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Co-existence of Various Types of Synchronization Between Hyper-chaotic Maps

Adel Ouannas *

Department of Mathematics and Computer Science,
Constantine University, Algeria;
Laboratory of Mathematics, Informatics and Systems (LAMIS), University of Larbi Tebessi,
Tebessa, 12002 Algeria.

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Abstract: In this paper, we propose a new type of hybrid synchronization combining projective synchronization (PS), full state hybrid projective synchronization (FSHPS) and generalized synchronization (GS). We present, based on nonlinear controllers, a new control scheme to study the co-existence of (PS), (FSHPS) and (GS) between general 3D hyperchaotic maps. The capability of the proposed approach is illustrated by numerical example.

Keywords: *hyperchaotic maps; synchronization; co-existence; Lyapunov stability.*

Mathematics Subject Classification (2010): 93C10, 93C55, 93D05.

1 Introduction

Historically, hyperchaos in discrete-time systems was firstly reported by Rössler [1]. A hyperchaotic system is usually defined as a chaotic system with more than one positive Lyapunov exponent. The occurrence of hyperchaotic behavior has been found in an electronic circuit [2], NMR laser [3], in a semi-conductor system [4] and in a chemical reaction system [5]. Some interesting hyperchaotic systems in discrete-time were presented in the past two decades such as Baier-Klain system [6], Hitzl-Zele map [7], Stefanski map [8], Wang map [9], Rössler discrete-time system [10] and Grassi-Miller map [11] etc. Since hyperchaotic maps are more complex than chaotic maps, their dynamics have been investigated extensively owing to their useful potential applications in

* Corresponding author: mailto:ouannas_adel@yahoo.fr

secure communications [12–17]. Thus it is a more important subject to study hyperchaos synchronization.

Recently, more and more attention has been paid to the synchronization of chaos (hyperchaos) in discrete-time dynamical systems [18–22]. Different synchronization types have been proposed for discrete-time chaotic and hyperchaotic maps such as projective synchronization [23], adaptive function projective synchronization [24, 25], function cascade synchronization [26], generalized synchronization [27, 28], lag synchronization [29], impulsive synchronization [30], hybrid synchronization [31], Q-S synchronization [32] and full state hybrid projective synchronization [33, 34]. Among all synchronization types, projective synchronization (PS), full-state hybrid projective synchronization (FSHPS) and generalized synchronization (GS) are effective approaches for achieving the synchronization of chaotic and hyperchaotic discrete-time systems. (PS) means that the drive chaotic system and the response chaotic system synchronize up to scaling constant, FSHPS means that each drive system state synchronizes with a linear combination of response system states and (GS) appears when there exists functional relationship between the states of the drive and the response chaotic systems.

In this paper, a new general scheme of synchronization which includes (PS), (FSHPS) and (GS) between coupled 3D hyperchaotic maps is constructed. Based on stability theory of linear discrete-time systems, Lyapunov stability theory and using nonlinear controllers, a new criterion of co-existence of (PS), (FSHPS) and (GS) is derived. The derived synchronization results can have an important effect in the application due to complexity of the proposed scheme and the difficulty of the prediction of the scaling factors. To validate the proposed approach numerically, we apply it to two hyperchaotic maps: the hyperchoatic Wang map and the hyperchoatic Stefanski map.

This paper is organized as follows. In Section 2, the problem of co-existence of synchronization types is introduced. Our approach of synchronization is described in Section 3. In Section 4, numerical example is used to show the effectiveness of the proposed synchronization method. In Section 5, conclusion is made.

2 Problem Statement

We consider the following drive and response chaotic systems

$$x_i(k+1) = f_i(X(k)), \quad 1 \leq i \leq 3, \tag{1}$$

$$y_i(k+1) = g_i(Y(k)) + u_i, \quad 1 \leq i \leq 3, \tag{2}$$

where $(x_1(k), x_2(k), x_3(k))^T$, $(y_1(k), y_2(k), y_3(k))^T$ are the states of the drive and the response systems, respectively, f_i, g_i : $\mathbf{R}^3 \to \mathbf{R}$, $1 \leq i \leq 3$, and u_i, $1 \leq i \leq 3$, are controllers to be determined.

The error system between the drive system (1) and the response system (2) is defined as

$$\begin{aligned} e_1(k) &= y_1(k) - \theta x_1(k), \\ e_2(k) &= y_2(k) - \sum_{j=1}^{3} \lambda_j x_j(k), \\ e_3(k) &= y_3(k) - \phi(x_1, x_2, x_3)(k), \end{aligned} \tag{3}$$

where $\theta \in \mathbf{R}^*$, $\lambda_j \in \mathbf{R}^*$ $j = 1, 2, 3$, and $\phi : \mathbf{R}^3 \to \mathbf{R}$ is a continuously bounded function.

We said that projective synchronization (PS), full-state hybrid projective synchronization (FSHPS) and generalized synchronization (GS) co-exist in the synchronization of the systems (1) and (2), if there exist controllers u_i, $1 \leq i \leq 3$, such that the synchronization errors (3) satisfy

$$\lim_{k \longrightarrow +\infty} e_i(k) = 0, \quad i = 1, 2, 3. \tag{4}$$

3 Synchronization Approach

As the drive system, we consider the following hyperchaotic map

$$x_i(k+1) = f_i(X(k)), \quad 1 \leq i \leq 3, \tag{5}$$

where $X(k) = (x_1(k), x_2(k), x_3(k))^T$ is the state vector of the drive system, $f_i : \mathbf{R}^3 \longrightarrow \mathbf{R}$, $1 \leq i \leq 3$. As the response, we consider the following chaotic system

$$y_i(k+1) = \sum_{j=1}^{3} b_{ij} y_j(k) + g_i(Y(k)) + u_i, \quad 1 \leq i \leq 3, \tag{6}$$

where $Y(k) = (y_1(k), y_2(k), y_3(k))^T$ is the state vector of the response systems, $(b_{ij}) \in \mathbf{R}^{3\times 3}$ is the linear part of the response system, $g_i : \mathbf{R}^3 \longrightarrow \mathbf{R}$, $1 \leq i \leq 3$, are nonlinear functions and u_i, $1 \leq i \leq 3$, are controllers to be designed.

The error system, according to (3), between the drive system (5) and the response system (6) can be derived as

$$\begin{aligned} e_1(k+1) &= y_1(k+1) - \theta x_1(k+1), \\ e_2(k+1) &= y_2(k+1) - \sum_{j=1}^{3} \lambda_j x_j(k+1), \\ e_3(k+1) &= y_3(k+1) - \phi(X(k+1)). \end{aligned} \tag{7}$$

Then, the error system (7) can be written as

$$\begin{aligned} e_1(k+1) &= \sum_{j=1}^{3} b_{1j} y_j(k) + g_1(Y(k)) + u_1 - \theta f_1(X(k)), \\ e_2(k+1) &= \sum_{j=1}^{3} b_{2j} y_j(k) + g_2(Y(k)) + u_2 - \sum_{j=1}^{3} \lambda_j f_j(X(k)), \\ e_3(k+1) &= \sum_{j=1}^{3} b_{3j} y_j(k) + g_3(Y(k)) + u_2 - \phi(f_1(X(k)),\ f_2(X(k)),\ f_3(X(k))). \end{aligned} \tag{8}$$

To achieve synchronization between the drive system (5) and the response system

(6), we propose the following synchronization controllers

$$\begin{aligned} u_1 &= N_1 - b_{11}\theta x_1(k) - b_{12}\left(\sum_{j=1}^{3}\lambda_j x_j(k)\right) - \sum_{j=1}^{3} l_{1j} e_j(k), \\ u_2 &= N_2 - b_{21}\theta x_1(k) - b_{22}\left(\sum_{j=1}^{3}\lambda_j x_j(k)\right) - \sum_{j=1}^{3} l_{2j} e_j(k), \\ u_3 &= N_3 - b_{31}\theta x_1(k) - b_{32}\left(\sum_{j=1}^{3}\lambda_j x_j(k)\right) - \sum_{j=1}^{3} l_{3j} e_j(k), \end{aligned} \tag{9}$$

where

$$\begin{aligned} N_1 &= \theta f_1(X(k)) - b_{13}\phi(X(k)) - g_1(Y(k)), \\ N_2 &= \sum_{j=1}^{3}\lambda_j f_j(X(k)) - b_{23}\phi(X(k)) - g_2(Y(k)), \\ N_3 &= \phi(f_1(X(k)),\ f_2(X(k)),\ f_3(X(k))) - b_{33}\phi(X(k)) - g_3(Y(k)), \end{aligned} \tag{10}$$

and $(l_{ij}) \in \mathbf{R}^{3\times 3}$ are control constants to be determined later.

By substituting the control law (9) into (8), the error system can be described as

$$\begin{aligned} e_1(k+1) &= \sum_{j=1}^{3}(b_{1j} - l_{1j})\, e_j(k), \\ e_2(k+1) &= \sum_{j=1}^{3}(b_{2j} - l_{2j})\, e_j(k), \\ e_3(k+1) &= \sum_{j=1}^{3}(b_{3j} - l_{3j})\, e_j(k). \end{aligned} \tag{11}$$

Now, rewrite the error system described in (11) in the compact form

$$e(k+1) = (B - L)\, e(k), \tag{12}$$

where $e(k) = (e_1(k), e_2(k), e_3(k))^T$, $B = (b_{ij})_{3\times 3}$ and $L = (l_{ij})_{3\times 3}$.

Hence, we have the following result.

Theorem 3.1 *If the control matrix L is chosen such that one of the following conditions is satisfied:*

(i) All eigenvalues of $B - L$ are strictly inside the unit disk.

(ii) $(B-L)^T(B-L) - I$ is negative definite matrix.

(iii) $(l_{ij})_{1\le i,\ j\le 3}$ are chosen such that

$$\begin{aligned} &\sum_{i=1}^{3}(b_{ip} - l_{ip})(b_{iq} - l_{iq}) = 0, \quad p, q = 1, 2, 3, \ p \neq q, \\ &\sum_{i=1}^{3}(b_{ij} - l_{ij})^2 < 1, \quad j = 1, 2, 3. \end{aligned} \tag{13}$$

Then, (PS), (FSHPS) and (GS) co-exist between the drive system (5) and the response system (6).

Proof. Firstly, according to stability theory of linear discrete-time systems, we can conclude that if condition (i) is satisfied it is immediate that $\lim_{k\longrightarrow+\infty} e_i(k)=0,\ i=1,2,3.$ Therefore, systems (5) and (6) are globally synchronized.

Secondly, we construct the Lyapunov function in the form $V(e(k))=e^T(k)e(k)$, we obtain

$$\begin{aligned}\Delta V(e(k)) &= e^T(k+1)e(k+1)-e^T(k)e(k)\\ &= e^T(k)(B-L)^T(B-L)e(k)-e^T(k)e(k)\\ &= e^T(k)\left[(B-L)^T(B-L)-I\right]e(k),\end{aligned}$$

and by using condition (ii) we get $\Delta V(e(k))<0$. Thus, from the Lyapunov stability theory, it is immediate that $\lim_{k\longrightarrow+\infty} e_i(k)=0\ \ (i=1,2,3)$ then the synchronization is achieved between systems (5) and (6).

Finally, consider the candidate Lyapunov function: $V(e(k))=\sum_{i=1}^{3} e_i^2(k)$, we get

$$\begin{aligned}\Delta V(e(k)) &= \sum_{i=1}^{3} e_i^2(k+1)-\sum_{i=1}^{3} e_i^2(k)\\ &= \sum_{j=1}^{3}\left(\sum_{i=1}^{3}(b_{ij}-l_{ij})^2-1\right)e_j^2(k)\\ &+ \sum_{\substack{p,\ q=1\\ p\neq q}}^{3}\left(\sum_{i=1}^{3}(b_{ip}-l_{ip})(b_{iq}-l_{iq})\right)e_p(k)e_q(k),\end{aligned}$$

and by using conditions (iii), we obtain $\Delta V(e(k))<0$. Then, it is immediate that $\lim_{k\longrightarrow+\infty} e_i(k)=0\ (i=1,2,3)$, and we conclude that the systems (4) and (5) are globally synchronized.

4 Numerical Example

We consider hyperchaotic Stefanski map as the drive system and the controlled hyperchaotic Wang map as the response system. The drive system is described as

$$\begin{aligned}x_1(k+1) &= 1+x_3(k)-\alpha x_2^2(k),\\ x_2(k+1) &= 1+\beta x_2(k)-\alpha x_1^2(k),\\ x_3(k+1) &= \beta x_1(k),\end{aligned}\tag{14}$$

which has a chaotic attractor, when $(\alpha,\ \beta)=(1.4,\ 0.2)$ [36]. The hyperchaotic attractor of Stefanski map is shown in Figure 1. The response system can be defined as

$$\begin{aligned}y_1(k+1) &= a_3\delta y_2(k)+(a_4\delta+1)y_1(k)+u_1,\\ y_2(k+1) &= a_1\delta y_1(k)+y_2(k)+a_2\delta y_3(k)+u_2,\\ y_3(k+1) &= (a_7\delta+1)y_3(k)+a_6\delta y_2(k)y_3(k)+a_5\delta+u_3,\end{aligned}\tag{15}$$

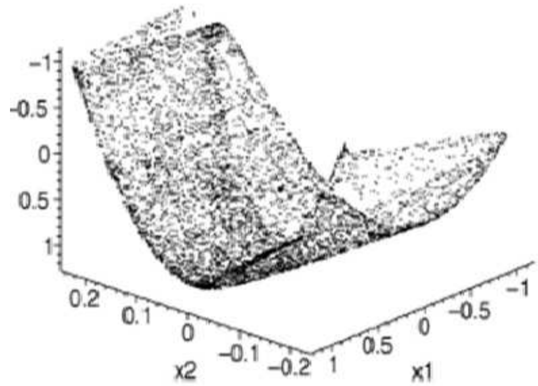

Figure 1: Hyperchaotic attractor of Stefanski map.

where $U = (u_1, u_2, u_3)^T$ is the vector controller. The hyperchaotic Wang map has a chaotic attractor, when $(a_1, a_2, a_3, a_4, a_5, a_6, a_7, \delta) = (-1.9, 0.2, 0.5, -2.3, 2, -0.6, -1.9, 1)$ [35]. The hyperchaotic attractor of Wang map is shown in Figure 2. According to our control scheme proposed in the previous section

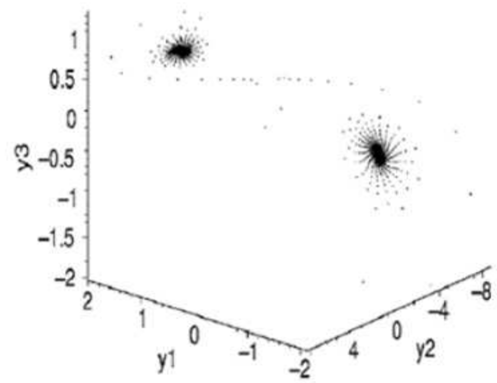

Figure 2: Hyperchaotic attractor of Wang map.

the synchronization errors between the drive system (14) and the response system (15) are defined as follows

$$\begin{aligned} e_1(k+1) &= y_1(k+1) - \theta x_1(k+1), \\ e_2(k+1) &= y_2(k+1) - \sum_{j=1}^{3} \lambda_j x_j(k+1), \\ e_3(k+1) &= y_3(k+1) - \phi(x_1(k+1), x_2(k+1), x_3(k+1)). \end{aligned} \tag{16}$$

In this example, the scaling constants $\theta, \lambda_1, \lambda_2$ and λ_3 are chosen as

$$\begin{cases} \theta = 2, \\ \lambda_1 = 1, \\ \lambda_2 = 2, \\ \lambda_3 = 3, \end{cases} \tag{17}$$

and the map $\phi : \mathbf{R}^3 \to \mathbf{R}$ is selected as

$$\phi(x_1(k), x_2(k), x_3(k)) = x_1(k) - x_2(k)x_3(k). \tag{18}$$

Then, the errors system (16) can be described as

$$\begin{aligned} e_1(k+1) &= (a_4\delta + 1) e_1(k) + R_1 + u_1,, \\ e_2(k+1) &= e_2(k) + R_2 + u_2, \\ e_3(k+1) &= (a_7\delta + 1) e_3(k) + u_3, \end{aligned} \tag{19}$$

where

$$\begin{aligned} R_1 &= a_3\delta y_2(k) + \sum_{j=1}^{3} \mu_{1j} x_j(k) + 2\alpha x_2^2(k) - 2, \\ R_2 &= a_1\delta y_1(k) + a_2\delta y_3(k) + \sum_{j=1}^{3} \mu_{2j} x_j(k) + \alpha x_2^2(k) + 2\alpha x_1^2(k) - 3, \\ R_3 &= a_6\delta y_2(k) y_3(k) + \sum_{j=1}^{3} \mu_{3j} x_j(k) - (a_7\delta + 1) x_2(k)x_3(k) + \beta x_1(k)x_2(k) \\ &- \alpha\beta x_1^3(k) + \alpha x_2^2(k) + a_5\delta - 1, \end{aligned} \tag{20}$$

where $\mu_{11} = 2(a_4\delta + 1)$, $\mu_{12} = 0$, $\mu_{13} = -2$, $\mu_{21} = -3\beta + 1$, $\mu_{22} = 2(1 - \beta)$, $\mu_{23} = 2$, $\mu_{31} = a_7\delta + 1 - \beta$, $\mu_{31} = a_7\delta + 1 + \beta$, $\mu_{32} = 0$, and $\mu_{33} = -1$.

To achieve synchronization between systems (14) and (15), we choose the synchronization controllers u_i $(i = 1, 2, 3)$, as

$$u_i = -R_i - l_i e_i, \quad i = 1, 2, 3, \tag{21}$$

where the control constants $(l_i)_{1 \le i \le 3}$ are selected as follows

$$\begin{aligned} l_1 &= a_4\delta, \\ |l_2| &< 1, \\ l_3 &= a_7\delta. \end{aligned} \tag{22}$$

Theorem 4.1 *The hyperchaotic Stefanski map (14) and the controlled hyperchaotic Wang map (15) are globally synchronized under the controllers (21).*

Proof. By substituting (21) into (19), the synchronization errors can be written as

$$\begin{aligned} e_1(k+1) &= e_1(k), \\ e_2(k+1) &= (1-l_2)\, e_2(k), \\ e_3(k+1) &= e_3(k). \end{aligned} \tag{23}$$

To prove the zero-stability of synchronization errors (23), we consider the quadratic Lyapunov function $V(e(k)) = \sum_{i=1}^{3} e_i^2(k)$, then we obtain

$$\begin{aligned} \Delta V(e(k)) &= \sum_{i=1}^{3} e_i^2(k+1) - \sum_{i=1}^{3} e_i^2(k) \\ &= e_1^2(k) + (1-l_2)^2 e_2^2(k) + e_3^2(k) - e_1^2(k) - e_2^2(k) - e_3^2(k) \\ &= (1-l_2)^2 e_2^2(k) < 0. \end{aligned}$$

Thus, by Lyapunov stability it is immediate that $\lim_{k\to\infty} e_i(k) = 0 \ \ (i = 1, 2, 3)$. Finally, we get the numeric results that are shown in Figure 3.

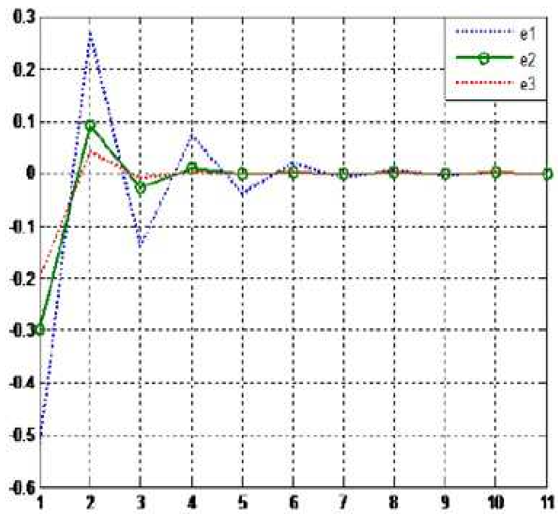

Figure 3: Time evolution of errors between systems (14) and (15).

5 Conclusion

In this paper, the co-existence of some synchronization types in general 3D coupled hyperchaotic maps has been investigated. Sufficient conditions have been derived for achieving a new synchronization scheme of co-existence of (PS), (FSHPS) and (GS) between hyperchaotic maps. The new synchronization criterion has been demonstrated using nonlinear controllers, stability theory of linear discrete-time systems and Lyapunov stability theory. An example of application and numerical simulations have been used to show the effectiveness of the derived result.

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