}$ defined by

$$I_\xi(x) = \int_{-\xi}^{\xi} \left[\frac{1}{2}|\dot{x}(t)|^2 + \frac{1}{2}(Ax(t).\dot{x}(t)) + K(t,x(t)) - W(t,x(t)) \right] dt.$$

It is easy to check that $I_\xi \in C^1(H_0^1([-\xi,\xi]),\mathbb{R})$ and by using the skew-symmetry of A, we have

$$I'_\xi(x)y = \int_{-\xi}^{\xi} [(\dot{x}(t).\dot{y}(t) - (A\dot{x}(t).y(t)) + K'(t,x(t)).y(t) - W'(t,x(t)).y(t)] \, dt. \tag{6}$$

Moreover, the critical points of I_ξ in $H_0^1([-\xi,\xi])$ are the classical solutions of (DS) in $[-\xi,\xi]$ satisfying $x(\xi) = x(-\xi) = 0$. We will obtain a critical point of I_ξ by using the Mountain Pass Theorem:

Lemma 2.1 *([16]) Let H be a real Banach space and $I \in C^1(H,\mathbb{R})$ satisfying the Palais-Smale condition. If I satisfies the following conditions:*

(i) $I(0) = 0$,

(ii) there exist constants ρ, $\alpha > 0$ such that $I_{|\partial B_\rho(0)} \geq \alpha$,

(iii) there exists $e \in H \backslash \overline{B}_\rho(0)$ such that $I(e) \leq 0$.

Then I possesses a critical value $c \geq \alpha$ given by

$$c = \inf_{g \in \Gamma} \max_{s \in [0,1]} I(g(s)),$$

where $B_\rho(0)$ is the open ball in H centered in 0, with radius ρ, $\partial B_\rho(0)$ as its boundary and

$$\Gamma = \{g \in C([0,1],H) : g(0) = 0, g(1) = e\}.$$

For a fixed $\xi > 0$, consider the subspace E_ξ of $H_0^1([-\xi,\xi])$ defined by

$$E_\xi = \left\{x \in H_0^1([-\xi,\xi]) | x(-t) = x(t),\ a.e.\ t \in]-\xi,\xi[\right\}.$$

We will proceed by successive lemmas.

Lemma 2.2 *The critical points of Φ_ξ on E_ξ are exactly the solutions of problem (5), where Φ_ξ is the restriction of I_ξ on E_ξ.*

Proof. Let

$$F_\xi = \left\{x \in H_0^1([-\xi,\xi]) / x(-t) = -x(t),\ a.e.\ t \in]-\xi,\xi[\right\}.$$

For every $x \in H_0^1([-\xi,\xi])$, set

$$y(t) = \frac{1}{2}\left(x(t) + x(-t)\right), \quad z(t) = \frac{1}{2}\left(x(t) - x(-t)\right),$$

then $y \in E_\xi$, $z \in F_\xi$ and $x = y + z$. So $H_0^1([-\xi,\xi]) = E_\xi + F_\xi$. Furthermore, for all $y \in E_\xi$, $z \in F_\xi$ we have

$$\langle y,z\rangle = \int_{-\xi}^{\xi}(y(t).z(t) + \dot{y}(t).\dot{z}(t))dt = \int_{\xi}^{-\xi}(y(-t).z(-t) + \dot{y}(-t).\dot{z}(-t))d(-t)$$

$$= \int_{-\xi}^{\xi}(y(t).(-z(t)) + (-\dot{y}(t)).\dot{z}(t))dt = -\langle y,z\rangle,$$

which implies that $\langle y,z\rangle = 0$ and then $E_\xi \perp F_\xi$. Hence $H_0^1([-\xi,\xi]) = E_\xi \oplus F_\xi$. If x is a critical point of Φ_ξ, for every $z \in E_\xi \subset C^0([-\xi,\xi],\mathbb{R}^N)$ (The space of continuous functions z on $[-\xi,\xi]$ such that $z(t) \to 0$ as $|t| \to +\infty$), then by (6) we have

$$\begin{aligned}\int_{-\xi}^{\xi}[\dot{x}(t).\dot{z}(t) - A\dot{x}(t).z(t)]dt &= \int_{-\xi}^{\xi}(\dot{x}(t) + Ax(t)).\dot{z}(t)dt \\ &= -\int_{\xi}^{\xi}(K'(t,x(t)) - W'(t,x(t))).z(t))dt\end{aligned}$$

which implies that $K'(t,x(t)) - W'(t,x(t))$ is the weak derivative of $\dot{x}(t) + Ax(t)$. Since $K, W \in C^1(\mathbb{R}\times\mathbb{R}^N,\mathbb{R})$ and $E_\xi \subset C^0([-\xi,\xi],\mathbb{R}^N)$, we see that $\dot{x}(t) + Ax(t)$ is continuous, which yields that $\dot{x}(t)$ is continuous and $x(t) \in C^2(\mathbb{R},\mathbb{R}^N)$; i.e $x \in E_\xi$ is a classical solutions of (5) if and only if it is a critical point of Φ_ξ on $H_0^1([-\xi,\xi])$. The proof of Lemma 2.2 is complete.

Lemma 2.3 *Assume that (H_2) holds. Then, for every $t \in [-\xi_0,\xi_0]$ and $x \in \mathbb{R}^N$, the following inequality holds:*

$$K(t,x) \leq M_1|x|^2 + M_2, \tag{7}$$

where M_1 is defined in (H_5) and $M_2 = \sup_{t\in[-\xi_0,\xi_0],|x|\leq 1} K(t,x)$.

Proof. To prove this lemma it suffices to show that for every $x \in \mathbb{R}^N$ and $t \in [-\xi_0, \xi_0]$ the function $(0, +\infty) \to \mathbb{R},\ s \mapsto K(t, s^{-1}x)s^2$ is nondecreasing; which is an immediate consequence of (H_2). The proof of Lemma 2.3 is complete.
By Sobolev's embedding theorem, $H^1(\mathbb{R}, \mathbb{R}^N)$ is continuously embedded into $L^p(\mathbb{R}, \mathbb{R}^N)$ for $p \in [2, +\infty]$. Thus there exists $\gamma_p > 0$ such that

$$\|x\|_{L^p(\mathbb{R},\mathbb{R}^N)} \leq \gamma_p \|x\|_{H^1(\mathbb{R},\mathbb{R}^N)}, \quad \forall\, p \in\ [2, +\infty], \forall\, x \in H^1(\mathbb{R}, \mathbb{R}^N).$$

Since $x \in H^1([-\xi, \xi])$ can be regarded as belonging to $H^1(\mathbb{R}, \mathbb{R}^N)$ if one extends it by zero in $\mathbb{R} \backslash [-\xi, \xi]$, then we have

$$\|x\|_{L^p([-\xi,\xi],\mathbb{R}^N)} \leq \gamma_p \|x\|, \quad \forall\, p \in\ [2, +\infty], \forall\, x \in H_0^1([-\xi, \xi]), \tag{8}$$

where γ_p is independent of $\xi > 0$.

Proposition 2.1 *Suppose that the conditions (H_1) - (H_6) or (H_1) - (H_4), (H_6) and (H_7) are satisfied, then for all $\xi \geq \xi_0$, the problem (5) possesses a nontrivial solution.*

Proof. **Step 1.** It is clear that $\Phi_\xi(0) = 0$. As shown in [2], a deformation lemma can be proved with condition (C) replacing the usual (PS) condition, and it turns out that the Mountain Pass Theorem in [16] holds true under condition (C), i.e., for every sequence $(y_j) \subset E_\xi$, (y_j) has a convergent subsequence if $\Phi_\xi(y_j)$ is bounded and $(1 + \|y_j\|) \left\|\Phi'_\xi(y_j)\right\|_{E^*_\xi} \to 0$ as $j \to +\infty$, where E^* is the dual space of E. Let $(y_j) \subset E_\xi$ be such that $\Phi_\xi(y_j)$ is bounded and $(1 + \|y_j\|) \left\|\Phi'_\xi(y_j)\right\|_{E^*_\xi} \to 0$ as $j \to +\infty$. Observe that for j large, it follows from (H_2) and (H_4) that there exists a constant M such that

$$M \geq \Phi_\xi(y_j) - \frac{1}{2}\Phi'_\xi(y_j)y_j =$$

$$\int_{-\xi}^{\xi} (\frac{1}{2}W'(t, y_j).y_j - W(t, y_j))dt + \int_{-\xi}^{\xi} (K(t, y_j) - \frac{1}{2}K'(t, y_j).y_j)dt$$

$$\geq \int_{-\xi}^{\xi} \overline{W}(t, y_j(t))dt. \tag{9}$$

By negation, if (y_j) is not bounded, passing to a subsequence if necessary we may assume that $\|y_j\| \to +\infty$ as $j \to +\infty$. Set $z_j = \frac{y_j}{\|y_j\|}$, then $\|z_j\| = 1$ and by (8) one has

$$\|z_j\|_{L^p([-\xi,\xi],\mathbb{R}^N)} \leq \gamma_p \|z_j\| = \gamma_p,\ \forall p \in\ [2, +\infty]. \tag{10}$$

By (H_2), (H_4) and (H_6) we have

$$\begin{aligned} 2M \geq 2\Phi_\xi(y_j) \quad &= \quad \int_{-\xi}^{\xi} |\dot{y_j}(t)|^2 dt - \int_{-\xi}^{\xi} (A\dot{y}_j(t).y_j(t))dt + 2\int_{-\xi}^{\xi} K(t, y_j(t))dt \\ &- \quad 2\int_{-\xi}^{\xi} W(t, y_j(t))dt \geq \overline{b}_1 \|y_j\|^2 - \|A\|\|y_j\|^2 - \int_{-\xi}^{\xi} W'(t, y_j(t)).y_j(t)dt \\ &\geq \quad \|y_j\|^2 \left(\overline{b}_1 - \frac{\overline{b}_1}{4} - \int_{-\xi}^{\xi} \frac{W'(t, y_j(t)).y_j(t)}{\|y_j\|^2} dt \right), \end{aligned}$$

where $\overline{b}_1 = \min\{1, 2b_1\} > 0$. Thus implies that

$$\lim_{j\to+\infty} \int_{-\xi}^{\xi} \frac{W'(t, y_j(t)).y_j(t)}{\|y_j\|^2} dt \geq \frac{3}{4}\overline{b}_1. \tag{11}$$

Set

$$f(r) := \inf\left\{\overline{W}(t,x)|\ t \in [-\xi,\xi]\ and\ x \in \mathbb{R}^N\ with\ |x| \geq r\right\}$$

for $r \geq 0$. By (H_4) one has

$$f(r) \to +\infty\ as\ r \to +\infty.$$

For $0 \leq a \leq b$ let

$$\Omega_j(a,b) = \{t \in [-\xi,\xi] \mid a < y_j(t) \leq b\}$$

and

$$C_b^a = \inf\left\{\frac{\overline{W}(t,x)}{|x|^2}, t \in [-\xi,\xi]\ and\ a < |x| \leq b\right\}.$$

Obviously, we have

$$\overline{W}(t, y_j(t)) \geq C_b^a |y_j(t)|^2,\ for\ all\ t \in\ \Omega_j(a,b). \tag{12}$$

By (9) and (12) it follows

$$M \geq \int_{-\xi}^{\xi} \overline{W}(t,y_j)dt = \int_{\Omega_j(0,a)} \overline{W}(t,y_j)dt + \int_{\Omega_j(a,b)} \overline{W}(t,y_j)dt + \int_{\Omega_j(b,\infty)} \overline{W}(t,y_j(t))dt$$

$$\geq \int_{\Omega_j(0,a)} \overline{W}(t,y_j)dt + C_b^a \int_{\Omega_j(a,b)} |y_j|^2 dt + f(b) meas(\Omega_j(b,\infty)), \tag{13}$$

which implies that

$$meas(\Omega_j(b,\infty)) \leq \frac{M}{f(b)} \to 0\ \ as\ b \to +\infty\ uniformly\ in\ j. \tag{14}$$

For any fixed $0 < a < b$ and by (8), (10) and (14) we have

$$\int_{\Omega_j(b,\infty)} |z_j|^2 dt \leq \|z_j\|^2_{L^\infty([-\xi,\xi])} meas(\Omega_j(b,\infty)) \tag{15}$$

$$\leq \gamma_\infty^2 meas(\Omega_j(b,\infty)) \to 0$$

as $b \to +\infty$ uniformly in j. Moreover, by (13) we obtain

$$\int_{\Omega_j(a,b)} |z_j|^2 dt = \frac{1}{\|y_j\|^2} \int_{\Omega_j(a,b)} |y_j|^2 dt \leq \frac{M}{C_b^a \|y_j\|^2} \to 0 \tag{16}$$

as $j \to +\infty$. Let $0 < \varepsilon < \frac{\overline{b}_1}{4}$, by (H_3) there exist $a_\varepsilon > 0$ such that

$$|W'(t,x)| \leq \frac{\varepsilon}{\gamma_2^2}|x|\ \ for\ all\ |x| \leq a_\varepsilon.$$

Consequently,

$$\int_{\Omega_j(0,a_\varepsilon)} \frac{|W'(t,y_j)||z_j|^2}{|y_j|} dt \leq \frac{\varepsilon}{\gamma_2^2} \int_{\Omega_j(0,a_\varepsilon)} |z_j|^2 dt \leq \varepsilon. \tag{17}$$

By (15) we can take b_ε large such that

$$\int_{\Omega_j(b_\varepsilon,\infty)} |z_j|^2 dt \leq \frac{\varepsilon}{C_0}.$$

Hence, by (H_3) we obtain

$$\int_{\Omega_j(b_\varepsilon,\infty)} \frac{|W'(t,y_j)||z_j|^2}{|y_j|} dt \leq C_0 \int_{\Omega_j(b_\varepsilon,\infty)} |z_j|^2 dt \leq \varepsilon. \tag{18}$$

By (16) there is j_0 such that

$$\int_{\Omega_j(a_\varepsilon,b_\varepsilon)} \frac{|W'(t,y_j)||z_j|^2}{|y_j|} dt \leq C_0 \int_{\Omega_j(a_\varepsilon,b_\varepsilon)} |z_j|^2 dt \leq \varepsilon, \tag{19}$$

for all $j \geq j_0$. Therefore, combining (17)-(19) we have

$$\int_{-\xi}^{\xi} \frac{W'(t,y_j).y_j}{\|y_j\|^2} dt \leq \int_{[-\xi,\xi]\setminus\{t\in[-\xi,\xi]/|y_j(t)|=0\}} \frac{|W'(t,y_j)||z_j|^2}{|y_j|} dt \leq 3\varepsilon < \frac{3}{4}\overline{b}_1,$$

which contradicts (11). Hence, (y_j) is bounded in E_ξ. Going if necessary to a subsequence, we can assume that there exists $y \in E_\xi$ such that $y_j \rightharpoonup y$ as $j \to +\infty$ in E_ξ, which implies that $y_j \to y$ as $j \to +\infty$ uniformly on $[-\xi,\xi]$. Hence $(\Phi'_\xi(y_j) - \Phi'_\xi(y))(y_j - y) \to 0$, $\|y_j - y\|_{L^2([-\xi,\xi],\mathbb{R}^N)} \to 0$ and $\int_{-\xi}^{\xi} (V'(t,y_j(t)) - V'(t,y(t))\,.(y_j(t) - y(t))dt \to 0$ and by the Hölder inequality, we have

$$\left| \int_{-\xi}^{\xi} (A\dot{y}_j(t) - A\dot{y}(t)).(y_j(t) - y(t))dt \right| \leq \|A\|\|\dot{y}_j - \dot{y}\|_{L^2}\|y_j - y\|_{L^2} \to 0$$

as $j \to +\infty$. On the other hand, an easy computation shows that

$$\begin{aligned}
&(\Phi'_\xi(y_j) - \Phi'_\xi(y))(y_j - y) \\
&= \|\dot{y}_j - \dot{y}\|^2_{L^2([-\xi,\xi],\mathbb{R}^N)} - \int_{-\xi}^{\xi} (A\dot{y}_j(t) - A\dot{y}(t).y_j(t) - y(t))dt \\
&\quad - \int_{-\xi}^{\xi} (V'(t,y_j(t))) - V'(t,y(t))).(y_j(t) - y(t))dt.
\end{aligned}$$

and so $\|\dot{y}_j - \dot{y}\|_{L^2([-\xi,\xi],\mathbb{R}^N)} \to 0$. Consequently, $\|y_j - y\| \to 0$ as $j \to +\infty$. Hence, Φ_ξ satisfies condition (C).

Step 2. Now, let us show that Φ_ξ satisfies assumption (ii) of Lemma 2.1. By (H_3) there exists a constant $\rho_0 > 0$ such that

$$|W'(t,x)| \leq \frac{\overline{b}_1}{2\gamma_2^2}|x|, \ \forall\, t \in \mathbb{R}, \ \forall\, |x| \leq \rho_0.$$

It follows that

$$|W(t,x)| = \left|\int_0^1 W'(t,sx).xds\right| \leq \int_0^1 |W'(t,sx).x|ds$$

$$\leq \frac{\overline{b}_1}{2\gamma_2^2}\int_0^1 |x|^2 sds = \frac{\overline{b}_1}{4\gamma_2^2}|x|^2, \forall\, t\in\mathbb{R},\ \forall\, |x|\leq\rho_0. \tag{20}$$

Let $\rho = \frac{\rho_0}{\gamma_\infty}$ and $S=\{x\in E_\xi/\|x\|=\rho\}$. By (8), we have $\|x\|_{L^\infty([-\xi,\xi],\mathbb{R}^N)}\leq\rho_0$, for all $x\in S$, which together with (20), (H_2) and (H_6) implies that

$$\begin{aligned}\Phi_\xi(x) &= \frac{1}{2}\int_{-\xi}^{\xi}|\dot{x}(t)|^2dt - \frac{1}{2}\int_{-\xi}^{\xi}(A\dot{x}(t).x(t))dt + \int_{-\xi}^{\xi}K(t,x(t))dt - \int_{-\xi}^{\xi}W(t,x(t))dt\\ &\geq \left(\frac{\overline{b}_1}{2}-\frac{\overline{b}_1}{8}-\frac{\overline{b}_1}{4}\right)\|x\|^2 = \frac{\overline{b}_1}{8}\rho^2 := \alpha,\ \forall\, x\in S.\end{aligned}$$

Step 3. It remains to prove that Φ_ξ satisfies assumption(iii) of Lemma 2.1. If (H_5) holds, let

$$e(t) = \begin{cases} m|\sin(\omega t)|e_1, if\ t\in[-\xi_0,\xi_0],\\ 0, if\ t\in[-\xi,\xi]\backslash[-\xi_0,\xi_0],\end{cases}$$

where $\omega=\frac{2\pi}{\xi_0}$, $e_1=(1,0,...,0)$ and $m\in\mathbb{R}\setminus\{0\}$. By the Hölder inequality, (H_6), Remark 1.1 and Lemma 2.3 we have

$$\begin{aligned}\Phi_\xi(e) &= \frac{1}{2}\int_{-\xi}^{\xi}|\dot{e}(t)|^2dt + \frac{1}{2}\int_{-\xi}^{\xi}(Ae(t).\dot{e}(t))dt + \int_{-\xi}^{\xi}K(t,e(t))dt - \int_{-\xi}^{\xi}W(t,e(t))dt\\ &= \frac{1}{2}m^2\omega^2\int_{-\xi_0}^{\xi_0}|\cos(\omega t)|^2dt + \frac{1}{2}m^2\omega\int_{-\xi_0}^{\xi_0}(A|\sin(\omega t)|e_1.|\cos(\omega t)|e_1)dt\\ &+ \int_{-\xi_0}^{\xi_0}K(t,m|\sin(\omega t)|e_1)dt - \int_{-\xi_0}^{\xi_0}W(t,m|\sin(\omega t)|e_1)dt\\ &\leq \frac{1}{2}m^2\omega^2\int_{-\xi_0}^{\xi_0}|\cos(\omega t)|^2dt + m^2\omega\ \|A\|\xi_0 + M_1m^2\int_{-\xi_0}^{\xi_0}|\sin(\omega t)|^2dt + 2\xi_0M_2\\ &- \left(\frac{2\pi^2+\frac{\pi}{2}\overline{b}_1\xi_0+a_1}{\xi_0^2}+M_1\right)m^2\int_{-\xi_0}^{\xi_0}|\sin(\omega t)|^2dt\\ &+ 2\xi_0\left(R^2\left(\frac{2\pi^2+\frac{\pi}{2}\overline{b}_1\xi_0+a_1}{\xi_0^2}+M_1\right)+M_3\right)\\ &\leq m^2\left(-\frac{\pi\overline{b}_1}{2}-\frac{2a_1}{\xi_0}\right) + 2\xi_0\left(M_2+R^2\left(\frac{2\pi^2+\frac{\pi}{2}\overline{b}_1\xi_0+a_1}{\xi_0^2}+M_1\right)+M_3\right)\to-\infty\end{aligned}$$

as $m\to\infty$. If (H_7) holds, set $g(s)=s^{-2}W(t,sx_0)$ for $s>0$. Then it follows from (H_4) that

$$g'(s) = s^{-3}[-2W(t,sx_0)+W'(t,sx_0).sx_0] > 0,\ \ for\ t\in\ \mathbb{R},\ s>0.$$

Integrating the above from 1 to $\lambda>1$, we obtain

$$W(t,\lambda x_0)\geq\lambda^2W(t,x_0), for\ t\in\mathbb{R},\ \lambda>1. \tag{21}$$

By (H_2), it is easy to show that

$$K(t, \lambda x_0) \leq \lambda^2 K(t, x_0), for\ t \in \mathbb{R},\ \lambda > 1. \tag{22}$$

From (21) and (22) we have

$$\Phi_\xi(\lambda x_0) = \int_{-\xi}^{\xi} [K(t, \lambda x_0) - W(t, \lambda x_0)]dt$$

$$\leq \lambda^2 \left(\int_{-\xi}^{\xi} K(t, x_0)dt - \int_{-\xi}^{\xi} W(t, x_0)dt \right). \tag{23}$$

Choose $\sigma > 1$ such that $|\sigma x_0|\sqrt{2\xi_0} > \rho$ and let

$$e(t) = \begin{cases} \sigma x_0, if\ t \in [-\xi_0, \xi_0], \\ 0, if\ t \in [-\xi, \xi] \backslash [-\xi_0, \xi_0]. \end{cases}$$

By (23) and (H_7) we have

$$\begin{aligned} \Phi_\xi(e) &= \int_{-\xi}^{\xi} (K(t, e(t)) - W(t, e(t)))dt \\ &= \int_{-\xi_0}^{\xi_0} (K(t, \sigma x_0) - W(t, \sigma x_0))dt \\ &\leq \sigma^2 \int_{-\xi_0}^{\xi_0} (K(t, x_0) - W(t, x_0))dt < 0. \end{aligned}$$

All the assumptions of Lemma 2.1 are satisfied, so for all $\xi \geq \xi_0$, Φ_ξ possesses a critical value $c_\xi \geq \alpha > 0$ defined by

$$c_\xi \equiv \inf_{g \in \Gamma_\xi} \max_{s \in [0,1]} \Phi_\xi(g(s)),$$

where

$$\Gamma_\xi = \{g(t) \in C([0, 1], E_\xi)/g(0) = 0, g(1) = e\}.$$

Hence, for every $\xi > 0$, there exists $x_\xi \in E_\xi$ such that

$$\Phi_\xi(x_\xi) = c_\xi,\ \ \Phi'_\xi(x_\xi) = 0.$$

Since $c_\xi > 0$, x_ξ is nontrivial. The proof of Proposition 2.1 is complete.

Take a sequence $(\xi_n)_{n\in\mathbb{N}}$ with $\xi_0 \leq \xi_1 \leq \xi_2 \leq ... \to \infty$ and consider problem (5) on E_{ξ_n}, i.e.

$$\begin{cases} \ddot{x}(t) + A\dot{x}(t) - K'(t, x(t)) + W'(t, x(t)) = 0, for\ \ t \in]-\xi_n, \xi_n[, \\ x(-t) = x(t), for\ \ t \in]-\xi_n, \xi_n[, x(-\xi_n) = x(\xi_n) = 0. \end{cases} \tag{24}$$

Then by Proposition 2.1, for each $n \in \mathbb{N}$, (24) possesses a nontrivial solution x_n. Let $C^p_{loc}(\mathbb{R}, \mathbb{R}^N)$ $(p \in \mathbb{N})$ denote the space of C^p functions under the topology of almost uniformly convergence of functions and all derivatives up to order p. We have the following result.

Lemma 2.4 *The sequence* (x_n) *possesses a subsequence also denoted by* (x_n) *which converges to a* C^2 *function* x *in* $C^2_{loc}(\mathbb{R}, \mathbb{R}^N)$.

Proof. Let $q > k$, as any function in E_{ξ_k} can be regarded as belonging to E_{ξ_q} if one extends it by zero in $[-\xi_q, \xi_q] \backslash [-\xi_k, \xi_k]$, we have $\Gamma_{\xi_k} \subset \Gamma_{\xi_q}$ which implies $c_{\xi_q} \le c_{\xi_k}$. Thus $c_{\xi_n} \le c_{\xi_0}$ for any $n \in \mathbb{N}$.

As $\Phi_{\xi_n}(x_n) \le c_{\xi_0}$ and $(1 + \|x_n\|) \left\| \Phi'_{\xi_n}(x_n) \right\| = 0$, just as in the proof of condition (C) in Proposition 2.1, it is easy to prove that (x_n) is bounded uniformly in n. Therefore, there is a constant $C_1 > 0$ such that:

$$\|x_n\| \le C_1, \forall n \in \mathbb{N}. \tag{25}$$

Arguing as in Theorem 2.1 in [11], we conclude from the fact

$$|x_n(t_2) - x_n(t_1)| \le \int_{t_1}^{t_2} |\dot{x}(t)| dt \le (t_2 - t_1)^{1/2} \left(\int_{t_1}^{t_2} |\dot{x}(t)|^2 dt \right)^{1/2}$$

that the sequence (x_n) is equicontinuous on every interval $[-\xi_n, \xi_n]$. By (25) and Arzela-Ascoli theorem, the sequence (x_n) has a uniformly convergent subsequence on each $[-\xi_n, \xi_n]$.

Let $(x^1_{n_k})$ be a subsequence of (x_n) that converges on $[-\xi_1, \xi_1]$. Then $(x^1_{n_k})$ is equicontinuous and uniformly bounded on $[-\xi_2, \xi_2]$. So we can choose a subsequence $(x^2_{n_k})$ of $(x^1_{n_k})$ that converges uniformly on $[-\xi_2, \xi_2]$. Repeat this procedure for all n and take the diagonal sequence $(x^k_{n_k})$. It is obvious that $(x^k_{n_k})_k$ is a subsequence of $(x^i_{n_k})$ for any $1 \le i \le k$. Hence, it converges uniformly to a function $x(t)$ on any bounded interval. In the following, for simplicity, we denote the subsequence $(x^k_{n_k})$ also by (x_n). As (x_n) satisfies

$$\ddot{x}_n(t) + A\dot{x}_n(t) + V'(t, x_n(t)) = 0, \tag{26}$$

we conclude that the sequence $(\ddot{x}_n)$ and then also $(\dot{x}_n)$ converge uniformly on any bounded intervals. It is easy to see that

$$x_n(t) = \int_{-\xi_n}^{t} (t - s)\ddot{x}_n(s) ds,$$

then $x \in C^2(\mathbb{R}, \mathbb{R}^N)$ and $\ddot{x}_n \to \ddot{x}$ uniformly on any bounded intervals. Hence, by passing to the limit in (26) we conclude that x solves (DS). As x_n is even, the same is true for their limit x. The proof of Lemma 2.4 is complete.

Proof of Theorem 1.1. We have shown that x satisfies (DS). It remains to prove that x is nontrivial and homoclinic to 0.

Step 1. Let us show that x is nontrivial. Consider the function Ψ defined by $\Psi(0) = 0$ and for $s > 0$

$$\Psi(s) = \max_{t \in \mathbb{R}, 0 < |x| \le s} \frac{W'(t, x).x}{|x|^2}.$$

Then Ψ is a continuous, nondecreasing function and $\Psi(s) \ge 0$ for $s \ge 0$. The definition of Ψ implies that

$$\int_{-\xi_n}^{\xi_n} W'(t, x_n(t)).x_n(t) dt \le \Psi(\|x_n\|_{L^\infty([-\xi_n, \xi_n], \mathbb{R}^N)}) \|x_n\|^2, \tag{27}$$

for every $n \in \mathbb{N}$. Since $\Phi'_{\xi_n}(x_n).x_n = 0$, we have

$$\int_{-\xi_n}^{\xi_n} W'(t, x_n(t)).x_n(t)dt =$$

$$\int_{-\xi_n}^{\xi_n} |\dot{x}_n(t)|^2 dt - \int_{-\xi_n}^{\xi_n} (A\dot{x}_n(t).x_n(t))dt + \int_{-\xi_n}^{\xi_n} K'(t, x_n(t)).x_n(t)dt. \tag{28}$$

From (27), (28), (H_2) and (H_6), we obtain

$$\begin{aligned}
\Psi(\|x_n\|_{L^\infty([-\xi_n,\xi_n],\mathbb{R}^N)})\|x_n\|^2 &\geq \int_{-\xi_n}^{\xi_n} |\dot{x}_n(t)|^2 dt - \int_{-\xi_n}^{\xi_n} (A\dot{x}_n(t).x_n(t))dt \\
&+ \int_{-\xi_n}^{\xi_n} K'(t, x_n(t)).x_n(t)dt \\
&\geq \int_{-\xi_n}^{\xi_n} |\dot{x}_n(t)|^2 dt + b_1 \int_{-\xi_n}^{\xi_n} |x_n(t)|^2 dt - \|A\|\|x_n\|^2 \\
&\geq (\min\{1, b_1\} - \|A\|)\|x_n\|^2.
\end{aligned}$$

Since $\|x_n\| > 0$, it follows that

$$\Psi(\|x_n\|_{L^\infty([-\xi_n,\xi_n],\mathbb{R}^N)}) \geq (\min\{1, b_1\} - \|A\|) > 0.$$

If $\|x_n\|_{L^\infty([-\xi_n,\xi_n],\mathbb{R}^N)} \to 0$ as $n \to \infty$, we would have $\Psi(0) \geq (\min\{1, b_1\} - \|A\|) > 0$, a contradiction. Passing to a subsequence of (x_n) if necessary, there is a constant $C_3 > 0$ such that

$$\|x_n\|_{L^\infty([-\xi_n,\xi_n],\mathbb{R}^N)} \geq C_3 \tag{29}$$

for every $n \in \mathbb{N}$. Now, suppose $x \equiv 0$ and let x_n be the function defined in Lemma 2.4, extended by 0 in $\mathbb{R} \setminus [-\xi_n, \xi_n]$. For $A > 0$ we have

$$\begin{aligned}
\|x_n\|^2 &= \int_{-\xi_n}^{\xi_n} (|\dot{x}_n(t)|^2 + |x_n(t)|^2)dt \\
&= \int_{\mathbb{R}} (|\dot{x}_n(t)|^2 + |x_n(t)|^2)dt \\
&= \int_{-A}^{A} (|\dot{x}_n(t)|^2 + |x_n(t)|^2)dt + \int_{\mathbb{R}\setminus[-A,A]} (|\dot{x}_n(t)|^2 + |x_n(t)|^2)dt \to 0 \ as\ A, n \to \infty.
\end{aligned}$$

which is in contradiction with (29). Hence x is nontrivial.

Step 2. We prove that $x(t) \to 0$ as $|t| \to +\infty$. By the argument of Lemma 2.4, for each $i \in \mathbb{N}$ there is $n_i \in \mathbb{N}$ such that for all $n \geq n_i$ we have

$$\int_{-\xi_i}^{\xi_i} (|x_n(t)|^2 + |\dot{x}_n(t)|^2)dt \leq \|x_n\|^2 \leq C_1^2.$$

Letting $n \to +\infty$, we obtain

$$\int_{-\xi_i}^{\xi_i} (|x(t)|^2 + |\dot{x}(t)|^2)dt \leq C_1^2.$$

As $i \to +\infty$, we have
$$\int_{-\infty}^{+\infty} (|x(t)|^2 + |\dot{x}(t)|^2)dt \le C_1^2.$$
Hence, we get
$$\int_{|t|\ge r} (|x(t)|^2 + |\dot{x}(t)|^2)dt \to 0 \ as\ r \to +\infty. \tag{30}$$

By Corollary 2.2 in [19], we have
$$|x(t)|^2 \le \int_{t-1}^{t+1} (|x(s)|^2 + |\dot{x}(s)|^2)ds \tag{31}$$
for every $t \in \mathbb{R}$. By (30) and (31) we conclude that
$$x(t) \to 0 \ \ as\ |t| \to \infty.$$

Step 3. We have to show that $\dot{x}(t) \to 0 \ \ as\ |t| \to \infty$. By Corollary 2.2 in [19] we have
$$|\dot{x}(t)|^2 \le \int_{t-1}^{t+1} (|x(s)|^2 + |\dot{x}(s)|^2)ds + \int_{t-1}^{t+1} |\ddot{x}(s)|^2 ds,$$
for every $t \in \mathbb{R}$. Since $x \in H^1(\mathbb{R}, \mathbb{R}^N)$, we get
$$\int_{t-1}^{t+1} (|x(s)|^2 + |\dot{x}(s)|^2)ds \to 0 \ as\ |t| \to \infty.$$
Hence, it suffices to prove that
$$\int_{t-1}^{t+1} |\ddot{x}(s)|^2 ds \to 0 \ as\ |t| \to \infty. \tag{32}$$
By (DS), we have
$$\begin{aligned}\int_{t-1}^{t+1} |\ddot{x}(s)|^2 ds &= \int_{t-1}^{t+1} |A\dot{x}(s) + V'(t, x(s))|^2 ds \\ &\le \|A\|^2 \int_{t-1}^{t+1} |\dot{x}(s)|^2 ds + \int_{t-1}^{t+1} |V'(t, x(s))|^2 ds \\ &+ 2\|A\| \left(\int_{t-1}^{t+1} |\dot{x}(s)|^2 ds\right)^{\frac{1}{2}} \left(\int_{t-1}^{t+1} |V'(t, x(s))|^2 ds\right)^{\frac{1}{2}}.\end{aligned}$$
Since $\int_{t-1}^{t+1} |\dot{x}(s)|^2 ds \to 0$ as $|t| \to \infty$, $x(t) \to 0$ as $|t| \to \infty$ and $V'(t, x) \to 0$ as $|x| \to 0$ uniformly in $t \in \mathbb{R}$, then (32) follows. The proof of Theorem 1.1 is complete.

Proof of Theorem 1.2. Let
$H^1_{nT}(\mathbb{R}, \mathbb{R}^N) = \{x : \mathbb{R} \to \mathbb{R}^N,\ 2nT - periodic,\ x,\ \dot{x} \in L^2([-nT, nT], \mathbb{R}^N)\ and$
$x(-nT) = x(nT) = 0\}$. Consider the family of functionals $(\Phi_n)_{n\ge1}$ defined on E_{nT} by
$$\Phi_n(x) = \int_{-nT}^{nT} \left[\frac{1}{2}|\dot{x}(t)|^2 + \frac{1}{2}(Ax(t).\dot{x}(t)) + K(t, x(t)) - W(t, x(t))\right] dt, \tag{33}$$

where

$$E_{nT} = \left\{x \in H^1_{nT}(\mathbb{R}, \mathbb{R}^N)/x(-t) = x(t),\ a.e.t\ \in \mathbb{R}\right\}.$$

Arguing as in the proof of Theorem 1.1, we prove that assumptions (H_1)-(H_4), (H_6) and (H_7) imply that for every positive integer n, the problem

$$\begin{cases} \ddot{x}(t) + A\dot{x}(t) - K'(t, x(t)) + W'(t, x(t)) = 0,\ for\ t \in]-nT, nT[, \\ x(-t) = x(t), for\ t \in]-nT, nT[, x(-nT) = x(nT) = 0, \end{cases} \tag{34}$$

possesses a solution x_n. Moreover, the sequence (x_n) converges uniformly on any bounded interval to a homoclinic solution $x \in H^1(\mathbb{R}, \mathbb{R}^N)$ satisfying $\dot{x}(t) \to 0$ as $|t| \to +\infty$. It remains to prove that $x(t) \not\equiv 0$. In the same way as in the proof of Theorem 1.1 it is easy to prove that there is a constant $C_4 > 0$ such that

$$\|x_n\|_{L^\infty([-nT,nT],\mathbb{R}^N)} \geq C_4 \tag{35}$$

for every $n \in \mathbb{N}$. Moreover, for all $j \in \mathbb{N}$, $t \mapsto x^j_n(t) = x_n(t + jT)$ is also a 2nT-periodic solution of problem (34). Hence, if the maximum of $|x_n|$ occurs in $\theta_n \in [-nT, nT]$ then the maximum of $|x^j_n|$ occurs in $\tau^j_n = \theta_n - jT$. Then there exists a $j_n \in \mathbb{Z}$ such that $\tau^{j_n}_n \in [-T, T]$. Consequently,

$$\|x^{j_n}_n\|_{L^\infty([-nT,nT],\mathbb{R}^N)} = \max_{t\in[-T,T]} |x^{j_n}_n(t)|.$$

Suppose contrary to our claim, that $x \equiv 0$. Then

$$\|x^{j_n}_n\|_{L^\infty([-nT,nT],\mathbb{R}^N)} = \max_{t\in[-T,T]} |x^{j_n}_n(t)| \to 0,$$

which contradicts (35). Then the proof of Theorem 1.2 is complete.

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Estimating the Bounds for the General 4-D Continuous-Time Autonomous System

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Abstract: In the present paper, the general 4-D continuous-time system is considered and the estimate of the upper bound of such a system is investigated, using the multivariable functions analysis. Especially, sufficient conditions for this system to be contained in a four-dimensional ellipsoidal surface are obtained. The results obtained in this investigation generalize all the existing results in the relevant literature concerning the finding of an upper bound for the fourth order dynamical system.

Keywords: *4-D continuous-time system; upper bounds.*

Mathematics Subject Classification (2010): 65P20, 65P30, 65P40.

1 Introduction

Since Lorenz discovered chaos in a simple system of three autonomous ordinary differential equations in order to describe the simplified Rayleigh–Benard problem in 1963 [12], the analysis of dynamics of 3-D chaotic and 4-D hyperchaotic systems has been a focal point of renewed interest for many researchers [2, 3, 5, 6, 8, 13, 15, 17, 19, 21, 22, 26, 27]. Hyperchaos is characterized as a chaotic system with more than one positive exponent, this implies that its dynamics are expended in several different directions simultaneously. Thus, hyperchaotic systems have more complex dynamical behaviors than ordinary chaotic systems. As we know, there are many hyperchaotic systems discovered in the four-dimensional social and economical systems. Typical examples are 4-D hyperchaotic Chua's circuit [1], 4-D hyperchaotic Rôsslor system [18] and 4-D hyperchaotic Lorenz-Haken system [14]. Since hyperchaotic system has the theoretical and practical applications in technological fields, such as secure communications, lasers, nonlinear

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circuits, neural networks, generation, control, synchronization, it has recently become a central topic in nonlinear sciences research.

The estimate of the bound for a chaotic system is of great importance for chaos control, chaos synchronization, and their applications [4], where the concepts of ultimate bound and attractive set of system serve as excellent tools for analysis of the qualitative behavior of a chaotic system. Such an estimation is quite difficult to achieve technically. Notwithstanding the difficulty, during the past 40 years or so, many good and interesting results on this topic have been obtained for some 3-D continuous-time systems [7, 9, 10, 16, 24].

In recent years, the study of the boundedeness of 4-D dynamical systems have attracted the attention of many engineers, physicists and mathematicians. For example in [11], the ultimate bound and positively invariant set for the 4-D hyperchaotic Lorenz-Haken system were investigated. In [20] the estimation of the bounds for the 4-D hyperchaotic Lorenz-Stenflo system was also obtained. Recently, the boundedness of the generalized 4-D hyperchaotic model containing Lorenz-Stenflo and Lorenz-Haken systems was done in [23] and the boundedness of a kind of hyperchaotic systems that have wide applications in the secure communications was also investigated in [25]. In the present paper, by using the multivariable functions analysis, we generalize all the existing results in the relevant literature concerning the finding of an upper bound for the general 4-D continuous-time system. In particular, we find sufficient conditions for this system to be contained in a four-dimensional ellipsoidal set.

Let us consider the general 4-D continuous-time autonomous system

$$\begin{cases} x^{'} = f\left(x, y, z, w, \delta\right), \\ y' = g\left(x, y, z, w, \delta\right), \\ z' = h\left(x, y, z, w, \delta\right), \\ w' = k\left(x, y, z, w, \delta\right), \end{cases} \tag{1}$$

where f, g, h and k are real functions and $\delta \in \mathbb{R}^m$ is the bifurcation parameter. Assume that system (1) has at least one equilibrium point, so bounded orbits are possible. Without loss of generality we can assume that the origin is an equilibrium point, i.e., $f\left(0,0,0,0,\delta\right) = g\left(0,0,0,0,\delta\right) = h\left(0,0,,0,\delta\right) = k\left(0,0,0,0,\delta\right) = 0$.

2 The Estimate of the Bound for the General 4-D Dynamical System

To study the estimate of the bound for the general system (1), we define the following Lyapunov function

$$V\left(x, y, z, w\right) =$$

$$\frac{\left(x - \alpha\left(x, y, z, w\right)\right)^2 + \left(y - \beta\left(x, y, z, w\right)\right)^2 + \left(z - \gamma\left(x, y, z, w\right)\right)^2 + \left(w - \theta\left(x, y, z, w\right)\right)^2}{2}, \tag{2}$$

where $\left(\alpha\left(x, y, z, w\right), \beta\left(x, y, z, w\right), \gamma\left(x, y, z, w\right), \theta\left(x, y, z, w\right)\right) \in \mathbb{R}^4$ are real functions, in which the derivative of (2) along the orbits of system (1) is given by

$$\frac{dV}{dt} = \left(x - \alpha\right)\left(x' - \alpha'\right) + \left(y - \beta\right)\left(y' - \beta'\right) + \left(z - \gamma\right)\left(z' - \gamma'\right) + \left(w - \theta\right)\left(w' - \theta'\right), \tag{3}$$

where

$$\begin{cases} \alpha^{'} = \dfrac{\partial\alpha}{\partial x}x' + \dfrac{\partial\alpha}{\partial y}y' + \dfrac{\partial\alpha}{\partial z}z' + \dfrac{\partial\alpha}{\partial w}w' = \psi_1 f + \psi_2 g + \psi_3 h + \psi_4 k, \\ \beta' = \dfrac{\partial\beta}{\partial x}x' + \dfrac{\partial\beta}{\partial y}y' + \dfrac{\partial\beta}{\partial z}z' + \dfrac{\partial\beta}{\partial w}w' = \mu_1 f + \mu_2 g + \mu_3 h + \mu_4 k, \\ \gamma' = \dfrac{\partial\gamma}{\partial x}x' + \dfrac{\partial\gamma}{\partial y}y' + \dfrac{\partial\gamma}{\partial z}z' + \dfrac{\partial\gamma}{\partial w}w' = \xi_1 f + \xi_2 g + \mu\xi_3 h + \xi_4 k, \\ \theta' = \dfrac{\partial\theta}{\partial x}x' + \dfrac{\partial\theta}{\partial y}y' + \dfrac{\partial\theta}{\partial z}z' + \dfrac{\partial\theta}{\partial w}w' = \zeta_1 f + \zeta_2 g + \zeta_3 h + \zeta_4 k. \end{cases} \tag{4}$$

Therefore, we have

$$\frac{dV}{dt} = c_1(x,y,z,w)\,x - \omega x^2 + c_2(x,y,z,w)\,y - \varphi y^2 + c_3(x,y,z,w)\,z - \phi z^2 + \\ c_4(x,y,z,w)\,w - \eta w^2 + c_5(x,y,z,w)\,, \tag{5}$$

where

$$\begin{cases} c_1(x,y,z,w) = f - \psi_1 f - \psi_2 g - \psi_3 h - \psi_4 k + \omega x, \\ c_2(x,y,z,w) = g - \mu_1 f - \mu_2 g - \mu_3 h - \mu_4 k + \varphi y, \\ c_3(x,y,z,w) = h - \xi_1 f - \xi_2 g - \mu\xi_3 h - \xi_4 k + \phi z, \\ c_4(x,y,z,w) = k - \zeta_1 f - \zeta_2 g - \zeta_3 h - \zeta_4 k + \eta w, \\ c_5(x,y,z,w) = c_6(x,y,z,w) + c_7(x,y,z,w)\,, \\ c_6(x,y,z,w) = -\alpha f - \beta g - \gamma h - \theta k + \alpha(\psi_1 f + \psi_2 g + \psi_3 h + \psi_4 k)\,, \\ c_7(x,y,z,w) = \beta(\mu_1 f + \mu_2 g + \mu_3 h + \mu_4 k) + \\ \gamma(\xi_1 f + \xi_2 g + \xi_3 h + \xi_4 k) + \theta(\zeta_1 f + \zeta_2 g + \zeta_3 h + \zeta_4 k)\,. \end{cases} \tag{6}$$

Assume that the equation (5) has the form

$$\frac{dV}{dt} = -\omega(x-\alpha_1)^2 - \varphi(y-\beta_1)^2 - \phi(z-\gamma_1)^2 - \eta(w-\theta_1)^2 + r, \tag{7}$$

where ω, φ, ϕ, η and r are strictly positive constants, α_1, β_1, γ_1, θ_1 are unknown constants and it should be determined in which the equation $\dfrac{dV}{dt} = 0$ determines an ellipsoid in $\mathbb{R}^4$.

Equation (7) is equivalent to

$$\frac{dV}{dt} = -\omega x^2 + 2\omega\alpha_1 x - \varphi y^2 + 2\varphi\beta_1 y - \phi z^2 + 2\phi\gamma_1 z - \eta w^2 + 2\eta\theta_1 w - \omega\alpha_1^2 - \varphi\beta_1^2 - \phi\gamma_1^2 - \eta\theta_1^2 + r. \tag{8}$$

By identification with (5) we get

$$\begin{cases} \alpha_1 = \dfrac{c_1(x,y,z,w)}{2\omega}, \\ \beta_1 = \dfrac{c_2(x,y,z,w)}{2\varphi}, \\ \gamma_1 = \dfrac{c_3(x,y,z,w)}{2\phi}, \\ \theta_1 = \dfrac{c_4(x,y,z,w)}{2\eta}, \\ r = \omega\alpha_1^2 + \varphi\beta_1^2 + \phi\gamma_1^2 + \eta\theta_1^2 + c_5(x,y,z,w)\,. \end{cases} \tag{9}$$

Since α_1, β_1, γ_1, θ_1 and r are real constants, the functions $\{c_i(x,y,z,w), i = 1,2,3,4,5\}$ are also constants, i.e.,

$$\frac{\partial c_i(x,y,z,w)}{\partial x} = \frac{\partial c_i(x,y,z,w)}{\partial y} = \frac{\partial c_i(x,y,z,w)}{\partial z} = \frac{\partial c_i(x,y,z,w)}{\partial w} = 0, \; i = \overline{1,5}. \tag{10}$$

Now, putting

$$H(x,y,z,w) = \frac{(x-\alpha_1)^2}{\frac{r}{\omega}} + \frac{(y-\beta_1)^2}{\frac{r}{\varphi}} + \frac{(z-\gamma_1)^2}{\frac{r}{\phi}} + \frac{(w-\theta_1)^2}{\frac{r}{\eta}} - 1. \tag{11}$$

In order to prove the boundedness of the system (1), we assume that it is bounded and then we will find its bound, i.e., assume that

$$\begin{cases} c_5(x,y,z,w) + \omega\alpha_1^2 + \varphi\beta_1^2 + \phi\gamma_1^2 + \eta\theta_1^2 > 0, \\ \omega > 0,\ \varphi > 0,\ \phi > 0,\ \eta > 0, \end{cases} \tag{12}$$

therefore, the equation $\frac{dV}{dt} = 0$, that means, the surface

$$\Gamma = \left\{(x,y,z,w) \in \mathbb{R}^4 : H(x,y,z,w) = 0,\ \omega,\ \varphi,\ \phi,\ \eta,\ r > 0\ \right\} \tag{13}$$

is an ellipsoid in four-dimensional space. If the system (1) is bounded, then the function (2) can reach its maximum value on Γ. Denote the maximum point as (x_0, y_0, z_0, w_0). In order to find it, we define the function F by

$$F(x,y,z,w) = G(x,y,z,w) + \lambda H(x,y,z,w), \tag{14}$$

where

$$G(x,y,z,w) = x^2 + y^2 + z^2 + w^2 \tag{15}$$

and $\lambda \in \mathbb{R}$ is a finite parameter. It is clear that $\max\limits_{(x,y,z,w)\in\Gamma} G = \max\limits_{(x,y,z,w)\in\Gamma} F$ and let

$$\begin{cases} \frac{\partial F(x,y,z,w)}{\partial x} = 2r^{-1}\left((\omega\lambda + r)x - \omega\lambda\alpha_1\right) = 0, \\ \frac{\partial F(x,y,z,w)}{\partial y} = 2r^{-1}\left((\varphi\lambda + r)y - \varphi\lambda\beta_1\right) = 0, \\ \frac{\partial F(x,y,z,w)}{\partial z} = 2r^{-1}\left((\phi\lambda + r)z - \phi\lambda\gamma_1\right) = 0, \\ \frac{\partial F(x,y,z,w)}{\partial w} = 2r^{-1}\left((\eta\lambda + r)w - \eta\lambda\theta_1\right) = 0. \end{cases} . \tag{16}$$

In the sequel, we can separate some cases to discuss the upper bounds of the system (1).

(i) If $\lambda \neq \frac{-r}{\omega}$, $\lambda \neq \frac{-r}{\varphi}$, $\lambda \neq \frac{-r}{\phi}$ and $\lambda \neq \frac{-r}{\eta}$, we get

$$(x_0, y_0, z_0, w_0) = \left(\frac{\omega\lambda\alpha_1}{r + \omega\lambda}, \frac{\varphi\lambda\beta_1}{r + \varphi\lambda}, \frac{\phi\lambda\gamma_1}{r + \phi\lambda}, \frac{\eta\lambda\theta_1}{r + \eta\lambda}\right) \tag{17}$$

and

$$\max_{(x,y,z,w)\in\Gamma} G = \frac{\omega^2\lambda^2\alpha_1^2}{(r + \omega\lambda)^2} + \frac{\varphi^2\lambda^2\beta_1^2}{(r + \varphi\lambda)^2} + \frac{\phi^2\lambda^2\gamma_1^2}{(r + \phi\lambda)^2} + \frac{\eta^2\lambda^2\theta_1^2}{(r + \eta\lambda)^2}. \tag{18}$$

In this case, there exists parametrized family (in λ) of bounds given by (18) of the system (1).

(*ii*) If $\lambda = \frac{-r}{\omega}$, ($\omega \neq \varphi$, $\omega \neq \phi$, $\omega \neq \eta$), $\lambda \neq \frac{-r}{\varphi}$, $\lambda \neq \frac{-r}{\phi}$, $\lambda \neq \frac{-r}{\eta}$, we obtain

$$(x_0, y_0, z_0, w_0) = \left(\pm\sqrt{\frac{r}{\omega}\left(1 - \frac{\xi_1}{\xi_2}\right)} + \alpha_1, \frac{-\varphi\beta_1}{\omega - \varphi}, \frac{-\phi\gamma_1}{\omega - \phi}, \frac{-\eta\theta_1}{\omega - \eta} \right), \tag{19}$$

where

$$\begin{cases} \xi_1 = \omega^2 \left[\varphi\beta_1^2 (\omega - \phi)^2 (\omega - \eta)^2 + \phi\gamma_1^2 (\omega - \varphi)^2 (\omega - \eta)^2 + \eta\theta_1^2 (\omega - \varphi)^2 (\omega - \phi)^2 \right] \\ \xi_2 = r(\omega - \varphi)^2 (\omega - \phi)^2 (\omega - \eta)^2 \\ \xi_2 \geq \xi_1. \end{cases} \tag{20}$$

The last condition of (20) confirms that the value x_0 in (19) is well defined. In this case, we have

$$\max_{(x,y,z,w)\in\Gamma} G = \left(\sqrt{\frac{r}{\omega}\left(1 - \frac{\xi_1}{\xi_2}\right)} + \alpha_1 \right)^2 + \frac{\varphi^2\beta_1^2}{(\omega - \varphi)^2} + \frac{\phi^2\gamma_1^2}{(\omega - \phi)^2} + \frac{\eta^2\theta_1^2}{(\omega - \eta)^2}. \tag{21}$$

(*iii*) If $\lambda = \frac{-r}{\varphi}$, ($\varphi \neq \omega$, $\varphi \neq \phi$, $\varphi \neq \eta$), $\lambda \neq \frac{-r}{\omega}$, $\lambda \neq \frac{-r}{\phi}$, $\lambda \neq \frac{-r}{\eta}$, we have

$$(x_0, y_0, z_0, w_0) = \left(\frac{-\alpha_1\omega}{\varphi - \omega}, \pm\sqrt{\frac{r}{\varphi}\left(1 - \frac{\xi_3}{\xi_4}\right)} + \beta_1, \frac{-\phi\gamma_1}{\varphi - \phi}, \frac{-\eta\theta_1}{\varphi - \eta} \right), \tag{22}$$

where

$$\begin{cases} \xi_3 = \varphi^2 \left[\omega\alpha_1^2 (\varphi - \phi)^2 (\varphi - \eta)^2 + \phi\gamma_1^2 (\varphi - \omega)^2 (\varphi - \eta)^2 + \eta\theta_1^2 (\varphi - \omega)^2 (\varphi - \phi)^2 \right], \\ \xi_4 = r(\varphi - \omega)^2 (\varphi - \phi)^2 (\varphi - \eta)^2, \\ \xi_4 \geq \xi_3. \end{cases} \tag{23}$$

By the last condition of (23), we can confirm that the value y_0 in (22) is well defined. In this case, we get

$$\max_{(x,y,z,w)\in\Gamma} G = \frac{\alpha_1^2\omega^2}{(\varphi - \omega)^2} + \left(\sqrt{\frac{r}{\varphi}\left(1 - \frac{\xi_3}{\xi_4}\right)} + \beta_1 \right)^2 + \frac{\phi^2\gamma_1^2}{(\varphi - \phi)^2} + \frac{\eta^2\theta_1^2}{(\varphi - \eta)^2}. \tag{24}$$

(*iv*) If $\lambda = \frac{-r}{\phi}$, ($\phi \neq \omega$, $\phi \neq \varphi$, $\phi \neq \eta$), $\lambda \neq \frac{-r}{\omega}$, $\lambda \neq \frac{-r}{\varphi}$, $\lambda \neq \frac{-r}{\eta}$, we obtain

$$(x_0, y_0, z_0, w_0) = \left(\frac{-\alpha_1\omega}{\phi - \omega}, \frac{-\varphi\beta_1}{\phi - \varphi}, \pm\sqrt{\frac{r}{\phi}\left(1 - \frac{\xi_5}{\xi_6}\right)} + \gamma_1, \frac{-\eta\theta_1}{\phi - \eta} \right), \tag{25}$$

where

$$\begin{cases} \xi_5 = \phi^2 \left[\omega\alpha_1^2 (\phi - \varphi)^2 (\phi - \eta)^2 + \varphi\beta_1^2 (\phi - \omega)^2 (\phi - \eta)^2 + \eta\theta_1^2 (\phi - \omega)^2 (\phi - \varphi)^2 \right], \\ \xi_6 = r(\phi - \omega)^2 (\phi - \varphi)^2 (\phi - \eta)^2, \\ \xi_6 \geq \xi_5. \end{cases} \tag{26}$$

Also, the last condition of (26) confirms that the value z_0 in (25) is well defined. In this case, we have

$$\max_{(x,y,z,w)\in\Gamma} G = \frac{\alpha_1^2\omega^2}{(\phi-\omega)^2} + \frac{\varphi^2\beta_1^2}{(\phi-\varphi)^2} + \left(\sqrt{\frac{r}{\phi}\left(1-\frac{\xi_5}{\xi_6}\right)} + \gamma_1\right)^2 + \frac{\eta^2\theta_1^2}{(\phi-\eta)^2}. \tag{27}$$

(v) If $\lambda = \dfrac{-r}{\eta}$, $(\eta \neq \omega,\ \eta \neq \varphi,\ \eta \neq \phi)$, $\lambda \neq \dfrac{-r}{\omega}$, $\lambda \neq \dfrac{-r}{\varphi}$, $\lambda \neq \dfrac{-r}{\phi}$, we get

$$(x_0, y_0, z_0, w_0) = \left(\frac{-\alpha_1\omega}{\eta-\omega}, \frac{-\varphi\beta_1}{\eta-\varphi}, \frac{-\phi\gamma_1}{\eta-\phi}, \pm\sqrt{\frac{r}{\eta}\left(1-\frac{\xi_7}{\xi_8}\right)} + \theta_1\right), \tag{28}$$

where

$$\begin{cases} \xi_7 = \eta^2\left[\omega\alpha_1^2(\eta-\varphi)^2(\eta-\phi)^2 + \varphi\beta_1^2(\eta-\omega)^2(\eta-\phi)^2 + \phi\gamma_1^2(\eta-\omega)^2(\eta-\varphi)^2\right], \\ \xi_8 = r(\eta-\omega)^2(\eta-\varphi)^2(\eta-\phi)^2, \\ \xi_8 \geq \xi_7. \end{cases} \tag{29}$$

The last condition of (29) confirms that the value w_0 in (28) is well defined. In this case, we obtain

$$\max_{(x,y,z,w)\in\Gamma} G = \frac{\alpha_1^2\omega^2}{(\eta-\omega)^2} + \frac{\varphi^2\beta_1^2}{(\eta-\varphi)^2} + \frac{\phi^2\gamma_1^2}{(\eta-\phi)^2} + \left(\sqrt{\frac{r}{\eta}\left(1-\frac{\xi_7}{\xi_8}\right)} + \theta_1\right)^2. \tag{30}$$

Finally, the other possible cases can be treated using the same technique.

Theorem 2.1 *Assume that conditions (9), (10) and (12) hold, then the general 4-D continuous-time autonomous system (1) is bounded, i.e., it is contained in the 4-D ellipsoid (13).*

Also, similar results can be found using the cases discussed above.

3 Example

We consider the Lorenz-Stenflo system studied in [20] and given by

$$\begin{cases} x' = ay - ax + dw, \\ y' = cx - xz - y, \\ z' = xy - bz, \\ w' = -x - aw. \end{cases} \tag{31}$$

We choose the Lyapunov function $V(x,y,z,w) = \lambda x^2 + y^2 + (z - \lambda a - c)^2 + \lambda d w^2$ as in [20]. Suppose that λ and d are strictly positive and denote $\sqrt{\lambda}x = \widetilde{x}$, $\sqrt{\lambda d}w = \widetilde{w}$, thus we get $V(\widetilde{x}, y, z, \widetilde{w}) = \widetilde{x}^2 + y^2 + (z - \lambda a - c)^2 + \widetilde{w}^2$ i.e., $\alpha = \beta = \theta = 0$, $\gamma = \lambda a + c$ and the system (31) becomes

$$\begin{cases} \widetilde{x}' = -a\widetilde{x} + \sqrt{\lambda}ay + \sqrt{d}\widetilde{w}, \\ y' = \frac{c}{\sqrt{\lambda}}\widetilde{x} - y - \frac{1}{\sqrt{\lambda}}\widetilde{x}z, \\ z' = -bz + \frac{1}{\sqrt{\lambda}}\widetilde{x}y, \\ \widetilde{w}' = -\sqrt{d}\widetilde{x} - a\widetilde{w}, \end{cases} \tag{32}$$

i.e., $f(\widetilde{x},y,z,\widetilde{w}) = -a\widetilde{x} + \sqrt{\lambda}ay + \sqrt{d}\widetilde{w}$, $g(\widetilde{x},y,z,\widetilde{w}) = \frac{c}{\sqrt{\lambda}}\widetilde{x} - y - \frac{1}{\sqrt{\lambda}}\widetilde{x}z$, $h(\widetilde{x},y,z,\widetilde{w}) = -bz + \frac{1}{\sqrt{\lambda}}\widetilde{x}y$, $k(\widetilde{x},y,z,\widetilde{w}) = -\sqrt{d}\widetilde{x} - a\widetilde{w}$. Thus, we have $\omega = a > 0$, $\varphi = 1 > 0$, $\phi = b > 0$, $\eta = a > 0$, $\alpha_1 = \beta_1 = \theta_1 = 0$, $\gamma_1 = \dfrac{\lambda a + c}{2}$ and $r = b\left(\dfrac{\lambda a + c}{2}\right)^2$. Then, we get $\dfrac{1}{2}\dfrac{dV}{dt} = -a\widetilde{x}^2 - y^2 - b\left(z - \frac{\lambda a+c}{2}\right)^2 - a\widetilde{w}^2 + b\left(\frac{\lambda a+c}{2}\right)^2$, i.e., $\dfrac{1}{2}\dfrac{dV}{dt} = -a\lambda x^2 - y^2 - bz^2 - a\lambda dw^2 + (\lambda a + c)\,bz$ which is the same as in [20]. Also, it is easy to verify that all conditions of Theorem 2.1 hold for this case. The 4-D ellipsoid Γ is given by

$$\Gamma = \left\{ \begin{array}{c} (x,y,z,w) \in \mathbb{R}^4 : \dfrac{\widetilde{x}^2}{\dfrac{b}{a}\left(\frac{\lambda a+c}{2}\right)^2} + \dfrac{y^2}{b\left(\frac{\lambda a+c}{2}\right)^2} + \dfrac{\left(z - \frac{\lambda a+c}{2}\right)^2}{\left(\frac{\lambda a+c}{2}\right)^2} + \dfrac{\widetilde{w}^2}{\dfrac{b}{a}\left(\frac{\lambda a+c}{2}\right)^2} = 1, \\ a > 0,\ b > 0,\ c > 0,\ d > 0,\ \lambda > 0 \end{array} \right\} \tag{33}$$

i.e.,

$$\Gamma = \left\{ \begin{array}{c} (x,y,z,w) \in \mathbb{R}^4 : \lambda a x^2 + y^2 + b\left(z - \dfrac{\lambda a + c}{2}\right)^2 + \lambda a d w^2 = \dfrac{b(\lambda a + c)^2}{4}, \\ a > 0,\ b > 0,\ c > 0,\ d > 0,\ \lambda > 0 \end{array} \right\} \tag{34}$$

which is also the same as in [20]. Finally, we have the result shown in [20] that confirms that if $a > 0$, $b > 0$, $c > 0$, $d > 0$, $\lambda > 0$, then the Lorenz-Stenflo system is contained in the following set

$$\Omega_\lambda = \left\{ (x,y,z,w) \in \mathbb{R}^4 : \lambda x^2 + y^2 + (z - \lambda a - c)^2 + \lambda d w^2 \le R^2 \right\}, \tag{35}$$

where

$$R^2 = \begin{cases} \dfrac{(\lambda a + c)^2 b^2}{4(b-1)}, & \text{if } a \ge 1,\ b \ge 2, \\ (\lambda a + c)^2, & \text{if } a > \frac{b}{2},\ b < 2, \\ \dfrac{(\lambda a + c)^2 b^2}{4a(b-a)}, & \text{if } 0 < a < 1,\ b \ge 2. \end{cases} \tag{36}$$

4 Conclusion

In this paper, based on the multivariable functions analysis, a generalization of all the existing results in the relevant literature for the upper bound of the general 4-D continuous-time system is investigated. Especially, sufficient conditions for this system to be contained in a four-dimensional ellipsoidal surface are determined.

The strategy presented in this work is sufficiently general, so it would be possible to apply the present method to consider other systems with high order and more complicated nonlinearity, which will be the topic for further papers.

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Approximate Controllability of Semilinear Stochastic Control System with Nonlocal Conditions

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Abstract: In this paper we study the approximate controllability of semilinear stochastic control system with nonlocal conditions in a Hilbert space. Nonlocal initial condition is a generalization of the classical initial condition and is motivated by physical phenomena. The results are obtained by using Sadovskii's fixed point theorem. At the end, an example is given to show the effectiveness of the result.

Keywords: *approximate controllability; semilinear systems; stochastic control system; Sadovskii's fixed point theorem.*

Mathematics Subject Classification (2010): 34K30, 34K35, 93C25.

1 Introduction

Controllability concepts play a vital role in deterministic control theory. It is well known that controllability of deterministic equation is widely used in many fields of science and technology. Kalman [23] introduced the concept of controllability for finite dimensional deterministic linear control systems. The basic concepts of control theory in finite and infinite dimensional spaces have been introduced in [31] and [24] respectively. However, in many cases, some kind of randomness can appear in the problem, so that the system should be modelled by a stochastic form. Only few authors have studied the extensions of deterministic controllability concepts to stochastic control systems. Klamka et al. [11]-[12] studied the controllability of linear stochastic systems in finite dimensional spaces with delay and without delay in control as well as in state using Rank theorem. In [17]-[22], Mahmudov et al. established results for controllability of linear and semilinear stochastic systems in Hilbert space. Instead of this, Sakthivel, Balachandran, Dauer and Bashirov et al. studied the approximate controllability of nonlinear stochastic systems

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in [25], [14], [13] and [1]. Shen and Sun [16] studied the controllability of stochastic first order nonlinear systems with delay in control in finite dimensional as well as in infinite dimensional spaces. In [26], Sakthivel et al. studied the approximate controllability of second order stochastic system with impulsive effects using Banach fixed point theorem. In [2]- [5] Anurag et al. obtained some sufficient conditions for controllability of integer and fractional order stochastic systems with delay in control and state term using different fixed point theorems.

On the other hand, Byszewski et al. [15] introduced nonlocal conditions into the initial value problems and argued that the corresponding models more accurately describe the phenomena since more information was taken into account at the oneset of the experiment, thereby reducing the ill effects incurred by a single initial measurement. Also, it has a better effect on the solution and is more precise for physical measurements than classical condition $x(0) = x_0$ alone. In [32], Y.K.Chang et al. obtained sufficient conditions for controllability of semilinear differential systems with nonlocal conditions in Banach spaces using Sadovskii fixed-point theorem.

Kumar [28]- [29] studied on the controllability of second order and fractional order systems with delays in Banach spaces using Sadovskii's Fixed point theorem. Also Farahi et al. [30] studied on the approximate controllability of fractional neutral stochastic evolution equations with nonlocal conditions using Sadovskii's fixed point theorem. Sanjukta [27] studied approximate controllability of a functional differential equation with deviated argument using fixed point theory.

Up to now, to the best of our knowledge , there are no results on the approximate controllability of semilinear stochastic control systems with nonlocal conditions using Sadovskii's fixed point theorem in the literature. So, the present paper is devoted to the study of approximate controllability of the semilinear stochastic control systems with nonlocal conditions using Sadovskii's fixed point theorem.

2 Problem Formulation and Preliminaries

Let $(\Omega, \Im, P)$ be a complete space equipped with a normal filtration $\Im_t$, $t \in J = [0, b]$. Let H, U and E be the separable Hilbert spaces and ω be a Q-Weiner process on $(\Omega, \Im_b, P)$ with the covariance operator Q such that $trQ < \infty$. We assume that there exists a complete orthonormal system e_n in E, a bounded sequence of nonnegative real numbers λ_n such that $Qe_n = \lambda_n e_n$, $n = 1, 2, \cdots$ and a sequence β_n of independent Brownian motions such that

$$w(t) = \sum_{n=1}^{\infty} \sqrt{\lambda_n}\beta_n(t)e_n, \quad t \in J,$$

and $\Im_t = {\Im_t}^{\omega}$, where ${\Im_t}^{\omega}$ is the σ-algebra generated by ω. Let ${L_2}^0 = L_2(Q^{1/2}E; H)$ be the space of all Hilbert-Schmidt operators from $Q^{1/2}E$ to H with the norm $||\psi||_Q^2 = tr[\psi Q\psi^*]$. Let $L_2^{\Im}(J, H)$ be the space of all $\Im_t$-adapted, H-valued measurable square integrable processes on $J \times \Omega$.Let $C([0, b]; L^2(\Im, H))$ be the Banach space of continuous maps from $[0, b]$ into $L^2(\Im, H)$ satisfying the condition $\sup_{t\in J} \mathbb{E}||x(t)||^2 < \infty$.

Let $H_2 = C_2([0, b]; H)$. Now H_2 is the closed subspace of $C([0, b]; L^2(\Im, H))$ consisting of measurable and $\Im_t$ - adapted H valued processes $\phi \in C([0, b]; L^2(\Im, H))$ endowed with the norm

$$||\phi||_{H_2} = \left(\sup_{t\in[0,b]} \mathbb{E}||\phi(t)||_H^2\right)^{1/2}.$$

In this paper, we consider a mathematical model given by the following nonlinear second order stochastic differential equations with variable delay in control and with nonlocal conditions of the form

$$\left.\begin{array}{rcl} dx(t) & = & [Ax(t) + Bu(t) + f(t, x(t))]dt + \sigma(t, x(t))d\omega(t), \quad t \in J \\ x(0) & = & x_0 + g(x), \end{array}\right\} \tag{1}$$

where $A : D(A) \subset H \to H$ is a closed, linear and densely defined operator on H which generates a compact semigroup $\{S(t) : t \in J\}$ on H. B is a bounded linear operator from the Hilbert space U into H. The control $u \in L^2_{\Im}([0, b], U)$; $f : J \times H \to H$; $\sigma : J \times H \to L^0_2$ are nonlinear suitable functions; x_0 is $\Im_0$ measurable H valued random variable independent of ω; g is continuous function from $C(J, H) \to H$.

For simplicity of considerations, we generally assume that the set of admissible controls is $U_a d = L^2_{\Im}(J, U)$.

Definition 2.1 A stochastic process $x \in H_2$ is a mild solution of (1) if for each $u \in L^2_{\Im}([0, b], U)$, it satisfies the following integral equation:

$$\begin{aligned} x(t) \;&=\; S(t)(x_0 + g(x)) + \int_0^t S(t-s)[Bu(s) + f(s, x(s))]ds \\ &\quad + \int_0^t S(t-s)\sigma(s, x(s))d\omega(s). \end{aligned}$$

Let us introduce the following operators and sets (see [15])
$L_b \in \mathfrak{L}(L^{\Im}_2(J \times \Omega, U), L_2(\Omega, \Im_b, H))$ is defined by

$$L_b u = \int_0^b S(b-s)Bu(s)ds,$$

where $\mathfrak{L}(X, Y)$ denotes the set of bounded linear operators from X to Y.

Then its adjoint operator $L^*_b : L_2(\Omega, \Im_b, H) \to L^{\Im}_2(J \times \Omega, U)$ is given by

$$L^*_b z = B^* S^*(b-t)\mathbb{E}\{z|\Im_t\}.$$

The set of all states reachable in time b from initial state $x(0) = x_0 \in L_2(\Omega, \Im_0, X)$, using admissible controls is defined as

$$\begin{aligned} R_b(U_{ad}) \;&=\; \{x(b; x_0, u) \in L_2(\Omega, \Im_b, H) : u \in U_{ad}\}, \\ x(b; x_0, u) \;&=\; S(b)(x_0 + g(x)) + \int_0^b S(b-s)Bu(s)ds + \int_0^b S(b-s)f(s, x(s))ds, \\ &\quad + \int_0^T S(T-s)\sigma(s, x(s)d\omega(s). \end{aligned}$$

Let us introduce the linear controllability operator $\Pi^b_0 \in \mathfrak{L}(L_2(\Omega, \Im_b, H), L_2(\Omega, \Im_b, H))$ as follows:

$$\begin{aligned} \Pi^b_0\{.\} \;&=\; L_b(L_b)^*\{.\} \\ &=\; \int_0^b S(b-t)BB^*S^*(b-t)\mathbb{E}\{.|\Im_t\}dt. \end{aligned}$$

The corresponding controllability operator for deterministic model is:

$$\begin{aligned}\Gamma_s^b &= L_b(s)L_b^*(s)\\ &= \int_s^b S(b-t)BB^*S^*(b-t)dt.\end{aligned}$$

Definition 2.2 The stochastic system (1) is approximately controllable on $[0,b]$ if $\overline{\Re(b)} = L_2(\Omega,\Im_b,H)$, where $\Re(b) = \{x(b;u) : u \in L_2(\Omega,\Im_b,H) : u \in U_ad\}$ and $L^2_{\Im}([0,b],U)$ is the closed subspace of $L^2_{\Im}([0,b]\times\Omega,U)$, consisting of all $\Im_t$ adapted, U valued stochastic processes.

Lemma 2.1 *[6] Let $G : J\times\Omega \to L_2^0$ be a strongly measurable mapping such that $\int_0^b \mathbb{E}||G(t)||^p_{L_2^0} < \infty$. Then*

$$\mathbb{E}\bigg|\bigg|\int_0^t G(s)d\omega(s)\bigg|\bigg|^p \le L_G\int_0^t \mathbb{E}||G(s)||^p_{L_2^0}ds$$

for all $t\in J$ and $p\ge 2$, where L_G is the constant involving p and b.

Lemma 2.2 *(Sadovskii's fixed point theorem [7]). Suppose that M is a nonempty, closed, bounded and convex subset of a Banach space X and $\Gamma : M\subseteq X\to X$ is a condensing operator. Then the operator Γ has a fixed point in M.*

To prove our main results, we list the following basic assumptions of this paper:

(i) A is the infinitesimal generator of a compact semigroup $\{S(t) : t\ge 0\}$ on H.

(ii) The function $f : J\times H\to H$ and $\sigma : J\times H\to L_2^0$ satisfy linear growth and Lipschitz conditions, i.e, there exist positive constants N_1,N_2,K_1 and K_2 such that

$$\begin{gathered}||f(t,x)-f(t,y)||^2 \le N_1||x-y||^2,\quad ||f(t,x)||^2\le N_2(1+||x||^2),\\ ||\sigma(t,x)-\sigma(t,y)||^2_{L_2^0}\le K_1||x-y||^2,\quad ||\sigma(t,x)||^2_{L_2^0}\le K_2(1+||x||^2).\end{gathered}$$

(iii) The function g is continuous function and there exists some positive constants M_g such that

$$||g(x)-g(y)||^2\le M_g||x-y||^2,\quad ||g(x)||^2\le M_g(1+||x||^2)$$

for all $x,y\in C(J,H)$.

(iv) For each $0\le t<b$, the operator $\alpha(\alpha I+\Gamma_t^b)^{-1}\to 0$ in the strong operator topology as $\alpha\to 0^+$, where

$$\Gamma_t^b = \int_t^b S(b-s)BB^*S^*(b-s)ds$$

is the controllability Grammian. Observe that the linear deterministic system corresponding to (1)

$$\left.\begin{aligned}dx'(t) &= [Ax(t)+Bu(t)]dt,\quad t\in J\\ x(0) &= x_0\end{aligned}\right\}\tag{2}$$

is approximately controllable on $[t,b]$ iff the operator $\alpha(\alpha I+\Gamma_t^b)^{-1}\to 0$ strongly as $\alpha\to 0^+$.

For simplicity, let us take $M_B = \max\{||B||\}$.

3 Main Result

Let us recall two lemmas concerning approximate controllability, which will be used in the proof.

The following lemma is required to define the control function.

Lemma 3.1 *[19] For any $x_b \in L_2(\Omega, \Im_b, H)$, there exists $\phi \in L_2^{\Im}(J, L_2^0)$ such that $x_b = \mathbb{E}x_b + \int_0^b \phi(s)d\omega(s)$.*

Now for any $\alpha > 0$ and $x_b \in L_2(\Omega, \Im_b, H)$, we define the control function in the following form

$$\begin{aligned} u^\alpha(t,x) &= B^*S^*(b-t)\big[(\alpha I + \Psi_0^b)^{-1}\big(\mathbb{E}x_b - S(b)(x_0 + g(x))\big) \\ &\quad + \int_0^t (\alpha I + \Psi_s^b)^{-1}\phi(s)dw(s)\big], \\ &\quad - B^*S^*(b-t)\int_0^t (\alpha I + \Psi_s^b)^{-1}S(b-s)f(s,x(s))ds, \\ &\quad - B^*S^*(b-t)\int_0^t (\alpha I + \Psi_s^b)^{-1}S(b-s)\sigma(s,x(s))dw(s). \end{aligned}$$

Lemma 3.2 *There exists a positive constant M_u such that for all $x, y \in H_2$, we have*

$$\mathbb{E}||u^\alpha(t,x) - u^\alpha(t,y)||^2 \le \frac{M_u}{\alpha^2}||x-y||^2, \tag{3}$$

$$\mathbb{E}||u^\alpha(t,x)||^2 \le \frac{M_u}{\alpha^2}\left(1 + ||x||^2\right). \tag{4}$$

Proof. Let $x, y \in H_2$. From Holder's inequality, Lemma 2.1 and the assumptions on the data, we obtain

$$\begin{aligned} \mathbb{E}||u^\alpha(t,x) - u^\alpha(t,y)||^2 &\le 3\mathbb{E}\left|\left|B^*S^*(b-t)(\alpha I + {\psi_0}^b)^{-1}S(b)[g(x) - g(y)]\right|\right|^2 \\ &+ 3\mathbb{E}\left|\left|B^*S^*(b-t)\int_0^t (\alpha I + \Psi_s^b)^{-1}S(b-s)[f(s,x(s)) - f(s,y(s))]ds\right|\right|^2 \\ &+ 3\mathbb{E}\left|\left|B^*S^*(b-t)\int_0^t (\alpha I + \Psi_s^b)^{-1}S(b-s)[\sigma(s,x(s)) - \sigma(s,y(s))]dw(s)\right|\right|^2 \\ &\le \frac{3}{\alpha^2}M_B^2M^4\left[M_g||x-y||_{H_2}^2 + b\int_0^t N_1\mathbb{E}||x(s)-y(s)||_H^2 ds + L_G\int_0^t K_1\mathbb{E}||x(s)-y(s)||_H^2 ds\right] \\ &\le \frac{3}{\alpha^2}M_B^2M^4\left[M_g + bN_1b\sup_{s\in[0,b]}\mathbb{E}||x(s)-y(s)||_H^2 + L_GK_1b\sup_{s\in[0,b]}\mathbb{E}||x(s)-y(s)||_H^2\right] \\ &\le \frac{3}{\alpha^2}M_B^2M^4[M_g + b^2N_1 + L_GK_1b]||x-y||_{H_2}^2 = \frac{M_u}{\alpha^2}||x-y||_{H_2}^2, \end{aligned}$$

where $M_u = 3M_B^2M^4[M_g + b^2N_1 + L_GK_1b]$ and p, q are conjugate indices.

The proof of the second inequality can be verified in a similar manner by putting $u^\alpha(t,y) = 0$. So, the proof of the lemma is completed.

For any $\alpha > 0$, define the operator $\mathbf{P}_\alpha : H_2 \to H_2$ by

$$\begin{aligned}(\mathbf{P}_\alpha x)(t) &= S(t)(x_0 + g(x)) + \int_0^t S(t-s)[Bu^\alpha(s,x) + f(s,x(s))]ds \\ &\quad + \int_0^t S(t-s)\sigma(s,x(s))d\omega(s).\end{aligned}$$

To prove the approximate controllability, we first prove in Theorem 3.1, the existence of a fixed point of the operator $\mathbf{P}_\alpha$ defined above, using the Sadovskii's fixed point theorem. Second, in Theorem 3.2, we show that under certain assumptions the approximate controllability of system (2) is implied by the approximate controllability of the corresponding deterministic linear system.

Theorem 3.1 *Assume hypothesis $(i)-(iv)$ are satisfied. Then the system (1) has a mild solution on $[0,b]$ provided that*

$$\begin{aligned}& 8M^2M_g + 4M^2\left(\frac{M_B^2 b^2 M_u}{\alpha^2} + b^2N_2 + L_\sigma K_2 b\right) < 1, \qquad (5) \\ & \frac{3M^2M_B^2 bM_u}{\alpha^2} + 3M^2bN_1 + 3M^2L_G < 1.\end{aligned}$$

Proof. The proof of this theorem is divided into several steps.
Step 1. For any $x \in H_2$, $\mathbf{P}_\alpha(x)(t)$ is continuous on J in the L^p sense.

Proof of Step 1: Let $0 \le t_1 \le t_2 \le b$. Then for any fixed $x \in H_2$, it follows from Holder's inequality, Lemma 2.1 and assumptions of the theorem that

$$\begin{aligned}& \mathbb{E}||(\mathbf{P}_\alpha x)(t_2) - (\mathbf{P}_\alpha x)(t_1)||^2 \\ \le\, & 7\Bigg[\mathbb{E}||(S(t_2)-S(t_1))(x_0+g(x))||^2 + \mathbb{E}\Big|\Big|\int_0^{t_1}[S(t_2-s)-S(t_1-s)]f(s,x(s))ds\Big|\Big|^2 \\ & +\mathbb{E}\Big|\Big|\int_{t_1}^{t_2} S(t_2-s)f(s,x(s))ds\Big|\Big|^2 + \mathbb{E}\Big|\Big|\int_0^{t_1}[S(t_2-s)-S(t_1-s)]\sigma(s,x(s))d\omega(s)\Big|\Big|^2 \\ & +\mathbb{E}\Big|\Big|\int_{t_1}^{t_2} S(t_2-s)\sigma(s,x(s))d\omega(s)\Big|\Big|^2 + \mathbb{E}\Big|\Big|\int_0^{t_1}[S(t_2-s)-S(t_1-s)]Bu^\alpha(s,x)ds\Big|\Big|^2 \\ & +\mathbb{E}\Big|\Big|\int_{t_1}^{t_2} S(t_2-s)Bu^\alpha(s,x)ds\Big|\Big|^2\Bigg] \\ \le\, & 7\Bigg[2\Big(\mathbb{E}||(S(t_2)-S(t_1))x_0||^2 + \mathbb{E}||(S(t_2)-S(t_1))g(x)||^2\Big) \\ & +t_1\int_0^{t_1}\mathbb{E}||[S(t_2-s)-S(t_1-s])f(s,x(s))||^2ds + M^2(t_2-t_1)\int_{t_1}^{t_2}\mathbb{E}||f(s,x(s))||^2ds \\ & +L_G\int_0^{t_1}\mathbb{E}||(S(t_2-s)-S(t_1-s))\sigma(s,x(s))||^2ds + M^2L_G\int_{t_1}^{t_2}\mathbb{E}||\sigma(s,x(s))||^2ds \\ & +t_1\int_0^{t_1}\mathbb{E}||[S(t_2-s)-S(t_1-s])Bu^\alpha(s,x)||^2ds + ||B||^2M^2(t_2-t_1)\int_{t_1}^{t_2}\mathbb{E}||u^\alpha(s,x)||^2ds\Bigg]\end{aligned}$$

Hence using Lebesgue's dominated convergence theorem, we conclude that the right hand side of the above inequality tends to zero as $t_2 - t_1 \to 0$. Thus we conclude $\mathbf{P}_\alpha(x)(t)$ is continuous from the right in $[0, b)$. A similar argument shows that it is also continuous from the left in $(0, b]$. Thus $\mathbf{P}_\alpha(x)(t)$ is continuous on J in the L^p sense.

Step 2: For each positive integer q, let $B_q = \{x \in H_2 : \mathbb{E}||x(t)||^2_H \leq q\}$, then the set B_q is clearly a bounded, closed and convex set in H_2. From Lemma 2.1, Holder's inequality and assumption (i), we derive that

$$\begin{aligned}
\mathbb{E}\Big|\Big|\int_0^t S(t-s)f(s,x(s))ds\Big|\Big|^2_H &\leq \mathbb{E}\Big(\int_0^t ||S(t-s)f(s,x(s))||_H ds\Big)^2 \\
&\leq M^2\mathbb{E}\Big(\int_0^t ||f(s,x(s))||_H ds\Big)^2 \\
&\leq M^2 b\int_0^t \mathbb{E}||f(s,x(s))||^2_H ds \\
&= M^2 b\int_0^t N_2(1+\mathbb{E}||x(s)||^2_H)ds \\
&\leq M^2 b N_2\int_0^t (1+\sup_{s\in[0,b]}\mathbb{E}||x(s)||^2_H)ds \\
&\leq M^2 b N_2 b(1+||x||^2_{H_2}) \\
&\leq M^2 b^2 N_2(1+||x||^2_{H_2}),
\end{aligned}$$

which deduces that $S(t-s)f(s,x(s))$ is integrable on J, by Bochner's theorem, P_α is well defined on B_q.

Similarly from (ii), we derive that

$$\begin{aligned}
\mathbb{E}\Big|\Big|\int_0^t S(t-s)\sigma(s,x(s))dw(s)\Big|\Big|^2 &\leq L_\sigma\int_0^t \mathbb{E}||S(t-s)\sigma(s,x(s))||^2_{L_2^0}ds \\
&\leq L_\sigma M^2\int_0^t \mathbb{E}||\sigma(s,x(s))||^2_{L_2^0}ds \\
&\leq L_\sigma M^2\int_0^t K_2(1+\mathbb{E}||x(s)||^2_H)ds \\
&\leq L_\sigma M^2 K_2\int_0^t (1+\sup_{s\in[0,b]}\mathbb{E}||x(s)||^2_H)ds \\
&\leq L_\sigma M^2 K_2 b(1+||x||^2_{H_2}) \\
&\leq L_\sigma M^2 K_2 b(1+||x||^2_{H_2}).
\end{aligned}$$

Now, we claim that there exists a positive number q such that $P_\alpha(B_q) \subseteq B_q$.

If it is not true, then for each positive number q, there is a function $x_q(.) \in B_q$ but $P_\alpha x_q$ does not belong to B_q, that is $\mathbb{E}||P_\alpha x_q(t)||^2_H > q$ for some $t \in J$.

On the other hand, from assumptions (ii), (iii) and Lemma 3.2, we have

$$\begin{aligned}
q \leq \mathbb{E}||P_\alpha x_q(t)||_H^2 &= \mathbb{E}\bigg|\bigg|S(t)(x_0+g(x)) + \int_0^t S(t-s)[Bu^\alpha(s,x)+f(s,x(s))]ds \\
&\quad + \int_0^t S(t-s)\sigma(s,x(s))dw(s)\bigg|\bigg|_H^2 \\
&\leq 4M^2\mathbb{E}||x_0+g(x)||^2 + 4M^2M_B^2b^2\frac{M_u}{\alpha^2}(1+||x||_{H_2}^2) \\
&\quad +4M^2b^2N_2(1+||x||_{H_2}^2) + 4M^2L_\sigma K_2 b(1+||x||_{H_2}^2) \\
&\leq 4M^2[2\mathbb{E}||x_0||^2 + 2\mathbb{E}||g(x)||^2] + 4M^2M_B^2b^2\frac{M_u}{\alpha^2}(1+||x||_{H_2}^2) \\
&+ 4M^2b^2N_2(1+||x||_{H_2}^2) + 4M^2L_\sigma K_2 b(1+||x||_{H_2}^2) \\
&\leq 4M^2[2\mathbb{E}||x_0||^2 + 2M_g(1+||x||_{H_2}^2)] + 4M^2M_B^2b^2\frac{M_u}{\alpha^2}(1+||x||_{H_2}^2) \\
&\quad +4M^2b^2N_2(1+||x||_{H_2}^2) + 4M^2L_\sigma K_2 b(1+||x||_{H_2}^2) \\
&\leq 4M^2[2\mathbb{E}||x_0||^2 + 2M_g(1+q)] + 4M^2M_B^2b^2\frac{M_u}{\alpha^2}(1+q) \\
&\quad +4M^2b^2N_2(1+q) + 4M^2L_\sigma K_2 b(1+q) \\
&\leq \bigg(8M^2\mathbb{E}||x_0||^2 + 8M^2M_g + \frac{4M^2M_B^2b^2M_u}{\alpha^2} \\
&+ 4M^2b^2N_2 + 4M^2L_\sigma K_2 b\bigg) \\
&+ \bigg(8M^2M_g + \frac{4M^2M_B^2b^2M_u}{\alpha^2} + 4M^2b^2N_2 + 4M^2L_\sigma K_2 b\bigg)q.
\end{aligned}$$

Dividing both sides by q and taking the limit as $q \to \infty$, we get

$$8M^2M_g + 4M^2\bigg(\frac{M_B^2b^2M_u}{\alpha^2} + b^2N_2 + L_\sigma K_2 b\bigg) > 1.$$

This contradicts condition (5). Hence for some positive number q, $P_\alpha B_q \subseteq B_q$.

Step 3. Now, we define operators P_{α_1} and P_{α_2} as

$$\begin{aligned}
(P_{\alpha_1}x)(t) &= S(t)[x_0+g(x)], \\
(P_{\alpha_2}x)(t) &= \int_0^t S(t-s)[Bu^\alpha(s,x)+f(s,x(s))]ds + \int_0^t S(t-s)\sigma(s,x(s))d\omega(s)
\end{aligned}$$

for $t \in J$. Here, we will prove that P_{α_1} is completely continuous, while P_{α_2} is a contraction operator.

By assumption (iii), it is clear that P_{α_1} is a completely continuous operator. Next we show that P_{α_2} is the contraction operator. For this, let $x, y \in B_q$, then for each $t \in J$, we have from assumptions (ii),(iii)

$$\begin{aligned}
\mathbb{E}||(P_{\alpha_2}x)(t)-(P_{\alpha_2}y)(t)||_H^2 &\leq 3\mathbb{E}\bigg|\bigg|\int_0^t S(t-s)B[u^\alpha(s,x)-u^\alpha(s,y)]ds\bigg|\bigg|_H^2\\
&\quad +3\mathbb{E}\bigg|\bigg|\int_0^t S(t-s)[f(s,x(s))-f(s,y(s))]ds\bigg|\bigg|_H^2\\
&\quad +3\mathbb{E}\bigg|\bigg|\int_0^t S(t-s)[\sigma(s,x(s))-\sigma(s,y(s))]d\omega(s)\bigg|\bigg|_H^2\\
&\leq 3M^2M_B^2\int_0^t \mathbb{E}||u^\alpha(s,x)-u^\alpha(s,y)||_H^2 ds\\
&+ 3M^2\int_0^t \mathbb{E}||f(s,x(s))-f(s,y(s))||^2 ds\\
&\quad +3M^2\int_0^t \mathbb{E}||\sigma(s,x(s))-\sigma(s,y(s))||^2 dw(s)\\
&\leq 3M^2M_B^2 b\frac{M_u}{\alpha^2}||x-y||_{H_2}^2+3M^2bN_1||x-y||_{H_2}^2\\
&+ 3M^2L_G||x-y||_{H_2}^2\\
&\leq \left(\frac{3M^2M_B^2bM_u}{\alpha^2}+3M^2bN_1+3M^2L_\sigma\right)||x-y||_{H_2}^2
\end{aligned}$$

therefore $||(P_{\alpha_2}x)-(P_{\alpha_2}y)||_{H_2}^2 \leq L_0||x-y||_{H_2}^2$, where

$$L_0=\left(\frac{3M^2M_B^2bM_u}{\alpha^2}+3M^2bN_1+3M^2L_G\right)<1.$$

Thus P_{α_2} is a contraction mapping.

Now we have $P_\alpha = P_{\alpha_1}+P_{\alpha_2}$ is a condensing map on B_q, so Sadovskii's fixed point theorem is satisfied. Hence we conclude that there exists a fixed point $x(.)$ for P_α on B_q, which is the mild solution of (1).

Theorem 3.2 *Assume assumptions $(i)-(iv)$ are satisfied and if f and σ are uniformly bounded, then the system* (1) *is approximately controllable on* $[0,b]$.

Proof. Let x_α be a fixed point of $\mathbf{P}_\alpha$ in H_2. By using the stochastic Fubini theorem, it is easy to see that

$$\begin{aligned}
x_\alpha(b) &= x_b-\alpha(\alpha I+\Gamma_0^b)^{-1}\Big(\mathbb{E}x_b-S(b)(x_0+g(x))\Big)\\
&\quad +\alpha\int_0^b(\alpha I+\Gamma_s^b)^{-1}S(b-s)f(s,x_\alpha(s))ds\\
&\quad +\alpha\int_0^b(\alpha I+\Gamma_s^b)^{-1}[S(b-s)\sigma(s,x_\alpha(s))-\phi(s)]d\omega(s).
\end{aligned}$$

By the assumption that f and σ are uniformly bounded, there exists $D>0$ such that

$$||f(s,x_\alpha(s))||^2+||\sigma(s,x_\alpha(s))||^2\leq D$$

in $[0,b]\times\Omega$. Then there is a subsequence denoted by $\{f(s,x_\alpha(s)),\sigma(s,x_\alpha(s))\}$ weakly converging to say $\{f(s,\omega),\sigma(s,\omega)\}$ in $H\times L_2^0$. Now, the compactness of $S(t)$ implies that $S(b-s)f(s,x_\alpha(s))\to S(b-s)f(s)$ and $S(b-s)\sigma(s,x_\alpha(s))\to S(b-s)\sigma(s)$ in $J\times\Omega$.

Now, from the above equation, we get

$$\begin{aligned}\mathbb{E}||x_\alpha(b)-x_b||^2 \quad\leq\quad & 6\Bigg|\Bigg|\alpha(\alpha I+\Gamma_0^b)^{-1}\Big[\mathbb{E}x_b - S(b)[x_0+g(x))]\Big]\Bigg|\Bigg|^2\\ &+6\mathbb{E}\bigg(\int_0^b ||\alpha(\alpha I+\Gamma_s^b)^{-1}\phi(s)||^2_{L_2^0}ds\bigg)\\ &+6\mathbb{E}\bigg(\int_0^b ||\alpha(\alpha I+\Gamma_s^b)^{-1}||\;||S(b-s)[f(s,x_\alpha(s))-f(s)]||ds\bigg)^2\\ &+6\mathbb{E}\bigg(\int_0^b ||\alpha(\alpha I+\Gamma_s^b)^{-1}S(b-s)f(s)||ds\bigg)^2\\ &+6\mathbb{E}\bigg(\int_0^b ||\alpha(\alpha I+\Gamma_s^b)^{-1}||\;||S(b-s)[\sigma(s,x_\alpha(s))-\sigma(s)]||^2_{L_2^0}ds\bigg)\\ &+6\mathbb{E}\bigg(\int_0^b ||\alpha(\alpha I+\Gamma_s^b)^{-1}S(b-s)\sigma(s)||^2_{L_2^0}ds\bigg)\bigg].\end{aligned}$$

Since by assumption (iv), for all $0\leq s<b$ the operator $\alpha(\alpha I+\Gamma_s^b)^{-1}\to 0$ strongly as $\alpha\to 0^+$ and moreover $||\alpha(\alpha I+\Gamma_s^b)^{-1}||\leq 1$. Thus by the Lebesgue dominated convergence theorem, we obtain $\mathbb{E}||x_\alpha(b)-x_b||^2\to 0^+$. This gives the approximate controllability.

4 Example

Consider the stochastic control system:

$$\left.\begin{array}{l} d_tz(t,\theta)=[z_{\theta\theta}+Bu(t,\theta)+p(t,z(t))]dt+k(t,z(t))d\omega(t),\\ z(t,0)=z(t,\pi)=0,\quad 0\leq t\leq T,\quad 0<\theta<\pi,\\ z(0,\theta)+\displaystyle\sum_{i=1}^n\alpha_i z(t_i,\theta)=z_0(\theta)\quad t\in J, \end{array}\right\}\tag{6}$$

where B is a bounded linear operator from a Hilbert space U into X; $p: J\times X\to X$, $k: J\times X\to L_2^0$ are all continuous and uniformly bounded, $u(t)$ is a feedback control and w is a Q-Wiener process.

Let $X=L_2[0,\pi]$, and let $A: D(A)\subset X\to X$ be an operator defined by

$$Az=z_{\theta\theta}$$

with domain

$$D(A)=\{z(.)\in X|z,z_\theta \text{ are absolutely continuous },z_{\theta\theta}\in X, z(0)=z(\pi)=0\}.$$

Furthermore, A has discrete spectrum, the eigenvalues are $-n^2, n=1,2,\cdots$ with the corresponding normalized characteristic vectors $e_n(s)=(2/\pi)^{1/2}\sin ns$, then

$$Az=\sum_{n=1}^\infty -n^2<z,e_n>e_n,\quad z\in X.$$

It is known that A generates a compact semigroup $S(t), t > 0$ in X and is given by

$$S(t)z = \sum_{n=1}^{\infty} e^{-n^2 t} < z, e_n > e_n(\theta), \quad z \in X.$$

Let $f : J \times X \to X$ be defined by

$$f(t, x(t))(\theta) = p(t, x(t))(\theta)), \quad (t, x_t) \in J \times X, \theta \in [0, \pi].$$

Let $\sigma : J \times X \to L_2^0$ be defined by

$$\sigma(t, x(t))(\theta) = k(t, x(t))(\theta)), \quad (t, x_t) \in J \times X, \theta \in [0, \pi].$$

The function $g : C(J, X) \to X$ is defined as

$$g(z)(\theta) = \sum_{i=1}^{n} \alpha_i z(t_i, \theta)$$

for $0 < t_i < T$ and $\theta \in [0, \pi]$.

With this choice of A, B, f, σ and g, (1) is the abstract formulation of (6) such that the conditions in (i) and (ii) are satisfied.

Now define an infinite-dimensional space

$$U = \left\{ u : u = \sum_{n=2}^{\infty} u_n e_n(\theta) \ \ | \sum_{n=2}^{\infty} u_n^2 < \infty \right\}$$

with the norm defined by

$$||u||_U = \left(\sum_{n=2}^{\infty} u_n^2 \right)^{1/2}$$

and a linear continuous mapping B from $U \to X$ as follows:

$$Bu = 2u_2 e_1(\theta) + \sum_{n=2}^{\infty} u_n(t) e_n(\theta).$$

It is obvious that for $u(t, \theta, \omega) = \sum_{n=2}^{\infty} u_n(t, \omega) e_n(\theta) \in L_2^{\Im}(J, U)$

$$Bu(t) = 2u_2(t) e_1(\theta) + \sum_{n=2}^{\infty} u_n(t) e_n(\theta) \in L_2^{\Im}(J, X).$$

Moreover,

$$B^* v = (2v_1 + v_2) e_2(\theta) + \sum_{n=3}^{\infty} v_n e_n(\theta),$$

$$B^* S^*(t) z = (2z_1 e^{-t} + z_2 e^{-4t}) e_2(\theta) + \sum_{n=3}^{\infty} z_n e^{-n^2 t} e_n(\theta),$$

for $v = \sum_{n=1}^{\infty} v_n e_n(\theta)$ and $z = \sum_{n=1}^{\infty} z_n e_n(\theta)$.

Let $||B^* S^*(t)z|| = 0, \quad t \in [0, T]$, it follows that

$$||2z_1 e^{-t} + z_2 e^{-4t}||^2 + \sum_{n=3}^{\infty} ||z_n e^{-n^2 t}||^2 = 0, \quad t \in [0, T]$$

$\Rightarrow z_n = 0, \quad n = 1, 2, \cdots \Rightarrow z = 0.$

Thus by Theorem 4.1.7 of [23], the deterministic linear system corresponding to (6) is approximate controllable on $[0, T]$. Therefore the system (6) is approximate controllable provided that f, σ and g satisfy the assumptions (i) and (ii).

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