Nonlinear Dynamics and Systems Theory, **16** (1) (2016) 1–11

Integral Estimates of Solutions to Nonlinear Systems and Their Applications

On the occasion of centenary of the birth of Professor A.N. Golubentsev

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Received: June 27, 2015;   Revised: January 25, 2016

March 29, 2016 marks the 100th birthday of Professor A.N. Golubentsev, the famous scientist in the field of machine mechanics and applied mathematics. For the detailed analysis of his scientific investigations and his contribution to the development of the Institute of Mechanics of NAS of Ukraine see the paper [13] and the book [14].

Abstract: The paper deals with the nonlinear systems of ordinary differential equations. New estimates of the norms of solutions for systems under consideration are established via nonlinear integral inequalities. The results are illustrated by the problems on boundedness of solutions, finite-time stability and exponential approximation of solution to a class of nonlinear systems.

Keywords: *nonlinear system; bounded solutions; finite-time stability.*

Mathematics Subject Classification (2010): 34A34, 34C11, 34C60, 93D05.

1 Introduction

For solution of problems of nonlinear dynamics different analytical and qualitative methods of general theory of equations are applied being adapted to a particular problem or a class of similar problems. For instance, in monograph [1] a method of dynamics analysis is considered for the systems described by the equations containing integrals with variable upper limit. The authors discuss physical meaning of the resolvent of integral equation and present basic analytical correlations relating the character of transient process in the system with its parameters. As to the dynamics of machines, a resolvent

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analytic expression is given for the systems of high order equations. In the investigation of nonlinear dynamics of machines the systems are treated which contain elastoplastic links, nonlinear couplings with hysteresis characteristic, etc.

Stability investigations of nonlinear system motions on finite and unbounded time interval for given estimates of the initial and subsequent perturbations were summarized in monograph [2].

In monographs [3 – 5] two classical theories of mathematics and mechanics have been developed. One was the theory of integral inequalities, and the other was a general theory of motion stability in terms of integral inequalities.

The present paper proposes estimates of norm of solutions to nonlinear systems based on the theory of nonlinear integral inequalities. Problems on boundedness of solutions, motion stability on finite time interval and exponential convergence of solutions for one class of nonlinear systems are considered as applications.

2 Statement of the Problem

Consider a model of some physical system described by a system of perturbed motion equations of the form

$$\frac{dx}{dt} = F(t,x), \tag{1}$$

$$x(t_0) = x_0, \tag{2}$$

where $x \in \mathbb{R}^n$, $F(t,x)$ is a vector-function definite and continuous with respect to $(t,x) \in \mathbb{R}^n_+$.

Further we shall assume that for the initial values $(t_0, x_0) \in J \times D$ the solution to the initial problem (1)–(2) is definite for all $t \in J$. Here $J \subset \mathbb{R}_+$ and $D \subseteq \mathbb{R}^n$ is an open domain in $\mathbb{R}^n$. It is known that the solution $x(t)$ of the initial problem (1)–(2) through the point (t_0, x_0) satisfies the integral equation

$$x(t) = x_0 + \int_{t_0}^{t} F(s, x(s))ds \tag{3}$$

on the interval where the solution $x(t) = x(t, t_0, x_0)$ is definite.

Assume that for the right-hand part of nonlinear system (1) there exist nonnegative continuous functions $a(t)$ and $b(t)$ on any finite interval J such that

$$\|F(t,x)\| \le a(t)\|x\| + b(t)\|x\|^k, \tag{4}$$

where $k > 1$ and $\|\cdot\|$ is an Euclidean norm of the vector.

It is of interest to estimate the norm of solutions $x(t)$ to system (1) and to study behavior of the solutions on unbounded or finite time interval when inequality (4) is satisfied.

3 New Estimate of Solutions

We shall obtain uniform estimate of the norm of solutions to nonlinear system (1) with the initial conditions (2) when the condition (4) is satisfied.

The following result holds.

Lemma 1 (see [7, 8]) *For the right-hand part of system* (1) *assume that estimate* (4) *of the domain of values* $(t, x) \in J \times D$ *is satisfied and, besides,*

$$L(t) = 1 - (k-1)\|x_0\|^{k-1} \int\limits_{t_0}^{t} b(s) \exp\left[(k-1) \int\limits_{t_0}^{s} a(\tau) d\tau\right] ds > 0 \tag{5}$$

for all $t \in J$*. Then for the norm of solutions* $x(t)$ *of system* (1)*, when* $(t_0, x_0) \in J \times D$ *the estimate*

$$\|x(t)\| \leq \|x_0\| \exp\left(\int\limits_{t_0}^{t} a(s) ds\right) (L(t))^{-\frac{1}{k-1}} \tag{6}$$

is valid for all $t \in J$*.*

Proof. From the integral equation (3) under condition (4) we have the estimate

$$\|x(t)\| \leq \|x_0\| + \int\limits_{t_0}^{t} (a(s)\|x(s)\| + b(s)\|x(s)\|^k) ds, \tag{7}$$

that is equivalent to the following one

$$\|x(t)\| \leq \|x_0\| + \int\limits_{t_0}^{t} (a(s) + b(s)\|x(s)\|^{k-1})\|x(s)\| ds \tag{8}$$

for all $t \in J$. Applying the Gronwall–Bellman lemma [6] to inequality (8) we get

$$\|x(t)\| \leq \|x_0\| \exp\left[\int\limits_{t_0}^{t} (a(s) + b(s)\|x(s)\|^{k-1}) ds\right]. \tag{9}$$

Then, we represent inequality (9) as

$$\|x(t)\|^{k-1} \leq \|x_0\|^{k-1} \exp\left[(k-1) \int\limits_{t_0}^{t} (a(s) + b(s)\|x(s)\|^{k-1}) ds\right] \tag{10}$$

and estimate from above the term

$$\exp\left[(k-1) \int\limits_{t_0}^{t} b(s)\|x(s)\|^{k-1} ds\right]. \tag{11}$$

Multiplying both parts of inequality (10) by the expression

$$-(k-1)b(t) \exp\left[-(k-1) \int\limits_{t_0}^{t} b(s)\|x(s)\|^{k-1} ds\right],$$

we arrive at

$$-(k-1)b(t)\|x(t)\|^{k-1}\exp\left[-(k-1)\int\limits_{t_0}^{t} b(s)\|x(s)\|^{k-1}ds\right]$$
$$\geq -(k-1)\|x_0\|^{k-1}b(t)\exp\left[(k-1)\int\limits_{t_0}^{t} a(s)ds\right].$$

Hence, it follows that

$$\begin{gathered}\frac{d}{dt}\left(\exp\left[-(k-1)\int\limits_{t_0}^{t} b(s)\|x(s)\|^{k-1}ds\right]\right)\\ \geq -(k-1)\|x_0\|^{k-1}b(s)\exp\left[(k-1)\int\limits_{t_0}^{t} a(s)ds\right].\end{gathered}\tag{12}$$

Integrating inequality (12) between t_0 and $t \in J$ we obtain

$$\exp\left[-(k-1)\int\limits_{t_0}^{t} b(s)\|x(s)\|^{k-1}ds\right] \geq L(t).$$

Under condition (5) the above inequality yields the estimate of the term (11) as follows

$$\exp\left[(k-1)\int\limits_{t_0}^{t} b(s)\|x(s)\|^{k-1}ds\right] \leq (L(t))^{-1} \quad \text{for all} \quad t \in J. \tag{13}$$

In view of estimate (13) we rewrite inequality (10) as

$$\|x(t)\|^{k-1} \leq \|x_0\|^{k-1}\exp\left[(k-1)\int\limits_{t_0}^{t} a(s)ds\right](L(t))^{-1}. \tag{14}$$

Since $k > 1$, we get from (14) the estimate (6), i.e.

$$\|x(t)\| \leq \|x_0\|\exp\left(\int\limits_{t_0}^{t} a(s)ds\right)(L(t))^{-\frac{1}{k-1}}$$

for all $t \in J$. This proves Lemma 1.

Corollary 1 In inequality (4) let the function $b(t) \equiv 0$ for all $t \in J$. Then estimate (6) becomes

$$\|x(t)\| \leq \|x_0\|\exp\left(\int\limits_{t_0}^{t} a(s)ds\right) \quad \text{for all} \quad t \in J. \tag{15}$$

This is the known Gronwall-Bellman estimate [6, p. 96].

Corollary 2 In inequality (4) let the function $a(t) \equiv 0$ for all $t \in J$. Then estimate (6) becomes

$$\|x(t)\| \le \|x_0\| \left\{ 1 - (k-1)\|x_0\|^{k-1} \int\limits_{t_0}^{t} b(s)ds \right\}^{-\frac{1}{k-1}} \tag{16}$$

for all $t \in J$ whenever

$$1 - (k-1)\|x_0\|^{k-1} \int\limits_{t_0}^{t} b(s)ds > 0.$$

Estimate (16) is obtained as well by the direct application of the Bihari lemma (see [9] to the inequality

$$\|x(t)\| \le \|x_0\| + \int\limits_{t_0}^{t} b(s)\|x(s)\|^k ds.$$

Remark 1 In paper [8] new estimates of the norm of solutions are presented for some characteristic types of nonlinear mechanics equations.

4 Applications

We shall make use of the estimate (6) to solve some problems of system dynamics.

4.1 Boundedness of Motion

In system (1) let the vector-function $F(t, x)$ be definite and continuous on $J \times \mathbb{R}^n$. We shall cite some definitions according to [10].

Definition 1 The solution $x(t) = x(t, t_0, x_0)$ of system (1) is bounded, if there exists $\beta > 0$ such that $\|x(t, t_0, x_0)\| < \beta$ for all $t \ge t_0$, where β can depend on every solution.

Definition 2 The solution $x(t)$ of system (1) is equi-bounded, if for any $\alpha > 0$ and $t_0 \in J$ there exists $\beta(t_0, \alpha) > 0$ such that if $\|x_0\| < \alpha$, then $\|x(t, t_0, x_0)\| < \beta(t_0, \alpha)$ for all $t \ge t_0$.

Estimate (6) provides the following results.

Theorem 1 *If for any $x_0 \in \mathbb{R}^n$, $\|x_0\| < \infty$, all conditions of Lemma 1 are satisfied and, in addition, there exists $\beta > 0$ such that*

$$\exp\left(\int\limits_{t_0}^{t} a(s)ds\right) (L(t))^{-\frac{1}{k-1}} < \frac{\beta}{\|x_0\|} \quad \textit{for all} \quad t \ge t_0,$$

then the motion described by the equation (1) *is bounded.*

Theorem 2 *If for $\|x_0\| < \alpha$ and*

$$L^*(t) = 1 - (k-1)\alpha^{k-1} \int\limits_{t_0}^{t} b(s) \exp\left[(k-1)\int\limits_{t_0}^{s} a(\tau)\, d\tau\right] ds > 0$$

all conditions of Lemma 1 are satisfied and there exists $\beta(t_0, \alpha) > 0$ *such that*

$$\exp\left(\int_{t_0}^{t} a(s)ds\right) (L^*(t))^{-\frac{1}{k-1}} < \frac{\beta(t_0, \alpha)}{\alpha} \quad \textit{for all} \quad t \geq t_0,$$

then the motion described by the equation (1) *is equi-bounded.*

Similar results can be established in terms of estimates (15) and (16) and Corollaries 1 and 2.

The proof of Theorems 1 and 2 follows immediately from the estimate (6) and Definitions 1 and 2.

4.2 Finite-Time Stability of Motion

For solution $x(t) = x(t, t_0, x_0)$ of the problem (1)–(2) we shall give the following definitions (see [2] and bibliography therein).

Definition 3 The motion of system (1) is:

(a) stable with respect to the values (λ, A, t_0, T), $0 < \lambda \leq A$, if for any solution $x(t)$ with the initial conditions $x_0\colon \|x_0\| < \lambda$ it follows that $\|x(t)\| < A$ for all $t \in [t_0, t_0 + T]$;

(b) uniformly stable with respect to the values (λ, A, t_0, T), $0 < \lambda \leq A$, if for any solution $x(t)$ the condition $\|x(t_1)\| < \lambda$ implies $\|x(t)\| < A$ for any $t \geq t_1$, $(t, t_1) \in [t_0, t_0 + T]$.

Based on Lemma 1 we shall formulate the following result.

Theorem 3 *The motion of system* (1) *is:*

(a) *stable with respect to the values* (λ, A, t_0, T)*, if all conditions of Lemma 1 are satisfied as well as the inequality*

$$\exp\left(\int_{t_0}^{t} a(s)ds\right) (L(t))^{-\frac{1}{k-1}} < \frac{A}{\lambda} \quad \textit{for all} \quad t \in [t_0, t_0 + T]; \tag{17}$$

(b) *uniformly stable with respect to the values* (λ, A, t_0, T)*, if the inequality* (17) *is satisfied for any* $t_1 \in [t_0, t_0 + T]$ *such that* $\|x(t_1)\| < \lambda$.

In terms of estimates (15) and (16) we obtain the following results.

Theorem 4 *Let all conditions of Corollary 1 be satisfied as well as the inequality*

$$\int_{t_0}^{t} a(s)ds \leq \ln\left(\frac{A}{\lambda}\right) \quad \textit{for all} \quad t \in [t_0, t_0 + T]. \tag{18}$$

Then the motion of system (1) *is stable with respect to the values* (λ, A, t_0, T).

Theorem 5 *Let all conditions of Corollary 2 be satisfied as well as the inequality*

$$\left\{1 - (k-1)\alpha^{k-1} \int_{t_0}^{t} b(s)ds\right\}^{-\frac{1}{k-1}} < \frac{A}{\lambda} \quad \textit{for all} \quad t \in [t_0, t_0 + T]. \tag{19}$$

Then the motion of system (1) *is stable with respect to the values* (λ, A, t_0, T).

The proof of Theorems 3–5 is based on the estimates (6), (15), (16) and Definition 3(a). The assumptions on motion uniform stability of system (1) with respect to the values (λ, A, t_0, T) are made in terms of estimates (18) and (19), provided that $\|x(t_1)\| < \lambda$ for any $t_1 \in [t_0, t_0 + T]$.

4.3 Exponential Convergence of Solutions to Systems with Quadratic Nonlinearity

Consider systems (1) with a particular type nonlinearity, namely, the systems with quadratic nonlinearity (see [11, 12] and bibliography therein)

$$\dot{x}(t) = Ax(t) + X^T(t)Bx(t), \quad x(0) = x_0. \tag{20}$$

Here $x \in \mathbb{R}^n$, A is a rectangular $n^2 \times n$-matrix consisting of symmetric square matrices B_i, $i = 1, 2, \dots, n$,

$$B_i = \begin{bmatrix} b_{11}^i & b_{12}^i & \dots & b_{1n}^i \\ b_{12}^i & b_{22}^i & \dots & b_{2n}^i \\ \dots & \dots & \dots & \dots \\ b_{1n}^i & b_{2n}^i & \dots & b_{nn}^i \end{bmatrix},$$

$X^T(t) = \{X_1(t), X_2(t), \dots, X_n(t)\}$ is a rectangular $n \times n^2$-matrix consisting of square $n \times n$-matrices $X_i(t)$ with vectors $x(t)$ on their i-th lines, and the other elements are zero, i.e.

$$X_1(t) = \begin{bmatrix} x_1(t) & x_2(t) & \dots & x_n(t) \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \end{bmatrix}, \quad X_2(t) = \begin{bmatrix} 0 & 0 & \dots & 0 \\ x_1(t) & x_2(t) & \dots & x_n(t) \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \end{bmatrix}, \quad \dots,$$

$$X_n(t) = \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ x_1(t) & x_2(t) & \dots & x_n(t) \end{bmatrix}.$$

Here and elsewhere the vector and matrix norms are specified by the formulas

$$\|x(t)\| = \left\{ \sum_{i=1}^{n} x_i^2(t) \right\}^{1/2}, \quad \|B\| = \{\lambda_{\max}(B^T B)\}^{1/2},$$

where $\lambda_{\max}(\cdot)$ and $\lambda_{\min}(\cdot)$ are extreme eigenvalues of the corresponding symmetric matrices.

Let the matrix A of the linear part of system (20) be asymptotically stable. Then, according to the theory of stability by first approximation (see [6]) the zero solution of nonlinear system (20) will also be asymptotically stable. We shall take the quadratic form $V(x) = x^T Hx$ as the Lyapunov function and compute its total derivative by virtue of system (20)

$$\begin{aligned} \frac{d}{dt}V(x(t)) &= [Ax(t) + X^T(t)Bx(t)]^T Hx(t) + x^t(t)H[Ax(t) + X^T(t)BNx(t)] \\ &= x^T(t)[(A^T H + HA) + (B^T X(t)H + HX^T(t)B)]x(t). \end{aligned} \tag{21}$$

Since the matrix A is asymptotically stable by assumption, for an arbitrary positive definite matrix C the matrix Lyapunov equation

$$A^T H + HA = -C \tag{22}$$

possesses a unique solution in the form of positive definite matrix H. In view of the fact that H is a solution of the Lyapunov equation (22) we get from (21) that

$$\frac{d}{dt}V(x(t)) = -x^T(t)[C - (B^T X(t)H + HX^T(t)B)]x(t). \tag{23}$$

The stability domain of the zero solution of system (20) is the interior of the surface of the level of the Lyapunov function $V(x) = r > 0$ located inside the domain

$$G_0 = \{x \in \mathbb{R}^n : \ C - B^T XH - HX^T B > \Theta\},$$

where the symbol

$$C - B^T XH - HX^T B > \Theta \tag{24}$$

is understood as positive definiteness of the matrix. We shall replace the condition (24) by a more "rough" one. Since for the chosen vector and matrix norms the correlation

$$\|X(t)\| = \|x(t)\|,$$

holds true, for the total derivative of the Lyapunov function (21) the estimate

$$\frac{d}{dt}V(x(t)) \le -\left[\lambda_{\min}(C) - 2\|H\|\|B\|\|x(t)\|\right]\|x(t)\|^2. \tag{25}$$

is satisfied.

We designate

$$G_0 = \left\{x \in \mathbb{R}^n : \ \|x\| < \frac{\lambda_{\min}(C)}{2\|H\|\|B\|}\right\}. \tag{26}$$

Then the domain of "guaranteed" stability is specified by the expression

$$G_{r_0} = \max_{r>0}\{C_r : \ G_r \subset G_0\}, \quad G_r = \{x \in \mathbb{R}^n : \ x^T Hx < r^2\}. \tag{27}$$

From this dependence it follows that for the "maximal" stability domain be defined, it is necessary to "imbed" the ellipsoid $x^T HX = r^2$ into a sphere of radius $R = \dfrac{\lambda_{\min}(C)}{2\|H\|\|B\|}$ and to extend it for $r \to \infty$ until the ellipsoid surface touches the sphere.

Theorem 6 *Let the matrix of the linear part of system* (20) *be asymptotically stable. Then the zero solution of system* (20) *is asymptotically stable and for the solutions of the system satisfying the initial conditions*

$$\|x_0\| < \frac{\gamma(H)}{2\|B\|\varphi(H)}, \tag{28}$$

where

$$\varphi(H) = \frac{\lambda_{\max}(H)}{\lambda_{\min}(H)}, \quad \gamma(H) = \frac{\lambda_{\min}(C)}{\lambda_{\max}(H)},$$

the convergence of solutions obeys the estimate

$$\|x(t)\| \leq \frac{\gamma(H)\sqrt{\lambda_{min}(Y)}\|x_0\|}{\Big[\gamma(H) - 2\|B\|\varphi(H)\|x_0\|\Big]\, e^{\frac{1}{2}\gamma(H)t} + 2\|B\|\varphi(H)\|x_0\|.} \tag{29}$$

Proof. In order to obtain estimate (29) we use the Lyapunov function $V(x) = x^T Hx$ with total derivative (25). Since for the quadratic function $V(x) = x^T Hx$ the two-sided inequality

$$\lambda_{\min}(H)\|x\|^2 \leq V(x) \leq \lambda_{\max}(H)\|x\|^2 \tag{30}$$

is valid, the inequality (25) can be rewritten as

$$\frac{d}{dt}V(x(t)) \leq -\frac{\lambda_{\min}(C)}{\lambda_{\max}(H)}\, V(x(t)) + 2\lambda_{\max}(H)\|B\|\, \frac{V^{3/2}(x(t))}{\lambda_{\min}^{3/2}(H)}\,. \tag{31}$$

Using the designation (28) we rewrite the obtained expression as

$$\frac{d}{dt}V(x(t)) \leq -\gamma(H)V(x(t)) + 2\frac{\|B\|\varphi(H)}{\sqrt{\lambda_{\min}(H)}}\, V^{3/2}(x(t)).$$

Dividing this inequality by the expression $V^{3/2}(x)$ we get the estimate

$$V^{-3/2}(x(t))\, \frac{dV(x(t))}{dt} \leq -\gamma(H)V^{-1/2}(x(t)) + 2\frac{\|B\|\varphi(H)}{\sqrt{\lambda_{\min}(H)}}\,.$$

Hence, having designated $V^{-1/2}(x(t)) = z(t)$, we arrive at

$$-2\, \frac{dz(t)}{dt} \leq -\gamma(H)z(t) + 2\, \frac{\|B\|\varphi(H)}{\sqrt{\lambda_{\min}(H)}}\,,$$

and then

$$\frac{dz(t)}{dt} \geq -\frac{1}{2}\,\gamma(H)z(t) - \frac{\|B\|\varphi(H)}{\sqrt{\lambda_{\min}(H)}}\,.$$

Solving this inequality (in the same way as the linear inhomogeneous Bernoulli equation) we get

$$z(t) \geq \left[z_0 - 2\, \frac{\|B\|\varphi(H)}{\gamma(H)\sqrt{\lambda_{\min}(H)}}\right] e^{\frac{1}{2}\gamma(H)t} + 2\, \frac{\|B\|\varphi(H)}{\gamma(H)\sqrt{\lambda_{\min}(H)}}\,.$$

Substitution $V^{-1/2}(x(t)) = z(t)$ yields the estimate

$$V^{-1/2}(x(t)) \geq \left[V^{-1/2}(x_0) - 2\, \frac{\|B\|\varphi(H)}{\gamma(H)\sqrt{\lambda_{\min}(H)}}\right] e^{\frac{1}{2}\gamma(H)t} + 2\, \frac{\|B\|\varphi(H)}{\gamma(H)\sqrt{\lambda_{\min}(H)}}\,.$$

Hence

$$V^{1/2}(x(t)) \leq \left\{\left[V^{-1/2}(x_0) - 2\, \frac{\|B\|\varphi(H)}{\gamma(H)\sqrt{\lambda_{\min}(H)}}\right] e^{\frac{1}{2}\gamma(H)t} + 2\, \frac{\|B\|\varphi(H)}{\gamma(H)\sqrt{\lambda_{\min}(H)}}\right\}^{-1}.$$

Application of the two-sided inequality for the quadratic form (30) gives

$$\sqrt{\lambda_{\min}(H)}(\|x(t)\| \le \left\{ \left[\frac{1}{\sqrt{V(\|x_0\|)}} - 2\frac{\|B\|\varphi(H)}{\gamma(H)\sqrt{\lambda_{\min}(H)}} \right] e^{\frac{1}{2}\gamma(H)t} + 2\frac{\|B\|\varphi(H)}{\gamma(H)\sqrt{\lambda_{\min}(H)}} \right\}^{-1}$$
$$\le \left\{ \left[\frac{1}{\sqrt{\lambda_{\min}(H)}\|x_0\|} - 2\,\frac{\|B\|\varphi(H)}{\gamma(H)\sqrt{\lambda_{\min}(H)}} \right] e^{\frac{1}{2}\gamma(H)t} + 2\,\frac{\|B\|\varphi(H)}{\gamma(H)\sqrt{\lambda_{\min}(H)}} \right\}^{-1}$$
$$= \frac{\gamma(H)\sqrt{\lambda_{\min}(H)}\|x_0\|}{\big[\gamma(H) - 2\|B\|\varphi(H)\|x_0\|\big] e^{\frac{1}{2}\gamma(H)t} + 2\|B\|\varphi(H)\|x_0\|}\,.$$

Thus, for solutions $x(t)$ of system (20) with the initial conditions from the domain (27), i. e. $x_0 \in G_0$, we obtain the estimate of solutions convergence of (29) type. This completes the proof.

Remark 2 Consider the first order scalar equation

$$\dot{x}(t) = -ax(t) + bx^2(t), \quad a > 0, \quad x(0) = x_0. \tag{32}$$

This equation is an equation with separating variables and its exact solution is the function

$$x(t) = \frac{ax_0e^{-at}}{a - bx_0[1 - e^{-at}]}\,. \tag{33}$$

Consider the application of the method of Lyapunov functions with the function $V(x) = x^2$ for the equation (32). For this function $\lambda_{\max}(H) = \lambda_{\min} = 1$. The total derivative by virtue of the linear part of system (32) is

$$\frac{d}{dt}V(x(t)) = -2ax^2(t).$$

Therefore, $\varphi(H) = 1$, $\gamma(H) = 2a$. The convergence estimate (29) for solutions of the equation with the initial conditions $\|x_0\| < a/|b|$ is of a similar form

$$\|x(t)\| \le \frac{a\|x_0\|}{[a - |b|\|x_0\|]e^{at} + |b|\|x_0\|} = \frac{a\|x_0\|e^{-at}}{a - |b|[1 - e^{-at}]} \to 0.$$

Thus, for the scalar equation (32) with the exact solution (33) the convergence estimate coincides with the estimate obtained by the application of the quadratic Lyapunov function.

5 Concluding Remarks

The proposed method for estimating the norm of solutions to nonlinear systems possesses a considerable potential for application in the investigation of particular mechanical and other nature systems. Efficiency of the proposed estimation is illustrated by the example of a first order scalar equation, for which the convergence estimate is obtained by means of the direct Lyapunov method.

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