Nonlinear Dynamics and Systems Theory, **15** (4) (2015) 409–417

A Simple Analytical Technique to Investigate Nonlinear Oscillations of an Elastic Two Degrees of Freedom Pendulum

Md. Abdur Razzak* and Md. Helal Uddin Molla

Dept of Mathematics, Rajshahi University of Engineering and Technology (RUET), Kazla, Rajshahi 6204, Bangladesh.

Received: September 6, 2014; Revised: October 28, 2015

Abstract: Based on the general Struble's technique, a simple analytical technique has been presented to investigate nonlinear oscillations of an elastic pendulum. The method is illustrated by swinging spring pendulum in the resonance cases (frequencies ratio is equal to 1 : 2). Solutions not only show a good coincidence with the corresponding numerical solution but also give better result than multiple scales (MS) method.

Keywords: *nonlinear oscillation; swinging spring pendulum; Struble's technique.*

Mathematics Subject Classification (2010): 34A34, 34C25.

1 Introduction

Struble's technique [1], Krylov-Bogoliubov-Mitropolskii (KBM) method [2, 3], multiple time-scales method [4] are usually applied to determine the approximation solutions of weakly nonlinear differential equations. Popov [5] extended the KBM method to a damped system. Bojadziev [6] studied second order nonlinear system with strong damping effect by the two time scales method and justified that the solution is similar to that obtained by Popov [5]. Sometimes, all classical perturbation techniques [1–3] are useless to solve some nonlinear differential equations. In this regard, Shamsul [7] presented a general Struble's techniques to determine approximate solution of n-th order weakly non-linear differential systems. It is easy to apply the general Struble's technique to solve nonlinear differential equations with various damping effect.

* Corresponding author: mailto:raz_math61@yahoo.com

In this paper, we have partially used this method [7] to solve nonlinear oscillations of elastic pendulum, in which the internal resonance occurs. In particular, a swinging spring pendulum without or with damping force has been investigated. Earlier Gorelik and Witt [8] studied this nonlinear oscillator in the case without damping. Then Kane and Kahn [9] studied the character of resonant case. Some authors studied similar type of swinging spring by the method of averaging [4, 10–12]. Latter, Nayfeh and Mook [13] studied two-degree-of freedom system by multiple scales (MS) method. Zaripov and Petrov [14]; Awrejcewicz and Petrov [15] investigated a spring type swinging pendulum in the resonance case by using Poincare–Birkhoff normal form method. Recently, some authors [16–19] have studied nonlinear differential equations. The solution obtained by the presented method is not only a better result than that by MS method [13] but also shows a nice coincidence with the corresponding numerical solution.

2 The Method

Consider a nonlinear oscillator of two degree-of-freedoms with strong damping effect

$$\ddot{x} + 2k_1\dot{x} + \omega_1^2 x = \varepsilon f(x,\theta,\dot{x},\dot{\theta}), \tag{1}$$

$$\ddot{\theta} + 2k_2\dot{\theta} + \omega_2^2 \theta = \varepsilon \Phi(x,\theta,\dot{x},\dot{\theta}), \tag{2}$$

where over dot denotes the derivatives with respect to t, $\omega_1, \omega_2 \geq 0$, k_1, k_2, ν are constants, ε denotes small parameter, ω_1 and ω_2 are natural frequency, $f(x,\theta,\dot{x},\dot{\theta})$ and $\Phi(x,\theta,\dot{x},\dot{\theta})$ are nonlinear functions.

When $\varepsilon = 0$, equations (1)–(2) become a linear equation and there are two eigenvalues of that two equations, say $\lambda_1 = -k_1 + i\omega_1^*$, $\lambda_2 = -k_1 - i\omega_1^*$, where $\omega_1^* = \sqrt{\omega_1^2 - k_1^2}$ and $\mu_1 = -k_2 + i\omega_2^*$, $\mu_2 = -k_2 - i\omega_2^*$, where $\omega_2^* = \sqrt{\omega_2^2 - k_2^2}$, respectively.

On the other hand when $\varepsilon \neq 0$, the first approximation solution of equations (1)–(2) is chosen in the form [7]

$$x = a_1 e^{\lambda_1 t} + a_2 e^{\lambda_2 t} + \varepsilon u_1 \tag{3}$$

and

$$\theta = b_1 e^{\mu_1 t} + b_2 e^{\mu_2 t} + \varepsilon v_1. \tag{4}$$

Equations (1)–(2) can be rewritten in the following form:

$$(D - \lambda_1)(D - \lambda_2)x = \varepsilon f, \tag{5}$$

$$(D - \mu_1)(D - \mu_2)\theta = \varepsilon \Phi. \tag{6}$$

Substituting equations (3)–(4) into equations (5)–(6), we obtain the following results, respectively as

$$(D - \lambda_1)(D - \lambda_2)(a_1 e^{\lambda_1 t} + a_2 e^{\lambda_2 t} + \varepsilon\, u_1) = \varepsilon f$$

or

$$(D - \lambda_2)(\dot{a}_1\, e^{\lambda_1 t}) + (D - \lambda_1)(\dot{a}_2\, e^{\lambda_2 t}) + (D - \lambda_1)(D - \lambda_2)(\varepsilon u_1) = \varepsilon f; \tag{7}$$

$$(D - \mu_1)(D - \mu_2)(b_1 e^{\mu_1 t} + b_2 e^{\mu_2 t} + \varepsilon\, v_1) = \varepsilon \Phi$$

or

$$(D - \mu_2)(\dot{b}_1\, e^{\mu_1 t}) + (D - \mu_1)(\dot{b}_2\, e^{\mu_2 t}) + (D - \mu_1)(D - \mu_2)(\varepsilon v_1) = \varepsilon \Phi, \tag{8}$$

since $(D - \lambda_1)(a_1\, e^{\lambda_1 t}) = \dot{a}_1\, e^{\lambda_1 t}$, $(D - \lambda_2)(a_2\, e^{\lambda_2 t}) = \dot{a}_2\, e^{\lambda_2 t}$, $(D - \mu_1)(b_1\, e^{\mu_1 t}) = \dot{b}_1\, e^{\mu_1 t}$ and $(D - \mu_2)(b_2\, e^{\mu_2 t}) = \dot{b}_2\, e^{\mu_2 t}$.

Herein the nonlinear functions f and Φ can be expanded in a Taylor series as

$$f = \sum_{m_1=0,\, m_2=0}^{\infty,\,\infty} F_{m_1,m_2} e^{(m_1\lambda_1+m_2\lambda_2)t}, \quad \Phi = \sum_{r_1=0,\, r_2=0}^{\infty,\,\infty} \Phi_{r_1,r_2} e^{(r_1\mu_1+r_2\mu_2)t}$$

and the unknown functions u_1 and v_1 can be obtained in terms of the variables a_1, a_2 and t; b_1, b_2 and t under the condition that u_1 and v_1 exclude the terms $F_{m_1,m_2}e^{(m_1\lambda_1+m_2\lambda_2)t}$ of f and $\Phi_{r_1,r_2}e^{(r_1\mu_1+r_2\mu_2)t}$ of Φ where, $m_1 - m_2 = \pm 1$ and $r_1 - r_2 = \pm 1$. On the other hand, both $\dot{a}_1$ and $\dot{a}_2$ respectively, contain the terms $F_{m_1,m_2}e^{(m_1\lambda_1+m_2\lambda_2)t}$ where $m_1 - m_2 = 1$ and $m_1 - m_2 = -1$. This assumption takes u_1 free from secular terms, *i.e.*, $t\cos t$, $t\sin t$. Similarly, both $\dot{b}_1$ and $\dot{b}_2$ respectively contain the terms $\Phi_{r_1,r_2}e^{(r_1\mu_1+r_2\mu_2)t}$ where $r_1 - r_2 = 1$ and $r_1 - r_2 = -1$. This assumption makes v_1 free from secular terms.

Now, separating equation (7) into three parts for $\dot{a}_1$, $\dot{a}_2$ and u_1 we get

$$(D-\lambda_2)(\dot{a}_1\, e^{\lambda_1 t}) = \sum_{m_1=0,\, m_2=0}^{\infty,\,\infty} F_{m_1,m_2} e^{(m_1\lambda_1+m_2\lambda_2)t}, \quad m_1 - m_2 = 1, \tag{9}$$

$$(D-\lambda_1)(\dot{a}_2\, e^{\lambda_2 t}) = \sum_{m_1=0,\, m_2=0}^{\infty,\,\infty} F_{m_1,m_2} e^{(m_1\lambda_1+m_2\lambda_2)t}, \quad m_1 - m_2 = -1, \tag{10}$$

$$(D-\lambda_1)(D-\lambda_2)u_1 = \sum_{m_1=0,\, m_2=0}^{\infty,\,\infty} F_{m_1,m_2} e^{(m_1\lambda_1+m_2\lambda_2)t}, \quad m_1 - m_2 \neq \pm 1. \tag{11}$$

Similarly, separating equation (8) into three parts for $\dot{b}_1$, $\dot{b}_2$ and p_1 we get

$$(D-\mu_2)(\dot{b}_1\, e^{\mu_1 t}) = \sum_{r_1=0,\, r_2=0}^{\infty,\,\infty} \Phi_{r_1,r_2} e^{(r_1\mu_1+r_2\mu_2)t}, \quad r_1 - r_2 = 1, \tag{12}$$

$$(D-\mu_1)(\dot{b}_2\, e^{\mu_2 t}) = \sum_{r_1=0,\, r_2=0}^{\infty,\,\infty} \Phi_{r_1,r_2} e^{(r_1\mu_1+r_2\mu_2)t}, \quad r_1 - r_2 = -1, \tag{13}$$

$$(D-\mu_1)(D-\mu_2)v_1 = \sum_{r_1=0,\, r_2=0}^{\infty,\,\infty} \Phi_{r_1,r_2} e^{(r_1\mu_1+r_2\mu_2)t}, \quad r_1 - r_2 \neq \pm 1. \tag{14}$$

Under transformation $a_1 = \frac{a}{2}\, e^{i\varphi_1}$, $a_2 = \frac{a}{2}\, e^{-i\varphi_1}$, $b_1 = \frac{b}{2}\, e^{i\varphi_2}$, $b_2 = \frac{b}{2}\, e^{-i\varphi_2}$, equations (9)–(14) are transformed to amplitude and phase equations. On the other hand, this transformation keeps u_1 and v_1 in an amplitude and phase form. Therefore, the first approximate solution is clearly found.

3 Example

Consider a swinging spring pendulum with damping force whose governing equation [4] is given by

$$\ddot{x} + \delta_1\dot{x} + \frac{k}{m}x + g(1-\cos\theta) - (l+x)\dot{\theta}^2 = 0, \tag{15}$$

$$\ddot{\theta} + \delta_2\dot{\theta} + \frac{g}{l+x}\sin\theta + \frac{2}{l+x}\dot{x}\dot{\theta} = 0, \tag{16}$$

where l is a length of swinging spring, $\omega_1^2 = \frac{k}{m} \approx 4\omega_2^2 = \frac{4g}{l}$ and k is constant.

If $x \ll l$, then equations (15) and (16) become

$$\ddot{x} + 2k_1\dot{x} + \omega_1^2 x + \omega_2^2\,\theta^2\,l/2 - l\,\dot{\theta}^2 = 0, \tag{17}$$

$$\ddot{\theta} + 2k_2\dot{\theta} + \omega_2^2\theta - \omega_2^2\,x\,\theta/l + 2\,\dot{x}\,\dot{\theta}/l = 0. \tag{18}$$

Substituting $x = \varepsilon\,x$ and $\theta = \varepsilon\,\theta$ in equations (17)–(4) , we obtain

$$\ddot{x} + 2k_1\dot{x} + \omega_1^2 x = -\varepsilon\,\omega_2^2\theta^2 l/2 + \varepsilon\,l\,\dot{\theta}^2, \tag{19}$$

$$\ddot{\theta} + 2k_2\dot{\theta} + \omega_2^2\theta = \varepsilon\omega_2^2 x\theta/l - 2\varepsilon\dot{x}\dot{\theta}/l, \tag{20}$$

where $\delta_1 = 2k_1$, $\delta_2 = 2k_2$.

Equations (19)–(20) can be written as

$$(D - \lambda_1)(D - \lambda_2)x = -\varepsilon(\omega_2^2\theta^2 l/2 - l\,\dot{\theta}^2), \tag{21}$$

$$(D - \mu_1)(D - \mu_2)\theta = \varepsilon\omega_2^2 x\theta/l - 2\varepsilon\dot{x}\dot{\theta}/l. \tag{22}$$

When $\varepsilon = 0$, equation (21) becomes a linear equation and there are two eigenvalues, say $\lambda_1 = -k_1 + i\omega_1^*$, $\lambda_2 = -k_1 - i\omega_1^*$, where $\omega_1^* = \sqrt{\omega_1^2 - k_1^2}$ and $x = a_1 e^{\lambda_1 t} + a_2 e^{\lambda_2 t} + \varepsilon u_1$; $\theta = b_1 e^{\mu_1 t} + b_2 e^{\mu_2 t} + \varepsilon p_1$; and

$$\begin{aligned} f = -(\omega_2^2\theta^2 l/2 - l\dot{\theta}^2) &= -l\omega_2^2 b_1^2 e^{2\mu_1 t}/2 - l\omega_2^2 b_2^2 e^{2\mu_2 t}/2 - 2lb_1 b_2\omega_2^2 e^{(\mu_1+\mu_2)t}/2 \\ &\quad + lb_1^2\mu_1^2 e^{2\mu_1 t} + 2lb_1 b_2\mu_1\mu_2 e^{(\mu_1+\mu_2)t} + lb_2^2\mu_2^2 e^{2\mu_2 t} + \cdots \end{aligned}$$

Therefore, equation (21) becomes

$$\begin{aligned} &(D - \lambda_2)(\dot{a}_1 e^{\lambda_1 t}) + (D - \lambda_1)(\dot{a}_2 e^{\lambda_2 t}) + \varepsilon(D - \lambda_1)(D - \lambda_2)u_1 \\ &= -\varepsilon l\omega_2^2 b_1^2 e^{2\mu_1 t}/2 - \varepsilon l\omega_2^2 b_2^2 e^{2\mu_2 t}/2 - 2\varepsilon lb_1 b_2\omega_2^2 e^{(\mu_1+\mu_2)t}/2 \\ &+ \varepsilon lb_1^2\mu_1^2 e^{2\mu_1 t} + 2\varepsilon lb_1 b_2\mu_1\mu_2 e^{(\mu_1+\mu_2)t} + \varepsilon lb_2^2\mu_2^2 e^{2\mu_2 t} + \cdots, \end{aligned} \tag{23}$$

It is mentioned that $\lambda_1 = -k_1 + i\omega_1^*$, $\lambda_2 = -k_1 - i\omega_1^*$, $\mu_1 = -k_2 + i\omega_2^*$, $\mu_2 = -k_2 - i\omega_2^*$ in the case of under-damped systems. For the resonance case, we have used $\omega_1^* \approx 2\omega_2^*$. Since $e^{\lambda_1 t}$ and $e^{2\mu_1 t}$ contain $e^{i\omega_1^* t}$, we equate the terms with $e^{\lambda_1 t}$ and $e^{2\mu_1 t}$ of equation (23). In a similar way, we equate the terms with $e^{\lambda_2 t}$ and $e^{2\mu_2 t}$ of equation (23). On the other hand, u_1 contains the term $e^{(\mu_1+\mu_2)t}$.

Now, separating equation (23) into three parts for $\dot{a}_1$, $\dot{a}_2$ and u_1 we get (see paper [7])

$$(D - \lambda_2)(\dot{a}_1 e^{\lambda_1 t}) = -\varepsilon l\omega_2^2 b_1^2 e^{2\mu_1 t}/2 + \varepsilon lb_1^2\mu_1^2 e^{2\mu_1 t}, \tag{24}$$

$$(D - \lambda_1)(\dot{a}_2 e^{\lambda_2 t}) = -\varepsilon l\omega_2^2 b_2^2 e^{2\mu_2 t}/2 + \varepsilon lb_2^2\mu_2^2 e^{2\mu_2 t}, \tag{25}$$

and

$$(D - \lambda_1)(D - \lambda_2)u_1 = -l\omega_2^2 b_1 b_2 e^{(\mu_1+\mu_2)t} + 2lb_1 b_2\mu_1\mu_2 e^{(\mu_1+\mu_2)t}. \tag{26}$$

From equation (24), we obtain

$$\dot{a}_1 e^{\lambda_1 t} = -\frac{\varepsilon l\omega_2^2 b_1^2 e^{2\mu_1 t}}{2(D - \lambda_2)} + \frac{\varepsilon lb_1^2\mu_1^2 e^{2\mu_1 t}}{(D - \lambda_2)} = \varepsilon lb_1^2\left(\mu_1^2 - \frac{\omega_2^2}{2}\right)e^{2\mu_1 t}/(2\mu_1 - \lambda_2). \tag{27}$$

Equation (27) can be written as

$$\dot{a}_1 = \varepsilon l b_1^2 \left(\mu_1^2 - \frac{\omega_2^2}{2}\right) e^{(2\mu_1-\lambda_1)t}/(2\mu_1 - \lambda_2). \tag{28}$$

Substituting $a_1 = \frac{a}{2}e^{i\varphi_1}$, $a_2 = \frac{a}{2}e^{-i\varphi_1}$, $b_1 = \frac{b}{2}e^{i\varphi_2}$, $b_2 = \frac{b}{2}e^{-i\varphi_2}$, $\lambda_1 = -k_1 + i\omega_1^*$, $\lambda_2 = -k_1 - i\omega_1^*$, and $\mu_1 = -k_2 + i\omega_2^*$, $\mu_2 = -k_2 - i\omega_2^*$ into equation (28), we obtain

$$\begin{aligned}(\dot{a} + ia\dot{\varphi}_1)/2 &= \frac{\varepsilon l b^2(2(-k_2+i\omega_2^*)^2 - \omega_2^2)e^{(2(-k_2+i\omega_2^*)-(-k_1+i\omega_1^*))t+2i\varphi_2-i\varphi_1}}{8(2(-k_2+i\omega_2^*)-(-k_1-i\omega_1^*))} \\ &= \frac{\varepsilon l b^2 e^{(k_1-2k_2)t}}{4((k_1-2k_2)^2+(2\omega_2^*+\omega_1^*)^2)}[(4k_2^2-3\omega_2^2)(k_1-2k_2) - 4k_2\omega_2^*(2\omega_2^*+\omega_1^*) \\ &\quad - i(4k_2\omega_2^*(k_1-2k_2) + (4k_2^2-3\omega_2^2)(2\omega_2^*+\omega_1^*))]e^{i\gamma},\end{aligned} \tag{29}$$

where $\gamma = (2\omega_2^* - \omega_1^*)t + 2\varphi_2 - \varphi_1$.

Separating the real and imaginary parts from both sides of equation (29), we obtain

$$\begin{aligned}\dot{a} &= \frac{\varepsilon l b^2 e^{(k_1-2k_2)t}}{4((k_1-2k_2)^2+(2\omega_2^*+\omega_1^*)^2)}[(4k_2^2-3\omega_2^2)(k_1-2k_2) - 4k_2\omega_2^*(2\omega_2^*+\omega_1^*)\cos\gamma \\ &\quad + (4k_2\omega_2^*(k_1-2k_2) + (4k_2^2-3\omega_2^2)(2\omega_2^*+\omega_1^*))\sin\gamma],\end{aligned} \tag{30}$$

$$\begin{aligned}\dot{\varphi}_1 &= \frac{\varepsilon l b^2 e^{(k_1-2k_2)t}}{4a((k_1-2k_2)^2+(2\omega_2^*+\omega_1^*)^2)}[(4k_2^2-3\omega_2^2)(k_1-2k_2) - 4k_2\omega_2^*(2\omega_2^*+\omega_1^*)\sin\gamma \\ &\quad - (4k_2\omega_2^*(k_1-2k_2) + (4k_2^2-3\omega_2^2)(2\omega_2^*+\omega_1^*))\cos\gamma].\end{aligned} \tag{31}$$

Similarly, equation (22) becomes

$$\begin{aligned}&(D-\mu_1)(\dot{b}_2 e^{\mu_2 t}) + (D-\mu_2)(\dot{b}_1 e^{\mu_1 t}) + \varepsilon(D-\mu_1)(D-\mu_2)v_1 \\ &= \varepsilon\omega_2^2(a_1b_1e^{(\lambda_1+\mu_1)t} + a_1b_2e^{(\lambda_1+\mu_2)t} + a_2b_1e^{(\lambda_2+\mu_1)t} \\ &+ a_2b_2e^{(\lambda_2+\mu_2)t})/l - 2\varepsilon(a_1b_1\lambda_1\mu_1e^{(\lambda_1+\mu_1)t} + a_1b_2\lambda_1\mu_2e^{(\lambda_1+\mu_2)t} \\ &+ a_2b_1\lambda_2\mu_1e^{(\lambda_2+\mu_1)t} + a_2b_2\lambda_2\mu_2e^{(\lambda_2+\mu_2)t})/l.\end{aligned} \tag{32}$$

Herein we have used $\dot{a}_1 = 0$, $\dot{b}_1 = 0$.

Applying the separation rule to equation (32), we obtain the following equations for $\dot{b}_1$, $\dot{b}_2$ and v_1

$$(D-\mu_2)(\dot{b}_1 e^{\mu_1 t}) = \varepsilon\omega_2^2 a_1 b_2 e^{(\lambda_1+\mu_2)t}/l - 2\varepsilon\, a_1 b_2 \lambda_1 \mu_2 e^{(\lambda_1+\mu_2)t}/l, \tag{33}$$

$$(D-\mu_1)(\dot{b}_2 e^{\mu_2 t}) = \varepsilon\omega_2^2 a_2 b_1 e^{(\lambda_2+\mu_1)t}/l - 2\varepsilon\, a_2 b_1 \lambda_2 \mu_1 e^{(\lambda_2+\mu_1)t}/l \tag{34}$$

and

$$\begin{aligned}(D-\mu_1)(D-\mu_2)v_1 &= (\omega_2^2 a_1b_1e^{(\lambda_1+\mu_1)t} + \omega_2^2 a_2b_2e^{(\lambda_2+\mu_2)t})/l \\ &- 2(a_1b_1\lambda_1\mu_1e^{(\lambda_1+\mu_1)t} + 2a_2b_2\lambda_2\mu_2e^{(\lambda_2+\mu_2)t})/l\end{aligned} \tag{35}$$

From equation (33), we obtain

$$\begin{aligned}\dot{b}_1 e^{\mu_1 t} &= \frac{\varepsilon\omega_2^2 a_1 b_2 e^{(\lambda_1+\mu_2)t}}{l(D-\mu_2)} - \frac{2\varepsilon\, a_1 b_2 \lambda_1 \mu_2 e^{(\lambda_1+\mu_2)t}}{l(D-\mu_2)} \\ &= \frac{\varepsilon\omega_2^2 a_1 b_2 e^{(\lambda_1+\mu_2)t}}{l\lambda_1} - \frac{2\varepsilon\, a_1 b_2 \lambda_1 \mu_2 e^{(\lambda_1+\mu_2)t}}{l\lambda_1}.\end{aligned} \tag{36}$$

From equation (36), we obtain

$$\dot{b}_1 = \frac{\varepsilon\omega_2^2 a_1 b_2 e^{(\lambda_1+\mu_2-\mu_1)t}}{l\lambda_1} - \frac{2\varepsilon\, a_1 b_2 \lambda_1 \mu_2 e^{(\lambda_1+\mu_2-\mu_1)t}}{l\lambda_1}. \tag{37}$$

Using a transformation $a_1 = \frac{a}{2}e^{i\varphi_1}$, $a_2 = \frac{a}{2}e^{-i\varphi_1}$, $b_1 = \frac{b}{2}e^{i\varphi_2}$, $b_2 = \frac{b}{2}e^{-i\varphi_2}$ for equation (37), we obtain

$$\dot{b} + ib\dot{\phi}_2 = \frac{\varepsilon a b e^{-k_1 t}}{2l\omega_1^2}\left[(2\omega_1^2 k_2 - \omega_2^2 k_1) + i(2\omega_1^2\omega_2^* - \omega_2^2\omega_1^*)\right] e^{-i\gamma}, \tag{38}$$

where $\gamma = (2\omega_2^* - \omega_1^*)t + 2\varphi_2 - \varphi_1$.

Separating the real and imaginary parts from both sides of equation (38), we obtain

$$\dot{b} = \frac{\varepsilon\, a\, b e^{-k_1 t}}{2l\omega_1^2}[(2\,\omega_1^2 k_2 - \omega_2^2 k_1)\cos\gamma + (2\,\omega_1^2\omega_2^* - \omega_2^2\omega_1^*)\sin\gamma], \tag{39}$$

$$\dot{\varphi}_2 = \frac{\varepsilon\, a\; e^{-k_1 t}}{2l\omega_1^2}[(2\,\omega_1^2\omega_2^* - \omega_2^2\omega_1^*)\cos\gamma - (2\,\omega_1^2 k_2 - \omega_2^2 k_1)\sin\gamma]. \tag{40}$$

Therefore, the first approximate solution of equations (15)–(16) becomes

$$x = \varepsilon a e^{-k_1 t}\cos(\omega_1 t + \varphi_1) + O(\varepsilon^2), \tag{41}$$

$$\theta = \varepsilon b e^{-k_2 t}\cos(\omega_2 t + \varphi_2) + O(\varepsilon^2). \tag{42}$$

If the damping force is absent i.e. $k = 0$, then equations (30)–(31) and (39)–(40) become

$$\dot{a} = -3\varepsilon l b^2\omega_2^2 \sin\psi/(4(2\omega_2 + \omega_1)), \tag{43}$$

$$\dot{\varphi}_1 = 3\varepsilon l b^2\omega_2^2\cos\psi/(4a(2\omega_2 + \omega_1)), \tag{44}$$

and

$$\dot{b} = \frac{\varepsilon a b\omega_2(2\omega_1 - \omega_2)\sin\psi}{2l\omega_1}, \tag{45}$$

$$\dot{\varphi}_2 = \frac{\varepsilon a\omega_2(2\omega_1 - \omega_2)\cos\psi}{2l\omega_1}, \tag{46}$$

where $\psi = (2\omega_2 - \omega_1)t + 2\varphi_2 - \varphi_1$.

In this case (undamped), the first approximate solution of equations (15)–(16) is

$$x = \varepsilon a\cos(\omega_1 t + \varphi_1) + O(\varepsilon^2), \tag{47}$$

$$\theta = \varepsilon b\cos(\omega_2 t + \varphi_2) + O(\varepsilon^2). \tag{48}$$

4 Results and Discussion

Usually a nonlinear problem is solved by a perturbation method [5, 20–23]. In this paper, a simple analytical technique has been developed based on the general Struble's technique [7] to investigate nonlinear oscillations of an elastic pendulum. The technique is very easy and straightforward. Nonlinear oscillations of the swinging spring pendulum in the case of resonance $\omega_1 : \omega_2 = 1 : 2$ have been considered. The solutions have been obtained without and with damping effect and presented respectively in Figure 1 and Figure 3.

On the other hand, the corresponding perturbation solutions have been obtained by MS method and shown in Figure 2 and Figure 4. To compare our solution with existing perturbation solutions, we have provided the numerical solutions in all the figures.

From Figure 2 and Figure 4, we see that the solutions by MS method deviate from numerical solution after a certain time. On the other hand, our solutions (see Figures 1, 3) show a good coincidence with the numerical solutions.

Comparing all the results of swinging spring pendulum in the case of resonance $\omega_1 : \omega_2 = 1 : 2$, we observe that the general Struble's technique provides more correct solution than other perturbation solutions especially those obtained by the multiple time scale method [13].

5 Conclusion

Based on the general Struble's technique [7], a simple analytical technique has been presented to investigate nonlinear oscillations of an elastic pendulum in which damping effect is present. Nonlinear oscillations of the swinging spring pendulum with or without damping effect in the case of resonance are considered. Previously, some authors (see [9, 14–15]) investigated swinging spring pendulum without damping effect. On the other hand, some perturbation methods especially MS methods are not suitable to investigate nonlinear oscillations of elastic pendulum. In this paper, a simple perturbation method has been presented and has given better result than MS method. The method also provides a good result compared to the numerical solution (considered to exact).

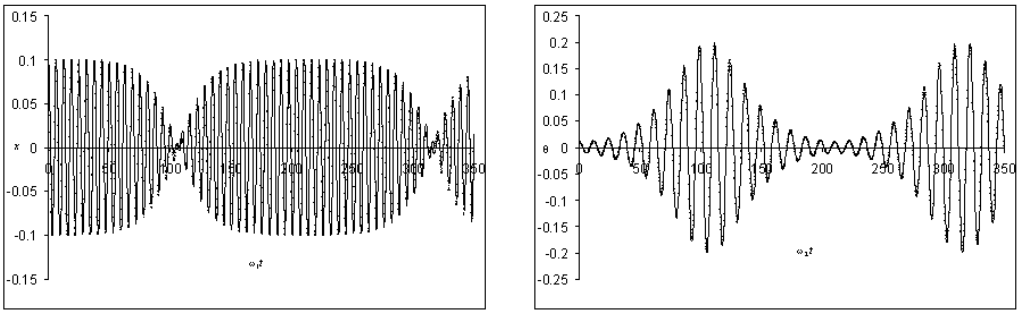

Fig 1: Solution of equations (15) and (16) obtained by the presented method has been presented (denoted by dots) when $k_1 = k_2 = 0, , \omega_2 = 0.5\,\omega_1, l = 1, \varepsilon = 0.1$ with initial conditions $[x(0) = 1,\, \dot{x}(0) = 0,\, \theta(0) = 0.1,\, \dot{\theta}(0) = 0]$. Corresponding numerical solution (obtained by fourth-order Runge-Kutta method) has been presented (represented by solid line) to be compared with the present solution.

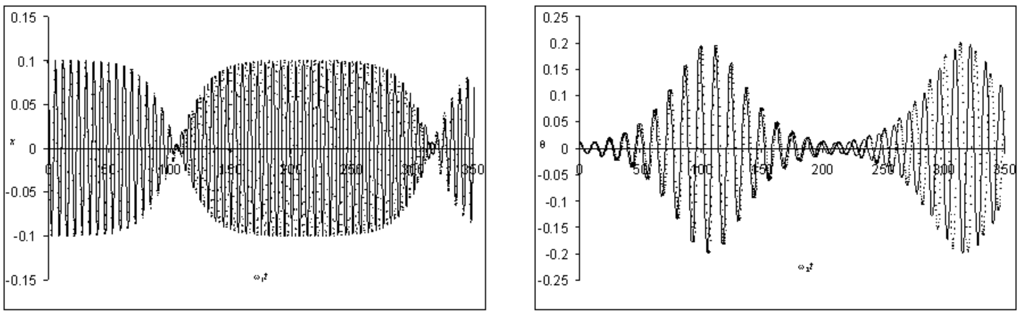

Fig 2: Solution of equations (15) and (16) obtained by MS method has been presented

(denoted by dots) when $k_1 = k_2 = 0, \omega_2 = 0.5\,\omega_1, l = 1, \varepsilon = 0.1$ with initial conditions $[x(0) = 1,\ \dot{x}(0) = 0,\ \theta(0) = 0.1,\ \dot{\theta}(0) = 0]$. Corresponding numerical solution (obtained by fourth-order Runge-Kutta method) has been presented (represented by solid line) to be compared with MS method solution.

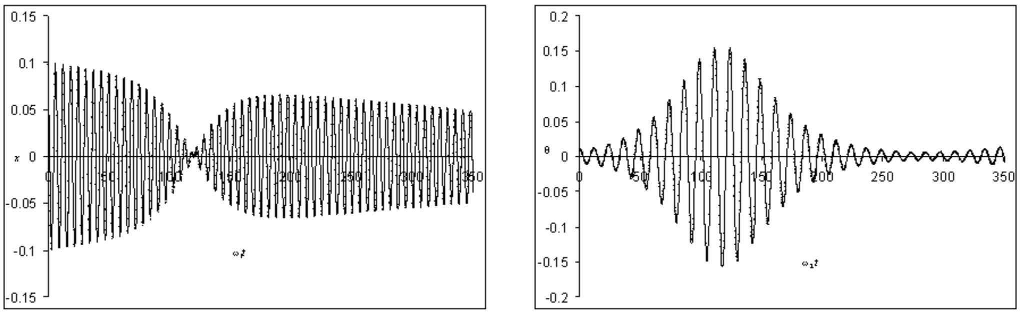

Fig 3: Solution of eqations (30) and (31) by the present method has been presented (denoted by dots) when $\omega_2 = 0.5\,\omega_1, l = 1, \varepsilon = 0.1, k_1 = \delta_1/2 = 0.002$, $k_2 = \delta_2/2 = 0.002$ and the initial conditions $[x(0) = 1,\ \dot{x}(0) = 0,\ \theta(0) = 0.1,\ \dot{\theta}(0) = 0]$. Corresponding numerical solution (obtained by fourth-order Runge-Kutta method) has been presented (represented by solid line) to be compared with the present solution.

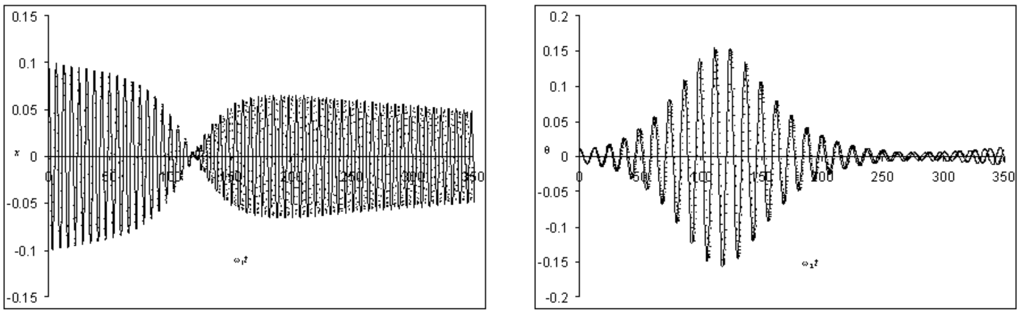

Fig 4: Solution of eqations (30) and (31) by MS method has been presented (denoted by dots) when $\omega_2 = 0.5\,\omega_1, l = 1, \varepsilon = 0.1, k_1 = \delta_1/2 = 0.002$, $k_2 = \delta_2/2 = 0.002$ and the initial conditions $[x(0) = 1,\ \dot{x}(0) = 0,\ \theta(0) = 0.1,\ \dot{\theta}(0) = 0]$. Corresponding numerical solution (obtained by fourth-order Runge-Kutta method) has been presented (represented by solid line) to be compared with MS method solution.

References

[1] Struble, R. A. The geometry of the orbits of artificial satellities. *Arch. Rational Mech. Anal.* **7** (1961) 87–104.

[2] Krylov, N. N. and Bogoliubov, N. N. *Introduction to Nonlinear Mechanics.* Princeton University Press, New Jersey, 1947.

[3] Bogoliubov, N. N. and Mitropolskii, Yu. A. *Asymptotic Methods in the Theory of Non-linear Oscillations.* Gordan and Breach, New York, 1961.

[4] Nayfeh, A. H. *Perturbation Methods.* John Wiley & Sons, New York, 1973.

[5] Popov, I. P. A generalization of the Bogoliubov asymptotic method in the theory of nonlinear oscillation. *Dokl. Akad. Nauk. SSSR* **111** (1956) 308–310.

[6] Bojadziev, G. N. Two variables expansion method applied to the study of damped nonlinear oscillations. *Nonlinear Vib. Probl.* **21** (1981) 11–18.

[7] Shamsul Alam, M., Azad, M. A. K., and Haque, M. A. A general Struble's technique for solving an n-th order weakly nonlinear differential system with damping. *Int. J. Nonlinear Mech.* **41** (2006) 905–918.

[8] Gorelik, G. and Witt, A. Swing of an elastic pendulum as an example of two parametrically bound linear vibration systems. *J. Tech. Phys. (USSR)* **3** (1933) 244–307.

[9] Kane, T. R. and Kahn, M. E. On a class of two-degree-of-freedom oscillations. *J. Appl. Mech.* **35** (1968) 547–552.

[10] Mettler, E. Stabilitatsfragen bei freien Schwingungen mechanischer Systeme. *Ingenieur-Archiv.* **28** (1959) 213–228.

[11] Sethna, P. R. Vibrations of dynamical systems with quadratic nonlinearities. *J. Appl. Mech.* **32** (1965) 576–582.

[12] Bogaevskii, V. N. and Povzner, A. Y. *Algebraic Methods in Non-linear Theory of Perturbation.* Nauka, Moscow, 1987.

[13] Nayfeh, A. H. and Mook, D. T. *Nonlinear Oscillations.* John Wiley & Sons, New York, 1979.

[14] Zaripov, M.N. and Petrov, A. G. Nonlinear oscillations of a swinging spring. *Doklady Physics* **49** (2004) 691–696.

[15] Awrejcewicz, J. and Petrov, A. G. Nonlinear oscillations of an elastic two-degrees-of-freedom pendulum. *Nonlinear Dyn.* **53** (2008) 19–30.

[16] Desale, B. S. and Dasre, N. R. Numerical solutions of system of non-linear ODEs by Euler modified method. *Nonlinear Dynamics and Systems Theory* **12** (3) (2012) 215–236.

[17] Tunc, C. Instability for nonlinear differential equations of fifth order subject to delay. *Nonlinear Dynamics and Systems Theory* **12** (2) (2012) 207–214.

[18] Denk, A. and Topal, S. Existence and uniqueness of a nontrivial solution for second order nonlinear m-point eigenvalue problems on time scales. *Nonlinear Dynamics and Systems Theory* **13** (4) (2013) 389–399.

[19] Korkmaz, E. and Tunc. C. On the convergence of solutions of some nonlinear differential equations of fourth order. *Nonlinear Dynamics and Systems Theory* **14** (4) (2014) 313–322.

[20] Bojadziev, G. N. Damped nonlinear oscillations modeled by a 3-dimensional differential system. *Acta Mech.* **48** (1983) 193–201.

[21] Osiniskii, Z. Longitudinal, torsional and bending vibrations of a uniform bar with nonlinear internal friction and relaxation. *Nonlinear Vib. Probl.* **4** (1962) 159–166.

[22] Mulholland, R. J. Nonlinear oscillations of third-order differential equation. *Int. J. Nonlinear Mech.* **6** (1971) 279–294.

[23] Murty, I. S. N., Deekshatulu, B. L., and Krisna, G. On asymptotic method of Krylov-Bogoliubov for over damped nonlinear systems. *J. Franklin Inst.* **288** (1969) 49–64.