Nonlinear Dynamics and Systems Theory, **16** (3) (2016) 246–259

Generalized Monotone Method for Multi-Order 2-Systems of Riemann-Liouville Fractional Differential Equations

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Received: August 9, 2015; Revised: June 20, 2016

Abstract: In this paper we develop a generalized monotone method for nonlinear multi-order 2-systems of Riemann-Liouville fractional differential equations. That is, the monotone method where the forcing function f can be decomposed into increasing and decreasing components, and applied to a hybrid system of nonlinear equations of orders q_1 and q_2 where $0 < q_1, q_2 < 1$. In the development of this method we recall any needed existence and comparison results along with any necessary changes; including results from needed linear theory. The monotone method is then developed via the construction of sequences of linear systems based on the upper and lower solutions, being then used to approximate the solution of the original nonlinear multi-order system. Finally we develop a numerical application to exemplify our results.

Keywords: *fractional differential systems; multi-order systems; lower and upper solutions; monotone method.*

Mathematics Subject Classification (2010): 34A08, 34A34, 34A45, 34A38.

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1 Introduction

Fractional differential equations have various applications in widespread fields of science, such as engineering [6], chemistry [7, 14, 15], physics [1, 2, 8], and others [9, 10]. Despite the number of existence theorems for nonlinear fractional differential equations this does not necessarily imply that calculating a solution explicitly will be possible. Therefore, it may be necessary to employ an iterative technique to numerically approximate a needed solution. In this paper we construct such a method.

Specifically, we construct a technique to approximate solutions to the nonlinear Riemann-Liouville (R-L) fractional differential multi-order 2-system. A multi-order system is a fractional differential system where each component is of unique order. That is, a fractional system of the type

$$\begin{aligned} D^{q_1}x_1 &= f_1(t, x_1, x_2), \\ D^{q_2}x_2 &= f_2(t, x_1, x_2). \end{aligned}$$

This is a generalization of normal R-L systems and yields a type of hybrid system of a fractional type. We note that various complications arise from systems of this type as many known properties used in the study of fractional differential equations require modification, but at the same time multi-order systems present far more possibilities for applications. For example, consider allowing each species in a population model to have their own order of derivative. Though we will consider a numerical example for this study, it will not be a specific physical application, we hope this will add to the groundwork of future studies.

The iterative technique we construct will be a generalization of the monotone method for multi-order R-L 2-systems of order q_1, q_2, where $0 < q_1, q_2 < 1$. The monotone method, in broad terms, is a technique in which unique solutions of linear differential equations are used to construct sequences that converge uniformly and monotonically, from above and below, to maximal and minimal solutions of the nonlinear equation. If the nonlinear DE considered has a unique solution then both sequences will converge uniformly and monotonically to that unique solution. The advantage of the monotone method is that it allows us to approximate solutions to nonlinear DEs using linear DEs. Further, the sequences are constructed initially using upper and lower solutions of the original DE, which guarantees the interval of existence. For more information on the monotone method for ordinary DEs see [11].

One notable complication when developing the monotone method for multi-order systems is that, unlike in the integer order case, the initially constructed sequences, $\{v_n\}, \{w_n\}$ do not converge uniformly on their own. Instead, the weighted sequences $\{t^{1-q_i}v_{n_i}\}$, $\{t^{1-q_i}w_{n_i}\}$ converge uniformly to $t^{1-q_i}v_i$ and $t^{1-q_i}w_i$ respectively, where $i \in \{1, 2\}$ and v, w are maximal and minimal solutions of the original equation. We note that there are other complications that derive from multi-order systems, but many of these were previously resolved in [3].

For our main method we consider the generalization of the monotone method where the nonlinear function can be split into two functions $f(t, x)+g(t, x)$ where f is increasing in x and g is decreasing in x. This generalization allows for various constructions utilizing different types of lower and upper solutions that we will detail in Section 3. Finally, in Section 4 we will develop a numerical application to exemplify our results. We note that the standard monotone method has been established for multi-order fractional systems in [3].

2 Preliminary Results

In this section, we will first consider basic results regarding scalar Riemann-Liouville differential equations of order q, $0 < q < 1$. We will recall basic definitions and results in this case for simplicity, and we note that many of these results carry over naturally to the multi-order case. Then we will consider existence and comparison results for multi-order systems of order $0 < q_1, q_2 < 1$ which will be used in our main result. In the next section, we will apply these preliminary results to develop the monotone method for these multi-order R-L systems. Note, for simplicity we only consider results on the interval $J = (0, T]$, where $T > 0$. Further, we will let $J_0 = [0, T]$, that is $J_0 = \bar{J}$.

Definition 2.1 Let $p = 1-q$, a function $\phi(t) \in C(J, \mathbb{R})$ is a C_p continuous function if $t^p\phi(t) \in C(J_0, \mathbb{R})$. The set of C_p functions is denoted $C_p(J, \mathbb{R})$. Further, given a function $\phi(t) \in C_p(J, \mathbb{R})$ we call the function $t^p\phi(t)$ the continuous extension of $\phi(t)$.

Now we define the R-L integral and derivative of order q on the interval J.

Definition 2.2 Let $\phi \in C_p(J, \mathbb{R})$, then $D_t^q\phi(t)$ is the q-th R-L derivative of ϕ with respect to $t \in J$ defined as

$$D_t^q\phi(t) = \frac{1}{\Gamma(1-q)}\frac{d}{dt}\int_0^t (t-s)^{-q}\phi(s)ds,$$

and $I_t^q\phi(t)$ is the q-th R-L integral of ϕ with respect to $t \in J$ defined as

$$I_t^q\phi(t) = \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\phi(s)ds.$$

Note that in cases where the initial value may be different or ambiguous, we will write out the definition explicitly. The next definition is related to the solution of linear R-L fractional differential equations and is also of great importance in the study of the R-L derivative.

Definition 2.3 The Mittag-Leffler function with parameters $\alpha, \beta \in \mathbb{R}$, denoted $E_{\alpha,\beta}$, is defined as

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)},$$

which is entire for $\alpha, \beta > 0$.

Of particular importance to the Riemann-Liouville derivative is the weighted Mittag-Leffler function of order q,

$$\mathcal{E} = t^{q-1}E_{q,q}(\lambda t^q) = \sum_{k=0}^{\infty} \frac{\lambda^k t^{qk+q-1}}{\Gamma(qk+q)},$$

where λ is a constant. $\mathcal{E}$ has the following properties which we present in the following remark.

Remark 2.1 We note that the weighted Mittag-Leffler function $\mathcal{E}$ is strictly positive, converges uniformly on compacta of J, and $D^q\mathcal{E} = \lambda\mathcal{E}$.

The next result gives us that the q-th R-L integral of a C_p continuous function is also a C_p continuous function. This result will give us that the solutions of R-L differential equations are also C_p continuous.

Lemma 2.1 *Let $f \in C_p(J, \mathbb{R})$, then $I_t^q f(t) \in C_p(J, \mathbb{R})$, i.e. the q-th integral of a C_p continuous function is C_p continuous.*

Note the proof of this theorem for $q \in R^+$ can be found in [5]. Now we consider results for the nonhomogeneous linear R-L differential equation,

$$D_t^q x(t) = \lambda x(t) + z(t), \tag{1}$$

with initial condition

$$t^p x(t)\big|_{t=0} = x^0,$$

where x^0 is a constant, $y \in C(J_0, \mathbb{R})$, and $z \in C_p(J, \mathbb{R})$, which has unique solution

$$x(t) = \Gamma(q) x^0 t^{q-1} E_{q,q}(\lambda t^q) + \int_0^t (t-s)^{q-1} E_{q,q}(\lambda (t-s)^q) z(s)\, ds.$$

For more details see [12].

Now, we will turn our attention to results for the nonlinear R-L fractional multi-order systems, and in doing so we must discuss any changes. First, we will consider systems of orders q_1 and q_2, $0 \leq q_1, q_2 < 1$. For simplicity we will let $q = (q_1, q_2)$, and when we write inequalities $x \leq y$, we mean it is true for both components. Further, from this point on, we will use the subscript i which we will always assume is in $\{1, 2\}$. For defining C_p continuity for multi-order systems we define $p_i = 1 - q_i$ and for simplicity of notation we will define the function x_p such that $x_{p_i}(t) = t^{p_i} x_i(t)$ for $t \in J_0$. We also note that at times it will be convenient to ephasize the product of t^p, therefore we will define $t^p x(t) = x_p(t)$ for $t \in J_0$. Now, we define the set of C_p continuous functions as

$$C_p(J, \mathbb{R}^2) = \{x \in C(J, \mathbb{R}^2) \,|\, x_p \in C(J_0, \mathbb{R}^2)\}.$$

For the rest of our results we will be considering the nonlinear R-L fractional multi-order system

$$\begin{aligned} D^{q_i} x_i &= f_i(t, x), \\ x_{p_i}(0) &= x_i^0, \end{aligned} \tag{2}$$

where $f \in C(J_0 \times \mathbb{R}^2, \mathbb{R}^2)$, and x^0 is a constant. Note that just as in the scalar case, a solution $x \in C_p(J, \mathbb{R}^2)$ of (2) also satisfies the equivalent R-L integral equation

$$x_i(t) = x_i^0 t^{q_i - 1} + \frac{1}{\Gamma(q_i)} \int_0^t (t-s)^{q_i - 1} f_i(s, x(s)) ds. \tag{3}$$

Thus, if $f \in C(J_0 \times \mathbb{R}^2, \mathbb{R}^2)$ then (2) is equivalent to (3). See [9, 12] for details.

The following comparison theorem is utilized throughout the construction of the monotone method. This theorem gives conditions for when lower and upper solutions v, w behave in an expected manner, that is $v \leq w$. This theorem is of great importance to the monotone method since it is used to prove that the constructed sequences in the method are actually monotone.

Theorem 2.1 *Let $v, w \in C_p(J, R^2)$ be lower and upper solutions of the nonlinear multiorder 2-system, i.e.*

$$D^{q_i} v_i \le f_i(t, v), \quad v_{p_i}(0) = v_i^0 \le x_i^0, \qquad (4)$$
$$D^{q_i} w_i \ge f_i(t, w), \quad w_{p_i}(0) = w_i^0 \ge x_i^0.$$

If f is quasimonotone nondecreasing and satisfies the following Lipschitz condition for $i = 1, 2$,

$$f_i(t, x) - f_i(t, y) \le L_i\big[(x_1 - y_1) + (x_2 - y_2)\big], \qquad (5)$$

for $x \ge y$, then $v(t) \le w(t)$ on J provided $v^0 \le w^0$.

We note that the proof of this theorem can be found in [3]. In the development of the monotone methods we will use a specific corollary from this theorem, which we give below.

Corollary 2.1 *Let $m \in C_p(J, \mathbb{R}^2)$ be such that*

$$D^{q_i} m_i(t) \le 0, \quad m_{p_i}(0) = 0.$$

Then we have from Theorem 2.1 that

$$m(t) \le 0,$$

for $t \in J$.

Now, if we know of the existence of lower and upper solutions v and w such that $v \le w$, we can prove the existence of a solution in the set

$$\Omega = \{(t, y) : v(t) \le y \le w(t),\, t \in J\}.$$

We consider this result in the following theorem.

Theorem 2.2 *Let $v, w \in C_p(J, \mathbb{R}^2)$ be lower and upper solutions of (2) such that $v(t) \le w(t)$ on J and let $f \in C(\Omega, \mathbb{R})$, where Ω is defined as above. Then there exists a solution $x \in C_p(J, \mathbb{R}^2)$ of (2) such that $v(t) \le x(t) \le w(t)$ on J.*

This theorem is proved in the same way as seen in [5], with only minor additions to apply it to multi-order 2-systems.

For our main results we will be considering the following generalized form of (2)

$$D^{q_i} x_i = f_i(t, x) + g_i(t, x), \quad x_{p_i}(0) = x_i^0, \qquad (6)$$

where $f, g \in C(J_0 \times \mathbb{R}^2, \mathbb{R}^2)$ such that f is increasing in x and g is decreasing in x. We will be constructing the generalized monotone methods for this nonlinear fractional differential equation. This generalization also allows us to consider various different types of lower and upper solutions given in the following definition.

Definition 2.4 Let $v, w \in C_p(J, \mathbb{R}^2)$ with $v_{p_i}(0) = v_i^0 \le x_i^0$ and $w_{p_i}(0) = w_i^0 \ge x_i^0$.

- v, w are natural lower and upper solutions of (6) if

$$D^{q_i} v_i \le f_i(t, v) + g_i(t, v), \quad D^{q_i} w_i \ge f_i(t, w) + g_i(t, w).$$

- v, w are Type I lower and upper solutions of (6) if

$$D^{q_i} v_i \leq f_i(t, v) + g_i(t, w), \quad D^{q_i} w_i \geq f_i(t, w) + g_i(t, v).$$

- v, w are Type II lower and upper solutions of (6) if

$$D^{q_i} v_i \leq f_i(t, w) + g_i(t, v), \quad D^{q_i} w_i \geq f_i(t, v) + g_i(t, w).$$

- v, w are unnatural lower and upper solutions of (6) if

$$D^{q_i} v_i \leq f_i(t, w) + g_i(t, w), \quad D^{q_i} w_i \geq f_i(t, v) + g_i(t, v).$$

Further we can define coupled quasisolutions of these types by incorporating equalities in the previous expressions. We give the two we use in our main results in the following definition.

Definition 2.5 Let $v, w \in C_p(J, \mathbb{R}^2)$ with $v_{p_i}(0) = w_{p_i}(0) = x_i^0$.

- v, w are Type I coupled quasisolutions of (6) if

$$D^{q_i} v_i = f_i(t, v) + g_i(t, w), \quad D^{q_i} w_i = f_i(t, w) + g_i(t, v).$$

- v, w are Type II coupled quasisolutions of (6) if

$$D^{q_i} v_i = f_i(t, w) + g_i(t, v), \quad D^{q_i} w_i = f_i(t, v) + g_i(t, w).$$

We can extend Theorem 2.2 to incorporate these coupled types of lower and upper solutions. We will only look at the cases for Type I and II since those will be the form we use in our monotone method constructions. We note that the proof of the following theorem is constructed in the same manner as Theorem 2.2, needing only very minor alterations.

Theorem 2.3 *Let $v, w \in C_p(J, \mathbb{R}^2)$ be Type I or Type II coupled lower and upper solutions such that $v(t) \leq w(t)$ on J and let $f + g \in C(\Omega, \mathbb{R})$, where Ω is defined as above. Then there exists a solution $x \in C_p(J, \mathbb{R}^2)$ of (6) such that $v(t) \leq x(t) \leq w(t)$ on J.*

3 Monotone Method

In this section we develop the generalized monotone method for fractional system (6). The first method we will construct is developed from Type I lower and upper solutions. The sequences are constructed as linear equations in a recursive manner resembling Type I quasisolutions.

Theorem 3.1 *Suppose that*

(A1) *$v_0, w_0 \in C_p(J, \mathbb{R}^2)$ are coupled lower and upper solutions of Type I for (6) with $v_0 \leq w_0$ on J.*

(A2) *$f, g \in C(J_0 \times \mathbb{R}^2, \mathbb{R}^2)$, where $f(t, x)$ is increasing in x and $g(t, x)$ is decreasing in x.*

Then the sequences defined by

$$D^{q_i} v_{n+1_i} = f_i(t, v_n) + g_i(t, w_n), \quad v_{n+1_{p_i}}(0) = x_i^0, \tag{7}$$

$$D^{q_i} w_{n+1_i} = f_i(t, w_n) + g_i(t, v_n), \quad w_{n+1_{p_i}}(0) = x_i^0, \tag{8}$$

are such that

$$t^p v_n \to t^p v, \quad t^p w_n \to t^p w$$

uniformly and monotonically on J_0, where v, w are Type I coupled minimal and maximal quasisolutions of (6) respectively, that is, if x is a solution of (6) such that that $v_0 \le x \le w_0$, then $v \le x \le w$.

Proof. We begin by considering v_1 and w_1. We note that both exist and are unique since both are linear in v_1 and w_1 respectively. Now letting $m = v_0 - v_1$, we get that $m_{p_i}(0) = 0$ and

$$D^{q_i} m_i \le 0,$$

implying by Corollary 2.1 that $m_i \le 0$ for each i. Therefore $v_0 \le v_1$, and similarly we can show that $w_1 \le w_0$. Now using a similar process by letting $m = v_1 - w_1$, we get that $m_{p_i}(0) = 0$ and

$$D^{q_i} m_i = f_i(t, v_0) - f_i(t, w_0) + g_i(t, w_0) - g_i(t, v_0) \le 0.$$

Thus, by Corollary 2.1 we have that $m_i \le 0$ for each i, giving us that $v_0 \le v_1 \le w_1 \le w_0$. Using these same arguments we can inductively show that

$$v_{n-1} \le v_n \le w_n \le w_{n-1}$$

on J for all $n \ge 1$, giving us that $\{v_n\}$ and $\{w_n\}$ are monotonic.

Now we will show that the weighted sequences $\{t^p v_n\}$ and $\{t^p w_n\}$ converge uniformly on J_0. To do so we will use the Arzela-Ascoli theorem. First we will show that these sequences are uniformly bounded on J_0. To do so, for each n and each i note that

$$|t^{p_i} v_{ni}| \le |t^{p_i}(v_{ni} - v_{0i})| + |t^{p_i} v_{0i}| \le |t^{p_i}(w_{0i} - v_{0i})| + |t^{p_i} v_{0i}|.$$

Therefore we can choose an $M \in R_+^2$ such that $|t^{p_i} v_{ni}| \le M_i$ for each n and each i, implying that $\{t^p v_n\}$ is uniformly bounded. Similarly we can prove the same result for $\{t^p w_n\}$.

Now we will show that the weighted sequences are equicontinuous. For simplicity, let F_n be defined as $F_n = f(t, v_n) + g(t, w_n)$ for each $n \ge 0$. Since f, g are continuous on J_0, and since each v_n, w_n are C_p continuous then there exist continuous functions $\tilde{f}, \tilde{g}$ such that

$$f(t, v_n) + g(t, w_n) = \tilde{f}(t, t^p v_n) + \tilde{g}(t, t^p w_n).$$

Given this, and that the weighted sequences are uniformly bounded we can choose an $N \in R_+^2$ such that $|F_{ni}| \le N_i$ for each i.

Now, choose t, τ such that $0 < t \le \tau \le T$. In the following proof of equicontinuity we use the fact that

$$\tau^{p_1}(\tau - s)^{q_1 - 1} - t^{p_1}(t - s)^{q_1 - 1} \le 0$$

for $0 < s < t$. To show why this is true, consider the function $\phi(t) = t^{p_1}(t-s)^{q_1-1} = t^{p_1}(t-s)^{-p_1}$ and note that

$$\begin{aligned}\tfrac{d}{dt}\phi(t) &= p_1 t^{p_1-1}(t-s)^{-p_1} - p_1 t^{p_1}(t-s)^{-p_1-1}\\ &= -t^{p_1-1}(t-s)^{-p_1-1}p_1 s \le 0.\end{aligned}$$

This implies that ϕ is nonincreasing, therefore $\phi(\tau) - \phi(t) \le 0$. Now consider,

$$\begin{aligned}|\tau^{p_i}v_{ni}(\tau) - t^{p_i}v_{ni}(t)| &\le \frac{\tau^{p_1}}{\Gamma(q_i)}\int_t^\tau (\tau-s)^{q_i-1}|F_{n-1_i}|ds + \frac{1}{\Gamma(q_i)}\int_0^t |\phi(\tau)-\phi(t)||F_{n-1_i}|ds\\ &\le \frac{N_i\tau^{p_i}}{\Gamma(q_i)}\int_t^\tau (\tau-s)^{q_i-1}ds + \frac{N_i}{\Gamma(q_i)}\int_0^t \big[\phi(t)-\phi(\tau)\big]ds\\ &= \frac{N_i}{\Gamma(q_i)}\Big[\frac{\tau^{p_i}}{q_i}(\tau-t)^{q_i} + t^{p_i}\int_0^t (t-s)^{q_i-1}ds - \tau^{p_i}\int_0^t(\tau-s)^{q_i-1}ds\Big]\\ &= \frac{N_i}{q_i\Gamma(q_i)}\Big[2\tau^{p_i}(\tau-t)^{q_i} + t - \tau\Big]\\ &\le \frac{2N_iT^{p_i}}{\Gamma(q_i+1)}(\tau-t)^{q_i}.\end{aligned}$$

In the case when $t = 0$, we note that

$$|\tau^{p_i}v_{ni}(\tau) - x_i^0/\Gamma(q_i)| \le \frac{N_iT^{p_i}}{\Gamma(q_i)}\int_0^\tau (\tau-s)^{q_i-1}ds = \frac{N_iT^{p_i}}{\Gamma(q_i+1)}\tau^{q_i}.$$

This result is not dependent on n or i, therefore if we define $K \ge 0$ such that

$$K = \max_{i\in\{1,2\}}\Big\{\frac{2N_iT^{p_i}}{\Gamma(q_i+1)}\Big\},$$

then we have that

$$|\tau^{p_i}v_{ni}(\tau) - t^{p_i}v_{ni}(t)| \le K|\tau - t|^{q_i},$$

for $0 \le t \le \tau \le T$, for each i and for all $n \ge 1$. With this, it is now routine to show that $\{t^p v_n\}$ is equicontinuous. Likewise, $\{t^p w_n\}$ is also equicontinuous. So by the Arzela-Ascoli theorem there exist subsequences of both weighted sequences that converge uniformly, but since both sequences are monotone we have that both $\{t^p v_n\}$ and $\{t^p w_n\}$ converge uniformly on J_0. Let $t^p v$ and $t^p w$ be the uniform limits of these weighted sequences respectively. We wish to show that v and w are Type 1 coupled minimal and maximal quasisolutions of (6). To do so, first note that for each i and $n \ge 1$ we have

$$t^{p_i}v_{ni} = x_i^0 + \frac{t^{p_i}}{\Gamma(q_i)}\int_0^t (t-s)^{q_i-1}\big[f_i(s, v_{n-1}) + g_i(s, w_{n-1})\big]ds.$$

Now, since the weighted sequences $\{t^p v_n\}, \{t^p w_n\}$ converge uniformly on J_0 we have that the non-weighted sequences converge pointwise on J. Therefore, by the continuity of f, g the above expression converges uniformly to

$$t^{p_i}v_i = x_i^0 + \frac{t^{p_i}}{\Gamma(q_i)}\int_0^t (t-s)^{q_i-1}\big[f_i(s, v) + g_i(s, w)\big]ds$$

on J_0. Thus

$$v_i = x_i^0 t^{q_i-1} + \frac{1}{\Gamma(q_i)} \int_0^t (t-s)^{q_i-1} \big[f_i(s,v) + g_i(s,w)\big] ds,$$

implying that v is a Type 1 coupled quasisolution of (6), similarly w is as well.

Now, to show that v and w are minimal and maximal, we let x be a solution of (6) such that $x_p(0) = 0$ and $v_0 \leq x \leq w_0$. We know such a solution exists thanks to Theorem 2.3. Now letting $m = v_1 - x$, $M = x - w_1$ and using the same method as we used above we have that $v_0 \leq v_1 \leq x \leq w_1 \leq w_0$. Further, as before, we can inductively prove that $v_n \leq x \leq w_n$ on J for all $n \geq 1$, therefore $v \leq x \leq w$ implying that v, w are minimal and maximal Type 1 coupled quasisolutions. This completes the proof. We note that if $f + g$ possesses an adequate condition for uniqueness then $v = w = x$ which is the unique solution. Now we will present more variations of the generalized monotone method, specifically incorporating Type II solutions. First, in the following theorem we construct the sequences in a manner resembling Type II coupled quasisolutions, but still beginning with Type I lower and upper solutions. In this case we get alternating sequences which are described in the statement of the theorem.

Theorem 3.2 *Suppose that conditions* $(A1)$ *and* $(A2)$ *of Theorem 3.1 are true. Then the sequences given by*

$$D^{q_i} v_{n+1_i} = f_i(t, w_n) + g_i(t, v_n), \quad v_{n+1_{p_i}}(0) = x_i^0, \tag{9}$$

$$D^{q_i} w_{n+1_i} = f_i(t, v_n) + g_i(t, w_n), \quad w_{n+1_{p_i}}(0) = x_i^0, \tag{10}$$

yield alternating monotone sequences $\{v_{2n}, w_{2n+1}\}$ *and* $\{v_{2n+1}, w_{2n}\}$ *that satisfy*

$$v_{2n} \leq w_{2n+1} \leq x \leq v_{2n+1} \leq w_{2n},$$

for each $n \geq 0$ *on* J*, provided* $v_0 \leq x \leq w_0$*. Further, the weighted sequences*

$$t^p v_{2n}, t^p w_{2n+1} \to t^p \rho, \quad t^p v_{2n+1}, t^p w_{2n} \to t^p r$$

uniformly and monotonically on J_0*, where* ρ, r *are Type 1 coupled minimal and maximal quasisolutions of (6).*

We note that the proof of this theorem follows in much the same way as that of Theorem 3.1, as do the proofs of the remaining monotone method proofs, therefore we will not show these proofs directly.

For the next form of the generalized monotone method we switch the initial lower and upper solutions to Type II, and the sequences are also constructed like Type II coupled quasisolutions, i.e. in the manner found in Theorem 3.2, and also yield alternating sequences. For this case to work we must further assume that $v_0 \leq w_1$ and $v_1 \leq w_0$.

Theorem 3.3 *Suppose that condition* $(A2)$ *of Theorem 3.1 is true. Further suppose that*

$(B1)$ *v_0, w_0 are coupled lower and upper solutions of Type II for (6) such that $v_0 \leq w_0$.*

Then the sequences defined by (9) and (10) yield alternating sequences $\{v_{2n}, w_{2n+1}\}$ *and* $\{v_{2n+1}, w_{2n}\}$ *satisfying*

$$v_{2n} \leq w_{2n+1} \leq x \leq v_{2n+1} \leq w_{2n},$$

for each $n \geq 0$ on J, provided that $v_0 \leq w_1 \leq x \leq v_1 \leq w_0$. Further, the weighted sequences

$$t^p v_{2n}, t^p w_{2n+1} \to t^p \rho, \quad t^p v_{2n+1}, t^p w_{2n} \to t^p r$$

uniformly and monotonically on J_0, where ρ, r are Type 1 coupled minimal and maximal quasisolutions of (6).

For our final construction of the monotone method we will also consider the case where we begin with Type II lower and upper solutions, but construct the sequences as Type I quasisolutions, i.e. in the manner found in Theorem 3.1. We do not get alternating sequences in this case, but for it to work we must further assume that $v_0 \leq v_1$ and $w_1 \leq w_0$.

Theorem 3.4 *Suppose that conditions $(B1)$ and $(A2)$ of Theorems 3.3 and 3.1 are true. Then the sequences defined by (7) and (8) are such that*

$$t^p v_n \to t^p v, \quad t^p w_n \to w$$

uniformly and monotonically on J_0 provided that $v_0 \leq v_1 \leq x \leq w_1 \leq w_0$, where v, w are Type I coupled minimal and maximal quasisolutions of (6) respectively.

4 Numerical Example

In this section we present an example that illustrates the result of Theorem 3.1.

Example 4.1 Consider the fractional system of the form (6) with $q_1 = \frac{1}{2}$ and $q_2 = \frac{1}{3}$,

$$\begin{array}{rcll} D^{\frac{1}{2}} x_1(t) &=& \frac{1}{2} + \frac{5}{8}t + \frac{1}{16}\left(x_1(t)^2 - \frac{1}{4}x_2(t)\right), & x_{p_1}(0) = 0, \\ D^{\frac{1}{3}} x_2(t) &=& \frac{1}{6} + \frac{1}{2}t + \frac{1}{20}\left(x_1(t) - x_2(t)\right), & x_{p_2}(0) = 0, \end{array} \tag{11}$$

where $p_1 = \frac{1}{2}$, $p_2 = \frac{2}{3}$ and call

$$\begin{array}{rcl} f_1(t, x_1(t), x_2(t)) &=& \dfrac{1}{2} + \dfrac{5}{8}t + \dfrac{1}{16}x_1(t)^2, \quad f_2(t, x_1(t), x_2(t)) = \dfrac{1}{6} + \dfrac{1}{2}t + \dfrac{1}{20}x_1(t), \\ g_1(t, x_1(t), x_2(t)) &=& -\dfrac{1}{16}\left(\dfrac{1}{4}x_2(t)\right) = -\dfrac{1}{64}x_2(t), \quad g_2(t, x_1(t), x_2(t)) = -\dfrac{1}{20}x_2(t). \end{array}$$

If $J = (0, 1]$ and $J_0 = [0, 1]$ then $f(t, x)$ and $g(t, x)$ satisfy condition $(A2)$ in Theorem 3.1. Now let

$$\begin{array}{c} v_{01} = \sqrt{t}/2, \quad v_{02} = 0, \\ w_{01} = 3, \quad w_{02} = 3 - t. \end{array}$$

We will illustrate graphically in Figures 1–4 that $v_0(t)$ and $w_0(t)$ satisfy $(A1)$. We have that

$$v_{0p_i}(0) = w_{0p_i}(0) = 0.$$

Since $D^{1/2} v_{01}(t) = \frac{\sqrt{\pi}}{4}$, then

$$\begin{array}{rcl} D^{1/2} v_{01}(t) = \dfrac{\sqrt{\pi}}{4} &\leq& \dfrac{1}{2} + \dfrac{5}{8}t + \dfrac{1}{16}\left(v_{01}(t)^2 - \dfrac{1}{4}w_{02}(t)\right) \\ &=& f_1(t, v_{01}(t), v_{02}(t)) + g_1(t, w_{01}(t), w_{02}(t)). \end{array}$$

Similarly,

$$D^{1/2}w_{01}(t) = \frac{3}{\sqrt{\pi t}} \geq \frac{1}{2} + \frac{5}{8}t + \frac{1}{16}\left(w_{01}(t)^2 - \frac{1}{4}v_{02}(t)\right)$$
$$= f_1(t, w_{01}(t), w_{02}(t)) + g_1(t, v_{01}(t), v_{02}(t)),$$

$$D^{1/3}v_{02}(t) = 0 \leq \frac{1}{6} + \frac{1}{2}t + \frac{1}{20}\left(v_{01}(t) - w_{02}(t)\right)$$
$$= f_2(t, v_{01}(t), v_{02}(t)) + g_2(t, w_{01}(t), w_{02}(t)),$$

and

$$D^{1/3}w_{02}(t) = \frac{6-3t}{2\sqrt[3]{t}\,\Gamma\left(\frac{2}{3}\right)} \geq \frac{1}{6} + \frac{1}{2}t + \frac{1}{20}\left(w_{01}(t) - v_{02}(t)\right)$$
$$= f_2(t, w_{01}(t), w_{02}(t)) + g_2(t, v_{01}(t), v_{02}(t)).$$

We show the graphs below.

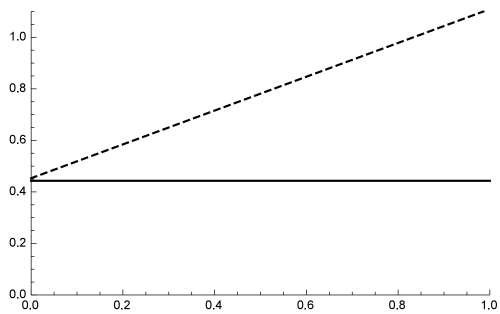

Figure 1: Solid: $D^{1/2}v_{01}(t)$, Dashed: $f_1(t, v_{01}(t), v_{02}(t)) + g_1(t, w_{01}(t), w_{02}(t))$.

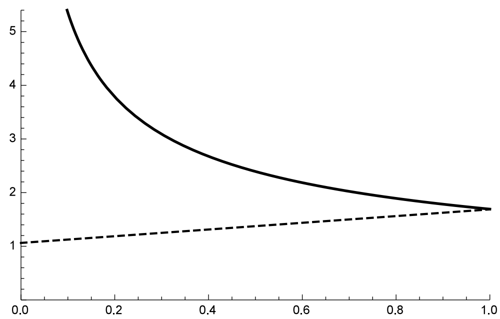

Figure 2: Solid: $D^{1/2}w_{01}(t)$, Dashed: $f_1(t, w_{01}(t), w_{02}(t)) + g_1(t, v_{01}(t), v_{02}(t))$.

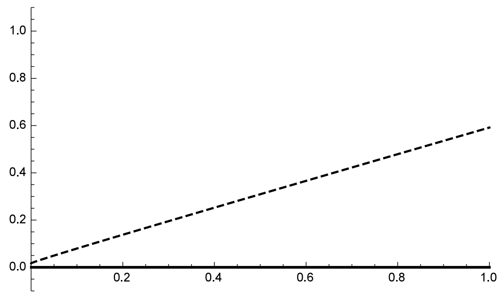

Figure 3: Solid: $D^{1/3}v_{02}(t)$, Dashed: $f_2(t, v_{01}(t), v_{02}(t)) + g_2(t, w_{01}(t), w_{02}(t))$.

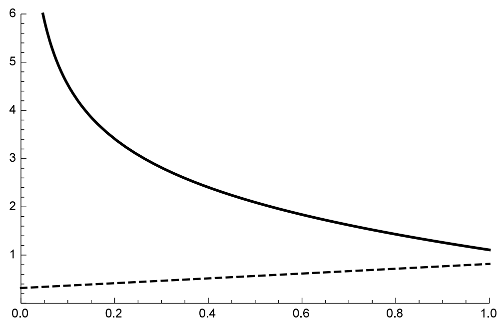

Figure 4: Solid: $D^{1/3}w_{02}(t)$, Dashed: $f_2(t, w_{01}(t), w_{02}(t)) + g_2(t, v_{01}(t), v_{02}(t))$.

After verifying that we have indeed coupled lower and upper solutions of Type I we computed four iterates of $\{t^{1/2}v_{n1}(t)\}$ and $\{t^{1/2}w_{n1}(t)\}$, as well as four iterates of $\{t^{1/3}v_{n2}(t)\}$ and $\{t^{1/3}w_{n2}(t)\}$ according to Theorem 3.1 for $t \in J_0 = [0, 1]$.

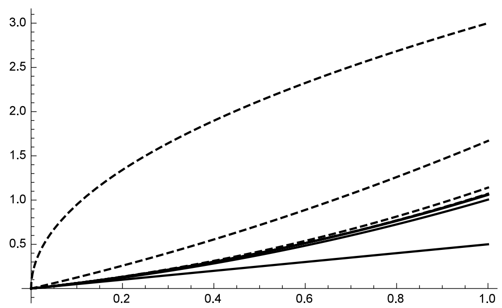

Figure 5: Solid: $\left\{t^{1/2}v_{n1}(t)\right\}$, Dashed: $\left\{t^{1/2}w_{n1}(t)\right\}$, $0 \le n \le 4$.

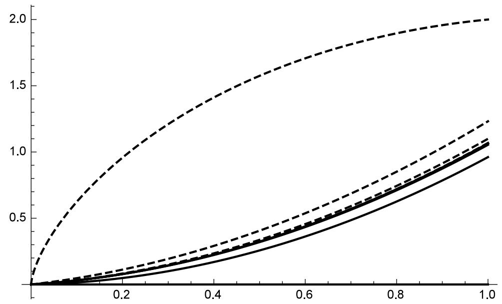

Figure 6: Solid: $\left\{t^{1/3}v_{n2}(t)\right\}$, Dashed: $\left\{t^{1/3}w_{n2}(t)\right\}$, $0 \le n \le 4$.

Finally we show a table of ten values of $\{t^{p_i}v_{4_i}(t)\}$ and $\{t^{p_i}w_{4_i}(t)\}$ on the interval $[0, 1]$.

t	$t^{1/2}v_{4,1}(t)$	$t^{1/2}w_{4,1}(t)$	$t^{1/3}v_{4,2}(t)$	$t^{1/3}w_{4,2}(t)$
0	0	0	0	0
0.1	0.0610930	0.0610933	0.0310197	0.0310208
0.2	0.1318091	0.1318108	0.0790014	0.0790066
0.3	0.2122992	0.2123045	0.1437895	0.1438034
0.4	0.3027222	0.3027352	0.2253221	0.2253509
0.5	0.4032596	0.4032874	0.3235653	0.3236175
0.6	0.5141177	0.5141722	0.4384997	0.4385866
0.7	0.6355296	0.6356297	0.5701140	0.5702515
0.8	0.7677574	0.7679318	0.7184090	0.7186130
0.9	0.9110939	0.9113858	0.8833827	0.8836781
1.0	1.0658661	1.0663374	1.0650431	1.0654591

We have developed a monotone iterative technique for multi-order 2-systems of Riemann-Liouville fractional differential equations with initial condition and presented an example that illustrates one of the main theorems. An advantage of this method is that the linear iterates do not require the computation of the Mittag-Leffler function. In our example the iterates appear to converge to a unique solution, we plan to work on establishing conditions for uniqueness in the near future. In the future we would also like to expand this method to N-systems as well as consider further generalizations of the monotone method. One such expansion would be the quasilinearization method, where the hypotheses are strengthened yet the convergence becomes quadratic, for more information see [4, 13]. And ultimately we hope that these results help further the study of R-L fractional multi-order systems.

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