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Periodic Solutions for a Class of Superquadratic Damped Vibration Problems

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Abstract: In the present paper, the following damped vibration problems

$$\begin{cases} \ddot{u}(t) + q(t)\dot{u}(t) - L(t)u(t) + \nabla W(t, u(t)) = 0, \\ u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0, \end{cases}$$

are studied, where $T > 0$, $q \in C(\mathcal{R}, \mathcal{R})$ is $T-$periodic with $\int_0^T q(t)dt = 0$, $L(t)$ is a continuous $T-$periodic and symmetric $N \times N$ matrix-valued function and $W \in C^1(\mathcal{R} \times \mathcal{R}^N, \mathcal{R})$ is $T-$periodic in the first variable. We use a new kind of superquadratic condition instead of the global Ambrosetti-Rabinowitz superquadratic condidition and we obtain a nontrivial $T-$periodic solution for the above system. The main idea here lies in the application of a variant of generalized weak linking theorem for strongly indefinite problem developed by Schechter and Zou.

Keywords: *periodic solutions; damped vibration problems; superquadradicity; weak linking theorem.*

Mathematics Subject Classification (2010): 34C25, 34B15.

1 Introduction

Consider the following damped vibration problems

$$(\mathcal{DV}) \qquad \begin{cases} \ddot{u}(t) + q(t)\dot{u}(t) - L(t)u(t) + \nabla W(t, u(t)) = 0, \\ u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0, \end{cases}$$

where $T > 0$, $q : \mathcal{R} \longrightarrow \mathcal{R}$ is a continuous $T-$periodic function with $\int_0^T q(t)dt = 0$, $Q(t) = \int_0^t q(s)ds$, $L(t)$ is a continuous $T-$periodic and symmetric $N \times N$ matrix-valued

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function and $W : \mathcal{R}\times\mathcal{R}^N \longrightarrow \mathcal{R}$ is a continuous function, $T-$periodic in the first variable and differentiable in the second variable with continuous derivative $\nabla W(t,x) = \frac{\partial W}{\partial x}(t,x)$. Equation $(\mathcal{DV})$ is a basic mathematical model for the representation of damped nonlinear oscillatory phenomena.

When $q(t) = 0$ for all $t \in \mathcal{R}$, $(\mathcal{DV})$ is just the following second-order Hamiltonian system

$$(\mathcal{HS}) \qquad \ddot{u}(t) - L(t)u(t) + \nabla W(t,u(t)) = 0,$$

which is a classical equation describing many mechanical systems, such as a pendulum. The system $(\mathcal{HS})$ has been thoroughly studied and a lot of existence results have been obtained, for example see [1-6] and references therein.

As far as the case $q(t) \neq 0$ is concerned, to our best knowledge, there are few research about the existence of periodic solutions for $(\mathcal{DV})$, see [7-9]. Recently, the existence of periodic solutions for $(\mathcal{DV})$ has been studied in [9] when W has a superquadratic growth at infinity satisfying the global Ambrosetti-Rabinowitz superquadratic condition: there exist constants $\mu > 2$ and $R > 0$ such that

$$(\mathcal{AR}) \qquad 0 < \mu W(t,x) \leq \nabla W(t,x).x$$

for all $t \in \mathcal{R}$ and $|x| \geq R$, where $x.y$ denotes the Euclidean inner product of $x, y \in \mathcal{R}^N$ and $|.|$ denotes the corresponding Euclidean norm. Our paper is motivated by the following reason: when dealing with superlinear differential equations, one often meets functionals which do not satisfy $(\mathcal{AR})$-condition. Without $(\mathcal{AR})$-condition, we do not know whether a Palais-Smale sequence is bounded. In the present paper, we shall study the existence of periodic solutions for $(\mathcal{DV})$ under a new kind of superquadratic condition given in [10] by Ding and Luan for Schrödinger's equation. Our approach is based on an application of a variant of generalized weak linking theorem for strongly indefinite problem developed by Schechter and Zou [11], where the authors developed the idea of monotonicity tric for strongly indefinite problems; the original idea is due to Struwe [12].

Our main result reads as follows:

Theorem 1.1 *Assume the following assumptions hold:*
$(\mathcal{L})$ *Zero is not an eigenvalue of* $\mathcal{L} = -\frac{d^2}{dt^2} + L(t)$;
(W_1) $\nabla W(t,x) = o(|x|)$ *as* $|x| \longrightarrow 0$, *uniformly on* $t \in [0,T]$;
(W_2) $\frac{W(t,x)}{|x|^2} \longrightarrow +\infty$, *as* $|x| \longrightarrow \infty$, $\forall t \in [0,T]$;
(W_3) $W(t,x) \geq 0$ *and* $\tilde{W}(t,x) = \frac{1}{2}\nabla W(t,x).x - W(t,x) > 0$, $\forall t \in [0,T], x \in \mathcal{R}^N - \{0\}$;
(W_4) *There exist constants* $c, r > 0$ *and* $\sigma > 1$ *such that*

$$\Big(\frac{|\nabla W(t,x)|}{|x|}\Big)^{\sigma} \leq c\tilde{W}(t,x), \ \forall\, |x| \geq r, \ \forall t \in [0,T].$$

Then $(\mathcal{DV})$ *has at least one nontrivial* $T-$ *periodic solution.*

Example 1.1 [10] Let $W(t,x) = a(t)(|x|^{\mu} + (\mu-2)\,|x|^{\mu-\epsilon}\, sin^2(\frac{|x|^{\epsilon}}{\epsilon}))$, where $a : \mathcal{R} \longrightarrow \mathcal{R}^*_+$ is a continuous $T-$ periodic function, $\mu > 2$ and $0 < \epsilon < \mu - 2$. A straigborhood calculation shows that W satisfies the conditions of Theorem 1.1, but does not satisfy the $(\mathcal{AR})$ condition.

2 Abstract Critical Point Theorem [11]

For the existence of periodic solutions for $(\mathcal{DV})$, we appeal to the following abstract critical point theorem. Let E be a Hilbert space with norm $\|.\|$ and have an orthogonal decomposition $E = N \oplus N^{\perp}$, $N \subset E$ is a closed and separable subspace. Since N is separable, there exists a norm $|.|_{\omega}$ that satisfies $|v|_{\omega} \leq \|v\|$ for all $v \in N$ and induces a topology equivalent to the weak topology of N on bounded subset of N. For $u = v + z \in N \oplus N^{\perp}$ with $v \in N$, $z \in N^{\perp}$, we define $|u|_{\omega}^2 = |v|_{\omega}^2 + |z|_{\omega}^2$, then $|u|_{\omega} \leq \|u\|$, $\forall u \in E$. Particularly, if $(u_n = v_n + z_n)$ is $\|.\|$ $-$bounded and $u_n \longrightarrow^{|.|_{\omega}} u$, then $v_n \rightharpoonup v$ weakly in N, $z_n \longrightarrow z$ strongly in $N^{\perp}$, $u_n \rightharpoonup v + z$ weakly in E. Next, let us recall some definitions:
(i) A functional $f : E \longrightarrow \mathcal{R}$ is said to be $|.|_{\omega}$ $-$upper semi-continuous, i.e., $u_n \longrightarrow^{|.|_{\omega}} u$ in E implies $\limsup_{n \longrightarrow \infty} f(u_n) \leq f(u)$.
(ii) Let $f \in C^1(E, \mathcal{R})$. f' is said to be weakly sequentially continuously, i.e., $u_n \longrightarrow u$ in E implies $\lim_{n \longrightarrow \infty} f'(u_n)w = f'(u)w$ for all $w \in E$.
Let $E = E^+ \oplus E^-$, $z_0 \in E^+$ with $\|z_0\| = 1$. Let $N = E^- \oplus \mathcal{R}z_0$ and $E_1^+ = N^{\perp} = (E^- \oplus \mathcal{R}z_0)^{\perp}$. For $R > 0$, let

$$M = \left\{ u = u^- + sz_0 / s \in \mathcal{R}^+, u^- \in E^-, \|u\| < R \right\}$$

with $P_0 = s_0 z_0 \in M$, $s_0 > 0$. We define

$$D = \left\{ u = sz_0 + z^+ / s \in \mathcal{R}, z^+ \in E_1^+, \left\| sz_0 + z^+ \right\| = s_0 \right\}.$$

For $f \in C^1(E, \mathcal{R})$, let Γ be the set of $\gamma : [0,1] \times \bar{M} \longrightarrow E$ satisfying

$$\begin{cases} \gamma \ is \ |.|_{\omega} - continuous, \\ \gamma(0,u) = u \ and \ f(\gamma(s,u)) \leq f(u) \ for \ all \ u \in \bar{M}, \\ for \ any \ (s_0, u_0) \in [0,1] \times \bar{M}, \ there \ is \ a \ |.|_{\omega} - neighborhood \\ U_{(s_0,u_0)} \ s.t. \ \left\{ U - \gamma(s,u) / (t,u) \in U_{(s_0,u_0)} \cap ([0,1] \cap \bar{M}) \right\} \subset E_{fin}, \end{cases}$$

where E_{fin} denotes various finite dimensional subspaces of E, Γ is not empty since $id \in \Gamma$.

Theorem 2.1 *Let (f_λ) be a family of C^1-functionals having the form*

$$f_\lambda(u) = g(u) - \lambda h(u), \ u \in E, \ \lambda \in [1,2].$$

a)
$$h(u) \geq 0, \ \forall u \in E, \ f_1 = f;$$

b)
$$g(u) \longrightarrow +\infty \ or \ h(u) \longrightarrow +\infty \ as \ \|u\| \longrightarrow \infty;$$

c) f_λ *is* $|.|_{\omega} - upper \ semi-continuous,$ $f_\lambda^{'}$ *is weakly sequentially continuous on* E. *Moreover,* f_λ *maps bounded sets into bounded sets;*

d)
$$\sup_{\partial M} f_\lambda < \inf_D f_\lambda, \ \forall \lambda \in [1,2].$$

Then for almost all $\lambda \in [1,2]$, there exists a sequence (u_n) such that

$$\sup_n \|u_n\| < \infty, \ f_\lambda(u_n) \longrightarrow c_\lambda, \ f_\lambda^{'}(u_n) \longrightarrow 0 \ as \ n \longrightarrow \infty,$$

where

$$c_\lambda = \inf_{\gamma \in \Gamma} \sup_{u \in M} f_\lambda(\gamma(1,u)) \in [\inf_D f_\lambda, \sup_{\bar{M}} f].$$

As usual, we say $f \in C^1(E, \mathcal{R})$ satisfies the Palais-Smale condition ((PS) in short) if any sequence $(u_n) \subset E$ for which $(f(u_n))$ is bounded and $f'(u_n) \longrightarrow 0$ as $n \longrightarrow \infty$, possesses a convergent subsequence.

3 Proof of Theorem 1.1

For $1 \leq s < \infty$, let $L^s_Q(0, T; \mathcal{R}^N)$ be the Banach space of measurable functions u defined on $[0, T]$ with values in $\mathcal{R}^N$ satisfying $\int_0^T e^{Q(t)} |u(t)|^s \, dt < \infty$, with the norm

$$\|u\|_{L^s_Q} = (\int_0^T e^{Q(t)} |u(t)|^s \, dt)^{\frac{1}{s}}$$

and $L^\infty_Q(0, T; \mathcal{R}^N)$ denote the Banach space of measurable functions u defined on $[0, T]$ with values in $\mathcal{R}^N$ under the norm

$$\|u\|_{L^\infty_Q} = esssup_{t\in[0,T]} e^{\frac{Q(t)}{2}} |u(t)| \, .$$

The space $L^2_Q(0, T; \mathcal{R}^N)$ provided with the inner product

$$< u, v >_{L^2_Q} = \int_0^T e^{Q(t)} u(t).v(t) dt, \;\; u, v \in L^2_Q(0, T; \mathcal{R}^N)$$

is a Hilbert space. Let E be the space defined by

$$E = \left\{ u \in L^2_Q(0, T; \mathcal{R}^N) : \dot{u} \in L^2_Q(0, T; \mathcal{R}^N), \;\; u(0) = u(T) \right\} .$$

The space E provided with the inner product

$$< u, v >_0 = \int_0^T e^{Q(t)} [u(t).v(t) + \dot{u}(t).\dot{v}(t)] dt, \;\; u, v \in E$$

and the associated norm

$$\|u\|_0 = (\int_0^T e^{Q(t)} [|u(t)|^2 + |\dot{u}(t)|^2] dt)^{\frac{1}{2}}, \;\; u \in E$$

is a Hilbert space. Define an operator $K : E \longrightarrow E$ by

$$< Ku, v >_0 = \int_0^T e^{Q(t)} (I_{N\times N} - L(t)) u(t).v(t) dt$$

for all $u, v \in E$, where $I_{N\times N}$ is the $N \times N$ identity matrix. Then it is easy to check that K is a bounded self-adjoint linear operator. By the assumption $(\mathcal{L})$ and the classical spectral theory, we can decompose E into the orthogonal sum of invariant subspaces for $I - K$: $E = E^- \oplus E^+$, where E^- (respectively E^+) is the subspace of E on which $I - K$ is negative (respectively positive) definite. Here, I denotes the identity operator. Besides, E^- is finite dimensional since K is compact. Furthermore, we introduce on E the equivalent new inner product

$$< u, v > = < (I - K) u^+, v^+ >_0 - < (I - K) u^-, v^- >_0$$

for $u = u^- + u^+$ and $v = v^- + v^+ \in E$ and the equivalent norm $\|.\| =< .,. >^{\frac{1}{2}}$. It is well known that E is compactly embedded in $L^s_Q(0,T;\mathcal{R}^N)$ for all $s \in [1,\infty]$ and as a consequence for all $s \in [1,\infty]$, there exists a constant $\mu_s > 0$ such that

$$\|u\|_{L^s_Q} \leq \mu_s \|u\|, \ \forall u \in E. \tag{3.1}$$

By definition of $< .,. >$, E^- and E^+ we have

$$< (I-K)u, u >_0 = \pm \|u\|^2, \ \forall u \in E^{\pm}.$$

For $(\mathcal{DV})$, we consider the functional $f(u) = \chi(u) - g(u)$ defined on the space E, where χ is the quadratic form

$$\chi(u) = \frac{1}{2}\int_0^T e^{Q(t)}[|\dot{u}(t)|^2 + L(t)u(t).u(t)]dt$$

and

$$g(u) = \int_0^T e^{Q(t)} W(t,u)dt.$$

By the definition of K, the functional f can be rewritten as

$$f(u) = \frac{1}{2} < (I-K)u, u >_0 - g(u) = \frac{1}{2}(\|u^+\|^2 - \|u^-\|^2) - g(u), \ u \in E.$$

By (W_4), for $|x| \geq r$ and $t \in [0,T]$, we have

$$|\nabla W(t,x)|^\sigma \leq c\tilde{W}(t,x)|x|^\sigma \leq \frac{c}{2}|\nabla W(t,x)||x|^{\sigma+1},$$

thus

$$|\nabla W(t,x)| \leq (\frac{c}{2})^{\frac{1}{\sigma-1}} |x|^{p-1},$$

where $p = \frac{2\sigma}{\sigma-1}$. Let $c_1 = \max_{t\in[0,T],|x|\leq r} |\nabla W(t,x)|$, then

$$|\nabla W(t,x)| \leq c_1 + (\frac{c}{2})^{\frac{1}{\sigma-1}} |x|^{p-1}, \ \forall t \in [0,T], \ x \in \mathcal{R}^N. \tag{3.2}$$

By (W_1), for all $\epsilon > 0$, there exists $r_\epsilon > 0$ such that

$$|\nabla W(t,x)| \leq 2\epsilon |x|, \ \forall t \in [0,T], \ |x| \leq r_\epsilon. \tag{3.3}$$

For $|x| \geq r_\epsilon$, we have by (3.2), $|\nabla W(t,x)| \leq pC_\epsilon |x|^{p-1}$, where $C_\epsilon = \frac{1}{p}(\frac{c_1}{r_\epsilon^{p-1}} + (\frac{c}{2})^{\frac{1}{\sigma-1}})$. So

$$|\nabla W(t,x)| \leq 2\epsilon|x| + pC_\epsilon |x|^{p-1}, \ \forall t \in [0,T], \ x \in \mathcal{R}^N. \tag{3.4}$$

Hence, for all $t \in [0,T]$ and $x \in \mathcal{R}^N$

$$W(t,x) = \int_0^1 \nabla W(t,sx).xds \leq \epsilon|x|^2 + C_\epsilon|x|^p, \ \forall t \in [0,T], \ x \in \mathcal{R}^N. \tag{3.5}$$

By Proposition B.37 in [13], the inequality (3.4) implies that the functional g is continuously differentiable on E and for all $u, v \in E$

$$g'(u)v = \int_0^T e^{Q(t)} \nabla W(t,u).vdt.$$

It is easy to see that the quadratic form χ is continuously differentiable and for all $u, v \in E$, we have

$$\chi'(u)v = \int_0^T e^{Q(t)}[\dot{u}.\dot{v} + L(t)u.v]dt.$$

Therefore the functional f is continuously differentiable on E and for all $u, v \in E$

$$f'(u)v = \int_0^T e^{Q(t)}[\dot{u}.\dot{v} + L(t)u.v - \nabla W(t,u).v]dt$$

$$=< u^+, v^+ > - < u^-, v^- > - \int_0^T e^{Q(t)} \nabla W(t,u).vdt.$$

Lemma 3.1 *If u is a $T-$periodic solution of the Euler equation $f'(u) = 0$, then u is a solution of problem $(\mathcal{DV})$.*

Proof. Since $f'(u) = 0$, then for all $v \in E$

$$0 = f'(u)v = \int_0^T e^{Q(t)}\dot{u}.\dot{v}dt + \int_0^T e^{Q(t)}[L(t)u - \nabla W(t,u)].vdt.$$

By the fundamental lemma and remarks in ([14], pages 6,9), we know that $e^Q\dot{u}$ has a weak derivative and

$$\frac{d}{dt}(e^Q\dot{u}) = e^Q(L(t)u - \nabla W(t,u)) \ a.e.\ t \in [0,T], \tag{3.6}$$

$$e^{Q(t)}\dot{u}(t) = \int_0^t e^{Q^{(s)}}[L(s)u(s) - \nabla W(s,u(s))]ds + c \ a.e.\ t \in [0,T], \tag{3.7}$$

$$\int_0^T e^{Q(s)}[L(s)u(s) - \nabla W(s,u(s))]ds = 0, \tag{3.8}$$

where c is a constant. We identify the equivalence class $e^{Q(t)}\dot{u}(t)$ and its continuous representation $\int_0^t e^{Q(s)}[L(s)u(s) - \nabla W(s,u(s))]ds + c$. Thus by (3.7), (3.8) and the existence of $\dot{u}$, one has

$$\dot{u}(0) - \dot{u}(T) = u(0) - u(T) = 0.$$

In order to apply Theorem 2.1, we consider the family of functionals

$$f_\lambda(u) = \frac{1}{2}\left\|u^+\right\|^2 - \lambda(\frac{1}{2}\left\|u^+\right\|^2 + \int_0^T e^{Q(t)}W(t,u)dt),$$

$\lambda \in [1,2]$. It is easy to see that f_λ satisfies conditions $a), b)$ in Theorem 2.1. To verify condition $c)$, let $u_n \longrightarrow^{|.|_\omega} u$, then $u_n^+ \longrightarrow u^+$ and $u_n^- \rightharpoonup u^-$ in E. Taking a subsequence if necessary, we have $u_n \longrightarrow u$ a.e. on $[0,T]$. By (W_3), Fatou's lemma and the weak lower semi-continuity of the norm, we have

$$\limsup_{n\longrightarrow\infty} f_\lambda(u_n) \leq f_\lambda(u),$$

which means that f_λ is $|.|_\omega$ −upper semi-continuous. f'_λ is weakly sequentially continuous on E is due to [15].

To continue the discussion, it remains to verify condition d) in Theorem 2.1.

Lemma 3.2 *Under assumptions* $(\mathcal{L})$, $(W_1)-(W_4)$, *we have*
(i) *There exists* $\rho>0$ *independent of* $\lambda\in[1,2]$ *such that* $m=\inf f_\lambda(S_\rho^+)>0$, *where*

$$S_\rho^+=\left\{u\in E^+/\ \|u\|=\rho\right\}.$$

(ii) *For fixed* $z_0\in E^+$ *with* $\|z_0\|=1$ *and any* $\lambda\in[1,2]$, *there is* $R>\rho>0$ *such that* $\sup f_\lambda(\partial M)\le 0$, *where*

$$M=\left\{u=u^-+sz_0/s\in\mathcal{R}^+,\ u^-\in E^-,\ \|u\|<R\right\}.$$

Proof. (i) By (3.5) and (2.1), for any $u\in E^+$, we have

$$\begin{aligned}f_\lambda(u)&\ge\frac{1}{2}\|u\|^2-\lambda\epsilon\|u\|^2_{L^2_Q}-\lambda C_\epsilon\|u\|^p_{L^p_Q}\\&\ge\frac{1}{2}\|u\|^2-2\epsilon\mu_2^2\|u\|^2-2C_\epsilon\mu_p^p\|u\|^p.\end{aligned}$$

Taking $\epsilon=\frac{1}{8\mu_2^2}$, we get

$$f_\lambda(u)\ge\frac{1}{4}\|u\|^2-2C_\epsilon\mu_p^p\|u\|^p.$$

Since $p>2$, there exists a constant $\rho>0$ independent of $\lambda\in[1,2]$ satisfying $\inf f_\lambda(S_\rho^+)>0$.
(ii) Assume by contradiction that there exists $u_n\in E^-\oplus\mathcal{R}^+z_0$ such that $f_\lambda(u_n)>0$ for all n and $\|u_n\|\longrightarrow\infty$ as $n\longrightarrow\infty$. Let $v_n=\frac{u_n}{\|u_n\|}=s_nz_0+v_n^-$, then

$$0<\frac{f_\lambda(u_n)}{\|u_n\|}=\frac{1}{2}(s_n^2-\lambda\left\|v_n^-\right\|^2)-\lambda\int_0^T e^{Q(t)}\frac{W(t,u_n)}{|u_n|^2}|v_n|^2\,dt.\tag{3.9}$$

It follows from (W_3) that

$$\left\|v_n^-\right\|^2\le\lambda\left\|v_n^-\right\|^2<s_n^2=1-\left\|v_n^-\right\|^2,$$

therefore $\left\|v_n^-\right\|^2\le\frac{1}{\sqrt{2}}$ and $1-\frac{1}{\sqrt{2}}\le s_n\le 1$. Taking a subsequence if necessary, we can assume that $s_n\longrightarrow s\ne 0$, $v_n\rightharpoonup v$ and $v_n\longrightarrow v$ almost everywhere on $[0,T]$. Hence $v=sz_0+v^-\ne 0$, and since $|u_n|\longrightarrow\infty$ almost everywhere on $[0,T]$, it follows from (W_2) and Fatou's lemma that

$$\int_0^T e^{Q(t)}\frac{W(t,u_n)}{|u_n|^2}|v_n|^2\,dt\longrightarrow\infty\ as\ n\longrightarrow\infty$$

which contradicts (3.9). The proof is finished.

Under assumptions $(\mathcal{L})$ and $(W_1)-(W_4)$, we obtain by applying Theorem 2.1, that for all $\lambda\in[1,2]$, there exists a sequence (u_n) such that

$$\sup_n\|u_n\|<\infty,\ f'_\lambda(u_n)=0,\ f_\lambda(u_n)\longrightarrow c_\lambda\in[m,\sup_{\bar M}f].\tag{3.10}$$

Lemma 3.3 *Under assumptions* $(\mathcal{L})$ *and* $(W_1)-(W_4)$, *for all* $\lambda\in[1,2]$, *there exists* $u_\lambda\in E-\{0\}$ *such that*

$$f'_\lambda(u_\lambda)=0,\ f_\lambda(u_\lambda)\le\sup_{\bar M}f.\tag{3.11}$$

Proof. Let (u_n) be the sequence obtained in (3.10), write $u_n = u_n^- + u_n^+$ with $u_n^\pm \in E^\pm$. Since (u_n) is bounded, then (u_n^+) is bounded, so $u_n \rightharpoonup u_\lambda$ and $u_n^+ \rightharpoonup u_\lambda^+$ in E, after going to a subsequence.
We claim that $u_\lambda^+ \neq 0$. If not, then after going to a subsequence, we can assume that $u_n^+ \longrightarrow 0$ in $L^s(\mathcal{R}, \mathcal{R}^N)$ for all $s \in [1, \infty]$ since E is compactly embedded in $L^s(\mathcal{R}, \mathcal{R}^N)$. It follows from inequality (3.4) and Hölder's inequality that

$$0 \leq \int_0^T e^{Q(t)} \left|\nabla W(t,u).u_n^+\right| dt \leq 2\epsilon \int_0^T |u_n| \left|u_n^+\right| dt + \rho C_\epsilon \int_0^T e^{Q(t)} |u_n|^{p-1} \left|u_n^+\right| dt$$

$$\leq 2\epsilon \|u_n\|_{L_Q^2} \left\|u_n^+\right\|_{L_Q^2} + \|u_n\|_{L_Q^p}^{p-1} \left\|u_n^+\right\|_{L_Q^p} \longrightarrow 0$$

as $n \longrightarrow \infty$. Hence by (3.10), we get

$$f_\lambda(u_n) \leq \left\|u_n^+\right\|^2 = f_\lambda^{'}(u_n)u_n^+ + \lambda \int_0^T e^{Q(t)} \nabla W(t,u).u_n^+ dt \longrightarrow 0$$

as $n \longrightarrow \infty$, which contradicts the fact that $f_\lambda(u_n) \geq m > 0$. Therefore $u_\lambda^+ \neq 0$ and thus $u_\lambda \neq 0$. Note that f_λ is weakly sequentially continuous on E, thus

$$f_\lambda^{'}(u_\lambda)w = \lim_{n\longrightarrow\infty} f_\lambda^{'}(u_n)w = 0, \ \forall w \in E,$$

which implies that $f_\lambda^{'}(u_\lambda) = 0$. By (3.10), (W_3) and Fatou's lemma, we have

$$\sup_{\bar{M}} f \geq c_\lambda = \lim_{n\longrightarrow\infty} (f_\lambda(u_n) - \frac{1}{2} f_\lambda^{'}(u_n)u_n)$$

$$= \lim_{n\longrightarrow\infty} \lambda \int_0^T e^{Q(t)} (\frac{1}{2} \nabla W(t,u_n).u_n - W(t,u_n))dt$$

$$\geq \lambda \int_0^T e^{Q(t)} (\frac{1}{2} \nabla W(t,u_\lambda).u_\lambda - W(t,u_\lambda))dt = f_\lambda(u_\lambda).$$

Thus we get $f_\lambda(u_\lambda) \leq \sup_{\bar{M}} f$.

Lemma 3.4 *Assume* $(\mathcal{L})$ *and* $(W_1) - (W_4)$ *hold, then there exist a sequence* (λ_n) *of* $[1,2]$ *converging to* 1 *and a bounded sequence* (u_{λ_n}) *on* E *such that*

$$f_{\lambda_n}^{'}(u_{\lambda_n}) = 0, \ f_{\lambda_n}(u_{\lambda_n}) \leq \sup_{\bar{M}} f.$$

Proof. Let $(\lambda_n) \subset [1,2]$ be a sequence such that $\lambda_n \longrightarrow 1$. By Lemma 3.3, there exists a sequence (u_{λ_n}) such that

$$f_{\lambda_n}^{'}(u_{\lambda_n}) = 0, \ f_{\lambda_n}(u_{\lambda_n}) \leq \sup_{\bar{M}} f.$$

It remains to prove the boundedness of (u_{λ_n}). Arguing by contradiction, suppose that $\|u_{\lambda_n}\| \longrightarrow \infty$ as $n \longrightarrow \infty$. Let $v_{\lambda_n} = \frac{u_{\lambda_n}}{\|u_{\lambda_n}\|}$, then $\|v_{\lambda_n}\| = 1$. By going to a subsequence if necessary, we can assume that $v_{\lambda_n} \rightharpoonup v$ in E and $v_{\lambda_n} \longrightarrow v$ almost everywhere on $[0, T]$. Since $f'_{\lambda_n}(u_{\lambda_n}) = 0$, then for any $w \in E$, we have

$$< u^+_{\lambda_n}, w > -\lambda_n < u^-_{\lambda_n}, w >= \lambda_n \int_0^T e^{Q(t)} \nabla W(t, u_{\lambda_n}).wdt. \tag{3.12}$$

Consequently, (v_{λ_n}) satisfies

$$< v^+_{\lambda_n}, w > -\lambda_n < v^-_{\lambda_n}, w >= \lambda_n \int_0^T e^{Q(t)} \frac{\nabla W(t, u_{\lambda_n}).w}{\|u_{\lambda_n}\|} dt. \tag{3.13}$$

Let $w = v^{\pm}_{\lambda_n}$ in (3.13) respectively. Then we have

$$\left\|v^+_{\lambda_n}\right\|^2 = \lambda_n \int_0^T e^{Q(t)} \frac{\nabla W(t, u_{\lambda_n}).v^+_{\lambda_n}}{\|u_{\lambda_n}\|} dt,$$

$$\left\|v^-_{\lambda_n}\right\|^2 = -\int_0^T e^{Q(t)} \frac{\nabla W(t, u_{\lambda_n}).v^-_{\lambda_n}}{\|u_{\lambda_n}\|} dt.$$

Since $1 = \|v_{\lambda_n}\|^2 = \left\|v^+_{\lambda_n}\right\|^2 + \left\|v^-_{\lambda_n}\right\|^2$, we have

$$1 = \int_0^T e^{Q(t)} \frac{\nabla W(t, u_{\lambda_n}).(\lambda_n v^+_{\lambda_n} - v^-_{\lambda_n})}{\|u_{\lambda_n}\|} dt. \tag{3.14}$$

For $s \geq 0$, let

$$\varphi(s) = \inf \left\{ \tilde{W}(t, x)/t \in [0, T],\ x \in \mathcal{R}^N,\ |x| \geq s \right\}.$$

By (W_3), we have $\varphi(s) > 0$ for all $s > 0$. By (W_3) and (W_4), we have for $t \in [0, T]$ and $|x| \geq r$

$$\tilde{W}(t, x) \geq \frac{1}{c} \Big(\frac{|\nabla W(t, x)|}{|x|} \Big)^{\sigma} \geq \frac{2^{\sigma}}{c} \Big(\frac{|W(t, x)|}{|x|^2} \Big)^{\sigma},$$

so by (W_2) we have $\varphi(s) \longrightarrow +\infty$ as $s \longrightarrow \infty$. For $0 \leq a < b$, let

$$A_n(a, b) = \{ t \in [0, T]/a \leq |u_{\lambda_n}(t)| \leq b \},$$

$$k_{a,b} = \inf \left\{ \frac{\tilde{W}(t, x)}{|x|^2} /t \in [0, T],\ x \in \mathcal{R}^N,\ a \leq |x| \leq b \right\}.$$

Since $W(t, x)$ depends periodically on t, then by (W_3), we have $k_{a,b} > 0$ for $a > 0$ and

$$\tilde{W}(t, u_{\lambda_n}(t)) \geq k_{a,b} |u_{\lambda_n}(t)|^2 \ \ for\ all\ t \in A_n(a, b).$$

Since $f'_{\lambda_n}(u_{\lambda_n}) = 0$ and $f_{\lambda_n}(u_{\lambda_n}) \leq \sup_{\bar{M}} f$, there exists a constant $c_0 > 0$ such that for all $n \in \mathcal{N}$

$$c_0 \geq f_{\lambda_n}(u_{\lambda_n}) - \frac{1}{2} f'_{\lambda_n}(u_{\lambda_n}) u_{\lambda_n} = \int_0^T e^{Q(t)} \tilde{W}(t, u_{\lambda_n}) dt$$

$$= \int_{A_n(0,a)} e^{Q(t)} \tilde{W}(t, u_{\lambda_n}) dt + \int_{A_n(a,b)} e^{Q(t)} \tilde{W}(t, u_{\lambda_n}) dt + \int_{A_n(b,\infty)} e^{Q(t)} \tilde{W}(t, u_{\lambda_n}) dt$$

$$\geq \int_{A_n(0,a)} e^{Q(t)}\tilde{W}(t,u_{\lambda_n})dt + k_{a,b}\int_{A_n(a,b)} e^{Q(t)}\left|u_{\lambda_n}\right|^2 dt + \varphi(b)\int_{A_n(b,\infty)} e^{Q(t)}dt. \quad (3.15)$$

Combining (3.15) with the fact that $\varphi(s) \longrightarrow \infty$ as $s \longrightarrow \infty$, yields

$$\int_{A_n(b,\infty)} e^{Q(t)}dt \longrightarrow 0 \ as\ b \longrightarrow \infty,\ uniformly\ in\ n. \quad (3.16)$$

Let $\gamma \in]p,\infty[$. By Hölder's inequality and (2.1), we have

$$\int_{A_n(b,\infty)} e^{Q(t)}\left|v_{\lambda_n}\right|^p dt \leq (\int_0^T e^{Q(t)}\left|v_{\lambda_n}\right|^\gamma dt)^{\frac{p}{\gamma}}(\int_{A_n(b,\infty)} e^{Q(t)}dt)^{1-\frac{p}{\gamma}}$$

$$\leq \mu_\gamma^p(\int_{A_n(b,\infty)} e^{Q(t)}dt)^{1-\frac{p}{\gamma}} \longrightarrow 0\ as\ b \longrightarrow \infty,\ uniformly\ in\ n. \quad (3.17)$$

By (3.15), we have

$$\int_{A_n(a,b)} e^{Q(t)}\left|v_{\lambda_n}\right|^2 dt = \frac{1}{\left\|u_{\lambda_n}\right\|^2}\int_{A_n(a,b)} e^{Q(t)}\left|u_{\lambda_n}\right|^2 dt \leq \frac{c_0}{k_{a,b}\left\|u_{\lambda_n}\right\|^2} \longrightarrow 0 \quad (3.18)$$

as $n \longrightarrow \infty$.

Let $0 < \epsilon < \frac{1}{3}$. By (W_1) there exists $a_\epsilon > 0$ such that $|\nabla W(t,x)| \leq \frac{\epsilon}{2\mu_2^2}|x|$ for all $|x| \leq a_\epsilon$. Consequently, by Hölder's inequality and (2.1)

$$\int_{A_n(0,a_\epsilon)} e^{Q(t)}\frac{\nabla W(t,u_{\lambda_n}).(\lambda_n v_{\lambda_n}^+ - v_{\lambda_n}^-)}{\|u_{\lambda_n}\|}dt$$

$$\leq \int_{A_n(0,a_\epsilon)} e^{Q(t)}\frac{|\nabla W(t,u_{\lambda_n})|}{|u_{\lambda_n}|}\left|v_{\lambda_n}\right|\left|\lambda_n v_{\lambda_n}^+ - v_{\lambda_n}^-\right|dt$$

$$\leq \frac{\epsilon}{2\mu_2^2}\int_{A_n(0,a_\epsilon)} e^{Q(t)}\left|v_{\lambda_n}\right|\left|\lambda_n v_{\lambda_n}^+ - v_{\lambda_n}^-\right|dt$$

$$\leq \frac{\epsilon}{2\mu_2^2}(\int_{A_n(0,a_\epsilon)} e^{Q(t)}\left|v_{\lambda_n}\right|^2 dt)^{\frac{1}{2}}(\int_{A_n(0,a_\epsilon)} e^{Q(t)}\left|\lambda_n v_{\lambda_n}^+ - v_{\lambda_n}^-\right|^2 dt)^{\frac{1}{2}}$$

$$\leq \frac{\epsilon}{2\mu_2^2}\lambda_n\left\|v_{\lambda_n}\right\|_{L_Q^2}^2 \leq \epsilon,\ \forall n \in \mathcal{N}. \quad (3.19)$$

Now, by Hölder's inequality, (W_4) and (3.17), we can take $b_\epsilon \geq r$ large enough so that

$$\int_{A_n(b_\epsilon,\infty)} e^{Q(t)}\frac{\nabla W(t,u_{\lambda_n}).(\lambda_n v_{\lambda_n}^+ - v_{\lambda_n}^-)}{\|u_{\lambda_n}\|}dt$$

$$\leq \int_{A_n(b_\epsilon,\infty)} e^{Q(t)}\frac{|\nabla W(t,u_{\lambda_n})|}{|u_{\lambda_n}|}\left|v_{\lambda_n}\right|\left|\lambda_n v_{\lambda_n}^+ - v_{\lambda_n}^-\right|dt$$

$$\leq (\int_{A_n(b_\epsilon,\infty)} e^{Q(t)}\Big(\frac{|\nabla W(t,u_{\lambda_n})|}{|u_{\lambda_n}|}\Big)^\sigma dt^{\frac{1}{\sigma}}(\int_{A_n(b_\epsilon,\infty)} e^{Q(t)}(\left|v_{\lambda_n}\right|\left|\lambda_n v_{\lambda_n}^+ - v_{\lambda_n}^-\right|)^{\sigma'}dt)^{\frac{1}{\sigma'}}$$

$$\leq (\int_{A_n(b_\epsilon,\infty)} e^{Q(t)}c\tilde{W}(t,u_{\lambda_n})dt)^{\frac{1}{\sigma}}(\int_{A_n(b_\epsilon,\infty)} e^{Q(t)}\left|v_{\lambda_n}\right|^{2\sigma'}dt)^{\frac{1}{2\sigma'}}$$

$$\cdot(\int_{A_n(b_\epsilon,\infty)} e^{Q(t)} \left|\lambda_n v_{\lambda_n}^+ - v_{\lambda_n}^-\right|^{2\sigma'} dt)^{\frac{1}{2\sigma'}}$$

$$\leq (cc_0)^{\frac{1}{\sigma}}(\int_{A_n(b_\epsilon,\infty)} e^{Q(t)} |v_{\lambda_n}|^p dt)^{\frac{2}{p}} < \epsilon \tag{3.20}$$

for all integer n, where $\frac{1}{\sigma}+\frac{1}{\sigma'}=1$. Since ∇W is continuous, there exists $d=d(\epsilon)$ such that $|\nabla W(t,x)| \leq d|x|$ for all $t\in[0,T]$ and $x\in[a_\epsilon,b_\epsilon]$. So, for all $t\in A_n(a_\epsilon,b_\epsilon)$, we have $|\nabla W(t,u_{\lambda_n})| \leq d|u_{\lambda_n}|$. Hence by Hölder's inequalitty and (3.18), there exists an integer n_0 such that

$$\int_{A_n(a_\epsilon,b_\epsilon)} e^{Q(t)} \frac{\nabla W(t,u_{\lambda_n}).(\lambda_n v_{\lambda_n}^+ - v_{\lambda_n}^-)}{\|u_{\lambda_n}\|} dt$$

$$\leq \int_{A_n(a_\epsilon,b_\epsilon)} e^{Q(t)} \frac{|\nabla W(t,u_{\lambda_n})|}{|u_{\lambda_n}|} |v_{\lambda_n}| \left|\lambda_n v_{\lambda_n}^+ - v_{\lambda_n}^-\right| dt$$

$$\leq d\int_{A_n(a_\epsilon,b_\epsilon)} e^{Q(t)} |v_{\lambda_n}| \left|\lambda_n v_{\lambda_n}^+ - v_{\lambda_n}^-\right| dt$$

$$\leq d(\int_{A_n(a_\epsilon,b_\epsilon)} e^{Q(t)} |v_{\lambda_n}|^2 dt)^{\frac{1}{2}}(\int_{A_n(a_\epsilon,b_\epsilon)} e^{Q(t)} \left|\lambda_n v_{\lambda_n}^+ - v_{\lambda_n}^-\right|^2 dt)^{\frac{1}{2}}$$

$$\leq 2d\int_{A_n(a_\epsilon,b_\epsilon)} e^{Q(t)} |v_{\lambda_n}|^2 dt < \epsilon \tag{3.21}$$

for all integer $n\geq n_0$. Therefore, combining $(3.19)-(3.21)$ yields for $n\geq n_0$

$$\int_0^T e^{Q(t)} \frac{\nabla W(t,u_{\lambda_n}).(\lambda_n v_{\lambda_n}^+ - v_{\lambda_n}^-)}{\|u_{\lambda_n}\|} dt \leq 3\epsilon < 1,$$

which contradicts (3.14). Hence (u_{λ_n}) is bounded.

Lemma 3.5 *Let (u_{λ_n}) be the sequence obtained in Lemma 3.4, then it is a (PS) sequence of f satisfying*

$$\lim_{n\longrightarrow\infty} f'(u_{\lambda_n}) = 0,\ \lim_{n\longrightarrow\infty} f(u_{\lambda_n}) \leq \sup_{\bar{M}} f.$$

Proof. We have

$$\lim_{n\longrightarrow\infty} f(u_{\lambda_n}) = \lim_{n\longrightarrow\infty}[f_{\lambda_n}(u_{\lambda_n}) + (\lambda_n-1)(\frac{1}{2}\left\|u_{\lambda_n}^-\right\|^2 + \int_0^T e^{Q(t)}W(t,u_{\lambda_n})dt)]. \tag{3.22}$$

By (3.5) and (2.1), we have

$$\int_0^T e^{Q(t)}W(t,u_{\lambda_n})dt \leq \epsilon\mu_2^2 \|u_{\lambda_n}\|^2 + C_\epsilon \mu_p^p \|u_{\lambda_n}\|^p. \tag{3.23}$$

It follows from (3.22), (3.23) and the boundedness of (u_{λ_n}) that

$$\lim_{n\longrightarrow\infty} f(u_{\lambda_n}) = \lim_{n\longrightarrow\infty} f_{\lambda_n}(u_{\lambda_n}) \leq \sup_{\bar{M}} f.$$

Similarly, for all $w \in E$, we have

$$\lim_{n \longrightarrow \infty} f'(u_{\lambda_n})w = \lim_{n \longrightarrow \infty}[f'_{\lambda_n}(u_{\lambda_n})w+(\lambda_n-1)(\frac{1}{2} < u^-_{\lambda_n}, w > + \int_0^T e^{Q(t)}\nabla W(t, u_{\lambda_n}).wdt)]$$

$$= \lim_{n \longrightarrow \infty} f'_{\lambda_n}(u_{\lambda_n})w = 0,$$

for all $w \in E$. The proof is complete.

Now, let (u_{λ_n}) be the bounded sequence obtained in Lemma 3.4. Taking a subsequence if necessary, we can assume that $u_{\lambda_n} \rightharpoonup u$ in E and $u_{\lambda_n} \longrightarrow u$ in $L^s_Q(0,T)$ for all $s \in [1, \infty]$ since E is compactly embedded in $L^s_Q(0,T)$. By $f'_{\lambda_n}(u_{\lambda_n}) = 0$, (3.4), Hölder's inequality and (2.1), we obtain

$$\left\|u^+_{\lambda_n}\right\|^2 = \lambda_n \int_0^T e^{Q(t)}\nabla W(t, u_{\lambda_n}).u^+_{\lambda_n}dt$$

$$\begin{aligned}
&\leq 4\epsilon \int_0^T e^{Q(t)}\left|u_{\lambda_n}\right|\left|u^+_{\lambda_n}\right|dt + 2pC_\epsilon \int_0^T e^{Q(t)}\left|u_{\lambda_n}\right|^{p-1}\left|u^+_{\lambda_n}\right|dt \\
&\leq 4\epsilon \left\|u_{\lambda_n}\right\|_{L^2_Q}\left\|u^+_{\lambda_n}\right\|_{L^2_Q} + 2pC_\epsilon \left\|u_{\lambda_n}\right\|^{p-1}_{L^p_Q}\left\|u^+_{\lambda_n}\right\|_{L^p_Q} \\
&\leq 4\epsilon \left\|u_{\lambda_n}\right\|_{L^2_Q}\left\|u^+_{\lambda_n}\right\|_{L^2_Q} + 2pC_\epsilon \left\|u_{\lambda_n}\right\|^{p-1}_{L^p_Q}\left\|u^+_{\lambda_n}\right\|_{L^p_Q} \\
&\leq 4\epsilon\mu_2^2 \left\|u_{\lambda_n}\right\|^2 + 2pC_\epsilon\mu_p^2 \left\|u_{\lambda_n}\right\|^{p-2}_{L^p_Q}\left\|u_{\lambda_n}\right\|^2. \qquad (2.24)
\end{aligned}$$

Similarly, we have

$$\left\|u^-_{\lambda_n}\right\|^2 \leq 4\epsilon\mu_2^2 \left\|u_{\lambda_n}\right\|^2 + 2pC_\epsilon\mu_p^2 \left\|u_{\lambda_n}\right\|^{p-2}_{L^p_Q}\left\|u_{\lambda_n}\right\|^2. \qquad (2.25)$$

Combining (3.24) and (3.25) yields

$$\left\|u_{\lambda_n}\right\|^2 \leq 8\epsilon\mu_2^2 \left\|u_{\lambda_n}\right\|^2 + 4pC_\epsilon\mu_p^2 \left\|u_{\lambda_n}\right\|^{p-2}_{L^p_Q}\left\|u_{\lambda_n}\right\|^2. \qquad (2.26)$$

Combining Lemma 3.3 and (3.26) yields

$$1 - 8\epsilon\mu_2^2 \leq 4pC_\epsilon\mu_p^2 \left\|u_{\lambda_n}\right\|^{p-2}_{L^p_Q}. \qquad (3.27)$$

Taking $\epsilon = \frac{1}{16\mu_p^2}$, we get $\left\|u_{\lambda_n}\right\|^{p-2}_{L^p_Q} \geq (8p\mu_p^2C_\epsilon)^{-1} > 0$, for all n. Since $u_{\lambda_n} \longrightarrow u$ in $L^p_Q([0,T])$ then $u \neq 0$. The fact that f' is weakly sequentially continuous on E and $u_{\lambda_n} \rightharpoonup u$ in E imply $f'(u) = 0$.
Let $K = \{u \in E/f'(u) = 0\}$ be the critical set of f and $m_0 = \inf\{f(u)/u \in K - \{0\}\}$. For any critical point u of f, assumption (W_3) implies that

$$f(u) = f(u) - \frac{1}{2}f'(u)u = \int_0^T e^{Q(t)}[\frac{1}{2}\nabla W(t,u).u - W(t,u)]dt \geq 0.$$

Therefore, $m_0 \geq 0$. Let $(u_j) \subset K - \{0\}$ be such that $f(u_j) \longrightarrow m_0$. Arguing as in the proof of Lemma 3.4, we can prove that (u_j) is bounded and by going to a subsequence if necessary, we can assume that $u_j \rightharpoonup u$ in E and $u_j \longrightarrow u$ almost everywhere on $[0,T]$, and as above $u \neq 0$. Thus by (W_3) and Fatou's lemma

$$m_0 = \lim_{j \longrightarrow \infty} f(u_j) = \lim_{j \longrightarrow \infty} \int_0^T e^{Q(t)} [\frac{1}{2} \nabla W(t,u_j).u_j - W(t,u_j)] dt$$

$$\geq \int_0^T e^{Q(t)} [\frac{1}{2} \nabla W(t,u).u - W(t,u)] dt = f(u) \geq m_0.$$

So $m_0 = f(u)$ and $m_0 > 0$ because $u \neq 0$.

References

[1] Fei, G. On periodic solutions of superquadratic Hamiltonian systems. *Elect. J. Diff. Eq.* **2012** (8) (2012) 1–12.

[2] He X. and Wu, X. Periodic solutions for a class of nonautonomous second order Hamiltonian systems. *J. Math. Anal. Appl.* **341** (2008) 1354–1364.

[3] Li, S. and Willem, M. Applications of local linking to critical point theory. *J. Math. Anal. Appl.* **189** (1995) 6–32.

[4] Long, Y.M. Multiple solutions of perturbed superquadratic second order Hamiltonian systems. *Trans. Amer. Math. Soc.* **311** (1989) 749–780.

[5] Tang, X.H. and Zhang, Q. New existence of periodic solutions for second-order nonautonomous Hamiltonian systems. *J. Math. Anal. Appl.* **369** (2010) 357–367.

[6] Tang, C.L. and Tao, Z.L. Periodic and subharmonic solutions of second-order Hamiltonian systems. *J. Math. Anal. Appl.* **293** (2004) 435–445.

[7] Chen, J. and Wu, X. Existence theorems of periodic solutions for a class of damped vibration problems. *Applied Mathematics and Computation.* **207** (2009) 230–235.

[8] Chen, S., Teng, K. and Wu, X. On variational methods for a class of damped vibration problems. *Nonlinear Analysis.* **68** (2008) 1432–1441.

[9] Wang, S. and Wu, X. On a class of damped vibration problems with obstacles. *Nonlinear Analysis* **11** (2010) 2973–2988.

[10] Ding, Y. and Luan, S. Multiple solutions for a class of nonlinear Schrödinger equations. *J. Differential Equations.* **207** (2004) 423–457.

[11] Schechter, M. and Zou, W. Weak linking theorems and Schrödinger equations with critical Sobolev exponent. *ESAIM: Contol Optim. Calc. Var.* **9** (2003) 601–619.

[12] Struwe, M. Variational Methods, Applications to Nonlinear Partial Differential Equations and Hamiltonian Systems. *Springer-Verlag*, Berlin,(2000).

[13] Rabinowitz, P.H. Minimax methods in critical point theory with applications to differential equations. *CBMS Regional Conf. Ser. in Math.* No. 65 AMs (1986).

[14] Mawhin, J. and Willem, M. Critical point theory and Hamiltonian systems. *Appl. Math. Sci.* Vol. 74, Springer, New York, (1989).

[15] Willem, M. Minimax Theorems. *Birkhäuser,* 1996.

[16] Khachnaoui, K. Existence of Even Homoclinic Solutions for a Class of Dynamical Systems. *Nonlinear Dynamics and Systems Theory* **15** (3) (2015) 298–312.