Nonlinear Dynamics and Systems Theory, **15** (4) (2015) 428–440

Observer Based Output Tracking Control for Bounded Linear Time Variant Systems

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Received: September 19, 2015; Revised: October 30, 2015

Abstract: In this paper, we propose a new approach to design a reduced observer based state feedback control for bounded linear time variant systems by means of shifted *Legendre* polynomials. The main objective is to force the controlled *LTV* system output to follow that of a linear reference model. On these grounds, augmented state modeling and useful Kronecker product properties are applied. Hence, an optimization problem is derived. Once the observation and control gains are determined by solving the latter problem, the stability of the closed loop system is checked through LMIs conditions. Simulation results illustrate the pertinence of the proposed method.

Keywords: *LTV systems; state observer; tracking; shifted Legendre polynomials, LMIs.*

Mathematics Subject Classification (2010): 93B50, 93C05.

1 Introduction

Modeling a physical process is a crucial step toward its analysis and adapted control synthesis. Indeed, the chosen mathematical model should be accurate enough in order to describe correctly dynamics of the system evolution. Moreover, most physical systems are described by nonlinear models which are not easy to study. A simplification alternative consists in linearizing the systems around some operating points, the procedure remains a very conservative approach. A global method consists in a linearization along a trajectory, that often leads to a linear time varying system (LTV) [2]. Thus, this type of models offer a good compromise between simplicity and ability to reproduce with fidelity the behavior of some real processes namely, highway vehicle [1], electronic circuit design [3] and biochemical systems [8]. Accordingly, several studies have focused on poles and zeros definition for these particular systems [4], also problems related to the controllability

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and observability analysis [9], the identification [13, 14], the stability analysis [7] and the control [11, 12] of LTV systems have been the subject of many publications.

Orthogonal functions are a well developed mathematical tool for dynamic systems analysis and control. In fact, it had been firstly introduced for optimal control [6] and identification [16] of LTI systems. Lately, in literature there appeared extended works to cover nonlinear systems identification [20], stabilization analysis [27] and optimal control [25]. LTV systems have been also among the fields of application of that wise approximation tool, namely, for model order reduction [22], state analysis [23], identification [15] and optimal control [24]. Indeed, the projection of the state differential equation of the dynamic system over an orthogonal basis and introducing useful properties of the latter tool such as operational matrices jointly used with the *Kronecker* product may transform the time depending differential equation into stationary algebraic relations.

In this work, shifted Legendre polynomials are basically used to deduce an observer based state feedback control, the latter tool may have advantages over other orthogonal functions. This was shown by examples [16] where shifted Legendre polynomials converge to the exact solution of a differential equation faster than the other types of orthogonal functions, as, for example Walsh functions, Hermite and Laguerre polynomials. We underline that the derived control law has to ensure, not only stability but also a performance level dictated by a linear reference model used for tracking purposes, which is effectively the main contribution of the proposed study compared to major methods in literature which focused only on stabilization problem [17]. Consequently, a mathematical development will be exposed which is based on the use of interesting properties of shifted Legendre polynomials. The final result is given as a nonlinear criterion whose minimization with optimization Toolbox routines of MATLAB leads to the desired control gains. The last step is to check the asymptotic stability of the closed loop system through Bounded Real Lemma.

The paper is organized as follows. In Section 2, we introduce the studied systems and explain the main objective of the work. In Section 3, the proposed development for a reduced observer based control law applied to time variant linear systems using shift Legendre polynomials is carried out. In Section 4, stability analysis is handled using existing LMIs results for polytopic systems. In Section 5, the effectiveness of the developed method is checked out by a DC motor benchmark.

2 Problem Statement

In this work we consider the linear time variant system described by the following state equations:

$$\begin{cases} \dot{x}(t) = A(t)x(t) + B(t)u(t), \\ y(t) = Cx(t), \\ x(0) = x_0, \end{cases} \tag{1}$$

where $u \in \mathbb{R}^m$ is the control vector, $x \in \mathbb{R}^n$ is the state vector and $y \in \mathbb{R}^l$ is the output vector. Matrices $A(t)$, $B(t)$, C and the vector $x(t)$ have the following forms:

$$A(t) = \begin{bmatrix} A_{11}(t) & A_{12}(t) \\ A_{21}(t) & A_{22}(t) \end{bmatrix}, B(t) = \begin{bmatrix} B_1(t) \\ B_2(t) \end{bmatrix}, x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, C = \begin{bmatrix} I_l & O_1 \end{bmatrix}$$

with

$$O_1 = 0(l, n-l), x_1(t) \in \mathbb{R}^l, B_1(t) \in \mathbb{R}^{l,m}.$$

System (1) matrices could be written as $\forall t \geqslant 0$, $A(t) = [a_{ij}(t)]$ and $B(t) = [b_{ij}(t)]$ and each term verifies the following boundedness:

$$\underline{a_{ij}} \leqslant a_{ij}(t) \leqslant \overline{a_{ij}} \quad and \quad \underline{b_{ij}} \leqslant b_{ij}(t) \leqslant \overline{b_{ij}}, \tag{2}$$

where $\underline{a_{ij}}$, $\underline{b_{ij}}$ and $\overline{a_{ij}}$, $\overline{b_{ij}}$ are some constant values corresponding respectively to the minimum and maximum of $a_{ij}(t)$ and $b_{ij}(t)$.

We assume that such system satisfies the controllability and observability conditions [9]. Our objective is then to design both reduced order observer and state feedback control law in order to ensure desired performances for the controlled system.

2.1 The state observer structure

We choose to design a reduced order observer which reproduces the non measurable state component $x_2(t)$. Such observer is then described by state model of the following form:

$$\begin{cases} \dot{w}(t) = (A_{22}(t) - L_r A_{12}(t))w(t) + (B_2(t) - L_r B_1(t))u(t) \\ +((A_{22}(t) - L_r A_{12}(t))L_r + (\bar{A}_{21}(t) - L_r A_{11}(t)))y(t), \\ \hat{x}_2(t) = w(t) + L_r y(t). \end{cases} \tag{3}$$

In these equations $w(t) \in \mathbb{R}^{n-l}$ is the state observer vector and $\hat{x}_2(t)$ is the observation of $x_2(t)$. L_r is the gain of the order observer.

Let us define the observation error by:

$$\varepsilon_r(t) = x_2(t) - \hat{x}_2(t). \tag{4}$$

It comes out then:

$$\dot{\varepsilon}_r(t) = \dot{x}_2(t) - \dot{\hat{x}}_2(t) = (A_{22}(t) - L_r A_{12}(t))\,\varepsilon_r(t). \tag{5}$$

The observation gain L_r is determined such that the observation error has the same dynamics as a chosen observation reference model described by a linear state equation

$$\dot{\varepsilon}_{r,ref}(t) = M_r \varepsilon_{r,ref}(t), \tag{6}$$

where M_r is a $((n-l) \times (n-l))$ matrix chosen such that the observer be faster than the controlled system.

2.2 Strategy of control

The control strategy that we plan to develop uses the desired output $y_c(t)$, measured and observed components of the state vector ($y(t)$ and $\hat{x}_2(t)$). It can be expressed in the following form:

$$u(t) = Ny_c(t) - K_1 x_1(t) - K_2 \hat{x}_2(t) \tag{7}$$

with

$$N \in \mathbb{R}^{m\times m}, K_1 \in \mathbb{R}^{m\times l} \quad and \quad K_2 \in \mathbb{R}^{m\times(n-l)}.$$

We can express the control law by the following equation:

$$u(t) = Ny_c(t) - K_1 x_1(t) - K_2 \hat{x}_2(t) = Ny_c(t) - K_1 x_1(t) - K_2 x_2(t) + K_2 \varepsilon_r(t). \tag{8}$$

The gain matrices N, K_1 and K_2 are determined such that the controlled system presents the same behavior of a chosen reference linear model:

$$\begin{cases} \dot{z}(t) = Ez(t) + Fy_c(t), \\ y_r(t) = Gz(t), \end{cases} \tag{9}$$

where $y_c \in \mathbb{R}^m$ is the input vector, $z \in \mathbb{R}^r$ is the state vector and $y_r \in \mathbb{R}^l$ is the output vector.

3 Proposed Observation and Control Approach

By taking into account, equations (1) and (8), state variables could be written as folows:

$$\begin{cases} \dot{x}_1(t) = (A_{11}(t) - B_1(t)K_1)x_1(t) + (A_{12}(t) - B_1(t)K_2)x_2(t) + B_1(t)Ny_c(t) + B_1(t)K_2\varepsilon_r(t), \\ \dot{x}_2(t) = (A_{21}(t) - B_2(t)K_1)x_1(t) + (A_{22}(t) - B_2(t)K_2)x_2(t) + B_2(t)Ny_c(t) + B_2(t)K_2\varepsilon_r(t), \\ y(t) = x_1(t). \end{cases} \tag{10}$$

The augmented state is defined by the concatenation of states related to the original system and the observation error:

$$\tilde{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \varepsilon_r(t) \end{bmatrix}.$$

Hence, equations of the closed loop system take the following form:

$$\begin{cases} \dot{\tilde{x}}(t) = \tilde{A}(t)\tilde{x}(t) + \tilde{B}(t)y_c(t), \\ \tilde{y}(t) = \tilde{C}\tilde{x}(t), \end{cases} \tag{11}$$

with

$$\tilde{A}(t) = \begin{bmatrix} \tilde{a}_{11} & \tilde{a}_{12} & \tilde{a}_{13} \\ \tilde{a}_{21} & \tilde{a}_{22} & \tilde{a}_{23} \\ 0(n-l,l) & 0(n-l,n-l) & \tilde{a}_{33} \end{bmatrix}, \tilde{B}(t) = \begin{bmatrix} B_1(t)N \\ B_2(t)N \\ O_2 \end{bmatrix}, \tilde{C} = \begin{bmatrix} I_l & O_3 \end{bmatrix},$$

where

$$\begin{aligned} &\tilde{a}_{11} = A_{11}(t) - B_1(t)K_1, \tilde{a}_{12} = A_{12}(t) - B_1(t)K_2, \tilde{a}_{21} = A_{21}(t) - B_2(t)K_1, \\ &\tilde{a}_{22} = A_{22}(t) - B_2(t)K_2, \tilde{a}_{13} = B_1(t)K_2, \tilde{a}_{23} = B_2(t)K_2, \tilde{a}_{33} = A_{22}(t) - L_rA_{12}(t), \\ &O_2 = 0(n-l,m), O_3 = 0(l,2(n-l)). \end{aligned}$$

The projection of the matrices $A_{11}(t)$, $A_{12}(t)$, $A_{21}(t)$, $A_{22}(t)$, $B_1(t)$ and $B_2(t)$ in a basis of *shifted* Legendre polynomials truncated to an order N (See Section 7.3 of the Appendix) can be written as:

$$\begin{aligned} &A_{11}(t) = \sum_{i=0}^{N-1} A_{11,iN}s_i(t), A_{12}(t) = \sum_{i=0}^{N-1} A_{12,iN}s_i(t), A_{21}(t) = \sum_{i=0}^{N-1} A_{21,iN}s_i(t), \\ &A_{22}(t) = \sum_{i=0}^{N-1} A_{22,iN}s_i(t), B_1(t) = \sum_{i=0}^{N-1} B_{1,iN}s_i(t), B_2(t) = \sum_{i=0}^{N-1} B_{2,iN}s_i(t). \end{aligned} \tag{12}$$

We can deduce now the projection of the matrice $\tilde{A}(t)$ and $\tilde{B}(t)$ in the same basis of shifted *Legendre* polynomials truncated to an order N, with:

$$\tilde{A}_{iN} = \begin{bmatrix} \tilde{a}_{11,iN} & \tilde{a}_{12,iN} & \tilde{a}_{13,iN} \\ \tilde{a}_{21,iN} & \tilde{a}_{22,iN} & \tilde{a}_{23,iN} \\ 0(n-l,l) & 0(n-l,n-l) & \tilde{a}_{33,iN} \end{bmatrix}, \tilde{B}_{iN} = \begin{bmatrix} B_{1,iN}N \\ B_{2,iN}N \\ O_2 \end{bmatrix},$$

where

$$\begin{aligned}&\tilde{a}_{11,iN} = A_{11,iN} - B_{1,iN}K_1, \tilde{a}_{12,iN} = A_{12,iN} - B_{1,iN}K_2, \tilde{a}_{21,iN} = A_{21,iN} - B_{2,iN}K_1,\\ &\tilde{a}_{22,iN} = A_{22,iN} - B_{2,iN}K_2, \tilde{a}_{13,iN} = B_{1,iN}K_2, \tilde{a}_{23,iN} = B_{2,iN}K_2,\\ &\tilde{a}_{33,iN} = A_{22,iN} - L_r A_{12,iN}.\end{aligned}$$

The control law defined by the equation (8) has to carry the dynamics of the closed loop system to reproduce as perfectly as possible that of a reference model which may be defined as follows:

$$\begin{cases} \dot{\tilde{z}}(t) = \tilde{E}\tilde{z}(t) + \tilde{F}y_c(t), \\ \tilde{y}_r(t) = \tilde{G}\tilde{z}(t), \end{cases} \tag{13}$$

with

$$\tilde{z} = \begin{bmatrix} z \\ \varepsilon_{ref} \end{bmatrix}, \tilde{E} = \begin{bmatrix} E & O_4 \\ O_4^T & M_r \end{bmatrix}, \tilde{F} = \begin{bmatrix} F \\ O_5 \end{bmatrix}, \tilde{G} = \begin{bmatrix} G & O_6 \end{bmatrix},$$

$O_4 = 0(n_r, n-l)$, $O_5 = 0(n-l, m)$ and $O_6 = 0(l, n_r + n - 2l)$, where n_r is the order of the reference model defined by the equation (9).

The integration of the equation (11) on the time interval $[0, t]$ leads to:

$$\tilde{x}(t) - \tilde{x}(0) = \int_0^t \tilde{A}(\tau)x(\tau)d\tau + \int_0^t \tilde{B}(\tau)y_c(\tau)d\tau, \tag{14}$$

where $\tilde{x}(0)$ denotes the initial conditions vector.

The projection of the state vector $\tilde{x}(t)$ and the order output $y_c(t)$ on the basis of shifted *Legendre* polynomials leads to:

$$\tilde{X}_N S_N(t) - \tilde{X}_{0N} S_N(t) = \int_0^t \sum_{i=0}^{N-1} \tilde{A}_{iN} s_i(t) \tilde{X}_N S_N(t) + \int_0^t \sum_{i=0}^{N-1} \tilde{B}_{iN} s_i(t) y_{cN} S_N(t) \tag{15}$$

Introducing now the product operational matrix (See Section 7.2 of Appendix) in equation (15) yields:

$$\tilde{X}_N S_N(t) - \tilde{X}_{0N} S_N(t) = \int_0^t \sum_{i=0}^{N-1} \tilde{A}_{iN} \tilde{X}_N M_i S_N(t) + \int_0^t \sum_{i=0}^{N-1} \tilde{B}_{iN} y_{cN} M_i S_N(t). \tag{16}$$

The use of the integration operational matrix (See Section 7.2 of the Appendix) yields:

$$\tilde{X}_N S_N(t) - \tilde{X}_{0N} S_N(t) = \sum_{i=0}^{N-1} \tilde{A}_{iN} \tilde{X}_N M_i P_N S_N(t) + \sum_{i=0}^{N-1} \tilde{B}_{iN} y_{cN} M_i P_N S_N(t). \tag{17}$$

Simplifying by the vector $S_N(t)$ and making use of the *vec* operator, which transforms a matrix structure into a vector one and the specific property [21]

$$vec(ABC) = \left(C^T \otimes A\right) vec(B), \tag{18}$$

equation (17) could be written as follows:

$$\begin{aligned} vec(\tilde{X}_N) - vec(\tilde{X}_{0N}) = \sum_{i=0}^{N-1} \left((M_i P_N)^T \otimes \tilde{A}_{iN} \right) vec(\tilde{X}_N) \\ + \sum_{i=0}^{N-1} \left((M_i P_N)^T \otimes \tilde{B}_{iN} \right) vec(y_{cN}), \end{aligned} \tag{19}$$

it comes out:

$$vec(\tilde{X}_N) = \begin{bmatrix} \left[I_{((n-l)+n)\times N} - \sum_{i=0}^{N-1} \left((M_i P_N)^T \otimes \tilde{A}_{iN} \right) \right]^{-1} \times \\ \left[vec(\tilde{X}_{0N}) + \left[\sum_{i=0}^{N-1} \left((M_i P_N)^T \otimes \tilde{B}_{iN} \right) \right] vec(y_{cN}) \right] \end{bmatrix}. \tag{20}$$

In the same way the projection of the closed loop reference model (13), and the use of the operational matrix of integration yield:

$$vec(\tilde{Z}_N) = \begin{bmatrix} \left[I_{(n_r+(n-l))\times N} - \sum_{i=0}^{N-1} \left({P_N}^T \otimes \tilde{E} \right) \right]^{-1} \times \\ \left[vec(\tilde{Z}_{0N}) + \left[\sum_{i=0}^{N-1} \left({P_N}^T \otimes \tilde{F} \right) \right] vec(y_{cN}) \right] \end{bmatrix}. \tag{21}$$

The condition permitting to have a similar behavior of the controlled system (11) and reference model (13) can be written mathematically as follows:

$$\tilde{y}(t) = \tilde{y}_r(t) \Leftrightarrow \tilde{C}\tilde{X}(t) = \tilde{G}Z(t) \Leftrightarrow \left(I_N \otimes \tilde{C} \right) vec(\tilde{X}_N) = \left(I_N \otimes \tilde{G} \right) vec(\tilde{Z}_N). \tag{22}$$

It comes out:

$$\begin{aligned} &\begin{bmatrix} \left(I_N \otimes \tilde{C} \right) \left[I_{((n-l)+n)\times N} - \sum_{i=0}^{N-1} \left((M_i P_N)^T \otimes \tilde{A}_{iN} \right) \right]^{-1} \times \\ \left[vec(\tilde{X}_{0N}) + \left[\sum_{i=0}^{N-1} \left((M_i P_N)^T \otimes \tilde{B}_{iN} \right) \right] vec(y_{cN}) \right] \end{bmatrix} \\ &= \begin{bmatrix} \left(I_N \otimes \tilde{G} \right) \left[I_{(n_r+(n-l))\times N} - \sum_{i=0}^{N-1} \left({P_N}^T \otimes \tilde{E} \right) \right]^{-1} \times \\ \left[vec(\tilde{Z}_{0N}) + \left[\sum_{i=0}^{N-1} \left({P_N}^T \otimes \tilde{F} \right) \right] vec(y_{cN}) \right] \end{bmatrix}. \end{aligned} \tag{23}$$

Notice that

$$\tilde{Z}_{0N} = \begin{bmatrix} Z_{0N} \\ \varepsilon_{r,ref,0N} \end{bmatrix} = \begin{bmatrix} I_{n_r} \\ O_4^T \end{bmatrix} Z_{0N} + \begin{bmatrix} O_4 \\ I_{n-l} \end{bmatrix} \varepsilon_{r,ref,0N}$$

and

$$\tilde{X}_{0N} = \begin{bmatrix} X_{0N} \\ \varepsilon_{r,0N} \end{bmatrix} = \begin{bmatrix} X_{1,0N} \\ X_{2,0N} \\ \varepsilon_{r,0N} \end{bmatrix} = \begin{bmatrix} I_l \\ O_7 \end{bmatrix} X_{1,0N} + \begin{bmatrix} O_8 \\ I_{n-l} \\ O_9 \end{bmatrix} X_{2,0N} \begin{bmatrix} O_{10} \\ I_{n-l} \end{bmatrix} \varepsilon_{r,0N},$$

where $O_7 = 0(2(n-l), l)$,$O_8 = 0(l, n-l)$, $O_9 = 0(n-l, n-l)$ and $O_{10} = 0(n, n-l)$.

In order to simplify the control problem, let us consider null initial condition for the reference model ($Z_{0N} = 0$) and null initial condition for the measurable components state part ($X_{1,0N} = 0$).

Moreover, intial conditions of the observation error, which are a priori unkown, should meet those of the reference observation error model in order to minimize the distance between both models ($\varepsilon_{r,0N} = \varepsilon_{r,ref,0N}$). Consquently, equation (23) could be written as follows:

$$\begin{bmatrix} \left(I_N \otimes \tilde{C}\right)\left[I_{((n-l)+n)\times N} - \sum_{i=0}^{N-1}\left((M_iP_N)^T \otimes \tilde{A}_{iN}\right)\right]^{-1} \times \\ \begin{bmatrix} \left(I_N \otimes \begin{bmatrix} O_8 \\ I_{n-l} \\ O_9 \end{bmatrix}\right) vec(X_{2,0N}) + \left(I_N \otimes \begin{bmatrix} O_{10} \\ I_{n-l} \end{bmatrix}\right) vec(\varepsilon_{r,ref,0N}) \\ + \left[\sum_{i=0}^{N-1}\left((M_iP_N)^T \otimes \tilde{B}_{iN}\right)\right] vec(y_{cN}) \end{bmatrix} \end{bmatrix} = \tag{24}$$

$$\begin{bmatrix} \left(I_N \otimes \tilde{G}\right)\left[I_{(n_r+(n-l))\times N} - \sum_{i=0}^{N-1}\left({P_N}^T \otimes \tilde{E}\right)\right]^{-1} \times \\ \left[\left(I_N \otimes \begin{bmatrix} O_4 \\ I_{n-l} \end{bmatrix}\right) vec(\varepsilon_{r,ref,0N}) + \left[\sum_{i=0}^{N-1}\left({P_N}^T \otimes \tilde{F}\right)\right] vec(y_{cN})\right] \end{bmatrix}.$$

The relation (24) can be written as:

$$\Delta_1 vec(y_{cN}) + \Delta_2 vec(X_{2,0N}) + \Delta_3 vec(\varepsilon_{r,ref,0N}) = 0, \tag{25}$$

where

$$\Delta_1 = \begin{bmatrix} \begin{bmatrix} \left(I_N \otimes \tilde{C}\right)\left[I_{((n-l)+n)\times N} - \sum_{i=0}^{N-1}\left((M_iP_N)^T \otimes \tilde{A}_{iN}\right)\right]^{-1} \times \\ \left[\sum_{i=0}^{N-1}\left((M_iP_N)^T \otimes \tilde{B}_{iN}\right)\right] \end{bmatrix} \\ -\left[\left(I_N \otimes \tilde{G}\right)\left[I_{(n_r+(n-l))\times N} - \sum_{i=0}^{N-1}\left({P_N}^T \otimes \tilde{E}\right)\right]^{-1}\left[\sum_{i=0}^{N-1}\left({P_N}^T \otimes \tilde{F}\right)\right]\right] \end{bmatrix},$$

$$\Delta_2 = \left[\left(I_N \otimes \tilde{C}\right)\left[I_{((n-l)+n)\times N} - \sum_{i=0}^{N-1}\left((M_iP_N)^T \otimes \tilde{A}_{iN}\right)\right]^{-1}\left(I_N \otimes \begin{bmatrix} O_8 \\ I_{n-l} \\ O_9 \end{bmatrix}\right)\right],$$

$$\Delta_3 = \begin{bmatrix} \left[\left(I_N \otimes \tilde{C}\right)\left[I_{((n-l)+n)\times N} - \sum_{i=0}^{N-1}\left((M_iP_N)^T \otimes \tilde{A}_{iN}\right)\right]^{-1}\left(I_N \otimes \begin{bmatrix} O_{10} \\ I_{n-l} \end{bmatrix}\right)\right] \\ -\left[\left(I_N \otimes \tilde{G}\right)\left[I_{(n_r+(n-l))\times N} - \sum_{i=0}^{N-1}\left({P_N}^T \otimes \tilde{E}\right)\right]^{-1}\left(I_N \otimes \begin{bmatrix} O_4 \\ I_{n-l} \end{bmatrix}\right)\right] \end{bmatrix}$$

In order to verify such relation for any initial conditions $X_{2,0N}$, $\varepsilon_{r,ref,0N}$ and for any output order y_{cN}, we must ensure:

$$\Delta_1 = 0, \quad \Delta_2 = 0 \quad and \quad \Delta_3 = 0.$$

However, these conditions could not be totally realized. Hence, we have to look for a pseudo-solution of this problem by minimizing the norms of matrices Δ_1, Δ_2 and Δ_3, denoted respectively δ_1, δ_2 and δ_3, using optimization MATLAB routines.

4 Stability Analysis of Closed Loop System

Once observation control parameters L_r, N, K_1 and K_2 are determined, the LTV model defined by the equation (11) can be expressed in the following polytopic form such as $\mathrm{M} = \left[\tilde{\mathrm{A}} \,\middle|\, \tilde{\mathrm{B}}\right]$ belongs to a polytope of matrices M defined by [5]:

$$M = \left\{ \mathrm{M} = \left[\tilde{A}(\theta) \,\middle|\, \tilde{B}(\theta)\right] / M(\theta) = \sum_{i=1}^{v} \left(\theta_i \left[\tilde{A}_i \,\middle|\, \tilde{B}_i\right]\right) \right\},$$

where

$$\theta \in \Theta = \left\{ \theta = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_v \end{bmatrix} / \sum_{i=1}^{v} \theta_i = 1 \right\}.$$

The closed loop system (11) is mean square asymptotically stable with an H_∞ disturbance attenuation γ if and only if there exists a $(n + (n - l)) \times (n + (n - l))$ matrix $P \succ 0$ such that $i = 1 \dots v$, [10]:

$$\begin{bmatrix} \tilde{A}_i^T P + P\tilde{A}_i & P\tilde{B}_i & \tilde{C}^T \\ \tilde{B}_i^T P & -\gamma^2 I_m & 0 \\ \tilde{C} & 0 & -I_p \end{bmatrix} \prec 0. \qquad (26)$$

5 Simulation Example

Let us consider the separated excitation DC motor described by the following equation [26]:

$$\begin{cases} \frac{d\Omega(t)}{dt} = -\frac{f\Omega(t)}{J} + \frac{K_m \Phi(t) I(t)}{J}, \\ \frac{dI(t)}{dt} = -\frac{K_e \Phi(t)\Omega(t)}{L} - \frac{RI(t)}{L} + \frac{V(t)}{L}, \\ x(0) = \begin{bmatrix} 0 & 0.2 \end{bmatrix}^T, \end{cases}$$

where $y = \Omega(t)$ denotes rotational speed of rotor as measured output, $I(t)$ and $V(t)$ are respectively the current and voltage of rotor, $\Phi(t)$ is the rotor flux. For this example, we will assume the values for the physical parameters given in Table 1.

The rotor flux is considered as a time depending function defined by the following relation:

$$\Phi(t) = \Phi_0(1 + 0.1\sin(\pi t))$$

with $\Phi_0 = 1$.

The considered reference model of the controlled system is a second order system characterized by the following parameter matrices:

$$E = \begin{bmatrix} -10.5 & -2.4 \\ 2 & -14.75 \end{bmatrix}, F = \begin{bmatrix} 0 \\ 1.5 \end{bmatrix}, G = \begin{bmatrix} 0 & 10.15 \end{bmatrix}.$$

The reference model of the observation error is characterized by the matrix:

$$M_r = -5.$$

Table 1: Motor parameters.

L: rotor inductance	0.5 H
R: rotor resistance	2 Ω
K_e: electromotive force against	0.1 $NmWb^{-1}A^{-1}$
K_m: electromagnetic torque	0.1 $NmWb^{-1}A^{-1}$
J: rotor and load inertias	0.006 $kgm^{-2}s^{-2}$
f: viscous friction coefficient	0.01 Nms

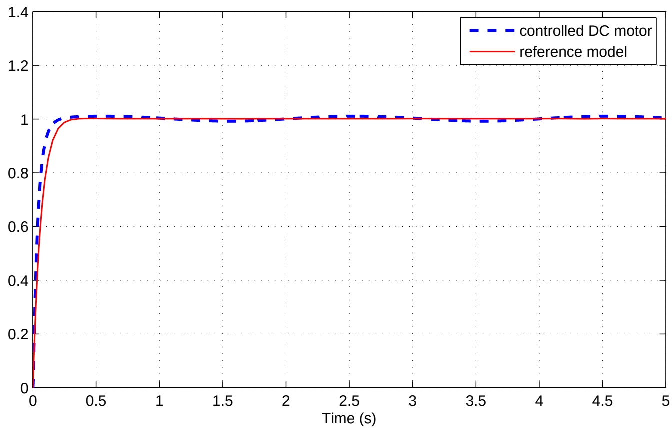

Figure 1: Step response of controlled DC motor and the considered reference model.

For N = 10 (number of elementary SLPs functions) and T=5s, the obtained control gains are the following:

$$N = 175.3, \quad K_1 = 158.6, \quad K_2 = 160.5, \quad \mathrm{L_r} = 1.8.$$

The norms of matrices Δ_1, Δ_2 and Δ_3 are given by the following:

$$\delta_1 = 0.0446, \quad \delta_2 = 0.0298, \quad and \quad \delta_3 = 0.0335.$$

Figure 1 illustrates step responses of controlled DC motor and the considered reference model over an interval $[0, T]$. Figure 2 shows the free motion of the current of rotor and the observer. The asymptotic stability with an H_∞ disturbance attenuation γ of the closed loop system is verified by the feasible solution of the LMI defined in relation (26). Obtained LMI variables were:

$$\gamma = 2.3547,$$

$$P = \begin{bmatrix} 0.8258 & 0.0418 & 0.0278 \\ 0.0418 & 0.0114 & 0.0048 \\ 0.0278 & 0.0048 & 0.0100 \end{bmatrix}.$$

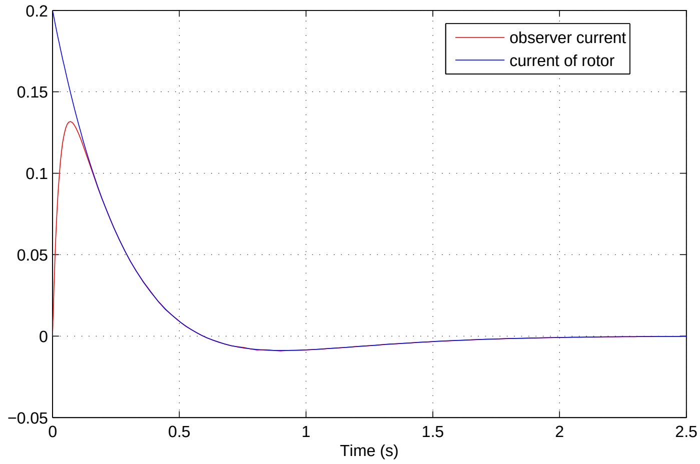

Figure 2: Free motion of the current of rotor and the observer.

6 Conclusions

In this paper, a new analytical approach was introduced for the synthesis of a reduced observed feedback control for linear time variant systems by using shift Legendre polynomials as an approximation tool. The use of the operational matrix of integration and operational matrix of product has allowed the transformation of differential equations into algebraic ones depending on gains of regulators. The main contribution of the paper can be summarized as the system performance guaranty jointly with stability which is obviously ensured. This is done by tracking a linear reference model. The effectiveness of the developed method is checked out by a DC motor benchmark. The simulations results obtained show clearly the accuracy of the synthesized control law. In future works, we intend to extend our development to handle the synthesis of observed state feedback control for nonlinear systems via orthogonal functions.

7 Appendix

7.1 Legendre polynomials

Legendre polynomials denoted by LPs in litterature, have been the most used ones in continuous control problems due to their high accuracy and a unit weighting function. That is why we use them in our work. They are defined over the time interval $\tau \in [-1, 1]$ and given by the recursive formula [18]:

$$(n+1)\, P_{n+1}(\tau) = (2n+1)\, \tau P_n(\tau) - nP_{n-1}(\tau), \; for \quad n = 1, 2, \cdots \tag{27}$$

with $P_0(\tau) = 1$ and $P_1(\tau) = \tau$.

In order to obtain orthogonal Legendre polynomials on the interval $[0, t_f]$, the following change of variable is performed:

$$\tau = \frac{2t}{t_f} - 1 \quad with \quad 0 \leq t \leq t_f \tag{28}$$

for $0 \leq t \leq t_f$, the shifted Legendre polynomials, denoted by SLPs, $s_n(t)$ are thus given by:

$$(n+1)\, s_{n+1}(t) = (2n+1)\left(\frac{2t}{t_f} - 1\right) s_n(t) - n s_{n-1}(t) \tag{29}$$

with $s_0(t) = 1$ and $s_1(t) = \frac{2t}{t_f} - 1$.

The principle of orthogonality of shifted Legendre polynomials is expressed by the following equation [19]:

$$\int_0^{t_f} s_i(t) s_j(t) dt = \frac{t_f}{2i+1} \delta_{ij}. \tag{30}$$

So, any integrable function on $0 \leq t \leq t_f$ can be developed into a series of shifted Legendre polynomials with a truncation to an order N under the following relation:

$$f(t) \cong \sum_{i=0}^{N-1} f_i s_i(t) = F_N S_N(t) \tag{31}$$

with

$$F_N = \begin{bmatrix} f_0 & f_1 & \cdots & f_{N-1} \end{bmatrix}$$

and

$$S_N(t) = \begin{bmatrix} s_0(t) & s_1(t) & \cdots & s_{N-1}(t) \end{bmatrix}^T.$$

7.2 Operational matrices

In the SLPs case, the operational matrix of integration P_N could be built through the following recurrent relation:

$$\int_0^t s_n(\tau) d\tau = \frac{t_f}{2} \times \frac{1}{2n+1} (s_{n+1}(t) - s_{n-1}(t)) \tag{32}$$

and

$$\int_0^t s_0(\tau) d\tau = \frac{t_f}{2} (s_0(t) + s_1(t)). \tag{33}$$

Hence, the following algebraic relation for integral calculus could be stated:

$$\int_0^t S_N(t) dt \cong P_N S_N(t). \tag{34}$$

The operational vectors of product K_{ij} have constant coefficients and verify the property [20]:

$$\forall i, j \in \{0, 1, \ldots, N-1\}, s_i(t) s_j(t) \cong K_{ij}^T S_N(t). \tag{35}$$

From the relationship (35), we can readily get the operational matrix of product:

$$M_{iN} = \begin{bmatrix} K_{i,0}^T \\ \vdots \\ K_{i,N-1}^T \end{bmatrix} \tag{36}$$

that allows the approximation:

$$s_i(t)S_N(t) \cong M_{iN}S_N(t). \tag{37}$$

7.3 Matrix functions approximation

Any time dependent matrix function $A(t) \in \mathbb{R}^{n\times m}$ given by $A(t) = [a_{ij}(t)]$ where $a_{ij}(t)$ are integrable over an interval $0 \leq t \leq t_f$ can be developed into a series of shifted Legendre polynomials with a truncation to an order N under the following relation:

$$A(t) \cong \sum_{i=0}^{N-1} A_{iN}s_i(t), \tag{38}$$

where $A_{iN} \in \mathbb{R}^{n\times m}$ for $i \in \{0, 1, ..., N-1\}$ are matrices with constant coefficients.

References

[1] Benton, R.E. and Smith, D. A static-output-feedback design procedure for robust emergency lateral control of a highway vehicle. *IEEE Transactions on Control Systems Technology* **13** (2005) 618–623.

[2] Marinescu, B. Output feedback pole placement for linear time-varying systems with application to the control of nonlinear systems. *Automatica* **64** (2010) 1524–1530.

[3] Darabi, H. Ibrahim, B. and Rofougaran, A. An analog GFSK modulator in 0.35-μ m CMOS. *IEEE Journal of Solid-State Circuits* **39** (2004) 2292–2296.

[4] Marinescu, B. and Bourlès, H. An intrinsic algebraic setting for poles and zeros of linear time-varying systems. *Systems & Control Letters* **58**(4) (2009) 248–253.

[5] Bosche, J. and El Hajjaji, A. An Output Feedback Controller Design for Lateral Vehicle Dynamic. *The International Federation of Automatic Control*, Seoul, Korea, 2008, 6–11.

[6] Chen, CF. and Hsiao, CH. Design of piecewise constant gains for optimal control via Walsh functions. *IEEE Trans. Automatic Control* **20**(5) (1975) 596–603.

[7] Pastravanu, O. and Matcovschi, M.H. Linear time-variant systems: Lyapunov functions and invariant sets defined by Holder norms. *Journal of the Franklin Institute* **347** (2010) 627-640.

[8] Zak, D.E. Stelling, J. and Doyle III, F.J. Sensitivity analysis of oscillatory (bio)chemical systems. *Computers and Chemical Engineering* **29** (2005) 663–673.

[9] Silverman, L.M. and Meadows, H.E. Controllability and observability in time-variable linear systems. *SIAMJ Control and optimization* **5** (1967) 64–73.

[10] Boyd, D. El Ghaoui, L. Peron, E. and Bala-krishnan, V. Linear Matrix Inequalities in System and Control Theory. *SIAM Society for Industrial and Applied Mathematics*, Philadelphia, 1994.

[11] Amato, F. Mattei, M. and Pironti, A. Solution of the state feedback singular H^∞ control problem for linear time-varying systems. *Automatica* **36** (2000) 1469–1479.

[12] Marino, R. and Tomei, P. Adaptive control of linear time-varying systems. *Automatica* **39** (2003) 651–659.

[13] Maciej, N. Identification of time-varying systems with abrupt parameter changes. *Automatica* **30** (1994) 447–459.

[14] Lozano, R. Dimogianopoulos, D. and Mahony, R. Identification of linear time-varying systems using a modified least-squares algorithm. *Automatica* **36** (2000) 1009–1015.

[15] Mouroutsos, S.G. and Paraskevopoulos, P.N. Identification of time-varying linear systems using orthogonal functions. *Journal of the Franklin Institute* **320** (1985) 249–258.

[16] Paraskevopoulos, P.N. Legendre series approach to identification and analysis of linear systems. *IEEE Trans. Automatic Control* (1985) 585–589.

[17] Zhoua, B. Caib,G.B. and Duana, G.R. Stabilisation of time-varying linear systems via Lyapunov differential equations. *Int. J. Control* **86** (2003) 332–347.

[18] Gradshteyn, I.S. and Ryzhik, I.M. Tables of Integrals, Series and Products, New York, *Academic Press*, 1979.

[19] Hwang, C. and Guo, T.Y. Transfer Function Matrix Identification in MIMO Systems via Shifted Legendre Polynomials. *Int. J. Control* **39** (1984) 807–814.

[20] Rottela, F. and G. Dauphin-Tanguy, G. Non-linear systems identification and optimal control. *Int. J. Control* **48** (1988) 525–544.

[21] Brewer, J.W. Kronecker products and matrix calculus in systems theory. *Trans. IEEE Circ. and Syst. CAS* **48** (1978).

[22] Ayadi, B. and Benhadj Braiek, N. Orthogonal Functions Approach for Model Order Reduction of LTI and LTV Systems. *Nonlinear Dynamics and Systems Theory* **13** (2013) 115–134.

[23] Hsiao, C. and Wang, W. State analysis and optimal control of linear time-varying systems via Haar wavelets. *Optimal Control Applications and Methods* **19** (1998) 423–433.

[24] Razzaghi, M. Optimal control of linear time-varying systems via Fourier series. *Journal of Optimization Theory and Applications* **65** (1990) 375–384.

[25] Mohan, B. M. and Sanjeeb Kumar Kar, Optimal Control of Nonlinear Systems via Block Pulse Functions and Legendre Polynomials. *Differential Equations and Dynamical Systems* **20** (2012) 149–159.

[26] Rotella, F. Ayadi, M. and Carrillo, F.J. Control and trajectory tracking by flatness of a time-variant stator flux motor. *Control Applications. Proceedings of the 2002 International Conference* **1** (2002) 102–107.

[27] Belhaouane, M.M. and Benhadj Braiek, N. Design of Stabilizing Control for Synchronous Machines via Polynomial Modelling and Linear Matrix Inequalities Approach. *International Journal of Control, Automation, and Systems – IJCAS* **9** (3) (2011) 425–436.