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On Exponential Domination of Some Graphs

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Abstract: Let G be a graph and $S \subseteq V(G)$. We denote by $\langle S \rangle$ the subgraph of G induced by S. For each vertex $u \in S$ and for each $v \in V(G) - S$, we define $\overline{d}(u,v) = \overline{d}(v,u)$ to be the length of the shortest path in $\langle V(G) - (S - \{u\}) \rangle$ if such a path exists, and ∞ otherwise. Let $v \in V(G)$. We define $w_s(v) = \sum_{u \in S} \frac{1}{2^{\overline{d}(u,v)-1}}$ if $v \notin S$, and $w_s(v) = 2$ if $v \in S$. If, for each $v \in V(G)$, we have $w_s(v) \geq 1$, then S is an *exponential dominating set*. The smallest cardinality of an exponential dominating set is the *exponential domination number* $\gamma_e(G)$. In this paper, we consider the *exponential domination number* in total graphs. We determine the *exponential domination number* of $T(G)$ for some specific graphs G.

Keywords: *graph vulnerability; network design and communication; domination; exponential domination number; total graph.*

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1 Introduction

In a communication network, the vulnerability measures the resistance of network to disruption of operation after the failure of certain stations or communication links. The stability of communication networks is of prime importance to network designers (see [9,10]). If we think of the graph as modeling a communication network, many graph theoretical parameters have been used to describe the stability of communication networks including connectivity, toughness, integrity, domination and its variations (see [1, 2, 4, 5]). The domination number is one of the measures of the graph vulnerability.

Domination in graphs is a well-studied concept in graph theory. Domination based parameters reveal an underlying efficient communication network in which a vertex can

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affect all its neighborhood vertices in some sense. In real life applications, we can encounter that a vertex can affect both its neighborhood vertices and all vertices within a given distance. Distance domination is a kind of this situation. There has been no framework yet in which the effect of a vertex broadens beyond its neighborhood while decreasing by distance. It has been suggested (see [7]) that exponential domination is a model for the reliability of the spread of information or gossip. In this model, the dominating strategy of a vertex decreases exponentially with a distance, by the factor 1/2. Therefore, it is possible that a vertex v is dominated by one of its neighbors or by some vertices that are closer to v. The assumption is that gossip heard directly from a source is totally reliable, while gossip passed from person to person loses half its credibility with each individual in the chain. Finding the exponential domination number in this application amounts to determining the minimum number of sources needed so that each person gets fully reliable information.

In this paper, we consider simple finite undirected graphs without loops and multiple edges. Let $G = (V, E)$ be a graph with vertex set $V = V(G)$ and an edge set $E = E(G)$. For vertices u of a graph G, the *open neighborhood* of u is $N(u) = \{v \in V(G)|(u, v) \in E(G)\}$. We define analogously for any $S \subseteq V(G)$ the open neighborhood $N(S) = \bigcup_{u\in S} N(u)$. The closed neighborhood of u is $N[u] = u \cup N(u)$. For a set $S \subseteq V$, its closed neighborhood $N[S] = N(S) \cup S$. A set S is dominating set of G if $N[S] = V$, or equivalently, every vertex in $V - S$ is adjacent to at least one vertex of S. The *dominating number* $\gamma(G)$ is the minimum cardinality of a dominating set of G.

The distance $d(u, v)$ between two vertices u and v in G is the length of the shortest path between them. If u and v are not connected, then $d(u, v) = \infty$, and for $u = v$, $d(u, v) = 0$. The diameter of G, denoted by $diam(G)$ is the largest distance between two vertices in $V(G)$ (see [3, 4]).

Throughout this paper, the largest integer not larger than x is denoted by $\lfloor x\rfloor$ and the smallest integer not smaller than x is denoted by $\lceil x\rceil$.

The paper proceeds as follows. In Sections 2 and 3, the definition of exponential domination number and known results are given, respectively. In Section 4, we give some results on the exponential domination number of total graphs. Formulas for the exponential domination number of the graphs obtained by binary graph operations are given in Section 5.

2 Exponential Domination Number

The exponential domination number of a graph is a new characteristic for graph vulnerability introduced in [7]. This definition is in the following:

This parameter is a variation of distance domination in which, as described in the motivation already given, the 'dominating power' radiating from a vertex declines exponentially with distance. Let G be a graph and $S \subseteq V(G)$. We denote by $\langle S\rangle$ the subgraph of G induced by S. For each vertex $u \in S$ and for each $v \in V(G)-S$, we define $\overline{d}(u, v) = \overline{d}(v, u)$ to be the length of the shortest path in $\langle V(G) - (S - \{u\})\rangle$ if such a path exists, and ∞ otherwise. Let $v \in V(G)$. The definition is

$$w_s(v) = \begin{cases} \sum_{u\in S} \frac{1}{2^{\overline{d}(u,v)-1}}, & if\ v \notin S, \\ 2, & if\ v \in S. \end{cases}$$

We refer to $w_s(v)$ as the weight of S at v (note that we define $w_s(v) = 2$ if $v \in S$ since then v contributes $w_s(v)/2^d$ to every vertex it exponentially dominates at distance d).

If, for each $v \in V(G)$, we have $w_s(v) \geq 1$, then S is an *exponential dominating set*. The smallest cardinality of an exponential dominating set is the *exponential domination number*, $\gamma_e(G)$, and such a set is a *minimum exponential dominating set*, or $\gamma_e(G)$ -set for short. If $u \in S$ and $v \in V(G) - S$ and $\frac{1}{2^{\bar{d}(u,v)-1}} > 1$, then we say that u *exponentially dominates* v. Note that if S is an exponential dominating set, then every vertex of $V(G) - S$ is exponentially dominated, but the converse is not true (see [7, 8]).

3 Basic Results

Theorem 3.1 *[7] For every positive integer n, $\gamma_e(P_n) = \lceil \frac{n+1}{4} \rceil$.*

Theorem 3.2 *[7] For every positive integer n,*

$$\gamma_e(C_n) = \begin{cases} 2, & if\ n = 4, \\ \lceil \frac{n}{4} \rceil, & if\ n \neq 4 \in S. \end{cases}$$

Theorem 3.3 *[7] If G is a connected graph of diameter d, then $\gamma_e(G) \geq \frac{\lceil d+2 \rceil}{4}$.*

Theorem 3.4 *[7] If G is a connected graph of order n, then $\gamma_e(G) \leq \frac{2}{5}(n+2)$.*

Theorem 3.5 *[7] Let G be a connected graph of order n and T be a spanning tree of G. Then $\gamma_e(G) \leq \gamma_e(T)$.*

4 Exponential Domination Number of Total Graphs

In this section, the exponential domination number of total graph of a graph is calculated and formula for the exponential domination number of $\gamma_e(\overline{T(G)})$ is given.

Definition 4.1 [3, 4] The vertices and edges of a graph are called its elements. Two elements of a graph are neighbors if they are either incident or adjacent. The *total graph* $T(G)$ of the graph $G = (V(G), E(G))$, has vertex set $V(G) \cup E(G)$, and two vertices of $T(G)$ are adjacent whenever they are neighbors in G.

Example 4.1

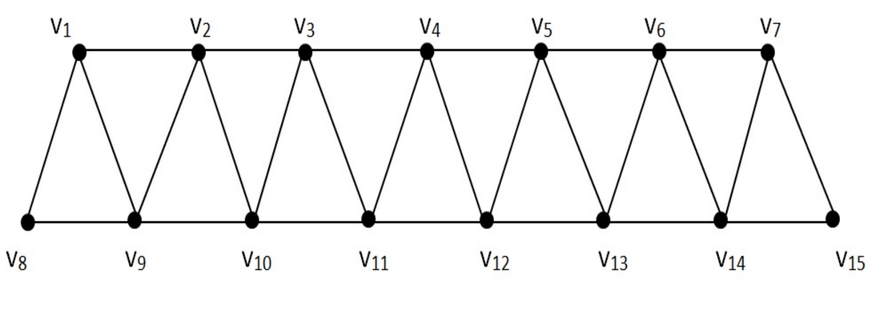

Figure 1: Total graph $T(P_8)$.

The following table shows us the weight of S_1 at all vertices of the graph $T(P_8)$, where $S_1 = \{v_2, v_8, v_{12}, v_{14}\}$.

v	$\overline{d}(v, v_2)$	$\overline{d}(v, v_8)$	$\overline{d}(v, v_{12})$	$\overline{d}(v, v_{14})$	$w_{s_1(v)}$
v_1	1	1	4	7	2.135
v_2	-	-	-	-	2
v_3	1	3	2	4	1.875
v_4	2	4	1	3	1.875
v_5	3	5	1	2	1.81
v_6	4	6	2	1	1.655
v_7	5	7	3	1	1.325
v_8	-	-	-	-	2
v_9	1	1	3	6	2.281
v_{10}	1	2	2	5	2.015
v_{11}	2	3	1	4	1.875
v_{12}	-	-	-	-	2
v_{13}	4	6	1	1	2.156
v_{14}	-	-	-	-	2
v_{15}	6	8	4	1	1.147

For S_1_*set*, $\forall v \in V(T(P_8))$, $w_s(v) \geq 1$ is satisfied. So, S_1_*set* is an exponential dominating set.

The following table shows us the weight of S_2 at all vertices of the graph $T(P_8)$, where $S_2 = \{v_5, v_{10}, v_{14}\}$.

v	$\overline{d}(v, v_5)$	$\overline{d}(v, v_{10})$	$\overline{d}(v, v_{14})$	$w_{s_2(v)}$
v_1	4	2	6	0.656
v_2	3	1	5	1.56
v_3	2	1	4	1.625
v_4	1	2	3	1.75
v_5	-	-	-	2
v_6	1	4	1	2.125
v_7	2	5	1	1.56
v_8	5	2	7	0.575
v_9	4	1	6	1.156
v_{10}	-	-	-	2
v_{11}	2	1	3	1.75
v_{12}	1	2	2	2
v_{13}	1	3	1	2.25
v_{14}	-	-	-	2
v_{15}	3	6	1	1.281

For S_2_*set*, $w_{S_2}(v_1) \geq 1$ and condition $w_{S_2}(v_8) \geq 1$ is not satisfied. So, S_2 is not an exponential dominating set.

The following table shows us the weight of S_3 at all vertices of the graph $T(P_8)$, where $S_3 = \{v_8, v_{11}, v_{14}\}$.

v	$\overline{d}(v, v_8)$	$\overline{d}(v, v_{11})$	$\overline{d}(v, v_{14})$	$w_{s3(v)}$
v_1	1	3	6	1.281
v_2	2	2	5	1.06
v_3	3	1	4	1.375
v_4	4	1	3	1.375
v_5	5	2	2	1.06
v_6	6	3	1	1.281
v_7	7	4	1	1.14
v_8	-	-	-	2
v_9	1	2	6	1.531
v_{10}	2	1	5	1.56
v_{11}	-	-	-	2
v_{12}	5	1	2	1.56
v_{13}	6	2	1	1.531
v_{14}	-	-	-	2
v_{15}	8	5	1	1.067

For S_3_*set*, $\forall v \in V(T(P_8))$, $w_s(v) \geq 1$ is satisfied. So, S_3_*set* is an exponential dominating set.

Similarly, we can get a lot of exponential dominating sets of the graph $T(P_8)$ but, for exponential domination number we need the minimum cardinality of among all exponential dominating sets. Here, $\gamma_e(T(P_n)) = min\{|S_1|, |S_3|\} = min\{4, 3\} = 3$.

Theorem 4.1 *Let P_n be a path graph with n vertices and $T(P_n) \cong G$ be a total graph of P_n with $2n-1$ vertices. Then $\gamma_e(G) = \lceil \frac{n}{3} \rceil$.*

Proof. The domination number of P_n is $\gamma(P_n) = \lceil \frac{n}{3} \rceil$. If we add the vertices of the domination set to $\gamma_e - set$, every vertex v in $\gamma_e - set$ is adjacent to four vertices in graph G. For $\forall u \in N_{\gamma_e - set}(v)$, $w_s(u) \geq 1$. The length of the shortest path, from $\forall u \in V(G) - N_{\gamma_e - set}[v]$ remaining vertices to exactly two vertices in $\gamma_e - set$ is 2. So, $w_s(u) \geq 1$. Consequently, exponential domination number of G is

$$\gamma_e(G) = \lceil \tfrac{n}{3} \rceil.$$

The proof is completed.

Theorem 4.2 *Let C_n be a cycle graph with n vertices and $T(C_n) \cong G$ be a total graph of C_n with $2n$ vertices. Then, for $n > 3$ $\gamma_e(G) = \lceil \frac{n}{3} \rceil$.*

Proof. The proof is similar to the proof of Theorem 7.

Theorem 4.3 *Let $S_{1,n}$ be a star graph with $n+1$ vertices and $T(S_{1,n}) \cong G$ be a total graph of $S_{1,n}$ with $2n+1$ vertices. Then $\gamma_e(G) = 1$.*

Proof. Every vertex in G is adjacent to centre vertex c in G. So, we can add only centre vertex c to $\gamma_e - set$. Hence, we have $w_s(v) = 1$ for $\forall v \in V(G) - \{c\}$ and $w_s(c) = 2$. Therefore, the result is obvious.

The proof is completed.

Theorem 4.4 *Let K_n be a complete graph with n vertices and $T(K_n) \cong G$ be a total graph of K_n with $(n^2 + n)/2$ vertices. Then, $\gamma_e(G) = 2$.*

Proof. Since in a complete graph all vertices are mutually adjacent, distance between each pair of vertex is 1. Distance between remaining vertices in $V(G) - V(K_n)$ and any vertex in K_n is at most 2. Hence, condition $w_s(v) \geq 1$ is not satisfied for some $v \in V(G) - V(K_n)$. As in the proof of Theorem 7, one more vertex in K_n should be added to $\gamma_e - set$ for the length of the path from vertices in $V(G) - V(K_n)$ to exactly two vertices in $\gamma_e - set$ to be 2. Hence, we have $\gamma_e(G) = 2$.

The proof is completed.

Theorem 4.5 *Let $W_{1,n}$ be a wheel graph with $n + 1$ vertices and $T(W_{1,n}) \cong G$ be a total graph of $W_{1,n}$ with $3n + 1$ vertices. Then, $\gamma_e(G) = \lceil \frac{n}{4} \rceil + 1$.*

Proof. Let the vertex-set of graph G be $V(G) = V_1(G) \cup V_2(G) \cup V_3(G) \cup V_4(G)$ where,

$V_1(G) =$ The set contains the center vertex c of graph $W_{1,n}$.

$V_2(G) =$ The set contains all vertices of graph $W_{1,n}$, except center vertex.

$V_3(G) =$ The set contains the edges of graph $W_{1,n}$, which are adjacent to center vertex; are the vertices of graph $T(W_{1,n})$.

$V_4(G) =$ The set contains the edges of the cycle of graph $W_{1,n}$ are the vertices of graph $T(W_{1,n})$.

The center vertex c is adjacent to every vertex in $V_2(G)$ and $V_3(G)$. So, the centre vertex c should be added to $\gamma_e - set$. But, the length of the path from $\forall u \in V_4(G)$ to every vertex in $\gamma_e - set$ is 2. Therefore, condition $w_s(u) \geq 1$ is not satisfied. As in the proof of Theorem 7, the length of the path from every vertex in $V_4(G)$ to exactly two vertices in $\gamma_e - set$ should be 2. The length of the path from every vertex in $V_2(G)$ to two vertices in $V_4(G)$ is 1 and two vertices in $V_4(G)$ is 2. Hence, $\lceil \frac{|V_2(G)|}{4} \rceil = \lceil \frac{n}{4} \rceil$ vertices in $V_2(G)$ should be added to $\gamma_e - set$. There is already the center vertex c in $\gamma_e - set$. Hence, we have $\gamma_e(G) = \lceil \frac{n}{4} \rceil + 1$.

The proof is completed.

5 Corona and join graphs, the exponential domination number

Definition 5.1 [3,4] The corona $G_1 o G_2$ is obtained by taking one copy of G_1 and $|G_1|$ copies of G_2 , and by joining each vertex of the *ith* copy of G_2 to the *ith* vertex of G_1, i=1,2,...,$|G_1|$.

Definition 5.2 [3,4] Let G_1 and G_2 be two disjoint graphs. The *join* of G_1 and G_2 with disjoint vertex sets $V(G_1)$ and $V(G_2)$ and edge sets $E(G_1)$ and $E(G_2)$ is the graph $G = G_1 + G_2$ with vertex set $V(G) = V(G_1) \cup V(G_2)$ and edge set $E(G) = E(G_1) \cup E(G_2) \cup \{(u, v) : u \in V(G_1), v \in V(G_2)\}$.

Theorem 5.1 *Let $G_1 \cong P_n$ be a path graph with n vertices and G be any connected graph. Then, $\gamma_e(G_1 o G) = \lfloor \frac{n-2}{2} \rfloor + 2$.*

Proof. If $|V(G)| = n_1$, then $|V(G_1oG)| = n(n_1+1)$. Every vertex in G_1 except the end vertices is adjacent to n_1 vertices and two vertices in G_1. The path between every vertex in G_1 except the end vertices and $n(n_1+1)-(n_1-2)$ vertices in G_1oG is at least 2. So, we obtain the minimum exponential domination set S by adding some vertices in G_1 to S and $S \subseteq V(G_1)$. Two end vertices of graph G_1 should be added to exponential domination set S of G_1oG. Otherwise, for $\forall v \in V(G_1oG) - V(G_1)$ that are adjacent to these end vertices, $w_s(v) \geq 1$ is not satisfied, since the length of the path between v and one vertex in G_1oG is 2; the length of the path between v and the other remaining vertices in G_1oG is at least 3. If we add $\lfloor \frac{n-2}{2} \rfloor$ vertices in G_1 except these end vertices, to S, for $\forall u \in V(G_1oG)$ $w_s(u) \geq 1$ is satisfied. There are already two end vertices in S. Hence, we have $\gamma_e(G_1oG) = \lfloor \frac{n-2}{2} \rfloor + 2$.

The proof is completed.

Theorem 5.2 *Let $G_1 \cong C_n$ be a cycle graph with n vertices and G be any connected graph. Then, $\gamma_e(G_1oG) = \lceil \frac{n}{2} \rceil$.*

Proof. If $|V(G)| = n_1$, then $|V(G_1oG)| = n(n_1+1)$. Every vertex in G_1 is adjacent to n_1 vertices and two vertices in G_1. The path between every vertex in G_1 and $n(n_1+1)-(n_1-2)$ vertices in G_1oG is at least 2. So, we obtain the minimum exponential domination set S by adding some vertices in G_1 to S and $S \subseteq V(G_1)$. We obtain S by adding $\forall v \in S$ satisfies $d(u,v) \leq 2$ or $d(u,v) = \infty$ for $\forall u \in (V(G_1oG) - S)$. So, there must be $\lceil \frac{n}{2} \rceil$ vertices from G_1 in S . Consequently, $w_s(x) \geq 1$ satisfying for $\forall x \in V(G_1oG)$ and we have

$$\gamma_e(G_1oG) = \lceil \tfrac{n}{2} \rceil.$$

The proof is completed.

Corollary 5.1 *Let $G_1 \cong C_n$. Then, $\gamma_e(G_1oG) = diam(G_1)$.*

Theorem 5.3 *Let $G_1 \cong S_{1,n}$ be a star graph with $n+1$ vertices and G be any connected graph. Then, $\gamma_e(G_1oG) = 4$.*

Proof. We denote the centre vertex of G_1 by c. In G_1oG, for $\forall u \in V(G)$ and $\forall v \in V(G_1 - \{c\})$ $d(u,v) \leq 3$. If we set S with vertices from $G_1 - \{c\}$ then vertex v contributes at least $\frac{1}{2^{\bar{d}(u,v)-1}} = \frac{1}{2^2}$ to $w_s(u)$. Hence, adding any 4 vertices from $G_1 - \{c\}$ to S is sufficient and we have

$$\gamma_e(G_1oG) = 4.$$

The proof is completed.

Theorem 5.4 *Let $G_1 \cong W_{1,n}$ be a wheel graph with $n+1$ vertices and G be any connected graph. Then, $\gamma_e(G_1oG) = 4$.*

Proof. The proof is similar to the proof of Theorem 17.

Theorem 5.5 *Let $G_1 \cong K_n$ be a complete graph with n vertices and G be any connected graph. Then $\gamma_e(G_1oG) = 2$.*

Proof. The length of the path between $\forall v \in G_1oG$ and every vertex in G_1 is at most 2. So, it is easy to see that $diam(G_1oG) = 3$. Hence, $S \subseteq V(G_1)$ satisfying $w_s(v) \geq 1$. It is sufficient to add any two vertices to S. Therefore, we have

$$\gamma_e(G_1 o G) = 2.$$

The proof is completed.

Corollary 5.2 *For any two graphs G_1 and G_2, $G_1 o G_2 \geq \lceil \frac{diam(G_1 o G_2)}{2} \rceil$.*

Theorem 5.6 *Let G_1 and G_2 be any two graphs having respectively diameters d_1 and d_2. If $diam(G_1) = d_1 < diam(G_2) = d_2$, then $\gamma_e(G_1 + G_2) = \gamma_e(G_1)$.*

Proof. We assume that $diam(G_1) = d_1 < diam(G_2) = d_2$. By the definition of $\gamma_e(G_1)$, we can not reduce any vertex from $\gamma_e(G_1)$ and every vertex in G_2 is adjacent to every vertex in $\gamma_e(G_1)$. If we add every vertex in $\gamma_e(G_1)$ to S minimum exponential number of $G_1 + G_2$, for $\forall u \in G_1$ $w_s(u) \geq 1$ and for $\forall v \in V(G_2)$ $w_s(v) = 1$ is satisfied.

The proof is completed.

6 Conclusion

In an administrative setup, decisions are taken by a small group who have effective communication links with other members of the organization. Domination in graphs provides a model for such a concept. The domination in graphs is one of the concepts in graph theory which has attracted many researchers to work on it because of its many and varied applications in such fields as linear algebra and optimization, design and analysis of communication networks, and social sciences and military surveillance. Many variants of dominating models are available in the existing literature. Dankelmann et al. (see [7]) recently defined exponential domination. Hence, in this paper, we investigate the exponential domination number of some total graphs. Moreover some results about exponential domination number of graphs obtained by graph operations are established.

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