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On Tractable Functionals in Antagonistic Games with a Constant Initial Condition

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Abstract: This paper continues modeling of an antagonistic game with two players initiated in Dshalalow and Ke [4] which dealt with a stochastic game with player A losing to player B. Theorem 1 in [4] gave an explicit functional of several key components of the game, including the ruin time of A and the total casualties to both players at the exit, i.e. at A's ruin time. The claim of why the formula in Theorem 1 of [4] for the above mentioned functional was explicit is fully justified. Here we work on a particular case calculating Laplace-Carson inverse transforms and probability density functions followed by numerics.

Keywords: *noncooperative stochastic games; fluctuation theory; marked point processes; Poisson process; ruin time; exit time; first passage time; modified Bessel functions.*

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1 Introduction

This paper models an antagonistic game with two players earlier initiated in Dshalalow and Ke [4]. The first part of [4] dealt with a basic game when player A lost the game to player B. Theorem 1 in [4] gave an explicit functional of several major components of the game, including the ruin (exit) time, the total casualties to both players at the exit. The claim of why the formula in Theorem 1 of [4] for the above mentioned functional was explicit is finally justified in this paper. Here we analyze a particular case evaluating Laplace-Carson inverse transforms and probability density functions followed by numerical calculations.

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In short, the game initiated in [4] was modeled by a complex marked point process. It included two marked Poisson processes representing incremental casualties to players A and B during the conflict as well as the hitting times. Both processes were supposed to be observed by a third party point process which preserved more or less crude information about the course of the game. So, the ruin time as well as other events were cumulative upon observation epochs. The literature on antagonistic games is very rich. We mention just a few articles and books: [1, 5, 7–8, 11, 12]. The contemporary work on antagonistic games finds its applications to economics [1, 7, 8, 11] and warfare [4, 5, 12]. The techniques used in this paper are based on fluctuation theory developed by the first author in [4] and his earlier papers. Related work on fluctuation theory is in [9,10].

The paper is organized as follows. In Section 2, we give a brief description of the model in [4]. Section 3 formalizes a special case making an assumption about the distributions of casualties and observation process. The double inverse of the Laplace-Carson transform is evaluated explicitly in terms of the modified Bessel functions of order zero and one. Section 4 deals with one marginal functional of the ruin time and casualties to player A, all in terms of the Laplace-Stieltjes transform. Other results, such as casualties to player B and inverse of the Laplace-Stieltjes transform (that yields associated probability density functions), are dealt in paper [6].

2 The Model

For consistency, we present some descriptional details of the model before we turn to the special case. Let $(\Omega, \mathcal{F}(\Omega), P)$ be a probability space and let $\mathcal{F}_A, \mathcal{F}_B, \mathcal{F}_\tau \subseteq \mathcal{F}(\Omega)$ be independent sub- σ-algebras. Suppose

$$\mathcal{A} := \sum_{j\geq 1} w_j \varepsilon_{s_j} \quad \text{and} \quad \mathcal{B} := \sum_{k\geq 1} z_k \varepsilon_{t_k}, \quad s_1,\, t_1 > 0, \tag{2.1}$$

are $\mathcal{F}_A$-measurable and $\mathcal{F}_B$-measurable marked Poisson random measures (ε_a is a point mass at a) with respective intensities λ_A and λ_B and position independent marking. They are specified by their transforms

$$Ee^{-\alpha\mathcal{A}(\cdot)} = e^{\lambda_A|\cdot|[g(\alpha)-1]}, \quad g(\alpha) = Ee^{-\alpha w_1}, \quad Re(\alpha) \geq 0, \tag{2.2}$$

$$Ee^{-\beta\mathcal{B}(\cdot)} = e^{\lambda_B|\cdot|[h(\beta)-1]}, \quad h(\beta) = Ee^{-\beta z_1}, \quad Re(\beta) \geq 0, \tag{2.3}$$

$|\cdot|$ is the Borel-Lebesgue measure, and w_j and z_k are nonnegative r.v.'s. Furthermore, let

$$\tau := \sum_{i\geq 0} \varepsilon_{\tau_i}, \quad \tau_0 > 0, \tag{2.4}$$

be an $\mathcal{F}_\tau$-measurable delayed renewal process.

If

$$(A(t), B(t)) := \mathcal{A} \otimes \mathcal{B}\left((-\infty, t]\right), \tag{2.5}$$

then

$$(A_j, B_j) := (A(\tau_j), B(\tau_j)) = \mathcal{A} \otimes \mathcal{B}\left((-\infty, \tau_j]\right), \quad j = 0, 1, \ldots, \tag{2.6}$$

forms the observation process upon $\mathcal{A} \otimes \mathcal{B}$ embedded over τ, with respective increments

$$(X_j, Y_j) = \mathcal{A} \otimes \mathcal{B}((\tau_{j-1}, \tau_j]), \quad j = 1, 2, \ldots, \tag{2.7}$$

and

$$X_0 = A_0, \quad Y_0 = B_0. \tag{2.8}$$

Obviously, the bivariate marked point process

$$\mathcal{A}_\tau \otimes \mathcal{B}_\tau := \sum_{j\geq 0}(X_j, Y_j)\varepsilon_{\tau_j}, \tag{2.9}$$

where

$$\mathcal{A}_\tau = \sum_{i\geq 0} X_i \varepsilon_{\tau_i} \quad \text{and} \quad \mathcal{B}_\tau = \sum_{i\geq 0} Y_i \varepsilon_{\tau_i}. \tag{2.10}$$

are with position dependent marking and with X_j and Y_j being interdependent. With the notation

$$\Delta_j := \tau_j - \tau_{j-1}, \quad j = 1, 2, \ldots, \tag{2.11}$$

we evaluate the functional

$$\gamma(\alpha, \beta, \theta) = Ee^{-\alpha X_j - \beta Y_j - \theta\Delta_j} = \delta\{\theta + \lambda_A(1 - g(\alpha)) + \lambda_B(1 - h(\beta))\}, \quad j = 1, 2, \ldots, \tag{2.12}$$

$$Re(\alpha) \geq 0, \quad Re(\beta) \geq 0, \quad Re(\theta) \geq 0, \tag{2.13}$$

where

$$\delta(\theta) = Ee^{-\theta\Delta_1}, \quad Re(\theta) \geq 0. \tag{2.14}$$

is the common marginal Laplace-Stieltjes transform of Δ_1, $\Delta_2, \ldots$.

Analogously,

$$\gamma_0(\alpha, \beta, \theta) = Ee^{-\alpha A_0 - \beta B_0 - \theta\tau_0} = \delta_0\{\theta + \lambda_A(1 - g(\alpha)) + \lambda_B(1 - h(\beta))\}, \tag{2.15}$$

where

$$\delta_0(\theta) = Ee^{-\theta\tau_0}. \tag{2.16}$$

The *game* in this case is stochastic process $\mathcal{A}_\tau \otimes \mathcal{B}_\tau$ describing the evolution of a conflict between players A and B known to an observer only upon process $\tau = \{\tau_0, \tau_1, \ldots\}$. The game is over when on the kth observation epoch τ_k (for some k), the cumulative damage to player A or B (A_k or B_k, respectively) exceeds its respective threshold M or N (some positive real numbers). But we are looking into the paths of the game where player A is losing first.

With the *exit indices*

$$\mu := \inf\{j \geq 0 : \ A_j = X_0 + X_1 + \ \ldots \ + X_j > M\} \tag{2.17}$$

and

$$\nu := \inf\{k \geq 0 : B_k = Y_0 + Y_1 + \ldots + Y_k > N\}, \tag{2.18}$$

A_μ and B_ν are the respective cumulative damages to players A and B at their ruin times. We will be concerned, however, with the ruin time of player A and thus restrict our game to the confined trace σ-algebra $\mathcal{F}(\Omega) \cap \{\mu < \nu\}$. In paper [4] (Dshalalow-Ke) the authors studied a game between two players, A and B, in particular, the functional

$$\varPhi_{\mu\nu} = \varPhi_{\mu\nu}(\alpha, \beta, \theta) = E\left[e^{-\alpha A_\mu - \beta B_\mu - \theta\tau_\mu}\mathbf{1}_{\{\mu<\nu\}}\right] \tag{2.19}$$

of the game. It represented the multivariate Laplace-Stieltjes transform of joint distribution of the exit time τ_μ of the game and the status of the casualties to both players at the exit. The evolution of the game is followed here when player A loses the game to player B.

Theorem 1 [4] below gives an explicit formula for $\varPhi_{\mu\nu}$. With (2.12) and (2.15) we abbreviate

$$\gamma_0(\alpha,\beta,\theta) := Ee^{-\alpha X_0-\beta Y_0-\theta\Delta_0}, \quad Re(\alpha)\geq 0, \quad Re(\beta)\geq 0, \quad Re(\theta)\geq 0, \tag{2.20}$$

$$\gamma(\alpha,\beta,\theta) := Ee^{-\alpha X_j-\beta Y_j-\theta\Delta_j}, \quad Re(\alpha)\geq 0, \quad Re(\beta)\geq 0, \quad Re(\theta)\geq 0, \quad j>0, \tag{2.21}$$

$$\varGamma_0 := \gamma_0(\alpha+x,\beta+y,\theta), \quad \varGamma_0^1 := \gamma_0(\alpha,\beta+y,\theta), \tag{2.22}$$

$$\varGamma := \gamma(\alpha+x,\beta+y,\theta), \quad \varGamma^1 := \gamma(\alpha,\beta+y,\theta). \tag{2.23}$$

The results are presented in terms of the inverse of the Laplace-Carson transform defined as

$$\mathcal{LC}_{pq}(\cdot)(x,y) := xy\int_{p=0}^{\infty}\int_{q=0}^{\infty} e^{-xp-yq}(\cdot)d(p,q), \quad Re(x)>0, \quad Re(y)>0. \tag{2.24}$$

Denote its inverse

$$\mathcal{LC}_{xy}^{-1}(\cdot)(p,q) = \mathcal{L}_{xy}^{-1}\left(\cdot\,\frac{1}{xy}\right), \tag{2.25}$$

where $\mathcal{L}^{-1}$ is the inverse of the bivariate Laplace transform.

Theorem 1 [4] *In light of abbreviations* (2.20)–(2.23)*, the functional* $\varPhi_{\mu\nu}$ *of the game on the trace* σ*-algebra* $\mathcal{F}(\Omega)\cap\{\mu<\nu\}$ *satisfies the following formula:*

$$\varPhi_{\mu\nu} = \mathcal{LC}_{xy}^{-1}\left(\varGamma_0^1-\varGamma_0+\frac{\varGamma_0}{1-\varGamma}(\varGamma^1-\varGamma)\right)(M,N), \tag{2.26}$$

which for the restricted functional (2.19) *of only three major components can be rewritten as*

$$\varPhi_{\mu\nu} = \mathcal{LC}_{xy}^{-1}\left(\varGamma_0^1-\varGamma_0\frac{1-\varGamma^1}{1-\varGamma}\right)(M,N). \tag{2.27}$$

3 A Special Case

We assume that the intervals $\Delta_1,\Delta_2,\ldots$ between the successive observation times $\tau_0,\tau_1,\ldots$ are exponentially distributed with parameter δ, i.e.

$$\delta(\theta) := Ee^{-\theta\Delta} = \frac{\delta}{\delta+\theta}. \tag{3.1}$$

We assume that the game starts from zero, i.e., X_0 and Y_0 are some constants and that

$$\Delta_0 := 0. \tag{3.2}$$

Furthermore, we assume that the marks in the processes $\mathcal{A}$ and $\mathcal{B}$ specified by g and h in (2.2) and (2.3), respectively, are exponential with parameters g and h, i.e.

$$g(\alpha) = \frac{g}{g+\alpha} \quad \text{and} \quad h(\beta) = \frac{h}{h+\beta}. \tag{3.3}$$

Our goal is to simplify $\Phi_{\mu\nu}$ of (2.27) for this special case to a form for which we can find the Laplace-Carson inverse explicitly. We start with the first factor, Γ_0^1 of (2.27) by unfolding notation (2.12):

$$\Gamma_0^1 = \gamma_0(\alpha, \beta + y, \theta) = E[e^{-\alpha X_0 - (\beta+y)Y_0 - \theta\Delta_0}] = Ee^{-\alpha X_0 - (\beta+y)Y_0}. \tag{3.4}$$

Now we apply the Laplace-Carson inverse to (3.4):

$$\begin{aligned}\mathcal{LC}_{xy}^{-1}(\Gamma_0^1)(p,q) &= \mathfrak{L}_{xy}^{-1}\left(\frac{1}{xy}e^{-\alpha X_0-(\beta+y)Y_0}\right)(p,q) = \mathfrak{L}_{y}^{-1}\left(\frac{1}{y}e^{-\alpha X_0-(\beta+y)Y_0}\right)(q)\\ &= e^{-\alpha X_0 - \beta Y_0}\mathbf{1}_{(Y_0,\infty)}(q) = \psi\mathbf{1}_{(Y_0,\infty)}(q).\end{aligned} \tag{3.5}$$

Turn to the second term $\Gamma_0\frac{1-\Gamma^1}{1-\Gamma}$ of (2.27). Firstly,

$$\Gamma_0 = \gamma_0(\alpha + x, \beta + y, \theta) = e^{-(\alpha+x)X_0 - (\beta+y)Y_0}. \tag{3.6}$$

Recall from (2.13) that $\gamma(\alpha,\beta,\theta) = \delta\big[\theta + \lambda_A(1-g(\alpha)) + \lambda_B(1-h(\beta))\big]$. Using (3.1) we get

$$1 - \gamma(\alpha,\beta,\theta) = \frac{\theta(g+\alpha)(h+\beta) + \lambda_A\alpha(h+\beta) + \lambda_B\beta(g+\alpha)}{(\delta+\theta)(g+\alpha)(h+\beta) + \lambda_A\alpha(h+\beta) + \lambda_B\beta(g+\alpha)}. \tag{3.7}$$

Denote $X := X(x) = g + \alpha + x$ and $Y := Y(y) = h + \beta + y$. Then

$$\begin{aligned}\frac{1-\Gamma^1}{1-\Gamma} &= \frac{1-\gamma(\alpha,\beta+y,\theta)}{1-\gamma(\alpha+x,\beta+y,\theta)}\\ &= \frac{\theta(g+\alpha)Y + \lambda_A\alpha Y + \lambda_B(\beta+y)(g+\alpha)}{(\delta+\theta)(g+\alpha)Y + \lambda_A\alpha Y + \lambda_B(\beta+y)(g+\alpha)} \cdot \frac{1}{\frac{\theta XY + \lambda_A(\alpha+x)Y + \lambda_B(\beta+y)X}{(\delta+\theta)XY + \lambda_A(\alpha+x)Y + \lambda_B(\beta+y)X}}.\end{aligned}$$

Continuing with calculations we have

$$\frac{1-\Gamma^1}{1-\Gamma} = \frac{GY - \lambda_B h(g+\alpha)}{G_\delta Y - \lambda_B h(g+\alpha)} \cdot \frac{(\delta+\Lambda)XY - \lambda_A gY - \lambda_B hX}{\Lambda XY - \lambda_A gY - \lambda_B hX} = f_1(Y)f_2(X,Y), \tag{3.8}$$

where

$$\Lambda := \theta + \lambda_A + \lambda_B, \quad G := \Lambda(g+\alpha) - \lambda_A g, \quad G_\delta := (\delta+\Lambda)(g+\alpha) - \lambda_A g, \tag{3.9}$$

$$f_1(Y) := \frac{GY - \lambda_B h(g+\alpha)}{G_\delta Y - \lambda_B h(g+\alpha)}, \quad f_2(X,Y) := \frac{(\delta+\Lambda)XY - \lambda_A gY - \lambda_B hX}{\Lambda XY - \lambda_A gY - \lambda_B hX}. \tag{3.10}$$

Here is how $f_2(X,Y)$ can be evaluated to separate x and $Y = Y(y)$:

$$f_2(X,Y) = \frac{(\delta+\Lambda)XY - \lambda_A gY - \lambda_B hX}{\Lambda XY - \lambda_A gY - \lambda_B hX} = f_3(Y) + \frac{\xi}{x+a}, \tag{3.11}$$

where

$$f_3(Y) := 1 + \frac{\delta}{\Lambda} + \frac{\lambda_B h\delta}{\Lambda} \cdot \frac{1}{\Lambda Y - \lambda_B h}, \tag{3.12}$$

$$\xi = \xi(Y) := \frac{\lambda_A g\delta Y^2}{(\Lambda Y - \lambda_B h)^2}, \tag{3.13}$$

$$a = a(Y) := g + \alpha - \frac{\lambda_A gY}{\Lambda Y - \lambda_B h}. \tag{3.14}$$

For the upcoming calculations we rewrite the function $f_3(Y)$ as

$$f_3(Y) = b + c \cdot \frac{1}{Y - r}, \tag{3.15}$$

where

$$b := 1 + \frac{\delta}{\Lambda}, \quad c := \frac{\lambda_B h \delta}{\Lambda^2}, \quad r := \frac{\lambda_B h}{\Lambda}. \tag{3.16}$$

(3.11) substituted in (3.8) gives

$$\frac{1 - \mathit{\Gamma}^1}{1 - \mathit{\Gamma}} = f_1(Y)\left(f_3(Y) + \frac{\xi}{x + a}\right). \tag{3.17}$$

Due to (3.5) and (3.6),

$$\mathit{\Gamma}_0 = e^{-(\alpha+x)X_0 - (\beta+y)Y_0} = e^{-\alpha X_0 - \beta Y_0} e^{-xX_0 - yY_0} = \psi \cdot e^{-xX_0 - yY_0}. \tag{3.18}$$

With (3.17) and (3.18) substituted in $\mathit{\Gamma}_0 \frac{1-\mathit{\Gamma}^1}{1-\mathit{\Gamma}}$, we arrive at

$$\mathit{\Gamma}_0 \frac{1 - \mathit{\Gamma}^1}{1 - \mathit{\Gamma}} = \psi \cdot e^{-xX_0 - yY_0} f_1(Y)\left(f_3(Y) + \frac{\xi}{x + a}\right). \tag{3.19}$$

Now we apply the Laplace-Carson inverse to (3.19):

$$\mathcal{LC}_{xy}^{-1}\left(\mathit{\Gamma}_0 \frac{1 - \mathit{\Gamma}^1}{1 - \mathit{\Gamma}}\right)(p, q) = \mathfrak{L}_{xy}^{-1}\left(\frac{1}{xy} \cdot \mathit{\Gamma}_0 \frac{1 - \mathit{\Gamma}^1}{1 - \mathit{\Gamma}}\right)(p, q)$$
$$= \mathfrak{L}_{xy}^{-1}\left\{\frac{1}{y} \cdot \psi \cdot e^{-yY_0} f_1(Y)\left(f_3(Y) \cdot \frac{1}{x} e^{-xX_0} + \frac{\xi}{a} \cdot e^{-xX_0}\left(\frac{1}{x} - \frac{1}{x + a}\right)\right)\right\}(p, q).$$

By Fubini's theorem, we can apply univariate Laplace inverses first in x and then in y. So

$$\begin{aligned} &\mathcal{LC}_{xy}^{-1}\left(\mathit{\Gamma}_0 \frac{1 - \mathit{\Gamma}^1}{1 - \mathit{\Gamma}}\right)(p, q) \\ &= \mathfrak{L}_y^{-1}\left\{\psi \cdot e^{-yY_0}\left[\frac{1}{y} f_1(Y) f_3(Y) + \frac{1}{y} f_1(Y) \frac{\xi}{a}\left(1 - e^{-a(p - X_0)}\right)\right] \mathbf{1}_{(X_0, \infty)}(p)\right\}(q). \end{aligned} \tag{3.20}$$

To make (3.20) inversely transformable in a closed form we decompose the underlying expressions with respect to y. The partial fraction decomposition will be rendered throughout.

Let

$$\sigma := \lambda_B h(g + \alpha) \quad \text{and} \quad f_1(Y) = \frac{GY - \lambda_B h(g + \alpha)}{G_\delta Y - \lambda_B h(g + \alpha)} = \frac{GY - \sigma}{G_\delta Y - \sigma}. \tag{3.21}$$

Then the partial fraction decomposition of $\frac{1}{y} f_1(Y)$ gives

$$\frac{1}{y} f_1(Y) = \frac{A}{y} + \frac{B}{G_\delta Y - \sigma}, \tag{3.22}$$

with

$$A = \frac{G(h + \beta) - \sigma}{G_\delta(h + \beta) - \sigma}, \quad B = \frac{\sigma(G_\delta - G)}{G_\delta(h + \beta) - \sigma}. \tag{3.23}$$

Continuing working on the first term $\frac{1}{y}f_1(Y)f_3(Y)$ of (3.20) and using (3.15) and (3.22) we get

$$\frac{1}{y}f_1(Y)f_3(Y) = \frac{Ab}{y} + \frac{Bb}{G_\delta Y - \sigma} + \frac{Ac}{y}\cdot\frac{1}{Y-r} + \frac{Bc}{G_\delta Y - \sigma}\cdot\frac{1}{Y-r},$$

in notation,

$$= \varphi_1(y) + \varphi_2(y) + \varphi_3(y) + \varphi_4(y). \tag{3.24}$$

Here is the partial fraction decomposition of $\varphi_3(y)$, and $\varphi_4(y)$. We distinguish three cases with various combinations of $\alpha \neq 0$, $\alpha = 0$, $\delta \neq \lambda_A$, and $\delta = \lambda_A$.

***(i)* Case $\boldsymbol{\alpha \neq 0}$.**

$$\varphi_3(y) = \frac{A_3}{y} + \frac{B_3}{Y-r} \quad \text{and} \quad \varphi_4(y) = \frac{A_4}{G_\delta Y - \sigma} + \frac{B_4}{Y-r}, \tag{3.25}$$

where

$$A_3 = -B_3 = \frac{Ac}{h+\beta-r}, \quad A_4 = (-G_\delta)B_4 = \frac{G_\delta Bc}{\sigma - rG_\delta}. \tag{3.26}$$

Substituting (3.25) into (3.24) we have

$$\begin{aligned}
&\frac{1}{y}f_1(Y)f_3(Y) = \frac{Ab}{y} + \frac{Bb/G_\delta}{Y-\sigma/G_\delta} + \left(\frac{A_3}{y} - \frac{A_3}{Y-r}\right) + \left(\frac{B_4}{Y-r} - \frac{B_4}{Y-\sigma/G_\delta}\right)\\
&= \Big(Ab + A_3\Big)\frac{1}{y} + \Big(B_4 - A_3\Big)\frac{1}{Y-r} + \left(\frac{Bb}{G_\delta} - B_4\right)\frac{1}{Y-\sigma/G_\delta}
\end{aligned} \tag{3.27}$$

***(ii)* Case $\boldsymbol{\alpha = 0}$ and $\boldsymbol{\delta \neq \lambda_A}$.** Here we have

$$\varphi_1(y) = \frac{\theta(h+\beta) + \lambda_B\beta}{(\delta+\theta)(h+\beta) + \lambda_B\beta}\left(1 + \frac{\delta}{\Lambda}\right)\cdot\frac{1}{y}, \tag{3.28}$$

$$\varphi_2(y) = \frac{\lambda_B h\delta}{(\delta+\theta)(h+\beta) + \lambda_B\beta}\cdot\frac{1}{\delta+\theta+\lambda_B}\left(1 + \frac{\delta}{\Lambda}\right)\cdot\frac{1}{Y - \frac{\lambda_B h}{\delta+\theta+\lambda_B}}, \tag{3.29}$$

$$\varphi_3(y) = \frac{\theta(h+\beta) + \lambda_B\beta}{(\delta+\theta)(h+\beta) + \lambda_B\beta}\cdot\frac{c}{h+\beta-r}\left(\frac{1}{y} + \frac{-1}{Y-r}\right), \tag{3.30}$$

$$\varphi_4(y) = \frac{\lambda_B h\delta^2}{(\delta+\theta)(h+\beta) + \lambda_B\beta}\cdot\frac{1}{\Lambda(\delta-\lambda_A)}\left(\frac{-1}{Y - \frac{\lambda_B h}{\delta+\theta+\lambda_B}} + \frac{1}{Y-r}\right). \tag{3.31}$$

Substituting (3.28)–(3.31) into (3.24) we obtain

$$\begin{aligned}
\frac{1}{y}f_1(Y)f_3(Y) &= \frac{\theta(h+\beta) + \lambda_B\beta}{(\delta+\theta)(h+\beta) + \lambda_B\beta}\left(1 + \frac{\delta(h+\beta)}{\Lambda(h+\beta-r)}\right)\frac{1}{y}\\
&+ \frac{-\lambda_A\lambda_B h\delta}{(\delta-\lambda_A)(\delta+\theta+\lambda_B)}\cdot\frac{1}{(\delta+\theta)(h+\beta) + \lambda_B\beta}\cdot\frac{1}{Y - \frac{\lambda_B h}{\delta+\theta+\lambda_B}}\\
&+ \frac{\lambda_A\lambda_B h\delta}{\Lambda(\delta-\lambda_A)}\cdot\frac{1}{\Lambda(h+\beta) - \lambda_B h}\cdot\frac{1}{Y-r}.
\end{aligned} \tag{3.32}$$

***(iii)* Case $\alpha = 0$ and $\delta = \lambda_A$.** Here we have

$$\varphi_1(y) = \frac{\theta(h+\beta) + \lambda_B\beta}{\Lambda(h+\beta) - \lambda_B h}\left(1 + \frac{\lambda_A}{\Lambda}\right) \cdot \frac{1}{y}, \tag{3.33}$$

$$\varphi_2(y) = \frac{\lambda_A\lambda_B h}{\Lambda(h+\beta) - \lambda_B h} \cdot \frac{1}{\Lambda}\left(1 + \frac{\lambda_A}{\Lambda}\right) \cdot \frac{1}{Y-r}, \tag{3.34}$$

$$\varphi_3(y) = \frac{\theta(h+\beta) + \lambda_B\beta}{(h+\beta-r)^2} \cdot \frac{\lambda_A\lambda_B h}{\Lambda^3}\left(\frac{1}{y} + \frac{-1}{Y-r}\right), \tag{3.35}$$

$$\varphi_4(y) = \frac{\lambda_A\lambda_B^2 h^2\delta}{\Lambda(h+\beta) - \lambda_B h} \cdot \frac{1}{\Lambda^3}\left(\frac{1}{Y-r}\right)^2. \tag{3.36}$$

Substituting (3.33)–(3.36) into (3.24) we finally have

$$\begin{aligned}\frac{1}{y}f_1(Y)f_3(Y) &= \frac{\theta(h+\beta) + \lambda_B\beta}{\Lambda(h+\beta) - \lambda_B h}\left(1 + \frac{\lambda_A(h+\beta)}{\Lambda(h+\beta) - \lambda_B h}\right)\frac{1}{y} \\ &+ \frac{\lambda_A^2\lambda_B h}{\Lambda^2} \cdot \frac{2\Lambda(h+\beta) - \lambda_B h}{[\Lambda(h+\beta) - \lambda_B h]^2} \cdot \frac{1}{Y-r} \\ &+ \frac{1}{\Lambda^3} \cdot \frac{\lambda_A^2\lambda_B^2 h^2}{\Lambda(h+\beta) - \lambda_B h}\left(\frac{1}{Y-r}\right)^2.\end{aligned} \tag{3.37}$$

Now turn to the factor $\frac{1}{y}f_1(Y)\frac{\xi}{a}$ in the second term of (3.20). We begin with evaluation of $\frac{\xi}{a}$ substituting (3.13) and (3.14),

$$\frac{\xi}{a} = \eta \cdot \frac{Y^2}{Y-r} \cdot \frac{1}{Y-R}, \tag{3.38}$$

where

$$\eta := \frac{\lambda_A g\delta}{\Lambda\big[\Lambda(g+\alpha) - \lambda_A g\big]} = \frac{\lambda_A g\delta}{\Lambda G} \quad \text{and} \quad R := \frac{\lambda_B h(g+\alpha)}{\Lambda(g+\alpha) - \lambda_A g} = \frac{\sigma}{G}. \tag{3.39}$$

Represent the last two factors $\frac{Y^2}{Y-r} \cdot \frac{1}{Y-R}$ of (3.38)–(3.39) as

$$\frac{Y^2}{Y-r} \cdot \frac{1}{Y-R} = 1 + \frac{r^2}{(Y-r)(r-R)} - \frac{R^2}{(Y-R)(r-R)}. \tag{3.40}$$

With (3.40) substituted in (3.38) we have

$$\frac{\xi}{a} = \eta \cdot \frac{Y^2}{Y-r} \cdot \frac{1}{Y-R} = \eta \cdot \left(1 + \frac{r^2}{(Y-r)(r-R)} - \frac{R^2}{(Y-R)(r-R)}\right). \tag{3.41}$$

Further, substituting (3.22) and (3.41) into the second term $\frac{1}{y}f_1(Y)\frac{\xi}{a}$ of (3.20), we arrive at

$$\begin{aligned}\frac{1}{y}f_1(Y)\frac{\xi}{a} = \eta \cdot \left(\frac{A}{y} + \frac{B/G_\delta}{Y-\sigma/G_\delta}\right) + \alpha_1 \cdot \frac{1}{y} \cdot \frac{1}{Y-r} + \alpha_2 \cdot \frac{1}{Y-\sigma/G_\delta} \cdot \frac{1}{Y-r} \\ + \alpha_3 \cdot \frac{1}{y} \cdot \frac{1}{Y-R} + \alpha_4 \cdot \frac{1}{Y-\sigma/G_\delta} \cdot \frac{1}{Y-R},\end{aligned} \tag{3.42}$$

where

$$\alpha_1 := \frac{Ar^2\eta}{r-R}, \quad \alpha_2 := \frac{Br^2\eta}{G_\delta(r-R)}, \quad \alpha_3 := -\frac{AR^2\eta}{r-R}, \quad \alpha_4 := -\frac{BR^2\eta}{G_\delta(r-R)}. \tag{3.43}$$

Continuing with calculations, after a decomposition and some algebra, we arrive at:

(*i*) Case $\alpha \neq 0$.

$$\frac{1}{y}f_1(Y)\frac{\xi}{a} = a_1 \cdot \frac{1}{y} + a_2 \cdot \frac{1}{Y-\sigma/G_\delta} + a_3 \cdot \frac{1}{Y-R} + a_4 \cdot \frac{1}{Y-r}, \tag{3.44}$$

where

$$a_1 = \eta A + \frac{\alpha_1}{h+\beta-r} + \frac{\alpha_3}{h+\beta-R}, \tag{3.45}$$

$$a_2 = \frac{\eta B}{G_\delta} + \frac{\alpha_2}{\sigma/G_\delta - r} + \frac{\alpha_4}{\sigma/G_\delta - R}, \tag{3.46}$$

$$a_3 = \frac{-\alpha_3}{h+\beta-R} + \frac{-\alpha_4}{\sigma/G_\delta - R}, \tag{3.47}$$

$$a_4 = \frac{-\alpha_1}{h+\beta-r} + \frac{-\alpha_2}{\sigma/G_\delta - r}. \tag{3.48}$$

(*ii*) Case $\alpha = 0$ and $\delta \neq \lambda_A$.

$$\begin{aligned}\frac{1}{y}f_1(Y)\frac{\xi}{a} &= \frac{\lambda_A\delta(h+\beta)^2}{\Lambda(h+\beta)-\lambda_B h} \cdot \frac{1}{(\delta+\theta)(h+\beta)+\lambda_B\beta} \cdot \frac{1}{y} \\ &+ \frac{-\lambda_A\lambda_B h\delta}{\Lambda(\delta-\lambda_A)} \cdot \frac{1}{\Lambda(h+\beta)-\lambda_B h} \cdot \frac{1}{Y-r} \\ &+ \frac{\lambda_A\lambda_B h\delta}{(\delta-\lambda_A)(\delta+\theta+\lambda_B)} \cdot \frac{1}{(\delta+\theta)(h+\beta)+\lambda_B\beta} \cdot \frac{1}{Y-\frac{\lambda_B h}{\delta+\theta+\lambda_B}}.\end{aligned} \tag{3.49}$$

(*iii*) Case $\alpha = 0$ and $\delta = \lambda_A$.

$$\begin{aligned}\frac{1}{y}f_1(Y)\frac{\xi}{a} = \frac{\lambda_A^2(h+\beta)^2}{[\Lambda(h+\beta)-\lambda_B h]^2} \cdot \frac{1}{y} + \frac{-\lambda_A^2\lambda_B h}{\Lambda^2} \cdot \frac{2\Lambda(h+\beta)-\lambda_B h}{[\Lambda(h+\beta)-\lambda_B h]^2} \cdot \frac{1}{Y-r} \\ + \frac{-\lambda_A^2\lambda_B^2 h^2}{\Lambda^3} \cdot \frac{1}{\Lambda(h+\beta)-\lambda_B h}\left(\frac{1}{Y-r}\right)^2\end{aligned} \tag{3.50}$$

With (3.32) and (3.44) substituted into (3.20) we have

(*i*) Case $\alpha \neq 0$.

$$\begin{aligned}\mathcal{L}\mathcal{C}_{xy}^{-1}\left(\Gamma_0\frac{1-\Gamma^1}{1-\Gamma}\right)(p,q) &= \mathfrak{L}_y^{-1}\Bigg\{\psi \cdot e^{-yY_0}\Bigg[\Big(Ab + A_3 + a_1\Big)\frac{1}{y} \\ &+ \left(\frac{Bb}{G_\delta} - B_4 + a_2\right)\frac{1}{Y-\sigma/G_\delta} + a_3 \cdot \frac{1}{Y-R} + \Big(B_4 - A_3 + a_4\Big)\frac{1}{Y-r} \\ &- e^{-a(p-X_0)}\left(a_1 \cdot \frac{1}{y} + a_2 \cdot \frac{1}{Y-\sigma/G_\delta} + a_3 \cdot \frac{1}{Y-R} + a_4 \cdot \frac{1}{Y-r}\right)\Bigg] \\ &\times \mathbf{1}_{(X_0,\infty)}(p)\Bigg\}(q).\end{aligned} \tag{3.51}$$

Correspondingly, we modify the above components in (3.51). After some algebra in (3.26) and (3.45) and the use of notation (3.16), (3.23), (3.39), and (3.43) we arrive at

$$A_3 = \frac{Ac}{h+\beta-r} = \frac{A}{h+\beta-r}\cdot\frac{\lambda_B h\delta}{\Lambda^2} = \frac{A}{h+\beta-r}\cdot\frac{r\delta}{\Lambda}, \tag{3.52}$$

$$a_1 = \eta A + \frac{\alpha_1}{h+\beta-r} + \frac{\alpha_3}{h+\beta-R} = \frac{\lambda_A g\delta(h+\beta)^2}{\Lambda(h+\beta-r)[\,G_\delta(h+\beta)-\sigma]}. \tag{3.53}$$

With (3.16) and (3.52)–(3.53) substituted into $Ab + A_3 + a_1$ we finally have

$$Ab + A_3 + a_1 = A(1+\frac{\delta}{\Lambda}) + \frac{A}{h+\beta-r}\cdot\frac{r\delta}{\Lambda} + \frac{\lambda_A g\delta(h+\beta)^2}{\Lambda(h+\beta-r)[\,G_\delta(h+\beta)-\sigma]} = 1. \tag{3.54}$$

We continue calculating $\frac{Bb}{G_\delta} - B_4 + a_2$ in (3.51). After some algebra we arrive at

$$\begin{aligned}\mathcal{LC}_{xy}^{-1}\Big(\mathit{\Gamma}_0\frac{1-\mathit{\Gamma}^1}{1-\mathit{\Gamma}}\Big)(p,q) &= \mathfrak{L}_y^{-1}\Big\{\psi\cdot e^{-yY_0}\Big[\frac{1}{y} - e^{-a(p-X_0)}\Big(a_1\cdot\frac{1}{y} \\ &+ a_2\cdot\frac{1}{Y-\sigma/G_\delta} + a_4\cdot\frac{1}{Y-r}\Big)\Big]\times\mathbf{1}_{(X_0,\infty)}(p)\Big\}(q),\end{aligned} \tag{3.55}$$

where

$$a_1 = \frac{\lambda_A g\delta(h+\beta)^2}{\Lambda(h+\beta-r)[\,G_\delta(h+\beta)-\sigma]},\quad a_2 = \frac{\lambda_A gB}{G_\delta G'_\delta}, \tag{3.56}$$

$$a_4 = \frac{-Ar^2\lambda_A g\delta}{\Lambda G(r-R)(h+\beta-r)} + \frac{Br\lambda_A g\delta}{\Lambda G G'_\delta(r-R)}. \tag{3.57}$$

***(ii)* Case $\boldsymbol{\alpha=0}$ and $\boldsymbol{\delta\neq\lambda_A}$.** Substituting (3.32) and (3.49) into (3.20), we have

$$\begin{aligned}&\mathcal{LC}_{xy}^{-1}\big(\mathit{\Gamma}_0\frac{1-\mathit{\Gamma}^1}{1-\mathit{\Gamma}}\big)(p,q)\\ &= \mathfrak{L}_y^{-1}\Big\{\psi\cdot e^{-yY_0}\Big[\frac{1}{y} + \Big(\frac{-\lambda_A\delta(h+\beta)^2}{\Lambda(h+\beta)-\lambda_B h}\cdot\frac{1}{(\delta+\theta)(h+\beta)+\lambda_B\beta}\cdot\frac{1}{y}\\ &+\frac{\lambda_A\lambda_B h\delta}{\Lambda(\delta-\lambda_A)}\cdot\frac{1}{\Lambda(h+\beta)-\lambda_B h}\cdot\frac{1}{Y-r}\\ &+\frac{-\lambda_A\lambda_B h\delta}{(\delta-\lambda_A)(\delta+\theta+\lambda_B)}\cdot\frac{1}{(\delta+\theta)(h+\beta)+\lambda_B\beta}\cdot\frac{1}{Y-\frac{\lambda_B h}{\delta+\theta+\lambda_B}}\Big)e^{-a(p-X_0)}\Big]\\ &\times\mathbf{1}_{(X_0,\infty)}(p)\Big\}(q).\end{aligned} \tag{3.58}$$

***(iii)* Case $\boldsymbol{\alpha=0}$ and $\boldsymbol{\delta=\lambda_A}$.** Substituting (3.37) and (3.50) into (3.20), we get

$$\begin{aligned}&\mathcal{LC}_{xy}^{-1}\big(\mathit{\Gamma}_0\frac{1-\mathit{\Gamma}^1}{1-\mathit{\Gamma}}\big)(p,q)\\ &= \mathfrak{L}_y^{-1}\Big\{\psi\cdot e^{-yY_0}\Big[\frac{1}{y} + \Big(\frac{-\lambda_A^2(h+\beta)^2}{[\Lambda(h+\beta)-\lambda_B h]^2}\cdot\frac{1}{y} + \frac{\lambda_A^2\lambda_B h}{\Lambda^2}\cdot\frac{2\Lambda(h+\beta)-\lambda_B h}{[\Lambda(h+\beta)-\lambda_B h]^2}\cdot\frac{1}{Y-r}\\ &+\frac{\lambda_A^2\lambda_B^2h^2}{\Lambda^3}\cdot\frac{1}{\Lambda(h+\beta)-\lambda_B h}\big(\frac{1}{Y-r}\big)^2\Big)e^{-a(p-X_0)}\Big]\mathbf{1}_{(X_0,\infty)}(p)\Big\}(q).\end{aligned} \tag{3.59}$$

Now we need to handle $e^{-a(p-X_0)}$ in (3.59) before inversely transforming the rest of the terms. Unfold (3.14) by using notation (3.16), we have

$$a = g + \alpha - \frac{\lambda_A g}{\Lambda} \cdot \frac{Y}{Y - \frac{\lambda_B h}{\Lambda}} = \xi_1 - \xi_2 \cdot \frac{1}{Y - r}, \tag{3.60}$$

where

$$\xi_1 = g + \alpha - \frac{\lambda_A g}{\Lambda}, \quad \xi_2 = \frac{\lambda_A \lambda_B h g}{\Lambda^2}. \tag{3.61}$$

We finally have

$$e^{-a(p-X_0)} = e^{-(\xi_1 - \xi_2 \cdot \frac{1}{Y-r})(p-X_0)} = e^{-\xi_1(p-X_0)} e^{\xi_2 (p-X_0) \cdot \frac{1}{Y-r}}, \tag{3.62}$$

where $\xi_2(p - X_0)$ is positive (if $p > X_0$) since $\xi_2 > 0$.

Now, we will apply the univariate Laplace-Carson inverse in y to (3.55), (3.58), and (3.59). We will make use of the following formulas for the Laplace inverse (cf. [2,3]):

$$\mathcal{L}_y^{-1}\left(e^{-\alpha y} \cdot \frac{1}{y+b}\right)(q) = e^{-b(q-\alpha)} \mathbf{1}_{(\alpha,\infty)}(q), \tag{3.63}$$

$$\mathcal{L}_y^{-1}\left(e^{-\alpha y} \cdot \frac{e^{\frac{a}{y+b}}}{y+b}\right)(q) = e^{-b(q-\alpha)} I_0\big(2\sqrt{a(q-\alpha)}\big)\mathbf{1}_{(\alpha,\infty)}(q), \tag{3.64}$$

$$\begin{aligned} \mathcal{L}_y^{-1}\left(e^{-\alpha y} \cdot \frac{e^{\frac{a}{y+b_1}}}{y+b_2}\right)(q) &= e^{-b_1(q-\alpha)} I_0\big(2\sqrt{a(q-\alpha)}\big)\mathbf{1}_{(\alpha,\infty)}(q) \\ &+ (b_1 - b_2) \cdot e^{-b_2(q-\alpha)} \int_{z=0}^{q-\alpha} e^{(b_2-b_1)z} I_0\big(2\sqrt{az}\big) dz \mathbf{1}_{(\alpha,\infty)}(q), \end{aligned} \tag{3.65}$$

$$\mathcal{L}_y^{-1}\left(e^{-\alpha y} \cdot \frac{e^{\frac{a}{y+b}}}{(y+b)^2}\right)(q) = \sqrt{\frac{q-\alpha}{a}} \cdot e^{-b(q-\alpha)} I_1\big(2\sqrt{a(q-\alpha)}\big)\mathbf{1}_{(\alpha,\infty)}(q), \tag{3.66}$$

where I_0 and I_1 are the modified Bessel functions of order zero and one, respectively. Equation (3.65) can be readily proved, while the rest of the above formulas can be found in references [2,3].

***(i)* Case $\boldsymbol{\alpha \neq 0}$.** Using (3.63)–(3.66) in (3.55), then combining it with (3.5) we finally have

$$\begin{aligned} &\mathcal{LC}_{xy}^{-1}\left(\Gamma_0^1 - \Gamma_0 \frac{1-\Gamma^1}{1-\Gamma}\right)(p,q) \\ &= \psi\bigg\{(a_1 + a_2 + a_4) e^{-\xi_1(p-X_0)} e^{-(h+\beta-r)(q-Y_0)} I_0\big(2\sqrt{\xi_2(p-X_0)(q-Y_0)}\big) \\ &\quad + a_1(h+\beta-r) e^{-\xi_1(p-X_0)} \int_{z=0}^{q-Y_0} e^{-(h+\beta-r)z} I_0\big(2\sqrt{\xi_2(p-X_0)z}\big) dz \\ &\quad + a_2\left(\frac{\sigma}{G_\delta} - r\right) e^{-\xi_1(p-X_0)} e^{-(h+\beta-\frac{\sigma}{G_\delta})(q-Y_0)} \\ &\quad \times \int_{z=0}^{q-Y_0} e^{(r-\frac{\sigma}{G_\delta})z} I_0\big(2\sqrt{\xi_2(p-X_0)z}\big) dz\bigg\} \mathbf{1}_{(X_0,\infty)}(p) \mathbf{1}_{(Y_0,\infty)}(q). \end{aligned} \tag{3.67}$$

Calculating $a_1 + a_2 + a_4$ and other terms we arrive at

$$
\begin{aligned}
&\mathcal{LC}_{xy}^{-1}\big(\mathit{\Gamma}_0^1 - \mathit{\Gamma}_0 \frac{1-\mathit{\Gamma}^1}{1-\mathit{\Gamma}}\big)(p,q) \\
&\quad = \psi \Big\{ \frac{\lambda_A g \delta}{\Lambda G_\delta} \cdot e^{-\xi_1(p-X_0)} e^{-(h+\beta-r)(q-Y_0)} I_0\big(2\sqrt{\xi_2(p-X_0)(q-Y_0)}\big) \\
&\qquad\quad + \frac{\lambda_A g \delta (h+\beta)^2}{\Lambda[\,G_\delta(h+\beta)-\sigma]} \cdot e^{-\xi_1(p-X_0)} \int_{z=0}^{q-Y_0} e^{-(h+\beta-r)z} I_0\big(2\sqrt{\xi_2(p-X_0)z}\big)dz \\
&\qquad\quad + \frac{-\lambda_A g \delta \sigma^2}{\Lambda G_\delta^2[G_\delta(h+\beta)-\sigma]} \cdot e^{-\xi_1(p-X_0)} e^{-(h+\beta-\frac{\sigma}{G_\delta})(q-Y_0)} \\
&\qquad\quad \times \int_{z=0}^{q-Y_0} e^{(r-\frac{\sigma}{G_\delta})z} I_0\big(2\sqrt{\xi_2(p-X_0)z}\big)dz \Big\} \mathbf{1}_{(X_0,\infty)}(p) \mathbf{1}_{(Y_0,\infty)}(q).
\end{aligned}
\tag{3.68}
$$

***(ii)* Case $\alpha = 0$ and $\delta \neq \lambda_A$.** Using (3.63)–(3.66) in (3.58) and then (3.5) we have

$$
\begin{aligned}
&\mathcal{LC}_{xy}^{-1}\big(\mathit{\Gamma}_0^1 - \mathit{\Gamma}_0 \frac{1-\mathit{\Gamma}^1}{1-\mathit{\Gamma}}\big)(p,q) \\
&\quad = e^{-\beta Y_0} \Big\{ \frac{\lambda_A \delta}{\Lambda(\delta+\theta+\lambda_B)} \cdot e^{-\xi_1(p-X_0)} e^{-(h+\beta-r)(q-Y_0)} I_0\big(2\sqrt{\xi_2(p-X_0)(q-Y_0)}\big) \\
&\qquad + \frac{\lambda_A \delta (h+\beta)^2}{\Lambda} \cdot \frac{1}{(\delta+\theta)(h+\beta)+\lambda_B \beta} \cdot e^{-\xi_1(p-X_0)} \\
&\qquad \times \int_{z=0}^{q-Y_0} e^{-(h+\beta-r)z} I_0\big(2\sqrt{\xi_2(p-X_0)z}\big)dz \\
&\qquad + \frac{-\lambda_A \lambda_B^2 h^2 \delta}{\Lambda(\delta+\theta+\lambda_B)^2} \cdot \frac{1}{(\delta+\theta)(h+\beta)+\lambda_B\beta} \cdot e^{-\xi_1(p-X_0)} \cdot e^{-(h+\beta-\frac{\lambda_B h}{\delta+\theta+\lambda_B})(q-Y_0)} \\
&\qquad \times \int_{z=0}^{q-Y_0} e^{(r-\frac{\lambda_B h}{\delta+\theta+\lambda_B})z} I_0\big(2\sqrt{\xi_2(p-X_0)z}\big)dz \Big\} \mathbf{1}_{(X_0,\infty)}(p) \mathbf{1}_{(Y_0,\infty)}(q).
\end{aligned}
\tag{3.69}
$$

***(iii)* Case $\alpha = 0$ and $\delta = \lambda_A$.** Using (3.63)–(3.66) in (3.59) then (3.5) we get

$$
\begin{aligned}
&\mathcal{LC}_{xy}^{-1}\left(\mathit{\Gamma}_0^1 - \mathit{\Gamma}_0 \frac{1-\mathit{\Gamma}^1}{1-\mathit{\Gamma}}\right)(p,q) \\
&\quad = e^{-\beta Y_0} \Big\{ \frac{\lambda_A^2}{\Lambda^2} \cdot e^{-\xi_1(p-X_0)} e^{-(h+\beta-r)(q-Y_0)} I_0\big(2\sqrt{\xi_2(p-X_0)(q-Y_0)}\big) \\
&\qquad + \frac{\lambda_A^2 (h+\beta)^2}{\Lambda} \cdot \frac{1}{\Lambda(h+\beta)-\lambda_B h} \cdot e^{-\xi_1(p-X_0)} \\
&\qquad \times \int_{z=0}^{q-Y_0} e^{-(h+\beta-r)z} I_0\ bigl(2\sqrt{\xi_2(p-X_0)z}\big)dz \\
&\qquad + \frac{-\lambda_A^2 \lambda_B^2 h^2}{\Lambda^3} \cdot \frac{1}{\Lambda(h+\beta)-\lambda_B h} \sqrt{\frac{q-Y_0}{\xi_2(p-X_0)}} \cdot e^{-\xi_1(p-X_0)} e^{-(h+\beta-r)(q-Y_0)} \\
&\qquad \times I_1\big(2\sqrt{\xi_2(p-X_0)(q-Y_0)}\big) \Big\} \mathbf{1}_{(X_0,\infty)}(p) \mathbf{1}_{(Y_0,\infty)}(q),
\end{aligned}
\tag{3.70}
$$

with the abbreviations:

$$\psi = e^{-\alpha X_0 - \beta Y_0}, \quad \Lambda = \theta + \lambda_A + \lambda_B, \quad \sigma = \lambda_B h(g+\alpha), \quad \xi_1 = g + \alpha - \frac{\lambda_A g}{\Lambda}, \tag{3.71}$$

$$\xi_2 = \frac{\lambda_A \lambda_B hg}{\Lambda^2}, \quad r = \frac{\lambda_B h}{\Lambda}, \quad G_\delta = (\delta + \Lambda)(g+\alpha) - \lambda_A g. \tag{3.72}$$

4 Marginal Functionals

Our next goal is to get the marginal transforms. This can be directly obtained from the version of $\Phi_{\mu\nu}(\alpha,\beta,\theta)$ in (3.68)–(3.70).

Case 1. With $\beta = \theta = 0$ we have the marginal Laplace–Stieltjes transform of the amount of casualties to player A at the A's ruin (which is the exit of the game):

$$\begin{aligned}
\Phi_{\mu\nu}(\alpha,0,0) &:= E\Big[e^{-\alpha A_\mu}\mathbf{1}_{\{\mu<\nu\}}\Big] \\
&= \Bigg\{ \frac{\lambda_A g\delta}{(\lambda_A+\lambda_B)(\delta+\lambda_A+\lambda_B)} \cdot \frac{1}{\alpha + \frac{(\delta+\lambda_B)g}{\delta+\lambda_A+\lambda_B}} \cdot e^{-\alpha M} e^{-(\frac{\lambda_B g}{\lambda_A+\lambda_B})(M-X_0)} \\
&\quad \times e^{-(\frac{\lambda_A h}{\lambda_A+\lambda_B})(N-Y_0)} I_0\left(2\sqrt{\frac{\lambda_A\lambda_B hg(M-X_0)(N-Y_0)}{(\lambda_A+\lambda_B)^2}}\right) + \frac{\lambda_A hg\delta}{(\lambda_A+\lambda_B)(\delta+\lambda_A)} \\
&\quad \times \frac{1}{\alpha + \frac{g\delta}{\delta+\lambda_A}} \cdot e^{-\alpha M} e^{-(\frac{\lambda_B g}{\lambda_A+\lambda_B})(M-X_0)} \int_{z=0}^{N-Y_0} e^{-(\frac{\lambda_A h}{\lambda_A+\lambda_B})z} \\
&\quad \times I_0\left(2\sqrt{\frac{\lambda_A\lambda_B hg(M-X_0)z}{(\lambda_A+\lambda_B)^2}}\right) dz + \int_{z=0}^{N-Y_0} \Bigg[\frac{-\lambda_A hg\delta}{(\lambda_A+\lambda_B)(\delta+\lambda_A)} \cdot \frac{1}{\alpha+\frac{g\delta}{\delta+\lambda_A}} \\
&\quad + \frac{\lambda_A hg\delta(\delta+\lambda_A+2\lambda_B)}{(\lambda_A+\lambda_B)(\delta+\lambda_A+\lambda_B)^2} \cdot \frac{1}{\alpha+\frac{(\delta+\lambda_B)g}{\delta+\lambda_A+\lambda_B}} \\
&\quad + \frac{\lambda_A^2\lambda_B hg^2\delta}{(\lambda_A+\lambda_B)(\delta+\lambda_A+\lambda_B)^3} \Big(\frac{1}{\alpha+\frac{(\delta+\lambda_B)g}{\delta+\lambda_A+\lambda_B}}\Big)^2 \Bigg] e^{-\alpha M} \\
&\quad \times e^{(\frac{\lambda_A\lambda_B hg}{(\delta+\lambda_A+\lambda_B)^2}\cdot\frac{1}{\alpha+\frac{(\delta+\lambda_B)g}{\delta+\lambda_A+\lambda_B}})(N-Y_0-z)} e^{-(\frac{\lambda_B g}{\lambda_A+\lambda_B})(M-X_0)} e^{-(\frac{(\delta+\lambda_A)h}{\delta+\lambda_A+\lambda_B})(N-Y_0)} \\
&\quad \times e^{(\frac{\lambda_B h\delta}{(\lambda_A+\lambda_B)(\delta+\lambda_A+\lambda_B)})z} I_0\Big(2\sqrt{\frac{\lambda_A\lambda_B hg(M-X_0)z}{(\lambda_A+\lambda_B)^2}}\Big) dz \Bigg\} \mathbf{1}_{(X_0,\infty)}(M)\mathbf{1}_{(Y_0,\infty)}(N).
\end{aligned} \tag{4.1}$$

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