Volume 16, Number 3, 2016 ISSN 1562-8353

NONLINEAR DYNAMICS AND SYSTEMS THEORY
An International Journal of Research and Surveys

Volume 16 Number 3 2016

CONTENTS

 ISSN 1562-8353 Printed in Ukraine
To receive contents and abstracts by e-mail, visit our Website at: **http://www.e-ndst.kiev.ua**

NONLINEAR DYNAMICS & SYSTEMS THEORY

Volume 16, No. 3, 2016

Nonlinear Dynamics and Systems Theory

An International Journal of Research and Surveys

InforMath Publishing Group
http://www.e-ndst.kiev.ua

Nonlinear Dynamics and Systems Theory

An International Journal of Research and Surveys

INSTRUCTIONS FOR CONTRIBUTORS

(1) General. Nonlinear Dynamics and Systems Theory (ND&ST) is an international journal devoted to publishing peer-refereed, high quality, original papers, brief notes and review articles focusing on nonlinear dynamics and systems theory and their practical applications in engineering, physical and life sciences. Submission of a manuscript is a representation that the submission has been approved by all of the authors and by the institution where the work was carried out. It also represents that the manuscript has not been previously published, has not been copyrighted, is not being submitted for publication elsewhere, and that the authors have agreed that the copyright in the article shall be assigned exclusively to InforMath Publishing Group by signing a transfer of copyright form. Before submission, the authors should visit the website:

http://www.e-ndst.kiev.ua

for information on the preparation of accepted manuscripts. Please download the archive Sample_NDST.zip containing example of article file (you can edit only the file Samplefilename.tex).

(2) Manuscript and Correspondence. Manuscripts should be in English and must meet common standards of usage and grammar. To submit a paper, send by e-mail a file in PDF format directly to

Professor A.A. Martynyuk, Institute of Mechanics,
Nesterov str.3, 03057, MSP 680, Kiev-57, Ukraine
e-mail: journalndst@gmail.com; amartynyuk@voliacable.com

or to one of the Regional Editors or to a member of the Editorial Board. Final version of the manuscript must typeset using LaTex program which is prepared in accordance with the style file of the Journal. Manuscript texts should contain the title of the article, name(s) of the author(s) and complete affiliations. Each article requires an abstract not exceeding 150 words. Formulas and citations should not be included in the abstract. AMS subject classifications and key words must be included in all accepted papers. Each article requires a running head (abbreviated form of the title) of no more than 30 characters. The sizes for regular papers, survey articles, brief notes, letters to editors and book reviews are: (i) 10-14 pages for regular papers, (ii) up to 24 pages for survey articles, and (iii) 2-3 pages for brief notes, letters to the editor and book reviews.

(3) Tables, Graphs and Illustrations. Each figure must be of a quality suitable for direct reproduction and must include a caption. Drawings should include all relevant details and should be drawn professionally in black ink on plain white drawing paper. In addition to a hard copy of the artwork, it is necessary to attach the electronic file of the artwork (preferably in PCX format).

(4) References. References should be listed alphabetically and numbered, typed and punctuated according to the following examples. Each entry must be cited in the text in form of author(s) together with the number of the referred article or in the form of the number of the referred article alone.

Journal: [1] Poincare, H. Title of the article. *Title of the Journal* **Vol. l** (No.l), Year, Pages. [Language]

Book: [2] Liapunov, A.M. *Title of the book.* Name of the Publishers, Town, Year.

Proceeding: [3] Bellman, R. Title of the article. In: *Title of the book.* (Eds.). Name of the Publishers, Town, Year, Pages. [Language]

(5) Proofs and Sample Copy. Proofs sent to authors should be returned to the Editorial Office with corrections within three days after receipt. The corresponding author will receive a sample copy of the issue of the Journal for which his/her paper is published.

(6) Editorial Policy. Every submission will undergo a stringent peer review process. An editor will be assigned to handle the review process of the paper. He/she will secure at least two reviewers' reports. The decision on acceptance, rejection or acceptance subject to revision will be made based on these reviewers' reports and the editor's own reading of the paper.

NONLINEAR DYNAMICS AND SYSTEMS THEORY
An International Journal of Research and Surveys
Published by InforMath Publishing Group since 2001

Volume 16 Number 3 2016

CONTENTS

Founded by A.A. Martynyuk in 2001.
Registered in Ukraine Number: KB 5267 / 04.07.2001.

NONLINEAR DYNAMICS AND SYSTEMS THEORY

An International Journal of Research and Surveys

Impact Factor from SCOPUS for 2013: SNIP – 1.108, IPP – 0.809, SJR – 0.496

Nonlinear Dynamics and Systems Theory (ISSN 1562–8353 (Print), ISSN 1813–7385 (Online)) is an international journal published under the auspices of the S.P. Timoshenko Institute of Mechanics of National Academy of Sciences of Ukraine and Curtin University of Technology (Perth, Australia). It aims to publish high quality original scientific papers and surveys in areas of nonlinear dynamics and systems theory and their real world applications.

AIMS AND SCOPE

Nonlinear Dynamics and Systems Theory is a multidisciplinary journal. It publishes papers focusing on proofs of important theorems as well as papers presenting new ideas and new theory, conjectures, numerical algorithms and physical experiments in areas related to nonlinear dynamics and systems theory. Papers that deal with theoretical aspects of nonlinear dynamics and/or systems theory should contain significant mathematical results with an indication of their possible applications. Papers that emphasize applications should contain new mathematical models of real world phenomena and/or description of engineering problems. They should include rigorous analysis of data used and results obtained. Papers that integrate and interrelate ideas and methods of nonlinear dynamics and systems theory will be particularly welcomed. This journal and the individual contributions published therein are protected under the copyright by International InforMath Publishing Group.

PUBLICATION AND SUBSCRIPTION INFORMATION

Nonlinear Dynamics and Systems Theory will have 4 issues in 2016, printed in hard copy (ISSN 1562–8353) and available online (ISSN 1813–7385), by InforMath Publishing Group, Nesterov str., 3, Institute of Mechanics, Kiev, MSP 680, Ukraine, 03057. Subscription prices are available upon request from the Publisher (mailto:journalndst@gmail.com), EBSCO Information Services (mailto:journals@ebsco.com), or website of the Journal: http://e-ndst.kiev.ua. Subscriptions are accepted on a calendar year basis. Issues are sent by airmail to all countries of the world. Claims for missing issues should be made within six months of the date of dispatch.

ABSTRACTING AND INDEXING SERVICES

Papers published in this journal are indexed or abstracted in: Mathematical Reviews / MathSciNet, Zentralblatt MATH / Mathematics Abstracts, PASCAL database (INIST–CNRS) and SCOPUS.

Nonlinear Dynamics and Systems Theory, **16** (3) (2016) 221–234

Dwell Time Stability Analysis for Nonlinear Switched Difference Systems

A.Yu. Aleksandrov [1*], A.A. Martynyuk [2] and A.V. Platonov [1]

[1] *Saint Petersburg State University, 7/9 Universitetskaya Nab., St. Petersburg, 199034, Russia*
[2] *Institute of Mechanics, National Academy of Science of Ukraine, Nesterov Str. 3, Kyiv, 03057, Ukraine*

Received: January 31, 2016;   Revised: June 8, 2016

Abstract: This paper addresses the stability problem for a set of switched nonlinear difference equations with parametric uncertainties. For the corresponding family of subsystems, a regularization procedure is suggested, and a multiple Lyapunov function is constructed. With the aid of the Lyapunov function, classes of switching signals are determined for which the asymptotic stability of a stationary solution of a given set of equations may be guaranteed. An application of the proposed approach to the stability analysis of multiconnected switched difference systems by nonlinear approximation is presented. An example is given to illustrate our results.

Keywords: *difference systems; switching law; stability; comparison equations; dwell-time; multiple Lyapunov functions; complex systems.*

Mathematics Subject Classification (2010): 39A22, 39A30.

1 Introduction

A general outline of the theory of set equations is presented in the monograph [18], where it is shown that classical results of qualitative theory of equations under an appropriate adaptation can be applied to the analysis of equations in metric spaces. The most effective methods are the method of integral inequalities [19], the Lyapunov direct method [22, 28] and the comparison method based on the use of scalar [11, 12], vector [25] and matrix-valued Lyapunov functions [22].

Difference equations are of great interest due to their wide applications in the modeling of real world processes in which states of systems are measured not continuously but at some fixed instants of time [1, 3, 16, 20]. Sets of difference equations with switching are

* Corresponding author: mailto:a.u.aleksandrov@spbu.ru

a new subject for research designed to describe more accurately situations where the phenomena under study possess variable structure. This paper focuses on the development of methods for analysis of such systems.

The stability problem of a stationary solution for a set of nonlinear switched difference equations with parametric uncertainties is studied. First, for the corresponding family of subsystems, a regularization procedure and an approach for finding partial Lyapunov functions are proposed. Next, with the aid of these partial functions, a multiple Lyapunov function [10] is constructed for the original set of switched equations. On the basis of a development of dwell-time approach [2, 10, 29], restrictions on the switching law are determined under which the asymptotic stability of the stationary solution can be guaranteed.

Furthermore, it is shown that the proposed approaches can be applied to the stability analysis of multiconnected switched difference systems describing interaction of essentially nonlinear homogeneous subsystems, and, for these systems, sufficient conditions of the asymptotic stability by nonlinear approximation can be obtained.

2 Preliminaries

Further we shall need the following notions and results, see [18] and the references cited therein.

Let $K_C(\mathbb{R}^q)$ denote a family of all nonempty compact and convex subsets in the Euclidean space $\mathbb{R}^q$; $K(\mathbb{R}^q)$ contain all nonempty compact subsets in $\mathbb{R}^q$, and $C(\mathbb{R}^q)$ be a subset of all nonempty closed subsets in $\mathbb{R}^q$. The distance between nonempty closed subsets A and B of the space $\mathbb{R}^q$ is specified by the formula

$$D[A,B] = \max\left\{d_H(A,B),\ d_H(B,A)\right\},$$

where $d_H(B,A) = \sup\{d(\mathbf{b},A):\ \mathbf{b}\in B\}$ is a Hausdorff separation of the sets A and B, and $d(\mathbf{b},A) = \inf\{\|\mathbf{b}-\mathbf{a}\|:\ \mathbf{a}\in A\}$ is a distance from the point $\mathbf{b}$ to the set A, $\|\cdot\|$ is the Euclidean norm of a vector.

The following operations can be defined on $K_C(\mathbb{R}^q)$:

$$A+B = \{\mathbf{a}+\mathbf{b}:\ \mathbf{a}\in A,\, \mathbf{b}\in B\}, \qquad \lambda A = \{\lambda\mathbf{a}:\ \mathbf{a}\in A\},$$

where $A,B\in K_C(\mathbb{R}^q)$, and λ is an arbitrary nonnegative number.

The pair $(C(\mathbb{R}^q), D)$ is a complete separable metric space, where $K(\mathbb{R}^q)$ and $K_C(\mathbb{R}^q)$ are closed subsets.

The set $W\in K_C(\mathbb{R}^q)$ is called the Hukuhara difference for the sets $A,B\in K_C(\mathbb{R}^q)$, if $A = B+W$.

Let F be a mapping of the domain Q of the space $\mathbb{R}^q$ into the metric space $(K_C(\mathbb{R}^q), D)$, i.e., $F:\ Q\to K_C(\mathbb{R}^q)$, which is equivalent to the inclusion $F(\mathbf{t})\in K_C(\mathbb{R}^q)$ for all $\mathbf{t}\in Q$. Such mappings are called the multivalued mappings of Q into $\mathbb{R}^q$.

Let $\mathbb{R}^q_+$ be the nonnegative cone of $\mathbb{R}^q$; $\mathbb{N}$ denote a set of positive integers, $\mathbb{N}_+ = \mathbb{N}\cup\{0\}$, and we designate by $\mathbb{N}_{n_0}$ the set

$$\mathbb{N}_{n_0} = \{n_0, n_0+1, \ldots, n_0+k, \ldots\},$$

where $k\in\mathbb{N}$ and $n_0\in\mathbb{N}_+$.

Next, let us introduce the concept of homogeneity, see [27, 30], for the following analysis.

Definition 2.1 A function $f(\mathbf{x}) : \mathbb{R}^q \to \mathbb{R}$ is called homogeneous of the order ν with respect to the dilation $(m_1, \ldots, m_q)$, where $\nu, m_1, \ldots, m_q$ are positive rationals with the odd denominators, if

$$f(\lambda^{m_1} x_1, \ldots, \lambda^{m_q} x_q) = \lambda^{\nu} f(\mathbf{x}) \tag{1}$$

for all $\lambda \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^q$. In the case when $\nu, m_1, \ldots, m_q$ are positive real numbers, and equality (1) holds for $\lambda \geq 0$ and $\mathbf{x} \in \mathbb{R}^q$, the function $f(\mathbf{x})$ is called positive homogeneous of the order ν with respect to the dilation $(m_1, \ldots, m_q)$.

Definition 2.2 A vector field $\mathbf{F}(\mathbf{x}) = (f_1(\mathbf{x}), \ldots, f_q(\mathbf{x}))^T : \mathbb{R}^q \to \mathbb{R}^q$ is called positive homogeneous of the order μ with respect to the dilation $(m_1, \ldots, m_q)$, where $m_i > 0$ and $\mu + m_i > 0$, $i = 1, \ldots, q$, if $f_i(\lambda^{m_1} x_1, \ldots, \lambda^{m_q} x_q) = \lambda^{\mu + m_i} f_i(x_1, \ldots, x_q)$, $i = 1, \ldots, q$, for all $\lambda \geq 0$ and $\mathbf{x} \in \mathbb{R}^q$.

Let the system of differential equations

$$\dot{\mathbf{x}}(t) = \mathbf{F}(\mathbf{x}(t)) \tag{2}$$

be given, where $\mathbf{x}(t) \in \mathbb{R}^q$ is the state vector, and components of the vector $\mathbf{F}(\mathbf{x})$ are continuous for all $\mathbf{x} \in \mathbb{R}^q$.

Definition 2.3 System (2) is called positive homogeneous if its vector field $\mathbf{F}(\mathbf{x})$ is positive homogeneous.

Moreover, we will use the following lemmas, see [6] and [14] respectively.

Lemma 2.1 *If a sequence $\{v_n\}$ satisfies the condition $0 \leq v_{n+1} \leq v_n - \alpha v_n^{1+\xi}$, $n \in \mathbb{N}_+$, with $\alpha > 0$, $\xi > 0$, $v_0 \geq 0$, and $\alpha(1+\xi) v_0^{\xi} \leq 1$, then*

$$v_n \leq v_0 \left(1 + \alpha \xi v_0^{\xi} n\right)^{-\frac{1}{\xi}} \quad for \quad n \in \mathbb{N}_+.$$

Lemma 2.2 *For any positive numbers x, y and ζ the estimate*

$$(x+y)^{\zeta} \geq 2^{\omega} \left(x^{\zeta} + y^{\zeta}\right)$$

holds, where $\omega = \min\{\zeta - 1; 0\}$.

3 Statement of the Problem

Consider a set of switched difference equations

$$X_{n+1} = F^{(\sigma)}(n, X_n, \alpha) \tag{3}$$

with initial conditions $X_{n_0} = X_0$, where $X_n \in K_C(\mathbb{R}^q)$ for all $n \geq n_0$; the function $\sigma = \sigma(n)$, with $\sigma(n) \in \{1, \ldots, S\}$, defines the switching law; $\alpha \in \Im \subset \mathbb{R}^d$ is the uncertainty parameter; the mappings $F^{(s)} : \mathbb{N}_+ \times K_C(\mathbb{R}^q) \times \Im \to K_C(\mathbb{R}^q)$ are continuous with respect to X_n for every $n \in \mathbb{N}_+$ and $\alpha \in \Im$.

Thus, we assume that the system under consideration depends on an uncertain parameter. Moreover, while operating, the system switches between several operation modes, and, for every $n \geq n_0$, one of the subsystems

$$X_{n+1} = F^{(s)}(n, X_n, \alpha), \quad s = 1, \ldots, S, \tag{4}$$

is active.

Let $X_n(n_0, X_0)$ be the solution of (3) satisfying the condition $X_{n_0} = X_0$.

For the set of equations (3) we introduce the following assumptions:

H$_1$. For equations (3) there exists a set of stationary solutions $\Theta_0 \in K_C(\mathbb{R}^q)$, i.e., $F^{(s)}(n, \Theta_0, \alpha) = \Theta_0$ for all $n \in \mathbb{N}_+$, $\alpha \in \Im$, $s = 1, \dots, S$.

H$_2$. For any $X_0 \in K_C(\mathbb{R}^q)$ and $Y_0 \in K_C(\mathbb{R}^q)$ there exists the Hukuhara difference $W_0 \in K_C(\mathbb{R}^q)$.

Definition 3.1 The stationary solution Θ_0 of the set of equations (3) is

(i) stable, if for any $n_0 \in \mathbb{N}_+$ and $\varepsilon > 0$ there exists a $\delta = \delta(n_0, \varepsilon) > 0$ such that the inequality $D[W_0, \Theta_0] < \delta$ implies the estimate $D[X_n, \Theta_0] < \varepsilon$ for all $n \geq n_0$, where $W_0 = X_0 - Y_0$, $X_0 \in K_C(\mathbb{R}^q)$, $Y_0 \in K_C(\mathbb{R}^q)$, and $X_n = X_n(n_0, X_0 - Y_0) = X_n(n_0, W_0)$ is the solution of (3);

(ii) attractive, if for any $n_0 \in \mathbb{N}_+$ there exists $\tilde{\delta}(n_0) > 0$, and for any $\xi > 0$ there exists $\tau(n_0, W_0, \xi) \in \mathbb{N}_+$ such that the inequality $D[W_0, \Theta_0] < \tilde{\delta}(n_0)$ implies the estimate $D[X_n, \Theta_0] < \xi$ for any $n \geq n_0 + \tau(n_0, W_0, \xi)$;

(iii) asymptotically stable, if it is both stable and attractive.

We will look for stability conditions for a stationary solution Θ_0 of the set of switched systems of difference equations (3).

It should be noted that the general stability theory of classical difference equations is well-developed, see [1, 3, 8, 15–17, 20] and references cited therein, whereas the stability theory of a set of difference equations is in a primitive state.

In particular, in [9] and [18] an extension of some results obtained for a set of continuous systems with Hukuhara derivative was proposed for a set of difference equations. Unsolved problem is that of constructing an appropriate Lyapunov function satisfying special properties providing the stability of a stationary solution.

In [4], an approach to the stability analysis for sets of difference equations of the form (3) has been developed in the case of absence of switching. In the present paper, we will extend this approach to the set of switched difference equations.

4 Construction of Matrix Lyapunov Functions and Comparison Equations

Let the symbol $\overline{\text{co}}$ mean the closure of convex shell of the corresponding set.

Together with subsystems (4) we will consider the following families of sets of difference equations

$$X_{n+1} = F_M^{(s)}(n, X_n), \quad s = 1, \dots, S, \tag{5}$$

where $F_M^{(s)}(n, X_n) = \overline{\text{co}} \bigcup\limits_{\alpha \in \Im} F^{(s)}(n, X_n, \alpha)$;

$$X_{n+1} = F_m^{(s)}(n, X_n), \quad s = 1, \dots, S, \tag{6}$$

where $F_m^{(s)}(n, X_n) = \overline{\text{co}} \bigcap\limits_{\alpha \in \Im} F^{(s)}(n, X_n, \alpha)$;

$$X_{n+1} = F_\beta^{(s)}(n, X_n), \quad s = 1, \dots, S, \tag{7}$$

where $F_\beta^{(s)}(n, X_n) = F_M^{(s)}(n, X_n)\beta + F_m^{(s)}(n, X_n)(1 - \beta)$, $\beta \in [0, 1]$.

In what follows it is assumed that $F_m^{(s)}$, $F_M^{(s)}$ and $F_\beta^{(s)} \in K_c(\mathbb{R}^q)$.

For every $s \in \{1, \dots, S\}$, we introduce an auxiliary matrix function, see [4],

$$\mathbf{U}^{(s)}(n, \beta, X_n) = [U_{ij}^{(s)}(n, \beta, X_n)], \quad i, j = 1, 2, \tag{8}$$

where the element $U_{11}^{(s)}(n, X_n)$ is associated with the s-th set from the family (5), $U_{22}^{(s)}(n, X_n)$ is associated with the s-th set from the family (6), $U_{12}^{(s)}(n, \beta, X_n) = U_{21}^{(s)}(n, \beta, X_n)$ is associated with the s-th set from the family (7).

In terms of function (8) we construct a scalar function [22]

$$V_s(n, X_n, \beta, \theta_s) = \theta_s^T \mathbf{U}^{(s)}(n, \beta, X_n)\theta_s, \quad \theta_s \in \mathbb{R}_+^2, \tag{9}$$

and assume that $V_s :\ \mathbb{N}_+ \times K_C(\mathbb{R}^q) \times [0, 1] \times \mathbb{R}_+^2 \to \mathbb{R}_+$.

Function (9) is a partial Lyapunov function for the s-th subsystem from (4) if, together with the first difference

$$\Delta V_s(n, X_n, \beta, \theta_s) = V_s(n + 1, X_{n+1}, \beta, \theta_s) - V_s(n, X_n, \beta, \theta_s),$$

it solves the problem of stability of the stationary solution Θ_0 for the s-th subsystem.

Let the following assumptions be fulfilled.

- H_3. For every $s \in \{1, \dots, S\}$, there exists $\tilde{\theta}_s \in \mathbb{R}_+^2$ such that for the function $V_s(n, X_n, \beta, \tilde{\theta}_s)$ and for its first difference along trajectories of the s-th set of equations from (4) the estimates

$$a_s(D[X_n, \Theta_0]) \le V_s(n, X_n, \beta, \tilde{\theta}_s) \le b_s(D[X_n, \Theta_0]), \tag{10}$$

$$\Delta V_s \le w^{(s)}(n, V_s) \tag{11}$$

are valid for $n \in \mathbb{N}_+$, $X_n \in S(\rho)$, $\beta \in [0, 1]$. Here $\rho = \text{const} > 0$; $S(\rho) = \{X \in K_c(\mathbb{R}^q) :\ D[X, \Theta_0] < \rho\}$; $a(\cdot)$ and $b(\cdot)$ are class $\mathcal{K}$ (in the sense of Hahn) functions [28]; functions $w^{(s)}(n, r)$ are continuous with respect to $r \in [0, \tilde{\rho}]$ for every value of $n \in \mathbb{N}_+$, and $w^{(s)}(n, r)/r \to 0$ as $r \to 0$; $\tilde{\rho} = \text{const} > 0$.

- H_4. The zero solutions of the equations

$$u_{n+1} = u_n + w^{(s)}(n, u_n), \qquad s = 1, \dots, S, \tag{12}$$

are asymptotically stable.

Equations (12) are comparison ones for subsystems from the family (4). It is known, see [4], that under assumptions H_3 and H_4 the stationary solution Θ_0 of each subsystem is asymptotically stable.

To obtain stability conditions for the set of switched systems of difference equations (3), we will use multiple Lyapunov functions and the dwell-time approach.

5 Dwell Time Stability Analysis

Let us impose additional restrictions on the Lyapunov functions (9) and on the comparison equations (12).

- H_5. There exist positive numbers c_{sl} such that

$$V_s(n, X_n, \beta, \tilde{\theta}_s) \le c_{sl} V_l(n, X_n, \beta, \tilde{\theta}_l) \tag{13}$$

for $n \in \mathbb{N}_+$, $X_n \in S(\rho)$, $\beta \in [0, 1]$; $s, l = 1, \dots, S$.

H_6. Let equations (12) be of the form

$$u_{n+1} = u_n - \alpha^{(s)} u_n^{1+\xi^{(s)}}, \qquad s = 1, \dots, S, \tag{14}$$

where $\alpha^{(s)}$ and $\xi^{(s)}$ are positive constants.

Remark 5.1 Equations (14) can be considered as equations of the nonlinear approximation for (12).

Remark 5.2 The case where $\xi^{(s)} = 0$, $s = 1, \dots, S$, is well-investigated, see, for instance, [10, 13, 21]. Therefore, in this section we assume that $\xi^{(s)} > 0$, $s = 1, \dots, S$, i.e., the switched comparison equations (14) are essentially nonlinear.

Remark 5.3 Using Lemma 2.1 and taking into account Assumptions H_3, H_4 and H_6, one can obtain estimates for solutions of subsystems (4).

Without loss of generality, we assume that the interval $(0, +\infty)$ contains an infinite number of switching instants. Let τ_i, $i \in \mathbb{N}$, be the switching times, $0 < \tau_1 < \tau_2 < \dots$, and $\tau_0 = 0$.

Denote, for brevity, $\hat{\xi}_i = \xi^{(\sigma(\tau_i))}$, $\hat{\alpha}_i = \alpha^{(\sigma(\tau_i))}$, $i \in \mathbb{N}_+$; $\hat{c}_i = c_{\sigma(\tau_i)\sigma(\tau_{i-1})}$, $i \in \mathbb{N}$.

For every $m \in \mathbb{N}$ and $L_m \in \mathbb{R}_+$, define a sequence $\chi_n(L_m, m)$ by the formulae

$$\chi_0(L_m, m) = L_m,$$

$$\chi_n(L_m, m) = \hat{c}_{m+n-1}^{-\hat{\xi}_{m+n-1}} \left(\chi_{n-1}(L_m, m)\right)^{\hat{\xi}_{m+n-1}/\hat{\xi}_{m+n-2}} + \hat{\alpha}_{m+n-1}\hat{\xi}_{m+n-1} T_{m+n}$$

for $n \in \mathbb{N}$, where $T_i = \tau_i - \tau_{i-1}$, $i \in \mathbb{N}$.

Theorem 5.1 *Let Assumptions* H_1–H_6 *be fulfilled. If there exists a positive constant* L *such that*

$$\chi_n(L, 1) \to +\infty \quad \textit{as} \quad n \to +\infty, \tag{15}$$

then the stationary solution Θ_0 *of the set of equations* (3) *is asymptotically stable.*

Proof. Using partial Lyapunov functions $V_1(n, X_n, \beta, \tilde{\theta}_1), \dots, V_S(n, X_n, \beta, \tilde{\theta}_S)$, construct a multiple Lyapunov function $V_{\sigma(n)}(n, X_n, \beta, \tilde{\theta}_{\sigma(n)})$ corresponding to the switching law $\sigma(n)$.

Choose a number ε such that $0 < \varepsilon < \rho$, and

$$\alpha^{(s)} \left(1 + \xi^{(s)}\right) V_s^{\xi^{(s)}}(n, X_n, \beta, \tilde{\theta}_s) \le 1, \qquad s = 1, \dots, S,$$

for $n \in \mathbb{N}_+$, $X_n \in S(\varepsilon)$, $\beta \in [0, 1]$.

Consider the solution X_n of (3) satisfying the condition $X_{n_0} = W_0$, where $n_0 \in \mathbb{N}_+$, $W_0 \in S(\varepsilon)$. Find a positive integer m such that $n_0 \in [\tau_{m-1}, \tau_m)$. Let $X_n \in S(\varepsilon)$ for $n = n_0, \dots, \tilde{n}$.

If $n_0 < \tilde{n} \le \tau_m$, then applying Lemma 2.1 to the $\sigma(\tau_{m-1})$-th inequality from (11), we obtain that

$$\begin{aligned} V_{\sigma(\tau_{m-1})}^{-\hat{\xi}_{m-1}} \left(\tilde{n}, X_{\tilde{n}}, \beta, \tilde{\theta}_{\sigma(\tau_{m-1})}\right) &\ge V_{\sigma(\tau_{m-1})}^{-\hat{\xi}_{m-1}} \left(n_0, W_0, \beta, \tilde{\theta}_{\sigma(\tau_{m-1})}\right) + \hat{\alpha}_{m-1}\hat{\xi}_{m-1}(\tilde{n} - n_0) \\ &\ge V_{\sigma(\tau_{m-1})}^{-\hat{\xi}_{m-1}} \left(n_0, W_0, \beta, \tilde{\theta}_{\sigma(\tau_{m-1})}\right). \end{aligned} \tag{16}$$

In the case of $\tilde{n} > \tau_m$, there exists a positive integer p satisfying the condition $\tau_{m+p-1} < \tilde{n} \le \tau_{m+p}$. It should be noted that $p \to +\infty$ as $\tilde{n} \to +\infty$. Applying successively Lemma 2.1 to the corresponding inequalities from (11) and taking into account estimates (13), we obtain

$$V^{-\hat{\xi}_{m+p-1}}_{\sigma(\tau_{m+p-1})}\left(\tilde{n}, X_{\tilde{n}}, \beta, \tilde{\theta}_{\sigma(\tau_{m+p-1})}\right) \ge V^{-\hat{\xi}_{m+p-1}}_{\sigma(\tau_{m+p-1})}\left(\tau_{m+p-1}, X_{\tau_{m+p-1}}, \beta, \tilde{\theta}_{\sigma(\tau_{m+p-1})}\right)$$
$$+\hat{\alpha}_{m+p-1}\hat{\xi}_{m+p-1}(\tilde{n} - \tau_{m+p-1})$$
$$\ge \hat{c}^{-\hat{\xi}_{m+p-1}}_{m+p-1}\left(V^{-\hat{\xi}_{m+p-2}}_{\sigma(\tau_{m+p-2})}\left(\tau_{m+p-1}, X_{\tau_{m+p-1}}, \beta, \tilde{\theta}_{\sigma(\tau_{m+p-2})}\right)\right)^{\hat{\xi}_{m+p-1}/\hat{\xi}_{m+p-2}}$$
$$\ge \ldots \ge \hat{c}^{-\hat{\xi}_{m+p-1}}_{m+p-1}\left(\chi_{p-1}\left(V^{-\hat{\xi}_{m-1}}_{\sigma(\tau_{m-1})}\left(n_0, W_0, \beta, \tilde{\theta}_{\sigma(\tau_{m-1})}\right), m\right)\right)^{\hat{\xi}_{m+p-1}/\hat{\xi}_{m+p-2}}. \tag{17}$$

From (10), (16) and (17), it follows that

$$D[X_{\tilde{n}}, \Theta_0] \le \max_{s=1,\ldots,S} a_s^{(-1)}\left(b_s\left(D[W_0, \Theta_0]\right)\right) \tag{18}$$

for $\tilde{n} = n_0, \ldots, \tau_m$, and

$$D[X_{\tilde{n}}, \Theta_0] \le \max_{s,k,j=1,\ldots,S} a_s^{(-1)}\left(c_{sk}\left(\chi_{p-1}\left(b_j^{-\xi^{(j)}}\left(D[W_0, \Theta_0]\right), m\right)\right)^{-1/\xi^{(k)}}\right) \tag{19}$$

for $\tilde{n} = \tau_{m+p-1} + 1, \ldots, \tau_{m+p}$ and $p \ge 1$. Here $a_s^{(-1)}(\cdot)$ is inverse of the function $a_s(\cdot)$, $s = 1, \ldots, S$.

Let there exist a positive constant L such that condition (15) is fulfilled. It is easy to check that if $L_m = \chi_{m-1}(L, 1)$, then $\chi_n(L_m, m) = \chi_{n+m-1}(L, 1)$. Hence, $\chi_n(L_m, m) \to +\infty$ as $n \to +\infty$.

Find a number δ_1 such that $0 < \delta_1 < \varepsilon$, and $b_j^{-\xi^{(j)}}\left(D[W_0, \Theta_0]\right) \ge L_m$ for $W_0 \in S(\delta_1)$, $j = 1, \ldots, S$. Using estimate (19), one can choose a positive integer p_0 satisfying the following condition: if $W_0 \in S(\delta_1)$ and $p \ge p_0$, then $X_{\tilde{n}} \in S(\varepsilon)$.

From (17) it follows that

$$D[X_{\tilde{n}}, \Theta_0] \le \max_{s,j=1,\ldots,S} a_s^{(-1)}\left(\bar{c}^p b_j\left(D[W_0, \Theta_0]\right)\right) \tag{20}$$

for $\tilde{n} = \tau_{m+p-1} + 1, \ldots, \tau_{m+p}$ and $p \ge 1$. Here $\bar{c} = \max_{s,k=1,\ldots,S} c_{sk}$. Taking into account (18) and (20), one can find a number δ_2, $0 < \delta_2 < \varepsilon$, such that if $W_0 \in S(\delta_2)$ and $p < p_0$, then $X_{\tilde{n}} \in S(\varepsilon)$.

Let $\delta = \min\{\delta_1; \delta_2\}$. We obtain that $D[W_0, \Theta_0] < \delta$ implies the estimate $D[X_n, \Theta_0] < \varepsilon$ for all $n \ge n_0$.

Moreover, from (19) it follows that $D[X_n, \Theta_0] \to 0$ as $n \to +\infty$. Thus, the stationary solution Θ_0 of the set of equations (3) is asymptotically stable. This completes the proof.

Corollary 5.1 *Let Assumptions* H_1–H_6 *be fulfilled. If there exists a positive constant* L *such that* $\chi_n(L, m) \to +\infty$ *as* $n \to +\infty$ *uniformly with respect to* $m \in \mathbb{N}$, *then the stationary solution* Θ_0 *of the set of equations* (3) *is uniformly asymptotically stable.*

Corollary 5.2 *Let Assumptions* H_1–H_6 *be fulfilled. If* $T_i \to +\infty$ *as* $i \to +\infty$, *then the stationary solution* Θ_0 *of the set of equations* (3) *is uniformly asymptotically stable.*

Next, let us show that the use of Lemma 2.2 permits us to replace condition (15) in Theorem 5.1 by a condition though more conservative but more convenient for applications.

Construct a sequence ψ_n by the formulae $\psi_1 = \hat{\alpha}_1\hat{\xi}_1 T_2$,

$$\psi_n = \hat{\alpha}_n\hat{\xi}_n T_{n+1} + \sum_{i=1}^{n-1} 2^{\omega_{n,n-1}+\ldots+\omega_{n,n-i}} \left(\hat{c}_n \ldots \hat{c}_{n-i+1}\right)^{-\hat{\xi}_n} \left(\hat{\alpha}_{n-i}\hat{\xi}_{n-i}T_{n-i+1}\right)^{\hat{\xi}_n/\hat{\xi}_{n-i}}$$

for $n = 2, 3, \ldots$, where $\omega_{n,j} = \min\{\hat{\xi}_n/\hat{\xi}_j - 1;\, 0\}$, $j = 1, \ldots, n-1$.

Corollary 5.3 *Let Assumptions* H_1–H_6 *be fulfilled. If*

$$\psi_n \to +\infty \quad as \quad n \to +\infty, \tag{21}$$

then the stationary solution Θ_0 *of the set of equations* (3) *is asymptotically stable.*

Proof. With the aid of Lemma 2.2, it is easy to check that $\chi_n(L,1) \geq \psi_n$ for any $L > 0$ and for any $n \in \mathbb{N}$.

Really, $\chi_0(L,1) = L > 0$,

$$\chi_1(L,1) = \hat{c}_1^{-\hat{\xi}_1} \left(\chi_0(L,1)\right)^{\hat{\xi}_1/\hat{\xi}_0} + \hat{\alpha}_1\hat{\xi}_1 T_2 = \hat{c}_1^{-\hat{\xi}_1} L^{\hat{\xi}_1/\hat{\xi}_0} + \psi_1 \geq \psi_1,$$

and, for $n > 1$, we obtain

$$\begin{aligned} \chi_n(L,1) &= \hat{c}_n^{-\hat{\xi}_n} \left(\chi_{n-1}(L,1)\right)^{\hat{\xi}_n/\hat{\xi}_{n-1}} + \hat{\alpha}_n\hat{\xi}_n T_{n+1} \\ &= \hat{c}_n^{-\hat{\xi}_n} \left(\hat{c}_{n-1}^{-\hat{\xi}_{n-1}} \left(\chi_{n-2}(L,1)\right)^{\hat{\xi}_{n-1}/\hat{\xi}_{n-2}} + \hat{\alpha}_{n-1}\hat{\xi}_{n-1}T_n\right)^{\hat{\xi}_n/\hat{\xi}_{n-1}} + \hat{\alpha}_n\hat{\xi}_n T_{n+1} \\ &\geq 2^{\omega_{n,n-1}} \left(\hat{c}_n\hat{c}_{n-1}\right)^{-\hat{\xi}_n} \left(\chi_{n-2}(L,1)\right)^{\hat{\xi}_n/\hat{\xi}_{n-2}} + 2^{\omega_{n,n-1}}\hat{c}_n^{-\hat{\xi}_n} \left(\hat{\alpha}_{n-1}\hat{\xi}_{n-1}T_n\right)^{\hat{\xi}_n/\hat{\xi}_{n-1}} \\ &+\hat{\alpha}_n\hat{\xi}_n T_{n+1} \geq \ldots \geq 2^{\omega_{n,n-1}+\ldots+\omega_{n,1}} \left(\hat{c}_n \ldots \hat{c}_1\right)^{-\hat{\xi}_n} L^{\hat{\xi}_n/\hat{\xi}_0} + \psi_n \geq \psi_n. \end{aligned}$$

Hence, from (21) follows the fulfilment of condition (15). This completes the proof.

Remark 5.4 The results of the present section can be extended to the case where Assumtion H_5 is replaced by the following one:

H_5'. There exist positive numbers c_{sl} and ν_{sl} such that

$$V_s(n, X_n, \beta, \tilde{\theta}_s) \leq c_{sl} V_l^{\nu_{sl}}(n, X_n, \beta, \tilde{\theta}_l)$$

for $n \in \mathbb{N}_+$, $X_n \in S(\rho)$, $\beta \in [0,1]$; $s, l = 1, \ldots, S$.

6 Stability Analysis of Multiconnected Switched Systems

Consider the system

$$\mathbf{x}_i(n+1) = \mathbf{x}_i(n) + \mathbf{F}_i^{(\sigma)}(\mathbf{x}_i(n)) + \sum_{j=1}^{k} \mathbf{\Psi}_{ij}^{(\sigma)}(n, \mathbf{x}(n)), \quad i = 1, \ldots, k, \tag{22}$$

which describes the dynamics of a complex system composed of k interconnected systems [19, 22]. Here $\mathbf{x}_i(n) = (x_{i1}(n), \ldots, x_{iq_i}(n))^T$, $\mathbf{x}(n) = (\mathbf{x}_1^T(n), \ldots, \mathbf{x}_k^T(n))^T$; $n \in \mathbb{N}_+$; function $\sigma = \sigma(n)$, with $\sigma(n) \in \{1, \ldots, S\}$, defines the switching law; vector fields $\mathbf{F}_i^{(s)}(\mathbf{x}_i)$ are continuous for $\mathbf{x}_i \in \mathbb{R}^{q_i}$ and positive homogeneous of the order $\mu_i^{(s)}$ with respect to the dilation $(m_{i1}, \ldots, m_{iq_i})$, where $\mu_i^{(s)}, m_{i1}, \ldots, m_{iq_i}$ are positive numbers; vector functions $\mathbf{\Psi}_{ij}^{(s)}(n, \mathbf{x}) = \left(\Psi_{ij1}^{(s)}(n, \mathbf{x}), \ldots, \Psi_{ijq_i}^{(s)}(n, \mathbf{x})\right)^T$ are defined for $n \in \mathbb{N}_+$, $\|\mathbf{x}\| < H$, $0 < H \leq +\infty$, and continuous with respect to $\mathbf{x}$ for every fixed n; $i, j = 1, \ldots, k$; $s = 1, \ldots, S$. We assume that the estimates

$$|\Psi_{ijg}^{(s)}(n, \mathbf{x})| \leq c_{ijg}^{(s)} r_j^{\alpha_{ijg}^{(s)}}(\mathbf{x}_j)$$

hold for $n \in \mathbb{N}_+$, $\|\mathbf{x}\| < H$, where $r_j(\mathbf{x}_j) = \sum_{p=1}^{q_j} |x_{jp}|^{1/m_{jp}}$, $c_{ijg}^{(s)} \geq 0$, $\alpha_{ijg}^{(s)} > 0$, $g = 1, \ldots, q_i$; $i, j = 1, \ldots, k$; $s = 1, \ldots, S$.

Thus, at each time instant, one of the subsystems

$$\mathbf{x}_i(n+1) = \mathbf{x}_i(n) + \mathbf{F}_i^{(s)}(\mathbf{x}_i(n)) + \sum_{j=1}^{k} \mathbf{\Psi}_{ij}^{(s)}(n, \mathbf{x}(n)), \quad i = 1, \ldots, k, \quad s = 1, \ldots, S, \tag{23}$$

is active.

From the properties of the right-hand sides of (22) it follows that the system admits the zero solution. We will look for conditions of asymptotic stability of the solution.

For every $i \in \{1, \ldots, k\}$, consider the family of isolated difference subsystems

$$\mathbf{x}_i(n+1) = \mathbf{x}_i(n) + \mathbf{F}_i^{(s)}(\mathbf{x}_i(n)), \quad s = 1, \ldots, S, \tag{24}$$

and the corresponding family of subsystems of differential equations

$$\dot{\mathbf{z}}_i(t) = \mathbf{F}_i^{(s)}(\mathbf{z}_i(t)), \quad s = 1, \ldots, S. \tag{25}$$

Let us impose some additional conditions on the right-hand sides of (22).

H_7. There exist numbers $h_1, \ldots, h_k$ such that $h_i \geq 2\max\{m_{i1}, \ldots, m_{iq_i}\}$, $i = 1, \ldots, k$, and, for every $s \in \{1, \ldots, S\}$, the inequalities

$$\frac{\alpha_{ijg}^{(s)}}{h_j + \mu_j^{(s)}} \geq \frac{\mu_i^{(s)} + m_{ig}}{h_i + \mu_i^{(s)}} \quad \text{for} \quad c_{ijg}^{(s)} \neq 0, \quad g = 1, \ldots, q_i, \quad i, j = 1, \ldots, k, \tag{26}$$

hold.

Remark 6.1 Assumption H_7 means that the orders of the right-hand sides of the isolated subsystems (24) are, in a certain sense, less than or equal to the orders of functions characterizing interconnections between the subsystems.

H_8. For every $i \in \{1, \ldots, k\}$, the zero solutions of all subsystems (25) are asymptotically stable.

Remark 6.2 It is known, see [7, 26], that the fulfilment of Assumption H_8 implies that the zero solutions of all difference subsystems (24) are asymptotically stable as well.

H_9. For every $i \in \{1, \dots, k\}$, for the family of subsystems (25), Lyapunov functions $v_{i1}(\mathbf{z}_i), \dots, v_{iS}(\mathbf{z}_i)$ are constructed so that $v_{is}(\mathbf{z}_i)$ is twice continuously differentiable for $\mathbf{z}_i \in \mathbb{R}^{q_i}$ positive definite and positive homogeneous of the order $\gamma_i \geq 2\max\{m_{i1}, \dots, m_{iq_i}\}$ with respect to the dilation $(m_{i1}, \dots, m_{iq_i})$ function, and the derivative of $v_{is}(\mathbf{z}_i)$ with respect to the s-th subsystem from the family (25) is negative definite, $s = 1, \dots, S$.

Remark 6.3 In [27, 30], it was proved that the fulfilment of Assumption H_8 implies the existence of the required Lyapunov functions.

Remark 6.4 In view of homogeneous functions properties, see [30], the estimates

$$a_{1i}^{(s)} r_i^{\gamma_i}(\mathbf{z}_i) \leq v_{is}(\mathbf{z}_i) \leq a_{2i}^{(s)} r_i^{\gamma_i}(\mathbf{z}_i), \qquad \left|\frac{\partial v_{is}(\mathbf{z}_i)}{\partial z_{ig}}\right| \leq a_{3ig}^{(s)} r_i^{\gamma_i - m_{ig}}(\mathbf{z}_i),$$

$$\left(\frac{\partial v_{is}(\mathbf{z}_i)}{\partial \mathbf{z}_i}\right)^T \mathbf{F}_i^{(s)}(\mathbf{z}_i) \leq -a_{4i}^{(s)} r_i^{\gamma_i + \mu_i^{(s)}}(\mathbf{z}_i)$$

hold for $\mathbf{z}_i \in \mathbb{R}^{q_i}$, where $a_{1i}^{(s)}$, $a_{2i}^{(s)}$, $a_{3ig}^{(s)}$, $a_{4i}^{(s)}$, $s = 1, \dots, S$, are positive constants depending on chosen Lyapunov functions; $g = 1, \dots, q_i$; $i = 1, \dots, k$.

In what follows, we will assume, without loss of generality, that $\gamma_i = h_i$, $i = 1, \dots, k$, where numbers $h_1, \dots, h_k$ satisfy the conditions specified in Assumption H_7.

H_{10}. For every $s \in \{1, \dots, S\}$, the inequality system

$$-a_{4i}^{(s)} \xi_i^{\gamma_i + \mu_i^{(s)}} + \sum_{g=1}^{q_i} a_{3ig}^{(s)} \xi_i^{\gamma_i - m_{ig}} \sum_{j=1}^{k} c_{ijg}^{(s)} \xi_j^{\alpha_{ijg}^{(s)}} < 0, \quad i = 1, \dots, k, \tag{27}$$

admits a positive solution.

Remark 6.5 Assumption H_{10} is the Martynyuk-Obolenskii condition [23, 24] of asymptotic stability for the zero solutions of the corresponding Wazewskij systems

$$\dot{z}_i(t) = -a_{4i}^{(s)} z_i^{\gamma_i + \mu_i^{(s)}}(t) + \sum_{g=1}^{q_i} a_{3ig}^{(s)} z_i^{\gamma_i - m_{ig}}(t) \sum_{j=1}^{k} c_{ijg}^{(s)} z_j^{\alpha_{ijg}^{(s)}}(t), \quad i = 1, \dots, k, \quad s = 1, \dots, S.$$

From the results of [5] it follows that if Assumptions H_7–H_{10} are fulfilled, then, for every $s \in \{1, \dots, S\}$, one can find positive numbers $\zeta_1^{(s)}, \dots, \zeta_k^{(s)}$ for which the first difference of the function

$$V_s(\mathbf{z}) = \sum_{i=1}^{k} \zeta_i^{(s)} v_{is}(\mathbf{z}_i) \tag{28}$$

with respect to solutions of the corresponding subsystem from family (23) will be negative definite.

It is easy to show the existence of positive numbers $\beta^{(1)}, \dots, \beta^{(S)}$, $\alpha^{(1)}, \dots, \alpha^{(S)}$ and $\bar{H}$ such that $\bar{H} \in (0, H)$, and for the first difference of $V_s(\mathbf{z})$ with respect to solutions of the s-th subsystem from (23) the inequalities

$$\Delta V_s\big|_{(s)} \leq -\beta^{(s)} \sum_{i=1}^{k} r_i^{\gamma_i + \mu_i^{(s)}}(\mathbf{x}_i(n)) \leq -\alpha^{(s)} V_s^{1+\xi^{(s)}}(\mathbf{x}(n))$$

hold for $\|\mathbf{x}(n)\| < \bar{H}$. Here $\xi^{(s)} = \max\limits_{i=1,\ldots,k} \mu_i^{(s)}/\gamma_i$, $s = 1, \ldots, S$.

Thus, for subsystems (23) we obtain comparison equations of the form (14). Hence, for the subsequent stability analysis of (22) one can use the results of Section 5.

7 Example

Let system (22) be of the form

$$\begin{cases} x_1(n+1) &= x_1(n) + x_2(n), \\ x_2(n+1) &= x_2(n) - a_\sigma x_1^3(n) - b_\sigma |x_2(n)|^{1/2}\, x_2(n) + \psi_1^{(\sigma)}(x_3(n)), \\ x_3(n+1) &= x_3(n) - d_\sigma x_3^{\lambda_\sigma}(n) + \psi_2^{(\sigma)}(x_2(n)). \end{cases} \tag{29}$$

Here $x_1(n), x_2(n), x_3(n)$ are scalar variables; $\sigma = \sigma(n) \in \{1, 2\}$; $a_1 = b_2 = 2$, $a_2 = b_1 = 1$, $d_1 = 8$, $d_2 = 4$, $\lambda_1 = 3$, $\lambda_2 = 5$; functions $\psi_1^{(s)}(z_3)$ and $\psi_2^{(s)}(z_2)$ are continuous for $|z_3| < H$ and $|z_2| < H$ respectively and satisfy the conditions

$$|\psi_1^{(s)}(z_3)| \le c_s |z_3|^{\alpha_s}, \qquad |\psi_2^{(s)}(z_2)| \le e_s |z_2|^{\beta_s}, \qquad s = 1, 2,$$

where $\alpha_1 = 12/5$, $\alpha_2 = 4$, $\beta_1 = 15/8$, $\beta_2 = 31/8$, and c_1, c_2, e_1, e_2 are positive parameters.

Thus, switching in (29) occurs between the subsystems

$$\begin{cases} x_1(n+1) &= x_1(n) + x_2(n), \\ x_2(n+1) &= x_2(n) - 2x_1^3(n) - |x_2(n)|^{1/2}\, x_2(n) + \psi_1^{(1)}(x_3(n)), \\ x_3(n+1) &= x_3(n) - 8x_3^3(n) + \psi_2^{(1)}(x_2(n)), \end{cases} \tag{30}$$

and

$$\begin{cases} x_1(n+1) &= x_1(n) + x_2(n), \\ x_2(n+1) &= x_2(n) - x_1^3(n) - 2|x_2(n)|^{1/2}\, x_2(n) + \psi_1^{(2)}(x_3(n)), \\ x_3(n+1) &= x_3(n) - 4x_3^5(n) + \psi_2^{(2)}(x_2(n)). \end{cases} \tag{31}$$

System (29) can be treated as a complex system describing the interaction of two ($k = 2$) systems

$$\begin{cases} x_1(n+1) &= x_1(n) + x_2(n), \\ x_2(n+1) &= x_2(n) - a_\sigma x_1^3(n) - b_\sigma |x_2(n)|^{1/2}\, x_2(n), \end{cases}$$

and

$$x_3(n+1) = x_3(n) - d_\sigma x_3^{\lambda_\sigma}(n).$$

The differential systems

$$\begin{cases} \dot{z}_1 &= z_2, \\ \dot{z}_2 &= -a_s z_1^3 - b_s |z_2|^{1/2}\, z_2, \quad s = 1, 2, \end{cases} \tag{32}$$

are homogeneous ones of the order $1/2$ with respect to the dilation $(1/2, 1)$, and the differential equations

$$\dot{z}_3 = -d_s z_3^{\lambda_s}, \quad s = 1, 2, \tag{33}$$

are homogeneous ones of the orders 2 and 4 with respect to the dilation 1.

Construct inequalities (26) corresponding to complex system (29). We obtain

$$\max\left\{\frac{8}{5(h_2+2)};\,\frac{5}{3(h_2+4)}\right\} \le \frac{2}{2h_1+1} \le \min\left\{\frac{8}{5(h_2+2)};\,\frac{8}{3(h_2+4)}\right\}.$$

These inequalities admit positive solutions. For example, one can choose $h_1 = h_2 = 2$. Hence, Assumption H_7 is fulfilled.

Lyapunov functions for systems (32) and equations (33) can be constructed in the forms

$$v_{1s}(z_1,z_2) = \frac{a_s}{4}z_1^4 + \frac{1}{2}z_2^2 + \frac{1}{10}\,|z_1|\,z_1 z_2, \quad s=1,2,$$

and

$$v_{2s}(z_3) = \frac{1}{2}z_3^2, \quad s=1,2,$$

respectively. Thus, Assumptions H_8 and H_9 are fulfilled as well.

In the present case inequalities (27) take the form

$$-0.1\xi_1^{5/2} + c_1\xi_1\xi_2^{12/5} < 0, \quad -8\xi_2^4 + e_1\xi_2\xi_1^{15/8} < 0 \tag{34}$$

for $s = 1$, and

$$-0.06\xi_1^{5/2} + c_2\xi_1\xi_2^4 < 0, \quad -4\xi_2^6 + e_2\xi_2\xi_1^3 < 0 \tag{35}$$

for $s = 2$. System (34) admits a positive solution if and only if

$$c_1 e_1^{4/5} < 8^{4/5}/10 \approx 0.52, \tag{36}$$

whereas system (35) admits a positive solution for any positive values of c_2 and e_2.

Assume that inequality (36) is valid. Let, for instance, $c_1 = e_2 = 1/2$, $c_2 = e_1 = 2/3$. Thus, Assumption H_{10} is fulfilled.

It is easy to check that if

$$V_s(\mathbf{z}) = \frac{a_s}{4}z_1^4 + \frac{1}{2}z_2^2 + \frac{1}{10}\,|z_1|\,z_1 z_2 + \frac{1}{4}z_3^2, \quad s=1,2,$$

then there exists $\bar{H} > 0$ such that

$$\Delta V_1\big|_{(30)} \le -0.004\,V_1^2(\mathbf{x}(n)), \quad \Delta V_2\big|_{(31)} \le -0.32\,V_2^3(\mathbf{x}(n))$$

for $\|\mathbf{x}(n)\| < \bar{H}$. Here $\mathbf{z} = (z_1, z_2, z_3)^T$, $\mathbf{x}(n) = (x_1(n), x_2(n), x_3(n))^T$.

Moreover, the estimates $V_1(\mathbf{z}) \le 2V_2(\mathbf{z})$, $V_2(\mathbf{z}) \le V_1(\mathbf{z})$ hold for all $\mathbf{z} \in \mathbb{R}^3$.

Next, with the aid of the results of Section 5, it easy to derive sufficient conditions of asymptotic stability of the zero solution of system (29).

Assume, for definiteness, that subsystem (30) is active for $n = \tau_{2i}, \ldots, \tau_{2i+1} - 1$, whereas subsystem (31) is active for $n = \tau_{2i+1}, \ldots, \tau_{2i+2} - 1$; $i \in \mathbb{N}_+$.

Consider the sequence $\chi_0 = L = \mathrm{const} > 0$,

$$\chi_{2i+1} = (\chi_{2i})^2 + 0.64\,T_{2i+2}, \quad \chi_{2i+2} = \frac{1}{2}\,(\chi_{2i+1})^{1/2} + 0.004\,T_{2i+3}, \quad i \in \mathbb{N}_+.$$

If there exists $L > 0$ such that

$$\chi_n \to +\infty \quad \text{as} \quad n \to +\infty, \tag{37}$$

then, by Theorem 5.1, the zero solution of system (29) is asymptotically stable.

For instance, condition (37) is fulfilled in the case when

$$T_1^2 + 0.64\,T_2 \geq 4\,p_1^2, \quad (p_i + 0.004\,T_{2i+1})^2 + 0.64\,T_{2i+2} \geq 4\,p_{i+1}^2, \quad i \in \mathbb{N},$$

where $\{p_i\}_{i=1}^{+\infty}$ is a sequence of positive numbers, such that $p_i \to +\infty$ as $i \to +\infty$.

8 Conclusion

In the present paper, for a set of switched difference equations, a regularization procedure with respect to the uncertainty parameter of the original system is developed. On the basis of the procedure, an approach to constructing Lyapunov functions and comparison systems for the corresponding family of subsystems is suggested. By means of the multiple Lyapunov function method, classes of switching law are determined for which the asymptotic stability of a stationary solution of the set of switched equations can be guaranteed. The developed approaches are applied to the stability analysis of a nonlinear multiconnected switched difference system.

An interesting problem for further research is that of estimating attraction domains of stationary solutions and finding restrictions on switching laws providing preassigned estimates.

Acknowledgment

This work was partially supported by the Saint Petersburg State University, project no. 9.42.1041.2016, and by the Russian Foundation for Basic Research, grant nos. 15-58-53017 and 16-01-00587.

References

[1] Agarwal, R.P. *Difference Equations and Inequalities.* Dekker, New York, 1992.

[2] Aleksandrov, A. and Aleksandrova, E. Asymptotic stability conditions for a class of hybrid mechanical systems with switched nonlinear positional forces. *Nonlinear Dynamics* **83**(4) (2016) 2427–2434.

[3] Aleksandrov, A.Yu., Chen, Y., Platonov, A.V. and Zhang, L. Stability analysis and uniform ultimate boundedness control synthesis for a class of nonlinear switched difference systems. *J. Difference Equ. Appl.* **18**(9) (2012) 1545–1561.

[4] Aleksandrov, A.Yu., Martynyuk, A.A. and Platonov, A.V. Analysis of a set of nonlinear dynamics trajectories: stability of difference equations. *J. Math. Anal. Appl.* **421**(1) (2015) 105–117.

[5] Aleksandrov, A.Yu. and Platonov, A.V. Conditions of ultimate boundedness of solutions for a class of nonlinear systems. *Nonlinear Dynamics and Systems Theory* **8**(2) (2008) 109–122.

[6] Aleksandrov, A.Yu. and Zhabko, A.P. On stability of solutions to one class of nonlinear difference systems. *Siberian Math. J.* **44**(6) (2003) 951–958.

[7] Aleksandrov, A.Yu. and Zhabko, A.P. On the stability of solutions of nonlinear difference systems. *Russian Mathematics (Izvestiya VUZ. Matematika)* **49**(2) (2005) 1–10.

[8] Aleksandrov, A.Yu. and Zhabko, A.P. Preservation of stability under discretization of systems of ordinary differential equations. *Siberian Math. J.* **51**(3) (2010) 383–395.

[9] Bhaskar, T.G. and Shaw, M. Stability results for set difference equations. *Dynamic Systems and Applications* **13**(3/4) (2004) 479–485.

[10] Branicky, M.S. Multiple Lyapunov functions and other analysis tools for switched and hybrid systems. *IEEE Transactions on Automatic Control* **43**(4) (1998) 475–482.

[11] Conti, R. Sulla prolungabilita delle soluzioni di un sistema di equazioni differenziali ordinarie. *Bolletino U.M.I.* **3**(11) (1956) 510–514. [Italian]

[12] Corduneanu, C. Application of differential inequalities to stability theory. *Analele Stiintifice Univ. Iasi* **VI** (1960) 47–58. [Russian]

[13] Decarlo, R.A., Branicky, M.S, Pettersson, S. and Lennartson, B. Perspectives and results on the stability and stabilizability of hybrid systems. *Proc. IEEE* **88** (2000) 1069–1082.

[14] Hardy, G.H., Littlewood, J.E. and Polya G. *Inequalities.* Cambridge University Press, Cambridge, 1952.

[15] Heinen, J.A. Difference inequalities and comparison theorems for stability of discrete systems. *Intern. J. Systems Science* **10**(6) (1979) 711–718.

[16] Hou, L., Zong, G. and Wu, Y. Robust exponential stability analysis of discrete-time switched Hopfield neural networks with time delay. *Nonlinear Analysis: Hybrid Systems* **5**(3) (2011) 525–534.

[17] Huang, Y., Luo, J., Huang, T. and Xiao, M. The set of stable switching sequences for discrete-time linear switched systems. *J. Math. Anal. Appl.* **377**(2) (2011) 732–743.

[18] Lakshmikantham, V., Bhaskar, T.G. and Vasundhara, D.J. *Theory of Set Differential Equations in Metric Spaces.* Cambridge Scientific Publishers, Cambridge, 2006.

[19] Lakshmikantham, V., Leela, S. and Martynyuk, A.A. *Stability Analysis of Nonlinear Systems.* Marcel Dekker, New York, 1989.

[20] LaSalle, J.P. *The Stability and Control of Discrete Process.* Springer, New York, 1986.

[21] Liberzon, D. *Switching in Systems and Control.* Birkhauser, Boston, MA, 2003.

[22] Martynyuk, A.A. *Stability of Motion. The Role of Multicomponent Liapunov's Functions.* Cambridge: Cambridge Scientific Publishers, 2007.

[23] Martynyuk, A.A. Asymptotic stability criterion for nonlinear monotonic systems and its applications (review). *Int. Applied Mechanics* **47**(5) (2011) 475–534.

[24] Martynyuk, A.A. and Obolenskij, A.Yu. Stability of solutions of autonomous Wazewskij systems. *Differ. Equ.* **16**(8) (1981) 890–901.

[25] Matrosov, V.M. *The Method of Vector Lyapunov Functions: Analysis of Dynamical Properties of Nonlinear Systems.* Fizmatlit, Moscow, 2001. [Russian]

[26] Minailo, A.V. On the stability of solutions of some classes of nonlinear difference systems. *Vestn. Samarskogo Gos. Tehn. Universiteta. Ser. Phiz.-Mat. Nauki* **43** (2006) 37–44. [Russian]

[27] Rosier, L. Homogeneous Lyapunov function for homogeneous continuous vector field. *Systems Control Lett.* **19** (1992) 467–473.

[28] Rouche, N., Habets, P. and Laloy, M. *Stability Theory by Liapunov's Direct Method.* Springer, New York etc., 1977.

[29] Zappavigna, A., Colaneri, P., Geromel, J.C. and Shorten, R. Dwell time analysis for continuous-time switched linear positive systems. In: *Proc. American Control Conf.* Baltimore, Maryland, USA, 2010, 6256–6261.

[30] Zubov, V.I. *Mathematical Methods for the Study of Automatical Control Systems.* Pergamon Press, New York etc.; Academic Press, Yerusalem, 1962.

Nonlinear Dynamics and Systems Theory, **16** (3) (2016) 235–245

Existence Results for Sobolev Type Fractional Differential Equation with Nonlocal Integral Boundary Conditions

Renu Chaudhary * and Dwijendra N. Pandey

Department of Mathematics,
Indian Institute of Technology Roorkee,
Roorkee-247667, Uttarakhand, India

Received: July 27, 2015; Revised: June 9, 2016

Abstract: In this paper, a Sobolev type fractional differential equation with nonlocal integral boundary condition is investigated. The theory of resolvent operators, fractional calculus and fixed point techniques are used to study the existence results to the given equation. In the end, an example is provided to illustrate the applications of the abstract results.

Keywords: *fractional differential equations; fixed point theorems; resolvent operator; nonlocal boundary conditions.*

Mathematics Subject Classification (2010): 34A08, 34B10, 34G20.

1 Introduction

In a few decades, fractional differential equations have received much attention of researchers mainly due to their extensive interesting applications in physics, mechanics and engineering such as electrochemistry, control theory, signal and image processing, porous media, electromagnetism etc.(see [23], [24], [29]). The fact, that fractional derivative (integral) is an operator which includes integer order derivatives (integrals) as special case and describes the hereditary properties and memory effects of various materials, is the reason why fractional differential equations are more precise in the modeling of many phenomena. Many physical phenomena such as seepage flow in porous media and in fluid dynamic traffic models [20] and nonlinear oscillations of earthquakes [21] can be described

* Corresponding author: mailto:rrenu94@gmail.com

by the fractional differential equations. For a good introduction and applications to fractional differential equations we refer the reader to [25], [30] and [33]. Recently, boundary value problems for nonlinear fractional differential equations have been investigated by many researchers, see [1]- [5], [26]- [28], [34] and [36].

The Sobolev type fractional differential equations can be considered as an abstract formulation of partial differential equations which occurs in various applications such as the flow of fluid through fissured rocks [6], thermodynamics [14], and shear in second order fluids [22], [35]. There are many papers dealing with the investigation on the existence of solutions for Sobolev type differential equations in Banach spaces see [7]- [11].

In [18] Hernàndez et al. talked about an error in some papers regarding the problem of existence of a solution for abstract fractional differential equation and proposed a different approach to treat a general class of abstract fractional differential equation based on the theory of resolvent operators. But the results in [18] were not relevant for the problems with nonlocal conditions. Then in [19] Hernàndez et al. studied the theory of abstract fractional differential equations with nonlocal conditions and proved the existence results using resolvent operators. In [10], [11] Balachandran et al. studied the existence of mild solution for fractional integro-differential equation with nonlocal conditions and abstract fractional integro-differential equation of Sobolev type respectively by using the theory of resolvent operator. In [12] Belmekki et al. established the sufficient conditions for existence and uniqueness results for semilinear fractional differential equations with finite delay via resolvent operators. In [13] Belmekki et al. extended the results given in [12] to cover the case of infinite delay. Recently in [16] Chadha et al. discussed the existence results of history valued neutral fractional differential equation with the help of the theory of resolvent operators. For more details on resolvent operators see [15], [17], [31].

Up to now, to the best of our knowledge, there is a little gap in the literature on the Sobolev type fractional differential equation of order $1 < \beta \leqslant 2$ with nonlocal integral boundary condition using resolvent operators. Motivated by the above papers, to fill this gap, in this paper we consider the following Sobolev type fractional differential equation with nonlocal integral boundary conditions

$$\begin{cases} {}^{C}\mathbf{D}^{\beta}[Bx(t)] = Ax(t) + \mathcal{F}(t, x(t)), \quad 1 < \beta \leqslant 2, \quad t \in (0,1), \\ x(0) = 0, \quad x(\varepsilon) = c\int_{\eta}^{1} x(s)ds, \qquad 0 < \varepsilon < \eta < 1, \end{cases} \tag{1}$$

where ${}^{C}\mathbf{D}^{\beta}$ is the Caputo fractional derivative of order β. A is a closed linear unbounded operator, B is linear operator. $\mathcal{F} : [0,1] \times X \to X$ is continuous function. c is a positive real constant. The nonlocal integral boundary condition $x(\varepsilon) = c\int_{\eta}^{1} x(s)ds$ shows that the value of the unknown function at a nonlocal point $\varepsilon \in (0,1)$ with $0 < \varepsilon < \eta < 1$ is proportional to the integration over a sub-strip $(\eta, 1)$ of an unknown function.

2 Preliminaries

In this segment, we have some basic notations, definitions, theorems and lemmas of fractional calculus and resolvent operators which will be used in the further sections. Let $(X, \|.\|)$ be a Banach space and $\mathcal{C} = C([0,1], X)$ be the Banach space of all continuous functions from $[0,1]$ to X equipped with the norm $\|x\| = \sup_{t\in[0,1]} \|x(t)\|_X$. X_H denotes the domain of $H := B^{-1}A$ endowed with the graph norm $\|x\|_H = \|x\| + \|Hx\|$. Let $L^p(J, X)$ be the Banach space of all Bochner measurable functions $x : J \to X$ such that $\|x(t)\|_X^p$

is integrable equipped with the norm

$$\|x\|_{L^p(J,X)} = \left(\int_J \|x(s)\|_X^p ds \right)^{1/p}.$$

Definition 2.1 [33] The fractional integral of order β for a function $\mathcal{F} \in L^1(\mathbb{R}^+)$ is defined by

$$I_{0+}^{\beta} \mathcal{F}(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \mathcal{F}(s) ds, \quad t > 0, \quad \beta > 0.$$

Definition 2.2 [24] The Caputo fractional derivative of order β for a function $\mathcal{F} \in C^{m-1}(\mathbb{R}^+) \cap L^1(\mathbb{R}^+)$ is defined by

$${}^c\mathbf{D}_{0+}^{\beta} \mathcal{F}(t) = \frac{1}{\Gamma(m-\beta)} \int_0^t (t-s)^{m-\beta-1} \mathcal{F}^m(s) ds,$$

where $m-1 < \beta < m,\ m = [\beta]+1$ and $[\beta]$ denotes the integral part of the real number β.

Lemma 2.1 [30] *Let $q > 0$, then*

$$D^{-\beta} D^{\beta} \mathcal{F}(t) = \mathcal{F}(t) + C_1 t^{\beta-1} + C_2 t^{\beta-2} + \ldots + C_n t^{\beta-1},$$

for some $C_i \in \mathbb{R}$, $i = 1, 2, \ldots, n$, $n = [\beta]+1$.

To prove the existence results we admit the following hypotheses:

(H1) The linear unbounded operator $A : D(A) \subset X \to X$ and linear bijective operator $B : D(B) \subset D(A) \subset X \to X$ are closed linear operators.

(H2) $B^{-1} : X \to D(B)$ is a continuous operator.

(H3) The function $\mathcal{F} : [0,1] \times X \to X$ is a continuous function such that

$$\|\mathcal{F}(t,x) - \mathcal{F}(t,y)\| \leqslant L\|x-y\|, \tag{2}$$

for all $x, y \in X$, $t \in [0,1]$ and L is a positive constant.

Lemma 2.2 *For any functions $\mathcal{F} \in C([0,1] \times X, X)$, the solution of Sobolev type fractional boundary value problem*

$$\begin{cases} {}^C\boldsymbol{D}^{\beta}[Bx(t)] = Ax(t) + \mathcal{F}(t, x(t)), & 1 < \beta \leqslant 2, \quad t \in (0,1), \\ x(0) = 0, \quad x(\varepsilon) = c\int_{\eta}^{1} x(s)ds, & 0 < \varepsilon < \eta < 1, \end{cases} \tag{3}$$

is given by

$$x(t) \;=\; C_1 t + \frac{1}{\Gamma\beta} \int_0^t (t-\tau)^{\beta-1} (B^{-1}Ax(\tau) + B^{-1}\mathcal{F}(\tau, x(\tau))) d\tau, \tag{4}$$

where

$$\begin{aligned} C_1 \;=\; & \frac{1}{\Lambda} \Big\{ \frac{c}{\Gamma\beta} \int_{\eta}^{1} \Big[\int_0^s (s-\tau)^{\beta-1} (B^{-1}Ax(\tau) + B^{-1}\mathcal{F}(\tau, x(\tau))) d\tau \Big] ds \\ & - \frac{1}{\Gamma\beta} \int_0^{\varepsilon} (\varepsilon-\tau)^{\beta-1} (B^{-1}Ax(\tau) + B^{-1}\mathcal{F}(\tau, x(\tau))) d\tau \Big\} \end{aligned} \tag{5}$$

with $\Lambda = \varepsilon - \frac{c}{2}(1-\eta^2) \neq 0$.

Proof. Using Lemma 2.1, the solution x of (3) can be written as

$$x(t) = C_1 t + C_2 + \frac{1}{\Gamma\beta}\int_0^t (t-\tau)^{\beta-1} B^{-1}Ax(\tau)d\tau + \frac{1}{\Gamma\beta}\int_0^t (t-\tau)^{\beta-1} B^{-1}\mathcal{F}(\tau, x(\tau))d\tau,$$

for some constants C_1, $C_2 \in \mathbb{R}$.
On applying boundary conditions, we get $C_2 = 0$ and

$$\begin{aligned} C_1 &= \frac{1}{\Lambda}\bigg\{\frac{c}{\Gamma\beta}\int_\eta^1 \bigg[\int_0^s (s-\tau)^{\beta-1}(B^{-1}Ax(\tau) + B^{-1}\mathcal{F}(\tau, x(\tau)))d\tau\bigg] ds \\ &\quad -\frac{1}{\Gamma\beta}\int_0^\varepsilon (\varepsilon-\tau)^{\beta-1}(B^{-1}Ax(\tau) + B^{-1}\mathcal{F}(\tau, x(\tau))d\tau\bigg\}. \end{aligned}$$

Equation (4) can also be written as

$$x(t) = k(t) + \frac{1}{\Gamma\beta}\int_0^t (t-\tau)^{\beta-1} B^{-1}Ax(\tau)d\tau, \tag{6}$$

where $k(t) = C_1 t + \frac{1}{\Gamma\beta}\int_0^t (t-\tau)^{\beta-1} B^{-1}\mathcal{F}(\tau, x(\tau))d\tau$.
Let $B^{-1}A = H$. To demonstrate existence results, let us assume that integral equation (6) has an associated resolvent operator $\{\mathcal{S}(t),\, t \geqslant 0\}$ on X.

Definition 2.3 [31] A one parameter family of bounded linear operators $\{\mathcal{S}(t),\, t \geqslant 0\}$ on X is called a resolvent operator for (6) if the following conditions are satisfied.

1. $\mathcal{S}(t)$ is strongly continuous on $\mathbb{R}_+$ and $\mathcal{S}(0) = I$,
2. $\mathcal{S}(t)D(H) \subset D(H)$ and $H\mathcal{S}(t)x = \mathcal{S}(t)Hx \;\forall\, x \in D(H)$ and $t \geqslant 0$,
3. for every $x \in D(H)$ and $t \geqslant 0$,

$$\mathcal{S}(t)x = x + \frac{1}{\Gamma\beta}\int_0^t (t-\tau)^{\beta-1} H\mathcal{S}(\tau)x d\tau. \tag{7}$$

Definition 2.4 [31] A resolvent operator $\{\mathcal{S}(t),\, t \geqslant 0\}$ for (6) is called differentiable if $\mathcal{S}(.)x \in W^{1,1}_{loc}(\mathbb{R}^+, X)$ ($W^{1,1}_{loc}(\mathbb{R}^+, X)$ is the space of all functions having distributional derivatives)for all $x \in D(H)$ and there exists $\phi_H \in L^1_{loc}(\mathbb{R}^+)$ such that $\|\mathcal{S}'(t)x\| \leqslant \phi_H(t)\|x\|_{X_H}$ $\forall x \in D(H)$.

Definition 2.5 [31] A resolvent operator $\{\mathcal{S}(t),\, t \geqslant 0\}$ for (6) is called analytic if the operator $S(t) : (0, \infty) \to L(X)$ ($L(X)$ denotes the space of all bounded linear operators from X to X) admits an analytic extension to a sector $\Sigma_{0,\theta} = \{\lambda \in \mathbb{C} : |arg(\lambda)| < \theta_0\}$ for some $0 < \theta_0 \leqslant \pi/2$.

Definition 2.6 A function $x \in \mathcal{C}$ is called a mild solution of the integral equation (6) if $\int_0^t (t-\tau)^{\beta-1}x(\tau)d\tau \in D(H)$ for all $t \in [0, 1]$, $k(t) \in \mathcal{C}$ and

$$x(t) = k(t) + \frac{H}{\Gamma\beta}\int_0^t (t-\tau)^{\beta-1} x(\tau)d\tau. \tag{8}$$

Lemma 2.3 [31] *If $\mathcal{S}(t)$ is the resolvent operator for (6).*

(i) *If x is a solution of (6) on $[0,1]$, then the function $t \to \int_0^t \mathcal{S}(t-s)k(s)ds$ is continuously differential on $[0,1]$ and*

$$x(t) = \frac{d}{dt}\int_0^t \mathcal{S}(t-s)k(s)ds, \forall t \in [0,1]. \tag{9}$$

(ii) *If $\mathcal{S}(t)$ is analytic and $k \in C^\alpha([0,1],X)$ for some $\alpha \in (0,1)$, then the function defined by*

$$x(t) = \mathcal{S}(t)(k(t)-k(0)) + \int_0^t \mathcal{S}'(t-s)[k(s)-k(t)]ds + \mathcal{S}(t)k(0), \forall t \in [0,1], \tag{10}$$

is a mild solution of (6).

(iii) *If $\mathcal{S}(t)$ is differentiable and $k \in C([0,1],X_H)$, then the function $x:[0,1]\to X$ given by*

$$x(t) = k(t) + \int_0^t \mathcal{S}'(t-s)k(s)ds, \forall t \in [0,1], \tag{11}$$

is a mild solution of (6).

3 Existence of Mild Solution

In this segment, we discuss the existence of mild solution for boundary value problem (1). Throughout this paper, we assume that the resolvent operator $\{\mathcal{S}(t), t \geqslant 0\}$ is a differential operator and function $\mathcal{F}$ is continuous in X_H.

By the help of Lemma (2.3)(iii), we introduce the mild solution of (6) given by

$$\begin{aligned} x(t) &= C_1 t + \frac{1}{\Gamma\beta}\int_0^t (t-\tau)^{\beta-1}B^{-1}\mathcal{F}(\tau,x(\tau))d\tau \\ &+ \int_0^t \mathcal{S}'(t-s)\Big(C_1 s + \frac{1}{\Gamma\beta}\int_0^s (s-\tau)^{\beta-1}B^{-1}\mathcal{F}(\tau,x(\tau))d\tau\Big)ds. \end{aligned} \tag{12}$$

For simplification, let $N = \max_{t\in[0,1]} \mathcal{F}(t,0)$, $R = \|B^{-1}\|$, $P = \|B^{-1}A\|$.

Theorem 3.1 *Let $(H1)-(H4)$ hold with*

$$\delta = (1+\|\phi_H\|_{L^1})\frac{(LR+P)}{|\Lambda|}\left[\frac{c(1-\eta^{\beta+1})}{\Gamma(\beta+2)} - \frac{\varepsilon^\beta}{\Gamma(\beta+1)}\right] < 1. \tag{13}$$

Then there exists a mild solution of (1) on $[0,1]$.

Proof. Let $\mathcal{B}_r = \{x \in \mathcal{C} : \|x\| \leqslant r\}$ such that

$$r \geqslant (1+\|\phi_H\|_{L^1})\left[\frac{(Pr + R(Lr+N))}{|\Lambda|}\left\{\frac{c(1-\eta^{1+\beta})}{\Gamma(\beta+2)} - \frac{\varepsilon^\beta}{\Gamma(\beta+1)}\right\} + \frac{R(Lr+N)}{\Gamma(\beta+1)}\right]. \tag{14}$$

Introduce the map $\Phi : \mathcal{C} \to \mathcal{C}$ by

$$\begin{aligned} \Phi x(t) &= C_1 t + \frac{1}{\Gamma\beta}\int_0^t (t-\tau)^{\beta-1}B^{-1}\mathcal{F}(\tau,x(\tau))d\tau \\ &+ \int_0^t \mathcal{S}'(t-s)\Big(C_1 s + \frac{1}{\Gamma\beta}\int_0^s (s-\tau)^{\beta-1}B^{-1}\mathcal{F}(\tau,x(\tau))d\tau\Big)ds. \end{aligned} \tag{15}$$

Decompose the map Φ into Φ_1 and Φ_2 on $\mathcal{B}_r$ for $t \in [0,1]$ such that

$$\begin{aligned}
\Phi_1 x(t) &= \frac{t}{\Lambda}\bigg\{\frac{c}{\Gamma\beta}\int_\eta^1 \Big(\int_0^s (s-\tau)^{\beta-1}(B^{-1}Ax(\tau)+B^{-1}\mathcal{F}(\tau,x(\tau)))d\tau\Big)ds \\
&\quad -\frac{1}{\Gamma\beta}\int_0^\varepsilon (\varepsilon-\tau)^{\beta-1}(B^{-1}Ax(\tau)+B^{-1}\mathcal{F}(\tau,x(\tau)))d\tau\bigg\} \\
&\quad +\int_0^t \mathcal{S}'(t-s)\bigg[\frac{s}{\Lambda}\bigg\{\frac{c}{\Gamma\beta}\int_\eta^1\Big(\int_0^v (v-\tau)^{\beta-1}(B^{-1}Ax(\tau)+B^{-1}\mathcal{F}(\tau,x(\tau)))d\tau\Big)dv \\
&\quad -\frac{1}{\Gamma\beta}\int_0^\varepsilon (\varepsilon-\tau)^{\beta-1}(B^{-1}Ax(\tau)+B^{-1}\mathcal{F}(\tau,x(\tau)))d\tau\bigg\}\bigg]ds. \\
\Phi_2 x(t) &= \frac{1}{\Gamma\beta}\int_0^t (t-\tau)^{\beta-1}B^{-1}\mathcal{F}(\tau,x(\tau))d\tau \\
&\quad +\int_0^t \mathcal{S}'(t-s)\bigg(\frac{1}{\Gamma\beta}\int_0^s (s-\tau)^{\beta-1}B^{-1}\mathcal{F}(\tau,x(\tau))d\tau\bigg)ds.
\end{aligned}$$

Step 1. We show that $\Phi_1 x + \Phi_2 y \in \mathcal{B}_r$ for every $x,\, y \in \mathcal{B}_r$, we have

$$\begin{aligned}
\|\Phi_1 x+\Phi_2 y\| &\leqslant \sup_{t\in[0,1]}\bigg\{\frac{t}{|\Lambda|}\bigg\{\frac{c}{\Gamma\beta}\int_\eta^1\Big(\int_0^s (s-\tau)^{\beta-1}(\|B^{-1}A\|\|x(\tau)\| \\
&\quad +\|B^{-1}\|\|\mathcal{F}(\tau,x(\tau))-\mathcal{F}(\tau,0)+\mathcal{F}(\tau,0)\|)d\tau\Big)ds \\
&\quad -\frac{1}{\Gamma\beta}\int_0^\varepsilon (\varepsilon-\tau)^{\beta-1}(\|B^{-1}A\|\|x(\tau)\| \\
&\quad +\|B^{-1}\|\|\mathcal{F}(\tau,x(\tau))-\mathcal{F}(\tau,0)+\mathcal{F}(\tau,0)\|)d\tau\bigg\} \\
&\quad +\int_0^t \mathcal{S}'(t-s)\bigg[\frac{s}{|\Lambda|}\bigg\{\frac{c}{\Gamma\beta}\int_\eta^1\Big(\int_0^v (v-\tau)^{\beta-1}(\|B^{-1}A\|\|x(\tau)\| \\
&\quad +\|B^{-1}\|\mathcal{F}(\tau,x(\tau))-\mathcal{F}(\tau,0)+\mathcal{F}(\tau,0)\|)d\tau\Big)dv \\
&\quad -\frac{1}{\Gamma\beta}\int_0^\varepsilon (\varepsilon-\tau)^{\beta-1}(\|B^{-1}A\|\|x(\tau)\| \\
&\quad +\|B^{-1}\|\|\mathcal{F}(\tau,x(\tau))-\mathcal{F}(\tau,0)+\mathcal{F}(\tau,0)\|)d\tau\bigg\}\bigg]ds \\
&\quad +\frac{\|B^{-1}\|}{\Gamma\beta}\int_0^t (t-\tau)^{\beta-1}\|\mathcal{F}(\tau,y(\tau))-\mathcal{F}(\tau,0)+\mathcal{F}(\tau,0)\|d\tau \\
&\quad +\int_0^t \|\mathcal{S}'(t-s)\|\bigg(\frac{\|B^{-1}\|}{\Gamma\beta}\int_0^s (s-\tau)^{\beta-1} \\
&\quad \|\mathcal{F}(\tau,y(\tau))-\mathcal{F}(\tau,0)+\mathcal{F}(\tau,0)\|d\tau\bigg)ds\bigg\} \\
&\leqslant (1+\|\phi_H\|_{L^1})\bigg[\frac{(Pr+R(Lr+N))}{|\Lambda|}\bigg\{\frac{c(1-\eta^{\beta+1})}{\Gamma(\beta+2)}-\frac{\varepsilon^\beta}{\Gamma(\beta+1)}\bigg\} \\
&+ \quad \frac{R(Lr+N)}{\Gamma(\beta+1)}\bigg] \leqslant r.
\end{aligned}$$

Thus $\Phi_1 x + \Phi_2 y \in \mathcal{B}_r$.

Step 2. We show that Φ_1 is a contraction. For x, $y \in \mathcal{B}_r$ and $t \in [0,1]$, we have

$$\begin{aligned}
\|\Phi_1 x - \Phi_1 y\| \leqslant & \sup_{t\in[0,1]} \Bigg\{ \frac{t}{|\Lambda|} \Bigg\{ \frac{c}{\Gamma\beta} \int_\eta^1 \Big(\int_0^s (s-\tau)^{\beta-1} (\|B^{-1}A\| \|x(\tau) - y(\tau)\| \\
& + \|B^{-1}\| \|\mathcal{F}(\tau, x(\tau)) - \mathcal{F}(\tau, y(\tau))\|) d\tau \Big) ds \\
& - \frac{1}{\Gamma\beta} \int_0^\varepsilon (\varepsilon - \tau)^{\beta-1} (\|B^{-1}A\| \|x(\tau) - y(\tau)\| \\
& + \|B^{-1}\| \|\mathcal{F}(\tau, x(\tau)) - \mathcal{F}(\tau, y(\tau))\|) d\tau \Bigg\} \\
& + \int_0^t \mathcal{S}'(t-s) \Bigg[\frac{s}{|\Lambda|} \Bigg\{ \frac{c}{\Gamma\beta} \int_\eta^1 \Big(\int_0^v (v-\tau)^{\beta-1} (\|B^{-1}A\| \|x(\tau) - y(\tau)\| \\
& + \|B^{-1}\| \mathcal{F}(\tau, x(\tau)) - \mathcal{F}(\tau, y(\tau))\|) d\tau \Big) dv \\
& - \frac{1}{\Gamma\beta} \int_0^\varepsilon (\varepsilon - \tau)^{\beta-1} (\|B^{-1}A\| \|x(\tau) - y(\tau)\| \\
& + \|B^{-1}\| \|\mathcal{F}(\tau, x(\tau)) - \mathcal{F}(\tau, y(\tau))\|) d\tau \Bigg\} \Bigg] ds \Bigg\} \\
\leqslant & (1 + \|\phi_H\|_{L^1}) \frac{(P + RL)}{|\Lambda|} \left(\frac{c(1-\eta^{\beta+1})}{\Gamma(\beta+2)} - \frac{\varepsilon^\beta}{\Gamma(\beta+1)} \right) \|x - y\| \\
\leqslant & \delta \|x - y\|.
\end{aligned}$$

By assumption, $\delta < 1$ and therefore Φ_1 is a contraction.

Step 3. Next, we prove that Φ_2 is continuous and compact. The continuity of map Φ_2 can be obtained from the continuity of $\mathcal{F}$. Also for $t \in [0,1]$

$$\begin{aligned}
\|\Phi_2\| \leqslant & \sup_{t\in[0,1]} \Bigg(\frac{1}{\Gamma\beta} \int_0^t (t-\tau)^{\beta-1} \|B^{-1}\| \|\mathcal{F}(\tau, x(\tau))\| d\tau \\
& + \int_0^t \|S'(t-s)\| \left(\frac{1}{\Gamma\beta} \int_0^s (s-\tau)^{\beta-1} \|B^{-1}\| \|\mathcal{F}(\tau, x(\tau))\| d\tau \right) ds \Bigg) \\
\leqslant & (1 + \|\phi_H\|_{L^1}) \frac{R(Lr + N)}{\Gamma(\beta+1)}.
\end{aligned}$$

i.e. Φ_2 is uniformly bounded $\mathcal{B}_r$. Now we show that the set $\{\Phi_2 x(t) : x \in \mathcal{B}_r\}$ is relatively compact in Y for all $t \in [0,1]$. Clearly the set $\{\Phi_2 x(0) : x \in \mathcal{B}_r\}$ is compact. Fix $t \in (0,1]$, let δ be a real number satisfying $0 < \delta < 1$. For $x \in \mathcal{B}_r$, define the operator Φ_2^δ by

$$\begin{aligned}
\Phi_2^\delta x(t) = & \frac{1}{\Gamma\beta} \int_0^{t-\delta} (t-\tau)^{\beta-1} B^{-1} \mathcal{F}(\tau, x(\tau)) d\tau \\
& + \int_0^{t-\delta} S'(t-s) \left(\frac{1}{\Gamma\beta} \int_0^s (s-\tau)^{\beta-1} B^{-1} \mathcal{F}(\tau, x(\tau)) d\tau \right) ds.
\end{aligned}$$

By assumption $(H4)$, $\mathcal{F}$ is completely continuous, the set $\{\Phi_2^\delta x(t) : x \in \mathcal{B}_r\}$ is precompact in X, for every $\delta \in (0,1]$. Furthermore, for every $x \in \mathcal{B}_r$, we have

$$\begin{aligned}
\|\Phi_2 x(t) - \Phi_2^\delta x(t)\| \leqslant & \frac{1}{\Gamma\beta} \int_{t-\delta}^t (t-\tau)^{\beta-1} \|B^{-1}\| \|\mathcal{F}(\tau, x(\tau))\| d\tau \\
& + \int_{t-\delta}^t \mathcal{S}'(t-s) \left(\frac{1}{\Gamma\beta} \int_0^s (s-\tau)^{\beta-1} \|B^{-1}\| \|\mathcal{F}(\tau, x(\tau))\| d\tau \right) ds.
\end{aligned}$$

It shows that the precompact sets $\{\Phi_2^{\delta}x(t) : x \in \mathcal{B}_r\}$ are arbitrary close to the set $\{\Phi_2 x(t) : x \in \mathcal{B}_r\}$. Hence the set $\{\Phi_2 x(t) : x \in \mathcal{B}_r\}$ is precompact in X.
Step 4. Now, we show that $\{\Phi_2 x(t) : x \in \mathcal{B}_r\}$ is equicontinuous. Clearly $\{\Phi_2 x(t) : x \in \mathcal{B}_r\}$ are equicontinuous at $t = 0$. For $t < t + h \leqslant 1$, $h > 0$, we have

$$\begin{aligned}
\|\Phi_2 x(t+h) - \Phi_2 x(t)\| \leqslant\ & \frac{1}{\Gamma\beta}\bigg\| \int_0^{t+h} (t+h-\tau)^{\beta-1} B^{-1}\mathcal{F}(\tau, x(\tau))d\tau \\
& - \int_0^{t} (t-\tau)^{\beta-1} B^{-1}\mathcal{F}(\tau, x(\tau))d\tau \bigg\| \\
& + \frac{1}{\Gamma\beta}\bigg\| \int_0^{t+h} \mathcal{S}'(t+h-s) \int_0^{s} (s-\tau)^{\beta-1} B^{-1}\mathcal{F}(\tau, x(\tau))d\tau ds \\
& - \int_0^{t} \mathcal{S}'(t-s) \int_0^{s} (s-\tau)^{\beta-1} B^{-1}\mathcal{F}(\tau, x(\tau))d\tau ds \bigg\| \\
\leqslant\ & \frac{1}{\Gamma\beta} \int_0^{t} \Big[(t+h-\tau)^{\beta-1} - (t-\tau)^{\beta-1} \Big] \|B^{-1}\| \|\mathcal{F}(\tau, x(\tau))\| d\tau \\
& + \frac{1}{\Gamma\beta} \int_t^{t+h} (t+h-\tau)^{\beta-1} \|B^{-1}\| \|\mathcal{F}(\tau, x(\tau))\| d\tau \\
& + \int_0^{h} \|\mathcal{S}'(t+h-s)\| \frac{1}{\Gamma\beta} \int_0^{s} (s-\tau)^{\beta-1} \|B^{-1}\| \|\mathcal{F}(\tau, x(\tau))\| d\tau ds \\
& + \int_0^{t} \|\mathcal{S}'(t-s)\| \frac{\|B^{-1}\|}{\Gamma\beta} \bigg\| \int_0^{s+h} (s+h-\tau)^{\beta-1} \mathcal{F}(\tau, x(\tau))d\tau \\
& - \int_0^{s} (s-\tau)^{\beta-1} \mathcal{F}(\tau, x(\tau))d\tau \bigg\| ds.
\end{aligned}$$

Which tends to zero as $h \to 0$, therefore the set $\{\Phi_2 x(t) : x \in \mathcal{B}_r\}$ is equicontinuous. Thus Φ_2 is relatively compact for $t \in [0, 1]$. By Arzela-Ascoli's theorem Φ_2 is compact. Hence by Krasnoselskii fixed point theorem [32] there exists a fixed point $x \in \mathcal{C}$ such that $\Phi x = x$ which is a mild solution of the boundary value problem (1).

4 Example

Let $X = L^2(0, \pi)$, $1 < \beta \leqslant 2$ and $t \in [0, 1]$. Consider the following partial differential equation with fractional derivative

$$\begin{cases} \frac{\partial^\beta}{\partial t^\beta}\Big(w(t,x) - \frac{\partial^2}{\partial x^2} w(t,x) \Big) = \frac{\partial^2}{\partial x^2} w(t,x) + \frac{w(t,x)}{1+w(t,x)}, \\ w(t,0) = w(t,\pi) = 0, \\ w(0,x) = 0,\ w(\varepsilon, x) = c\int_\eta^1 w(t,s)ds. \end{cases} \tag{16}$$

Define the operators $A : D(A) \subset X \to X$ and $B : D(B) \subset X \to X$ by

$$Aw = w'', \quad Bw = w - w'',$$

where

$$D(A) = D(B) = \{w \in X, w, w' \text{are absolutely continuous}, w'' \in X, w(0) = w(\pi) = 0\}.$$

Then A and B can be written as

$$\begin{aligned} Aw &= \sum_{n=1}^{\infty} n^2 (w, w_n) w_n, \quad w \in D(A), \\ Bw &= \sum_{n=1}^{\infty} (1+n^2)(w, w_n) w_n, \quad w \in D(B), \end{aligned}$$

where $w_n(x) = \sqrt{2/\pi} \sin nx$, $n = 1, 2, \ldots$, is the original set of vectors A. Moreover, we have

$$B^{-1} w = \sum_{n=1}^{\infty} \frac{1}{1+n^2} (w, w_n) w_n,$$

$$Hw = B^{-1} A w = \sum_{n=1}^{\infty} \frac{-n^2}{1+n^2} (w, w_n) w_n.$$

The equation (16) can be reformulated as the following Sobolev type fractional differential equation with nonlocal integral boundary condition

$$\begin{cases} D^{\beta}(Bw(t)) = Aw(t) + \mathcal{F}(t, w(t)), \quad 1 < \beta \leqslant 2, \, t \in (0,1), \\ w(0) = 0, \, w(\varepsilon) = c \int_{\eta}^{1} w(s) ds, \quad 0 < \varepsilon < \eta < 1. \end{cases} \tag{17}$$

Clearly all the assumptions $(H1) - (H4)$ are satisfied.

Theorem 4.1 *Suppose* $(H1) - (H4)$ *hold and* A *generates a differential resolvent operator* $\{\mathcal{S}(t)\}$ *with*

$$\delta = (1 + \|\phi_H\|_{L^1}) \frac{(LR + P)}{|\Lambda|} \left[\frac{c(1 - \eta^{\beta+1})}{\Gamma(\beta+2)} - \frac{\varepsilon^{\beta}}{\Gamma(\beta+1)} \right] < 1.$$

Then the problem (17) has a solution.

Acknowledgment

The authors would like to thank the referee for valuable comments and suggestions. The work of the first author is supported by the "Ministry of Human Resource and Development, India under grant number: MHR-02-23-200-44".

References

[1] Ahmad, B., Alsaedi, A., Assolami, A. and Agarwal, R. P. A new class of fractional boundary value problems. *Adv. Diff. Equ.* **273** (2013) 1–8.

[2] Ahmad, B., Alsaedi, A., Assolami, A. and Agarwal, R. P. A study of Riemann-Liouville fractional nonlocal integral boundary value problems. *Adv. Diff. Equ.* **274** (2013) 1–9.

[3] Ahmad, B., Ntouyas, S. K. and Alsaedi, A. A study of nonlinear fractional differential equations of arbitrary order with Riemann-Liouville type multistrip boundary conditions. *Math. Prob. Engin.* (2013), 9 pages.

[4] Ahmad, B. and Ntouyas, S. K. Existence results for higher order fractional differential inclusions with multi-strip fractional integral boundary conditions. *Electron. J. Qual. Theory Differ. Equ.* **20** (2013) 1–19.

[5] Akiladevi, K. S., Balachandran, K. and Kim, J. K. Existence results for neutral fractional integrodifferential equations with fractional integral boundary conditions. *Nonlinear Func. Anal. and App.* **19** (2) (2014) 251–270.

[6] Barenblat, G., Zheltor, J. and Kochiva, I. Basic concepts in the theory of seepage of homogeneous liquids in fissured rocks. *J. Appl. Math. Mech.* **24** (1960) 1286–1303.

[7] Balachandran, K., Park, D. G. and Kwun, Y. C. Nonlinear integrodifferential equations of Sobolev type with nonlocal conditions in Banach spaces. *Commun. Korean Math. Soc.* **14** (1) (1999) 223–231.

[8] Balachandran, K. and Uchiyama, K. Existence of solutions of nonlinear integrodifferential equations of Sobolev type with nonlocal condition in Banach spaces. *Proc. Indian Acad. Sci. Math. Sci.* **110** (2) (2000) 225–232.

[9] Balachandran, K., Park, J. Y. and Chandrasekaran, M. Nonlocal Cauchy problem for delay integrodifferential equations of Sobolev type in Banach spaces. *Appl. Math. Lett.* **15** (7) (2002) 845–854.

[10] Balachandran, K. and Kiruthika, S. Existence results for fractional integrodifferential equations with nonlocal condition via resolvent operators. *Comput. Math. Appl.* **62** (3) (2011) 1350–1358.

[11] Balachandran, K. and Kiruthika, S. Existence of solutions of abstract fractional integrodifferential equations of Sobolev type. *Computers and Mathematics with Applications* **64** (10) (2012) 3406–3413.

[12] Belmekki, M., Mekhalfi, K. and Ntouyas, S. K. Semilinear functional differential equations with fractional order and finite delay. *Malaya J. of Mate.* **1** (2012) 73–81.

[13] Belmekki, M., Mekhalfi, K. and Ntouyas, S. K. Existence and uniqueness for semilinear fractional differential equations with infinite delay via resolvent operators. *J. of Frac. Cal. and App.* **62** (4) (2013) 267–282.

[14] Chen, P.J. and Curtin, M.E. On a theory of heat conduction involving two temperatures. *Z. Angew. Math. Phys.* **19** (1968) 614–627.

[15] Chen, C., Li, M. On fractional resolvent operator functions. *Semigroup Forum* **80** (2010) 121–142.

[16] Chadha, A. and Pandey, D. N. Existence of a solution for history valued neutral fractional differential equation with a nonlocal condition. *J. Non. Evol. Eqn. and App.* **2** (2014) 13–28.

[17] Desch, W., Prüss, J. Counterexamples for abstract linear Volterra equations. *J. of Int. Equ. and Appl.* **5**, (1) (1993) 29–45.

[18] Hernández, E. , O'Regan, D. and Balachandran, K. On recent developments in the theory of abstract differential equations with fractional derivatives. *Nonlinear Anal.* **73** (10) (2010) 3462–3471.

[19] Hernández, E., O'Regan, D. and Balachandran, K. Existence results for abstract fractional differential equations with nonlocal conditions via resolvent operators. *Indag. Math.* (N.S.) **24** (1) (2013) 68-82.

[20] He, J. H. Some applications of nonlinear fractional differential equations and their approximations. *Bull. of Sci. Tech. and Soci.* **15** (2) (1999) 86–90.

[21] He, J. H. Nonlinear oscillation with fractional derivative and its applications. In:*Proceedings of the International Conference on Vibrating Engineering*. Dalian, Chaina, 1998, 288–291.

[22] Huilgol, R. R. A second order fluid of the differential type. *Internat. J. Non-Linear Mech.* **3** (4) (1968) 471–482.

[23] Hilfer, R. Applications of Fractional Calculus in Physics. *World Scientific*, Singapore, 2000.

[24] Kilbas, A. A., Srivastava, H. M. and Trujillo, J. J. Theory and Applications of Fractional Differential Equations. *North-Holland Mathematics Studies*, Vol. **204**, 2006.

[25] Miller, K. S. and Ross, B. An introduction to the fractional calculus and fractional differential equations. *A Wiley-Interscience Publication*. Wiley, New York, 1993.

[26] Murad, S. A. and Hadid, S.B. Existence and uniqueness theorem for fractional differential equation with integral boundary condition. *J. Frac. Calc. Appl.* **3** (2012) 1–9.

[27] Ntouyas, S.K. Existence results for first order boundary value problems for fractional differential equations and inclusions with fractional integral boundary conditions. *J. Frac. Calc. Appl.* **3** (2012) 1–14.

[28] Ntouyas, S.K. Boundary value problems for nonlinear fractional differential equations and inclusions with nonlocal and fractional integral boundary conditions. *Opuscula Math.* **33** (2013) 117–138.

[29] Oldham, K. B. and Spanier, J. The fractional calculus. *Academic Press*, New York, 1974.

[30] Podlubny, I. Fractional differential equations. *Mathematics in Science and Engineering*, **198**, Academic Press, San Diego, CA, 1999.

[31] Prüss, J. Evolutionary integral equations and applications. *Monographs in Mathematics*, **87**, Birkhäuser, Basel, 1993.

[32] Smart, D. R. Fixed point theorems. *Cambridge Univ. Press, London*, 1974.

[33] Samko, S. G., Kilbas, A.A. and Marichev, O.I. Fractional Integrals and Derivatives. Theory and Applications. *Gordon and Breach*, Yverdon, 1993.

[34] Sitthiwirattham, T., Tariboon, J. and Ntouyas, S. K. Existence results for fractional difference equations with three-point fractional sum boundary conditions. *Discrete Dyn. Nat. Soc.* **2013**, 10 pages.

[35] Ting, T. W. Certain non-steady flows of second-order fluids. *Arch. Rational Mech. Anal.* **14** (1963) 1–26.

[36] Zhong, W. and Lin, W. Nonlocal and multiple-point boundary value problem for fractional differential equations. *Comput. Math. Appl.* **39** (2010) 1345–1351.

Nonlinear Dynamics and Systems Theory, **16** (3) (2016) 246–259

Generalized Monotone Method for Multi-Order 2-Systems of Riemann-Liouville Fractional Differential Equations

Z. Denton [1*] and J.D. Ramírez [2]

[1] *Department of Mathematics, North Carolina A&T State University*
Greensboro, NC, 27411 USA
[2] *Department of Mathematics and Statistics, South Dakota State University*
Brookings, SD 57007 USA

Received: August 9, 2015;   Revised: June 20, 2016

Abstract: In this paper we develop a generalized monotone method for nonlinear multi-order 2-systems of Riemann-Liouville fractional differential equations. That is, the monotone method where the forcing function f can be decomposed into increasing and decreasing components, and applied to a hybrid system of nonlinear equations of orders q_1 and q_2 where $0 < q_1, q_2 < 1$. In the development of this method we recall any needed existence and comparison results along with any necessary changes; including results from needed linear theory. The monotone method is then developed via the construction of sequences of linear systems based on the upper and lower solutions, being then used to approximate the solution of the original nonlinear multi-order system. Finally we develop a numerical application to exemplify our results.

Keywords: *fractional differential systems; multi-order systems; lower and upper solutions; monotone method.*

Mathematics Subject Classification (2010): 34A08, 34A34, 34A45, 34A38.

* Corresponding author: mailto:zdenton@ncat.edu

1 Introduction

Fractional differential equations have various applications in widespread fields of science, such as engineering [6], chemistry [7, 14, 15], physics [1, 2, 8], and others [9, 10]. Despite the number of existence theorems for nonlinear fractional differential equations this does not necessarily imply that calculating a solution explicitly will be possible. Therefore, it may be necessary to employ an iterative technique to numerically approximate a needed solution. In this paper we construct such a method.

Specifically, we construct a technique to approximate solutions to the nonlinear Riemann-Liouville (R-L) fractional differential multi-order 2-system. A multi-order system is a fractional differential system where each component is of unique order. That is, a fractional system of the type

$$\begin{aligned} D^{q_1} x_1 &= f_1(t, x_1, x_2), \\ D^{q_2} x_2 &= f_2(t, x_1, x_2). \end{aligned}$$

This is a generalization of normal R-L systems and yields a type of hybrid system of a fractional type. We note that various complications arise from systems of this type as many known properties used in the study of fractional differential equations require modification, but at the same time multi-order systems present far more possibilities for applications. For example, consider allowing each species in a population model to have their own order of derivative. Though we will consider a numerical example for this study, it will not be a specific physical application, we hope this will add to the groundwork of future studies.

The iterative technique we construct will be a generalization of the monotone method for multi-order R-L 2-systems of order q_1, q_2, where $0 < q_1, q_2 < 1$. The monotone method, in broad terms, is a technique in which unique solutions of linear differential equations are used to construct sequences that converge uniformly and monotonically, from above and below, to maximal and minimal solutions of the nonlinear equation. If the nonlinear DE considered has a unique solution then both sequences will converge uniformly and monotonically to that unique solution. The advantage of the monotone method is that it allows us to approximate solutions to nonlinear DEs using linear DEs. Further, the sequences are constructed initially using upper and lower solutions of the original DE, which guarantees the interval of existence. For more information on the monotone method for ordinary DEs see [11].

One notable complication when developing the monotone method for multi-order systems is that, unlike in the integer order case, the initially constructed sequences, $\{v_n\}, \{w_n\}$ do not converge uniformly on their own. Instead, the weighted sequences $\{t^{1-q_i} v_{n_i}\}$, $\{t^{1-q_i} w_{n_i}\}$ converge uniformly to $t^{1-q_i} v_i$ and $t^{1-q_i} w_i$ respectively, where $i \in \{1, 2\}$ and v, w are maximal and minimal solutions of the original equation. We note that there are other complications that derive from multi-order systems, but many of these were previously resolved in [3].

For our main method we consider the generalization of the monotone method where the nonlinear function can be split into two functions $f(t, x) + g(t, x)$ where f is increasing in x and g is decreasing in x. This generalization allows for various constructions utilizing different types of lower and upper solutions that we will detail in Section 3. Finally, in Section 4 we will develop a numerical application to exemplify our results. We note that the standard monotone method has been established for multi-order fractional systems in [3].

2 Preliminary Results

In this section, we will first consider basic results regarding scalar Riemann-Liouville differential equations of order q, $0 < q < 1$. We will recall basic definitions and results in this case for simplicity, and we note that many of these results carry over naturally to the multi-order case. Then we will consider existence and comparison results for multi-order systems of order $0 < q_1, q_2 < 1$ which will be used in our main result. In the next section, we will apply these preliminary results to develop the monotone method for these multi-order R-L systems. Note, for simplicity we only consider results on the interval $J = (0, T]$, where $T > 0$. Further, we will let $J_0 = [0, T]$, that is $J_0 = \bar{J}$.

Definition 2.1 Let $p = 1-q$, a function $\phi(t) \in C(J, \mathbb{R})$ is a C_p continuous function if $t^p\phi(t) \in C(J_0, \mathbb{R})$. The set of C_p functions is denoted $C_p(J, \mathbb{R})$. Further, given a function $\phi(t) \in C_p(J, \mathbb{R})$ we call the function $t^p\phi(t)$ the continuous extension of $\phi(t)$.

Now we define the R-L integral and derivative of order q on the interval J.

Definition 2.2 Let $\phi \in C_p(J, \mathbb{R})$, then $D_t^q\phi(t)$ is the q-th R-L derivative of ϕ with respect to $t \in J$ defined as

$$D_t^q\phi(t) = \frac{1}{\Gamma(1-q)} \frac{d}{dt} \int_0^t (t-s)^{-q}\phi(s)ds,$$

and $I_t^q\phi(t)$ is the q-th R-L integral of ϕ with respect to $t \in J$ defined as

$$I_t^q\phi(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1}\phi(s)ds.$$

Note that in cases where the initial value may be different or ambiguous, we will write out the definition explicitly. The next definition is related to the solution of linear R-L fractional differential equations and is also of great importance in the study of the R-L derivative.

Definition 2.3 The Mittag-Leffler function with parameters $\alpha, \beta \in \mathbb{R}$, denoted $E_{\alpha,\beta}$, is defined as

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)},$$

which is entire for $\alpha, \beta > 0$.

Of particular importance to the Riemann-Liouville derivative is the weighted Mittag-Leffler function of order q,

$$\mathcal{E} = t^{q-1}E_{q,q}(\lambda t^q) = \sum_{k=0}^{\infty} \frac{\lambda^k t^{qk+q-1}}{\Gamma(qk+q)},$$

where λ is a constant. $\mathcal{E}$ has the following properties which we present in the following remark.

Remark 2.1 We note that the weighted Mittag-Leffler function $\mathcal{E}$ is strictly positive, converges uniformly on compacta of J, and $D^q\mathcal{E} = \lambda\mathcal{E}$.

The next result gives us that the q-th R-L integral of a C_p continuous function is also a C_p continuous function. This result will give us that the solutions of R-L differential equations are also C_p continuous.

Lemma 2.1 *Let $f \in C_p(J, \mathbb{R})$, then $I_t^q f(t) \in C_p(J, \mathbb{R})$, i.e. the q-th integral of a C_p continuous function is C_p continuous.*

Note the proof of this theorem for $q \in R^+$ can be found in [5]. Now we consider results for the nonhomogeneous linear R-L differential equation,

$$D_t^q x(t) = \lambda x(t) + z(t), \tag{1}$$

with initial condition

$$t^p x(t)\big|_{t=0} = x^0,$$

where x^0 is a constant, $y \in C(J_0, \mathbb{R})$, and $z \in C_p(J, \mathbb{R})$, which has unique solution

$$x(t) = \Gamma(q)x^0 t^{q-1} E_{q,q}(\lambda t^q) + \int_0^t (t-s)^{q-1} E_{q,q}(\lambda(t-s)^q) z(s)\, ds.$$

For more details see [12].

Now, we will turn our attention to results for the nonlinear R-L fractional multi-order systems, and in doing so we must discuss any changes. First, we will consider systems of orders q_1 and q_2, $0 \le q_1, q_2 < 1$. For simplicity we will let $q = (q_1, q_2)$, and when we write inequalities $x \le y$, we mean it is true for both components. Further, from this point on, we will use the subscript i which we will always assume is in $\{1, 2\}$. For defining C_p continuity for multi-order systems we define $p_i = 1 - q_i$ and for simplicity of notation we will define the function x_p such that $x_{p_i}(t) = t^{p_i} x_i(t)$ for $t \in J_0$. We also note that at times it will be convenient to ephasize the product of t^p, therefore we will define $t^p x(t) = x_p(t)$ for $t \in J_0$. Now, we define the set of C_p continuous functions as

$$C_p(J, \mathbb{R}^2) = \{x \in C(J, \mathbb{R}^2) \,|\, x_p \in C(J_0, \mathbb{R}^2)\}.$$

For the rest of our results we will be considering the nonlinear R-L fractional multi-order system

$$\begin{aligned} D^{q_i} x_i &= f_i(t, x), \\ x_{p_i}(0) &= x_i^0, \end{aligned} \tag{2}$$

where $f \in C(J_0 \times \mathbb{R}^2, \mathbb{R}^2)$, and x^0 is a constant. Note that just as in the scalar case, a solution $x \in C_p(J, \mathbb{R}^2)$ of (2) also satisfies the equivalent R-L integral equation

$$x_i(t) = x_i^0 t^{q_i - 1} + \frac{1}{\Gamma(q_i)} \int_0^t (t-s)^{q_i - 1} f_i(s, x(s)) ds. \tag{3}$$

Thus, if $f \in C(J_0 \times \mathbb{R}^2, \mathbb{R}^2)$ then (2) is equivalent to (3). See [9, 12] for details.

The following comparison theorem is utilized throughout the construction of the monotone method. This theorem gives conditions for when lower and upper solutions v, w behave in an expected manner, that is $v \le w$. This theorem is of great importance to the monotone method since it is used to prove that the constructed sequences in the method are actually monotone.

Theorem 2.1 *Let $v, w \in C_p(J, R^2)$ be lower and upper solutions of the nonlinear multiorder 2-system, i.e.*

$$\begin{aligned} D^{q_i} v_i &\leq f_i(t, v), \quad v_{p_i}(0) = v_i^0 \leq x_i^0, \\ D^{q_i} w_i &\geq f_i(t, w), \quad w_{p_i}(0) = w_i^0 \geq x_i^0. \end{aligned} \tag{4}$$

If f is quasimonotone nondecreasing and satisfies the following Lipschitz condition for $i = 1, 2$,

$$f_i(t, x) - f_i(t, y) \leq L_i\big[(x_1 - y_1) + (x_2 - y_2)\big], \tag{5}$$

for $x \geq y$, then $v(t) \leq w(t)$ on J provided $v^0 \leq w^0$.

We note that the proof of this theorem can be found in [3]. In the development of the monotone methods we will use a specific corollary from this theorem, which we give below.

Corollary 2.1 *Let $m \in C_p(J, \mathbb{R}^2)$ be such that*

$$D^{q_i} m_i(t) \leq 0, \quad m_{p_i}(0) = 0.$$

Then we have from Theorem 2.1 that

$$m(t) \leq 0,$$

for $t \in J$.

Now, if we know of the existence of lower and upper solutions v and w such that $v \leq w$, we can prove the existence of a solution in the set

$$\Omega = \{(t, y) \,:\, v(t) \leq y \leq w(t),\, t \in J\}.$$

We consider this result in the following theorem.

Theorem 2.2 *Let $v, w \in C_p(J, \mathbb{R}^2)$ be lower and upper solutions of (2) such that $v(t) \leq w(t)$ on J and let $f \in C(\Omega, \mathbb{R})$, where Ω is defined as above. Then there exists a solution $x \in C_p(J, \mathbb{R}^2)$ of (2) such that $v(t) \leq x(t) \leq w(t)$ on J.*

This theorem is proved in the same way as seen in [5], with only minor additions to apply it to multi-order 2-systems.

For our main results we will be considering the following generalized form of (2)

$$D^{q_i} x_i = f_i(t, x) + g_i(t, x), \quad x_{p_i}(0) = x_i^0, \tag{6}$$

where $f, g \in C(J_0 \times \mathbb{R}^2, \mathbb{R}^2)$ such that f is increasing in x and g is decreasing in x. We will be constructing the generalized monotone methods for this nonlinear fractional differential equation. This generalization also allows us to consider various different types of lower and upper solutions given in the following definition.

Definition 2.4 Let $v, w \in C_p(J, \mathbb{R}^2)$ with $v_{p_i}(0) = v_i^0 \leq x_i^0$ and $w_{p_i}(0) = w_i^0 \geq x_i^0$.

- v, w are natural lower and upper solutions of (6) if

$$D^{q_i} v_i \leq f_i(t, v) + g_i(t, v), \quad D^{q_i} w_i \geq f_i(t, w) + g_i(t, w).$$

- v, w are Type I lower and upper solutions of (6) if

$$D^{q_i} v_i \le f_i(t,v) + g_i(t,w), \quad D^{q_i} w_i \ge f_i(t,w) + g_i(t,v).$$

- v, w are Type II lower and upper solutions of (6) if

$$D^{q_i} v_i \le f_i(t,w) + g_i(t,v), \quad D^{q_i} w_i \ge f_i(t,v) + g_i(t,w).$$

- v, w are unnatural lower and upper solutions of (6) if

$$D^{q_i} v_i \le f_i(t,w) + g_i(t,w), \quad D^{q_i} w_i \ge f_i(t,v) + g_i(t,v).$$

Further we can define coupled quasisolutions of these types by incorporating equalities in the previous expressions. We give the two we use in our main results in the following definition.

Definition 2.5 Let $v, w \in C_p(J, \mathbb{R}^2)$ with $v_{p_i}(0) = w_{p_i}(0) = x_i^0$.

- v, w are Type I coupled quasisolutions of (6) if

$$D^{q_i} v_i = f_i(t,v) + g_i(t,w), \quad D^{q_i} w_i = f_i(t,w) + g_i(t,v).$$

- v, w are Type II coupled quasisolutions of (6) if

$$D^{q_i} v_i = f_i(t,w) + g_i(t,v), \quad D^{q_i} w_i = f_i(t,v) + g_i(t,w).$$

We can extend Theorem 2.2 to incorporate these coupled types of lower and upper solutions. We will only look at the cases for Type I and II since those will be the form we use in our monotone method constructions. We note that the proof of the following theorem is constructed in the same manner as Theorem 2.2, needing only very minor alterations.

Theorem 2.3 *Let $v, w \in C_p(J, \mathbb{R}^2)$ be Type I or Type II coupled lower and upper solutions such that $v(t) \le w(t)$ on J and let $f + g \in C(\Omega, \mathbb{R})$, where Ω is defined as above. Then there exists a solution $x \in C_p(J, \mathbb{R}^2)$ of (6) such that $v(t) \le x(t) \le w(t)$ on J.*

3 Monotone Method

In this section we develop the generalized monotone method for fractional system (6). The first method we will construct is developed from Type I lower and upper solutions. The sequences are constructed as linear equations in a recursive manner resembling Type I quasisolutions.

Theorem 3.1 *Suppose that*

($A1$) *$v_0, w_0 \in C_p(J, \mathbb{R}^2)$ are coupled lower and upper solutions of Type I for (6) with $v_0 \le w_0$ on J.*

($A2$) *$f, g \in C(J_0 \times \mathbb{R}^2, \mathbb{R}^2)$, where $f(t,x)$ is increasing in x and $g(t,x)$ is decreasing in x.*

Then the sequences defined by

$$D^{q_i} v_{n+1_i} = f_i(t, v_n) + g_i(t, w_n), \quad v_{n+1_{p_i}}(0) = x_i^0, \tag{7}$$

$$D^{q_i} w_{n+1_i} = f_i(t, w_n) + g_i(t, v_n), \quad w_{n+1_{p_i}}(0) = x_i^0, \tag{8}$$

are such that

$$t^p v_n \to t^p v, \quad t^p w_n \to t^p w$$

uniformly and monotonically on J_0, where v, w are Type I coupled minimal and maximal quasisolutions of (6) respectively, that is, if x is a solution of (6) such that that $v_0 \le x \le w_0$, then $v \le x \le w$.

Proof. We begin by considering v_1 and w_1. We note that both exist and are unique since both are linear in v_1 and w_1 respectively. Now letting $m = v_0 - v_1$, we get that $m_{p_i}(0) = 0$ and

$$D^{q_i} m_i \le 0,$$

implying by Corollary 2.1 that $m_i \le 0$ for each i. Therefore $v_0 \le v_1$, and similarly we can show that $w_1 \le w_0$. Now using a similar process by letting $m = v_1 - w_1$, we get that $m_{p_i}(0) = 0$ and

$$D^{q_i} m_i = f_i(t, v_0) - f_i(t, w_0) + g_i(t, w_0) - g_i(t, v_0) \le 0.$$

Thus, by Corollary 2.1 we have that $m_i \le 0$ for each i, giving us that $v_0 \le v_1 \le w_1 \le w_0$. Using these same arguments we can inductively show that

$$v_{n-1} \le v_n \le w_n \le w_{n-1}$$

on J for all $n \ge 1$, giving us that $\{v_n\}$ and $\{w_n\}$ are monotonic.

Now we will show that the weighted sequences $\{t^p v_n\}$ and $\{t^p w_n\}$ converge uniformly on J_0. To do so we will use the Arzela-Ascoli theorem. First we will show that these sequences are uniformly bounded on J_0. To do so, for each n and each i note that

$$|t^{p_i} v_{ni}| \le |t^{p_i}(v_{ni} - v_{0i})| + |t^{p_i} v_{0i}| \le |t^{p_i}(w_{0i} - v_{0i})| + |t^{p_i} v_{0i}|.$$

Therefore we can choose an $M \in R_+^2$ such that $|t^{p_i} v_{ni}| \le M_i$ for each n and each i, implying that $\{t^p v_n\}$ is uniformly bounded. Similarly we can prove the same result for $\{t^p w_n\}$.

Now we will show that the weighted sequences are equicontinuous. For simplicity, let F_n be defined as $F_n = f(t, v_n) + g(t, w_n)$ for each $n \ge 0$. Since f, g are continuous on J_0, and since each v_n, w_n are C_p continuous then there exist continuous functions $\tilde{f}, \tilde{g}$ such that

$$f(t, v_n) + g(t, w_n) = \tilde{f}(t, t^p v_n) + \tilde{g}(t, t^p w_n).$$

Given this, and that the weighted sequences are uniformly bounded we can choose an $N \in R_+^2$ such that $|F_{ni}| \le N_i$ for each i.

Now, choose t, τ such that $0 < t \le \tau \le T$. In the following proof of equicontinuity we use the fact that

$$\tau^{p_1}(\tau - s)^{q_1 - 1} - t^{p_1}(t - s)^{q_1 - 1} \le 0$$

for $0 < s < t$. To show why this is true, consider the function $\phi(t) = t^{p_1}(t-s)^{q_1-1} = t^{p_1}(t-s)^{-p_1}$ and note that

$$\begin{aligned}\tfrac{d}{dt}\phi(t) &= p_1 t^{p_1-1}(t-s)^{-p_1} - p_1 t^{p_1}(t-s)^{-p_1-1}\\ &= -t^{p_1-1}(t-s)^{-p_1-1}p_1 s \leq 0.\end{aligned}$$

This implies that ϕ is nonincreasing, therefore $\phi(\tau) - \phi(t) \leq 0$. Now consider,

$$\begin{aligned}|\tau^{p_i}v_{ni}(\tau) - t^{p_i}v_{ni}(t)| &\leq \frac{\tau^{p_1}}{\Gamma(q_i)}\int_t^\tau (\tau-s)^{q_i-1}|F_{n-1_i}|ds + \frac{1}{\Gamma(q_i)}\int_0^t |\phi(\tau)-\phi(t)||F_{n-1_i}|ds\\ &\leq \frac{N_i\tau^{p_i}}{\Gamma(q_i)}\int_t^\tau (\tau-s)^{q_i-1}ds + \frac{N_i}{\Gamma(q_i)}\int_0^t \big[\phi(t)-\phi(\tau)\big]ds\\ &= \frac{N_i}{\Gamma(q_i)}\Big[\frac{\tau^{p_i}}{q_i}(\tau-t)^{q_i} + t^{p_i}\int_0^t (t-s)^{q_i-1}ds - \tau^{p_i}\int_0^t (\tau-s)^{q_i-1}ds\Big]\\ &= \frac{N_i}{q_i\Gamma(q_i)}\Big[2\tau^{p_i}(\tau-t)^{q_i} + t - \tau\Big]\\ &\leq \frac{2N_iT^{p_i}}{\Gamma(q_i+1)}(\tau-t)^{q_i}.\end{aligned}$$

In the case when $t = 0$, we note that

$$|\tau^{p_i}v_{ni}(\tau) - x_i^0/\Gamma(q_i)| \leq \frac{N_iT^{p_i}}{\Gamma(q_i)}\int_0^\tau (\tau-s)^{q_i-1}ds = \frac{N_iT^{p_i}}{\Gamma(q_i+1)}\tau^{q_i}.$$

This result is not dependent on n or i, therefore if we define $K \geq 0$ such that

$$K = \max_{i\in\{1,2\}}\Big\{\frac{2N_iT^{p_i}}{\Gamma(q_i+1)}\Big\},$$

then we have that

$$|\tau^{p_i}v_{ni}(\tau) - t^{p_i}v_{ni}(t)| \leq K|\tau-t|^{q_i},$$

for $0 \leq t \leq \tau \leq T$, for each i and for all $n \geq 1$. With this, it is now routine to show that $\{t^pv_n\}$ is equicontinuous. Likewise, $\{t^pw_n\}$ is also equicontinuous. So by the Arzela-Ascoli theorem there exist subsequences of both weighted sequences that converge uniformly, but since both sequences are monotone we have that both $\{t^pv_n\}$ and $\{t^pw_n\}$ converge uniformly on J_0. Let t^pv and t^pw be the uniform limits of these weighted sequences respectively. We wish to show that v and w are Type 1 coupled minimal and maximal quasisolutions of (6). To do so, first note that for each i and $n \geq 1$ we have

$$t^{p_i}v_{ni} = x_i^0 + \frac{t^{p_i}}{\Gamma(q_i)}\int_0^t (t-s)^{q_i-1}\big[f_i(s,v_{n-1}) + g_i(s,w_{n-1})\big]ds.$$

Now, since the weighted sequences $\{t^pv_n\}, \{t^pw_n\}$ converge uniformly on J_0 we have that the non-weighted sequences converge pointwise on J. Therefore, by the continuity of f, g the above expression converges uniformly to

$$t^{p_i}v_i = x_i^0 + \frac{t^{p_i}}{\Gamma(q_i)}\int_0^t (t-s)^{q_i-1}\big[f_i(s,v) + g_i(s,w)\big]ds$$

on J_0. Thus

$$v_i = x_i^0 t^{q_i - 1} + \frac{1}{\Gamma(q_i)} \int_0^t (t-s)^{q_i - 1} \big[f_i(s, v) + g_i(s, w) \big] ds,$$

implying that v is a Type 1 coupled quasisolution of (6), similarly w is as well.

Now, to show that v and w are minimal and maximal, we let x be a solution of (6) such that $x_p(0) = 0$ and $v_0 \leq x \leq w_0$. We know such a solution exists thanks to Theorem 2.3. Now letting $m = v_1 - x$, $M = x - w_1$ and using the same method as we used above we have that $v_0 \leq v_1 \leq x \leq w_1 \leq w_0$. Further, as before, we can inductively prove that $v_n \leq x \leq w_n$ on J for all $n \geq 1$, therefore $v \leq x \leq w$ implying that v, w are minimal and maximal Type 1 coupled quasisolutions. This completes the proof. We note that if $f+g$ possesses an adequate condition for uniqueness then $v = w = x$ which is the unique solution. Now we will present more variations of the generalized monotone method, specifically incorporating Type II solutions. First, in the following theorem we construct the sequences in a manner resembling Type II coupled quasisolutions, but still beginning with Type I lower and upper solutions. In this case we get alternating sequences which are described in the statement of the theorem.

Theorem 3.2 *Suppose that conditions* $(A1)$ *and* $(A2)$ *of Theorem 3.1 are true. Then the sequences given by*

$$D^{q_i} v_{n+1_i} = f_i(t, w_n) + g_i(t, v_n), \quad v_{n+1_{p_i}}(0) = x_i^0, \tag{9}$$

$$D^{q_i} w_{n+1_i} = f_i(t, v_n) + g_i(t, w_n), \quad w_{n+1_{p_i}}(0) = x_i^0, \tag{10}$$

yield alternating monotone sequences $\{v_{2n}, w_{2n+1}\}$ *and* $\{v_{2n+1}, w_{2n}\}$ *that satisfy*

$$v_{2n} \leq w_{2n+1} \leq x \leq v_{2n+1} \leq w_{2n},$$

for each $n \geq 0$ *on* J*, provided* $v_0 \leq x \leq w_0$*. Further, the weighted sequences*

$$t^p v_{2n}, t^p w_{2n+1} \to t^p \rho, \quad t^p v_{2n+1}, t^p w_{2n} \to t^p r$$

uniformly and monotonically on J_0*, where* ρ, r *are Type 1 coupled minimal and maximal quasisolutions of (6).*

We note that the proof of this theorem follows in much the same way as that of Theorem 3.1, as do the proofs of the remaining monotone method proofs, therefore we will not show these proofs directly.

For the next form of the generalized monotone method we switch the initial lower and upper solutions to Type II, and the sequences are also constructed like Type II coupled quasisolutions, i.e. in the manner found in Theorem 3.2, and also yield alternating sequences. For this case to work we must further assume that $v_0 \leq w_1$ and $v_1 \leq w_0$.

Theorem 3.3 *Suppose that condition* $(A2)$ *of Theorem 3.1 is true. Further suppose that*

$(B1)$ *v_0, w_0 are coupled lower and upper solutions of Type II for (6) such that $v_0 \leq w_0$.*

Then the sequences defined by (9) and (10) yield alternating sequences $\{v_{2n}, w_{2n+1}\}$ *and* $\{v_{2n+1}, w_{2n}\}$ *satisfying*

$$v_{2n} \leq w_{2n+1} \leq x \leq v_{2n+1} \leq w_{2n},$$

for each $n \geq 0$ on J, provided that $v_0 \leq w_1 \leq x \leq v_1 \leq w_0$. Further, the weighted sequences

$$t^p v_{2n}, t^p w_{2n+1} \to t^p \rho, \quad t^p v_{2n+1}, t^p w_{2n} \to t^p r$$

uniformly and monotonically on J_0, where ρ, r are Type 1 coupled minimal and maximal quasisolutions of (6).

For our final construction of the monotone method we will also consider the case where we begin with Type II lower and upper solutions, but construct the sequences as Type I quasisolutions, i.e. in the manner found in Theorem 3.1. We do not get alternating sequences in this case, but for it to work we must further assume that $v_0 \leq v_1$ and $w_1 \leq w_0$.

Theorem 3.4 *Suppose that conditions $(B1)$ and $(A2)$ of Theorems 3.3 and 3.1 are true. Then the sequences defined by (7) and (8) are such that*

$$t^p v_n \to t^p v, \quad t^p w_n \to w$$

uniformly and monotonically on J_0 provided that $v_0 \leq v_1 \leq x \leq w_1 \leq w_0$, where v, w are Type I coupled minimal and maximal quasisolutions of (6) respectively.

4 Numerical Example

In this section we present an example that illustrates the result of Theorem 3.1.

Example 4.1 Consider the fractional system of the form (6) with $q_1 = \frac{1}{2}$ and $q_2 = \frac{1}{3}$,

$$\begin{aligned} D^{\frac{1}{2}} x_1(t) &= \tfrac{1}{2} + \tfrac{5}{8}t + \tfrac{1}{16}\left(x_1(t)^2 - \tfrac{1}{4}x_2(t)\right), \qquad x_{p_1}(0) = 0, \\ D^{\frac{1}{3}} x_2(t) &= \tfrac{1}{6} + \tfrac{1}{2}t + \tfrac{1}{20}\left(x_1(t) - x_2(t)\right), \qquad x_{p_2}(0) = 0, \end{aligned} \tag{11}$$

where $p_1 = \frac{1}{2}$, $p_2 = \frac{2}{3}$ and call

$$\begin{aligned} f_1(t, x_1(t), x_2(t)) &= \frac{1}{2} + \frac{5}{8}t + \frac{1}{16}x_1(t)^2, \quad f_2(t, x_1(t), x_2(t)) = \frac{1}{6} + \frac{1}{2}t + \frac{1}{20}x_1(t), \\ g_1(t, x_1(t), x_2(t)) &= -\frac{1}{16}\left(\frac{1}{4}x_2(t)\right) = -\frac{1}{64}x_2(t), \quad g_2(t, x_1(t), x_2(t)) = -\frac{1}{20}x_2(t). \end{aligned}$$

If $J = (0, 1]$ and $J_0 = [0, 1]$ then $f(t, x)$ and $g(t, x)$ satisfy condition $(A2)$ in Theorem 3.1. Now let

$$v_{01} = \sqrt{t}/2, \quad v_{02} = 0,$$
$$w_{01} = 3, \quad w_{02} = 3 - t.$$

We will illustrate graphically in Figures 1–4 that $v_0(t)$ and $w_0(t)$ satisfy $(A1)$. We have that

$$v_{0p_i}(0) = w_{0p_i}(0) = 0.$$

Since $D^{1/2} v_{01}(t) = \frac{\sqrt{\pi}}{4}$, then

$$\begin{aligned} D^{1/2} v_{01}(t) = \frac{\sqrt{\pi}}{4} &\leq \frac{1}{2} + \frac{5}{8}t + \frac{1}{16}\left(v_{01}(t)^2 - \frac{1}{4}w_{02}(t)\right) \\ &= f_1(t, v_{01}(t), v_{02}(t)) + g_1(t, w_{01}(t), w_{02}(t)). \end{aligned}$$

Similarly,

$$\begin{aligned} D^{1/2}w_{01}(t) = \frac{3}{\sqrt{\pi t}} \quad &\geq \quad \frac{1}{2} + \frac{5}{8}t + \frac{1}{16}\left(w_{01}(t)^2 - \frac{1}{4}v_{02}(t)\right) \\ &= \quad f_1(t, w_{01}(t), w_{02}(t)) + g_1(t, v_{01}(t), v_{02}(t)), \end{aligned}$$

$$\begin{aligned} D^{1/3}v_{02}(t) = 0 \quad &\leq \quad \frac{1}{6} + \frac{1}{2}t + \frac{1}{20}\left(v_{01}(t) - w_{02}(t)\right) \\ &= \quad f_2(t, v_{01}(t), v_{02}(t)) + g_2(t, w_{01}(t), w_{02}(t)), \end{aligned}$$

and

$$\begin{aligned} D^{1/3}w_{02}(t) = \frac{6-3t}{2\sqrt[3]{t}\,\Gamma\left(\frac{2}{3}\right)} \quad &\geq \quad \frac{1}{6} + \frac{1}{2}t + \frac{1}{20}\left(w_{01}(t) - v_{02}(t)\right) \\ &= \quad f_2(t, w_{01}(t), w_{02}(t)) + g_2(t, v_{01}(t), v_{02}(t)). \end{aligned}$$

We show the graphs below.

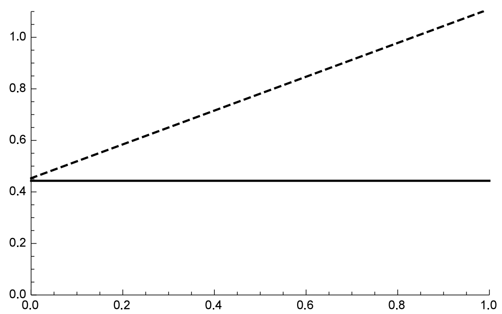

Figure 1: Solid: $D^{1/2}v_{01}(t)$, Dashed: $f_1(t, v_{01}(t), v_{02}(t)) + g_1(t, w_{01}(t), w_{02}(t))$.

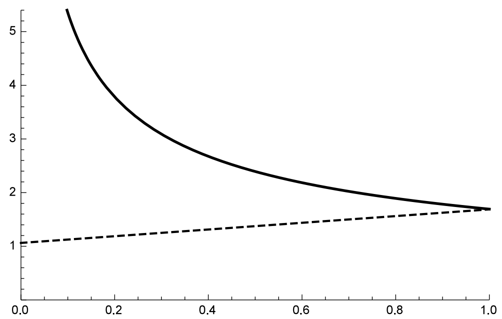

Figure 2: Solid: $D^{1/2}w_{01}(t)$, Dashed: $f_1(t, w_{01}(t), w_{02}(t)) + g_1(t, v_{01}(t), v_{02}(t))$.

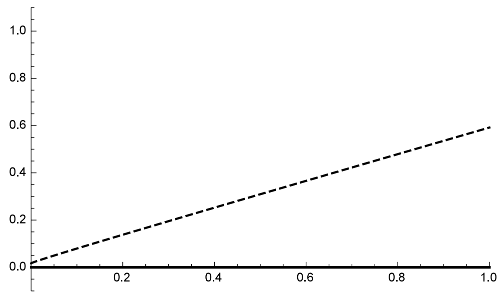

Figure 3: Solid: $D^{1/3}v_{02}(t)$, Dashed: $f_2(t, v_{01}(t), v_{02}(t)) + g_2(t, w_{01}(t), w_{02}(t))$.

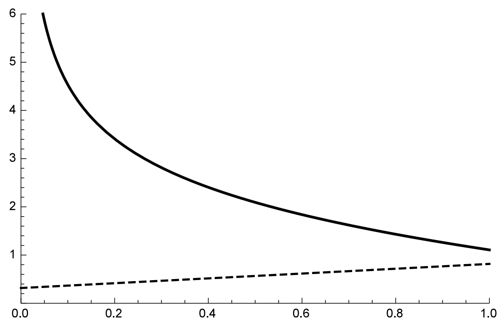

Figure 4: Solid: $D^{1/3}w_{02}(t)$, Dashed: $f_2(t, w_{01}(t), w_{02}(t)) + g_2(t, v_{01}(t), v_{02}(t))$.

After verifying that we have indeed coupled lower and upper solutions of Type I we computed four iterates of $\{t^{1/2}v_{n1}(t)\}$ and $\{t^{1/2}w_{n1}(t)\}$, as well as four iterates of $\{t^{1/3}v_{n2}(t)\}$ and $\{t^{1/3}w_{n2}(t)\}$ according to Theorem 3.1 for $t \in J_0 = [0, 1]$.

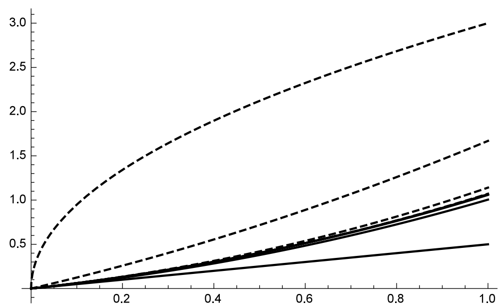

Figure 5: Solid: $\left\{t^{1/2}v_{n1}(t)\right\}$, Dashed: $\left\{t^{1/2}w_{n1}(t)\right\}$, $0 \leq n \leq 4$.

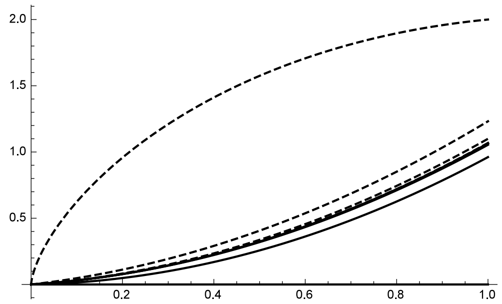

Figure 6: Solid: $\left\{t^{1/3} v_{n2}(t)\right\}$, Dashed: $\left\{t^{1/3} w_{n2}(t)\right\}$, $0 \leq n \leq 4$.

Finally we show a table of ten values of $\{t^{p_i} v_{4_i}(t)\}$ and $\{t^{p_i} w_{4_i}(t)\}$ on the interval $[0, 1]$.

t	$t^{1/2}v_{4,1}(t)$	$t^{1/2}w_{4,1}(t)$	$t^{1/3}v_{4,2}(t)$	$t^{1/3}w_{4,2}(t)$
0	0	0	0	0
0.1	0.0610930	0.0610933	0.0310197	0.0310208
0.2	0.1318091	0.1318108	0.0790014	0.0790066
0.3	0.2122992	0.2123045	0.1437895	0.1438034
0.4	0.3027222	0.3027352	0.2253221	0.2253509
0.5	0.4032596	0.4032874	0.3235653	0.3236175
0.6	0.5141177	0.5141722	0.4384997	0.4385866
0.7	0.6355296	0.6356297	0.5701140	0.5702515
0.8	0.7677574	0.7679318	0.7184090	0.7186130
0.9	0.9110939	0.9113858	0.8833827	0.8836781
1.0	1.0658661	1.0663374	1.0650431	1.0654591

We have developed a monotone iterative technique for multi-order 2-systems of Riemann-Liouville fractional differential equations with initial condition and presented an example that illustrates one of the main theorems. An advantage of this method is that the linear iterates do not require the computation of the Mittag-Leffler function. In our example the iterates appear to converge to a unique solution, we plan to work on establishing conditions for uniqueness in the near future. In the future we would also like to expand this method to N-systems as well as consider further generalizations of the monotone method. One such expansion would be the quasilinearization method, where the hypotheses are strengthened yet the convergence becomes quadratic, for more information see [4, 13]. And ultimately we hope that these results help further the study of R-L fractional multi-order systems.

References

[1] Caputo, M. Linear models of dissipation whose Q is almost independent, II. *Geophy. J. Roy. Astronom.* **13** (5) (1967) 29–539.

[2] Chowdhury, A. and Christov, C.I. Memory effects for the heat conductivity of random suspensions of spheres. *Proc. R. Soc. A* **466** (2010) 3253–3273.

[3] Denton, Z. Monotone method for multi-order 2-systems of Riemann-Liouville fractional differential equations. *Communications in Applied Analysis* **19** (2015) 353–368.

[4] Denton, Z., Ng, P.W. and Vatsala, A.S. Quasilinearization method via lower and upper solutions for Riemann-Liouville fractional differential equations. *Nonlinear Dynamics and Systems Theory* **11** (3) (2011) 239–251.

[5] Denton, Z. and Vatsala, A.S. Monotone iterative technique for finite systems of nonlinear Riemann-Liouville fractional differential equations. *Opuscula Mathematica* **31** (3) (2011) 327–339.

[6] Diethelm, K. and Freed, A.D. On the solution of nonlinear fractional differential equations used in the modeling of viscoplasticity. In: *Scientific Computing in Chemical Engineering II: Computational Fluid Dynamics, Reaction Engineering, and Molecular Properties* (Eds: F. Keil, W. Mackens, H. Vob, and J. Werther). Heidelberg: Springer, 1999, 217–224.

[7] Glöckle, W.G. and Nonnenmacher, T.F. A fractional calculus approach to self similar protein dynamics. *Biophy. J.* **68** (1995) 46–53.

[8] Hilfer, R. (editor). *Applications of Fractional Calculus in Physics*. World Scientific Publishing, Germany, 2000.

[9] Kilbas, A.A., Srivastava, H.M. and Trujillo, J.J. *Theory and Applications of Fractional Differential Equations*. Elsevier, North Holland, 2006.

[10] Kiryakova, V. *Generalized Fractional Calculus and Applications*. Longman-Wiley, New York, 1994.

[11] Ladde, G.S.Lakshmikantham, V. and Vatsala, A.S. *Monotone Iterative Techniques for Nonlinear Differential Equations*. Pitman Publishing Inc, 1985.

[12] Lakshmikantham, V.,Leela, S. and Vasundhara, D.J. *Theory of Fractional Dynamic Systems*. Cambridge Scientific Publishers, 2009.

[13] Lakshmikantham, V. and Vatsala, A.S. *Generalized Quasilinearization for Nonlinear Problems*. Kluwer Academic Publishers, Boston, 1998.

[14] Metzler, R.,Schick, W., Kilian, H.G. and Nonnenmacher, T.F. Relaxation in filled polymers: A fractional calculus approach. *J. Chem. Phy.* **103** (1995) 7180–7186.

[15] Oldham, B. andSpanier, J. *The Fractional Calculus*. Academic Press, New York – London, 2002.

Nonlinear Dynamics and Systems Theory, **16** (3) (2016) 260–267

The Jacobi Elliptic Method and Its Applications to the Generalized Form of the Phi-Four Equation

R.B. Djob[1*], E. Tala-Tebue [1], A. Kenfack-Jiotsa[2] and T.C. Kofane[1]

[1] *Laboratory of Mechanics, Department of Physics, Faculty of Science, University of Yaoundé 1, P.O. Box 812, Yaoundé, Cameroon*
[2] *Nonlinear Physics and Complex Systems Group, Department of Physics, The Higher Teachers' Training College, University of Yaoundé I, P.O. Box 47 Yaoundé, Cameroon*

Received: June 30, 2015; Revised: June 9, 2016

Abstract: In order to investigate the generalized periodic solutions of the generalized phi-four equation, we use the Jacobi elliptic functions. Many kinds of solutions are obtained. For some parameters, these envelope periodic solutions can degenerate to the envelope shock wave solutions (dark solitons) and the envelope solitary wave solutions (bright solitons). The existence of these solutions is determined by the parameters of the initial equation. The solutions found in this work can be used in many areas of physics such as telecommunications.

Keywords: *generalized periodic solutions, generalized phi-four equation; Jacobi elliptic functions; envelope periodic solutions.*

Mathematics Subject Classification (2010): 00A69, 35Q51.

1 Introduction

Before the discovery of solitons, scientists had taken the nonlinear terms in an equation as perturbations. The history of solitons (the wave of translation), in fact, dates back to 1834, the year in which John Scott Russell observed that a heap of water in a canal propagated undistorted over several kilometers. The results obtained in the linear theory of waves, by ignoring the nonlinear parts, are most frequently too far from reality to be useful. The transition from linear to nonlinear description is justified by the necessity to take into account all the details of the observed phenomena. The wave of translation was regarded as a curiosity until the 1960s, when scientists began to use computers to study nonlinear wave propagation. The discovery of mathematical solutions started

* Corresponding author: mailto:rogerdjob@yahoo.fr

with the analysis of nonlinear partial differential equations, such as in the works of Boussinesq and Rayleigh, carried out independently. Recently, a new direction related to the investigation of nonlinear phenomena and processes has been actively developed in various areas, including hydrodynamics, nonlinear optics, plasma physics, and biology [1–8], to mention a few. A remarkable number of evolution equations (sine-Gordon, Korteweg de Vries, Boussinesq, Schrodinger and others) considered by the end of the 19^{th} century, radically changed the thinking of scientists about the nature of nonlinearity. It then becomes necessary to solve these nonlinear equations. The exact analytical solutions of nonlinear equations are hardly obtained. In recent years, quite a few methods for obtaining explicit traveling and solitary wave solutions of nonlinear evolution equations in mathematics and physics have been proposed. We can list the generalized iterative methods [9], computational methods [10], travelling wave solutions method [11], the sine-cosine method [12,13], Backlund transform method [14], the sinc-collocation method [15], Darboux transform method [16], Painleve's singularity structure analysis [17], homotopy perturbation method [18], variational iteration method [19], inverse scattering transform method [20], the (G'/G)-expansion method [21], the Hirota's bilinear method [22], exp-function method [23], tanh method [24, 25], extended three-wave method [26]. These methods, however, can only obtain the shock and solitary wave solutions or the periodic solutions with the elementary functions [27–32], but cannot get the generalized periodic solutions of nonlinear equations. The objective of this work is to use the Jacobi elliptic method [33] to obtain the generalized periodic solutions with the phi-four equation.

The standard form of phi-four equation

$$u_{tt} - u_{xx} + u^3 - u = 0 \tag{1}$$

arises in many branches of mathematical physics. Its special solutions are known as kink and antikink solitons. In our investigations, we consider the following form of equation (1):

$$(u^l)_{tt} - a(u^n)_{xx} - bu^m + cu^n = 0, \tag{2}$$

where a, b and c are arbitrary nonzero constants and l, m and n are integers; $u(x,t)$ is the unknown function depending on the spatial variable x and the temporal variable t. The subscripts x and t denote partial derivatives with respect to these variables. The technique that will be used is the most effective direct method to construct generalized wave solutions of nonlinear evolution equations.

2 Jacobi Elliptic sn Function

By means of the Jacobi elliptic function, $u(x,t)$ can be expressed as follows:

$$u(x,t) = Asn^p\xi, \quad \xi = q(x - v_0 t), \tag{3}$$

where $p > 0$ is a constant which will be determined later. A represents the amplitude of the wave, while v_0 is the velocity of the wave; q can represent the inverse width of the wave. From equation (3), we have:

$$\begin{cases} u^m = A^m sn^{pm}\xi, \\ u^n = A^n sn^{pn}\xi, \\ (u^l)_{tt} = A^l plq^2 v_0^2 \Big[(pl-1)sn^{pl-2}\xi - pl(1+k^2)sn^{pl}\xi + (pl+1)k^2 sn^{pl+2}\xi\Big], \\ (u^n)_{xx} = A^n pnq^2 \Big[(pn-1)sn^{pn-2}\xi - pn(1+k^2)sn^{pn}\xi + (pn+1)k^2 sn^{pn+2}\xi\Big]. \end{cases} \tag{4}$$

The following relations are taken into account:

$$cn^2(\xi,k) + sn^2(\xi,k) = 1, \quad dn^2(\xi,k) + k^2 sn^2(\xi,k) = 1. \tag{5}$$

Substituting the expression (4) into (2) yields

$$\begin{aligned} A^l plq^2 v_0^2 &\Big[(pl-1)sn^{pl-2}\xi - pl(1+k^2)sn^{pl}\xi + (pl+1)k^2 sn^{pl+2}\xi\Big] \\ -aA^n pnq^2 &\Big[(pn-1)sn^{pn-2}\xi - pn(1+k^2)sn^{pn}\xi + (pn+1)k^2 sn^{pn+2}\xi\Big] \\ &-bA^m sn^{pm}\xi + cA^n sn^{pn}\xi = 0. \end{aligned} \tag{6}$$

From equation (6), equating the exponents of $sn^{pn+2}\xi$ and $sn^{pm}\xi$ functions we get

$$p = \frac{2}{m-n}. \tag{7}$$

Also from equation (6), equating the exponents of $sn^{pl}\xi$ and $sn^{pn}\xi$ functions we have

$$l = n. \tag{8}$$

If we make the same gymnastic with the exponents of $sn^{pl+2}\xi$ and $sn^{pn+2}\xi$ and for $sn^{pl-2}\xi$ and $sn^{pn-2}\xi$ functions, we also obtain $l = n$. Now, in view of equation (8), the functions $sn^{pl+j}\xi$ with $j = -2, 0, 2$ in (6) are linearly independent. Thus, their respective coefficients must vanish. Setting their coefficients to zero gives the system of algebraic equations:

$$A^n pnq^2(pn-1)(v_0^2 - a) = 0, \tag{9}$$

$$A^n p^2 n^2 q^2(1+k^2)(a - v_0^2) + cA^n = 0, \tag{10}$$

$$A^n pnq^2 k^2(pn+1)(v_0^2 - a) - bA^m = 0. \tag{11}$$

If $v_0^2 - a \neq 0$, then equation (9) gives the relation between the two parameters p and n, that is

$$p = \frac{1}{n}, \tag{12}$$

and using relation (7), we have:

$$m = 3n. \tag{13}$$

From equation (10), one obtains

$$q^2 = \frac{c}{(1+k^2)(v_0^2 - a)}. \tag{14}$$

Inserting (14) into (11) yields

$$A = \Big[\frac{2k^2c}{b(1+k^2)}\Big]^{\frac{1}{2n}}. \tag{15}$$

Thus, the generalized solutions of equation (3) are given by

$$u(x,t) = \Big\{\sqrt{\frac{2k^2c}{b(1+k^2)}}sn\Big[\sqrt{\frac{c}{(1+k^2)(v_0^2-a)}}(x-v_0t)\Big]\Big\}^{\frac{1}{n}}. \tag{16}$$

We clearly observe that these solutions exist if and only if $c(v_0^2 - a) > 0$ and $bc > 0$. As $k \to 1$, corresponding envelope solitary wave solutions are

$$u(x,t) = \Big\{\sqrt{\frac{c}{b}}\tanh\Big[\sqrt{\frac{c}{2(v_0^2-a)}}(x-v_0t)\Big]\Big\}^{\frac{1}{n}}. \tag{17}$$

Namely dark solitons of equation (17) look like those found by Triki and Wazwaz in [34]. This justifies the fact that the present method is more explicit.

3 Jacobi Elliptic cn Function

In this section, $u(x,t)$ is expressed as follows:

$$u(x,t) = Acn^p\xi, \quad \xi = q(x-v_0t). \tag{18}$$

In this equation, $p > 0$. From equation (18), we get:

$$\begin{cases} u^m = A^m cn^{pm}\xi, \\ u^n = A^n cn^{pn}\xi, \\ (u^l)_{tt} = A^l plq^2v_0^2\Big[(pl-1)(1-k^2)cn^{pl-2}\xi + pl(2k^2-1)cn^{pl}\xi - (pl+1)k^2cn^{pl+2}\xi\Big], \\ (u^n)_{xx} = A^n pnq^2\Big[(pn-1)(1-k^2)cn^{pn-2}\xi + pn(2k^2-1)cn^{pn}\xi - (pn+1)k^2cn^{pn+2}\xi\Big]. \end{cases} \tag{19}$$

Inserting (19) into (2), one obtains:

$$\begin{aligned} A^l plq^2v_0^2\Big[(pl-1)(1-k^2)cn^{pl-2}\xi + pl(2k^2-1)cn^{pl}\xi - (pl+1)k^2cn^{pl+2}\xi\Big] \\ -aA^n pnq^2\Big[(pn-1)(1-k^2)cn^{pn-2}\xi + pn(2k^2-1)cn^{pn}\xi - (pn+1)k^2cn^{pn+2}\xi\Big] \\ -bA^m cn^{pm}\xi + cA^n cn^{pn}\xi = 0. \end{aligned} \tag{20}$$

In equation (20), equating the exponents of $cn^{pn+2}\xi$ and $cn^{pm}\xi$ functions gives

$$p = \frac{2}{m-n}. \tag{21}$$

Also from equation (20), equating the exponents of $cn^{pl}\xi$ and $cn^{pn}\xi$ functions we get

$$l = n. \tag{22}$$

The same work can be done with the exponents of $cn^{pl+2}\xi$ and $cn^{pn+2}\xi$ and for $cn^{pl-2}\xi$ and $cn^{pn-2}\xi$ functions; we also obtain $l = n$. Now, the functions $cn^{pl+j}\xi$ with $j = -2, 0, 2$

in (20) are linearly independent. Thus, their respective coefficients must vanish. Setting their coefficients to zero gives the system of algebraic equations:

$$A^n pnq^2(pn-1)(1-k^2)(v_0^2-a)=0, \tag{23}$$

$$A^n p^2n^2q^2(2k^2-1)(v_0^2-a)+cA^n=0, \tag{24}$$

$$A^n pnq^2k^2(pn+1)(a-v_0^2)-bA^m=0. \tag{25}$$

If $v_0^2-a\neq 0$, then equation (23) gives the following two relations, that is

$$\begin{cases} p=\frac{1}{n},\\ k^2=1. \end{cases} \tag{26}$$

Equation (24) gives

$$q^2=\frac{c}{p^2n^2(2k^2-1)(a-v_0^2)}, \tag{27}$$

and (25) yields

$$A=\Big[\frac{(pn+1)k^2c}{bpn(2k^2-1)}\Big]^{\frac{1}{m-n}}. \tag{28}$$

Thus, the generalized solutions of equation(3) are given by:

Case 1: $p=\frac{1}{n}$, *i.e.* $m=3n$; $q^2=\frac{c}{(2k^2-1)(a-v_0^2)}$, $A=\left[\frac{2k^2c}{b(2k^2-1)}\right]^{\frac{1}{2n}}$ and

$$u(x,t)=\Big\{\sqrt{\frac{2k^2c}{b(2k^2-1)}}cn\Big[\sqrt{\frac{c}{(2k^2-1)(a-v_0^2)}}(x-v_0t)\Big]\Big\}^{\frac{1}{n}}. \tag{29}$$

Case 2: $k^2=1$; $q^2=\frac{c(m-n)^2}{4n^2(a-v_0^2)}$, $A=\left[\frac{(m+n)c}{2nb}\right]^{\frac{1}{m-n}}$ and

$$u(x,t)=\Big\{\frac{(m+n)c}{2nb}sech^2\Big[\sqrt{\frac{c(m-n)^2}{4n^2(a-v_0^2)}}(x-v_0t)\Big]\Big\}^{\frac{1}{m-n}}. \tag{30}$$

It is evident that these solutions have a physical sense if and only if $c(a-v_0^2)>0$ and $bc>0$. k must be different from zero in (29). Equation (30) is an exact bright soliton solution of (3).

4 Jacobi Elliptic dn Function

In this section, $u(x,t)$ is expressed as follows:

$$u(x,t)=Adn^p\xi,\quad \xi=q(x-v_0t). \tag{31}$$

Here again, p has to be positive. From (31), we get:

$$\begin{cases} u^m=A^m dn^{pm}\xi,\\ u^n=A^n dn^{pn}\xi,\\ (u^l)_{tt}=A^l plq^2v_0^2\Big[(pl-1)(k^2-1)dn^{pl-2}\xi+pl(2-k^2)dn^{pl}\xi-(pl+1)dn^{pl+2}\xi\Big],\\ (u^n)_{xx}=A^n pnq^2\Big[(pn-1)(k^2-1)dn^{pn-2}\xi+pn(2-k^2)dn^{pn}\xi-(pn+1)dn^{pn+2}\xi\Big]. \end{cases} \tag{32}$$

Inserting (32) into (3), one obtains:

$$A^l plq^2v_0^2\Big[(pl-1)(k^2-1)dn^{pl-2}\xi + pl(2-k^2)dn^{pl}\xi - (pl+1)dn^{pl+2}\xi\Big]$$
$$-aA^n pnq^2\Big[(pn-1)(k^2-1)dn^{pn-2}\xi + pn(2-k^2)dn^{pn}\xi - (pn+1)dn^{pn+2}\xi\Big]$$
$$-bA^m dn^{pm}\xi + cA^n dn^{pn}\xi = 0. \tag{33}$$

In (33), equating the exponents of $dn^{pn+2}\xi$ and $dn^{pm}\xi$ functions gives

$$p = \frac{2}{m-n}. \tag{34}$$

Also from (33), equating the exponents of $dn^{pl}\xi$ and $dn^{pn}\xi$ functions we have

$$l = n \tag{35}$$

which is also obtained by equating the exponents' pairs $pl+2$ and $pn+2$, $pl-2$ and $pn-2$. Setting the coefficients of the linearly independent functions $dn^{pl+j}\xi$, where $j=-2,0,2$, to zero gives the system of algebraic equations:

$$A^n pnq^2(pn-1)(k^2-1)(v_0^2-a) = 0, \tag{36}$$
$$A^n p^2n^2q^2(2-k^2)(v_0^2-a) + cA^n = 0, \tag{37}$$
$$A^n pnq^2(pn+1)(a-v_0^2) - bA^m = 0. \tag{38}$$

If $v_0^2 - a \neq 0$, then equation (36) gives the following two relations

$$\begin{cases} p = \frac{1}{n}, \\ k^2 = 1. \end{cases} \tag{39}$$

Equation (37) gives

$$q^2 = \frac{c}{p^2n^2(2-k^2)(a-v_0^2)}, \tag{40}$$

and (38) yields

$$A = \Big[\frac{(pn+1)c}{bpn(2-k^2)}\Big]^{\frac{1}{m-n}}. \tag{41}$$

The generalized solutions of equation (3) are given by:

Case 1: $p=\frac{1}{n}$, *i.e.* $m=3n$; $q^2=\frac{c}{(2-k^2)(a-v_0^2)}$, $A=\Big[\frac{2c}{b(2-k^2)}\Big]^{\frac{1}{2n}}$ and

$$u(x,t) = \Big\{\sqrt{\frac{2c}{b(2-k^2)}}dn\Big[\sqrt{\frac{c}{(2-k^2)(a-v_0^2)}}(x-v_0t)\Big]\Big\}^{\frac{1}{n}}. \tag{42}$$

Case 2: $k^2=1$; $q^2=\frac{c(m-n)^2}{4n^2(a-v_0^2)}$, $A=\Big[\frac{(m+n)c}{2nb}\Big]^{\frac{1}{m-n}}$ and

$$u(x,t) = \Big\{\frac{(m+n)c}{2nb}sech^2\Big[\sqrt{\frac{c(m-n)^2}{4n^2(a-v_0^2)}}(x-v_0t)\Big]\Big\}^{\frac{1}{m-n}}, \tag{43}$$

with $c(a-v_0^2)>0$ and $bc>0$.

These soliton solutions can be controlled well by adjusting the parameters of the system. From one ansatz, we carry out many types of solutions, and we conclude that the present method is straightforward and concise.

5 Conclusion

In this work, we have considered a generalized phi-four equation with arbitrary constant coefficients and general values of the exponents in the dissipation and nonlinear terms. With the aid of Jacobi elliptic functions, the generalized periodic solutions are obtained. We have noted that the existence of these solutions depends on whether $c(v_0^2 - a) > 0$ or $c(a - v_0^2) > 0$ and $bc > 0$. We have also pointed out that for some parameters, these envelope periodic solutions can degenerate to the non-topological and topological solitons.

References

[1] Tabiryan, N.V., Sukhov, A.V. and Zeldovich, B.Y. The orientational optical nonlinearity of liquid crystals. *Mol Cryst Liq Cryst.* **136** (1986) 1–140.

[2] Khoo, I.C. *Progress in Optics*. Elsevier, Amsterdam, 1988.

[3] Khoo, I.C. and Wu, N.T. *Optics and Nonlinear Optics of Liquid Crystals.* World Scientific, Singapore, 1993.

[4] Braun, E., Faucheux, L.P., Libchaber, A., McLaughlin, D.W., Muraki, D.J. and Shelley, M.J. Filamentation and Undulation of Self-Focused Laser Beams in Liquid Crystals. *Europhys. Lett.* **23** (1993) 239–244.

[5] Braun, E., Faucheux, L.P. and Libchaber, A. Strong self-focusing in nematic liquid crystals. *Phys. Rev. A* **48** (1993) 611–622.

[6] McLaughlin, D.W., Muraki, D.J., Shelley, M.J. and Wang, X. A paraxial model for optical self focussing in a nematic liquid crystal. *Physica D* **88** (1995) 55–81.

[7] McLaughlin, D.W., Muraki, D.J. and Shelley, M.J. Self-focussed optical structures in a nematic liquid crystal. *Physica D* **97** (1996) 471–497.

[8] Janossy, I. Molecular Interpretation of the Absorption-Induced Optical Reorientation of. Nematic Liquid Crystals. *Phys. Rev. E* **49** (1994) 2957–2963.

[9] Sowmya, M. and Vatsala, A.S. Generalized Iterative Methods for Caputo Fractional Differential Equations via Coupled Lower and Upper Solutions with Superlinear Convergence. *Nonlinear Dynamics and Systems Theory* **15** (2) (2015) 198–208.

[10] Jafari, H. and Azad, A. A Computational Method for Solving a System of Volterra Integro-differential Equations. *Nonlinear Dynamics and Systems Theory* **12** (4) (2012) 389–396.

[11] Danylenko, V.A. and Skurativskyi, S.I. Travelling Wave Solutions of Nonlocal Models for Media with Oscillating Inclusions. *Nonlinear Dynamics and Systems Theory* **12** (4) (2012) 365–374.

[12] Bekir, A. New solitons and periodic wave solutions for some nonlinear physical models by using the sine-cosine method. *Phys. Scr.* **77** (2008) 501–504.

[13] Gambo, B., Bouetou, T.B., Kuetche, V.K. and Kofane, T.C. Explicit Series Solutions to Nonlinear Evolution Equations: The Sine-Cosine Method. *Appl. Math. Comput.* **215** (2010) 4239–4247.

[14] Tsigaridas, G., Fragos, A., Polyzos, I., Fakis, M., Ioannou, A., Giannetas, V. and Persephonis, P. Evolution of near-soliton initial profiles in nonlinear wave equations through their Backlund transforms. *Chaos Solitons Fract.* **23** (2005) 1841–1854.

[15] Yeganeh, S., Saadatmandi, A., Soltanian, F., Dehghan, M. The numerical solution of differential-algebraic equations by sinc-collocation method. *Comput. and Appl. Math.* **32** (2013) 343–354.

[16] Suzo, A.A. Intertwining technique for the matrix Schrodinger equation. *Phys. Lett. A* **335** (2005) 88–102.

[17] Banerjee, R.S. Painlev analysis of the K(m,n) equations which yield compactons. *Phys. Scr.* **57** (1998) 598–600.

[18] Mehdi, D., Fatemeh, S. Solution of a partial differential equation subject to temperature overspecification by He's homotopy perturbation method. *Phys. Scr.* **75** (2007) 778–787.

[19] He, J.H. The variational iteration method for eighth-order initial-boundary value problems. *Phys. Scr.* **76** (2007) 680–682.

[20] Vakhnenko, V.O., Parkes, E.J. and Morrison, A.J. A Backlund transformation and the inverse scattering transform method for the generalised Vakhnenko equation. *Chaos Solitons Fract.* **17** (2003) 683–692.

[21] Wang, M., Li, X. and Zhang, J. The (G'/G)-expansion method and travelling wave solutions of nonlinear evolution equations in mathematical physics. *Phys. Lett. A* **372** (2008) 417–423.

[22] Hirota, R. Exact solution of the Korteweg-de Vries equation for multiple collisions of solitons, *Phys Rev Lett* **27** (1971) 1192–1194.

[23] Zayed, E.M., Abdelaziz, M.A. Exact solutions for the nonlinear Schrdinger equation with variable coefficients using the generalized extended tanh-function, the sine-cosine and the exp-function methods. *Appl. Math. Comput.* **218** (2011) 2259–2268.

[24] Wazwaz, A.M. The tanh method: solitons and periodic solutions for the Dodd-Bullough-Mikhailov and the Tziteica-Dodd-Bullough equations. *Chaos Solitons Fract.* **25** (2005) 55–63.

[25] Zayed, E.M. and Abdelaziz, M.A. The tanh-function method using a generalized wave transformation for nonlinear equations. *Int. J. Nonlinear Sci. Numer. Simula.* **11** (2010) 595–601.

[26] Li, Z., Dai, Z., Liu, J. Exact three-wave solutions for the(3+1)-dimensional Jimbo-Miwa equation. *Comput. Math. Appl.* **61** (2011) 2062–2066.

[27] Wang, M.L. Solitary wave solution for variant Boussinesq equations. *Phys. Lett. A* **199** (1995) 169–172.

[28] Wang, M.L., Zhou, Y.B. and Li, Z.B. Application of a homogeneous balance method to exact solutions of nonlinear equations in mathematical physics. *Phys. Lett. A* **216** (1996) 67–75.

[29] Yang, L., Zhu, Z. and Wang, Y. Exact Solutions of Nonlinear Equations. *Phys. Lett. A* **260** (1999) 55–59.

[30] Hirota, R. Exact N-solutions of the wave equation of long waves in shallow water and in nonlinear lattices. *J. Math. Phys.* **14** (1973) 810–814.

[31] Kudryashov, N.A. Exact solutions of the generalized Kuramoto Sivashinsky equation. *Phys. Lett. A* **147** (1990) 287–291.

[32] Otwinowski, M., Paul, R. and Laidlaw, W.G. Exact Traveling Wave Solutions of a Class of Nonlinear Diffusion Equations by Reduction to Quadrature. *Phys. Lett. A* **128** (1988) 483–487.

[33] Huang, W.H., Liu, Y.L. Jacobi elliptic function solutions of the Ablowitz-Ladik discrete nonlinear Schrdinger system. *Chaos Solitons Fract.* **40** (2009) 786–792.

[34] Triki, H., Wazwaz, A.M. Bright and dark soliton solutions for a $K(m,n)$ equation with t-dependent coefficients. *Phys. Lett. A* 373 (2009) 2162–2165.

Nonlinear Dynamics and Systems Theory, **16** (3) (2016) 268–275

On Antagonistic Game With a Constant Initial Condition. Marginal Functionals and Probability Distributions

J.H. Dshalalow [1*], W. Huang [1], H.-J. Ke [1], and A. Treerattrakoon [2]

[1] *Department of Mathematics, Florida Institute of Technology*
[2] *Department of Industrial Engineering Kasetsart University, Thailand*

Received: January 11, 2016; Revised: June 10, 2016

Abstract: This paper continues dealing with a class of antagonistic games with two players initiated in Dshalalow et al. [1]. Here we validate our claim of analytic tractability in the results obtained in [1] under various transforms.

Keywords: *noncooperative stochastic games; fluctuation theory; marked point processes; Poisson process; ruin time; exit time; first passage time; modified Bessel functions.*

Mathematics Subject Classification (2010): 82B41, 60G51, 60G55, 60G57, 91A10, 91A05, 91A60, 60K05.

1 Introduction

In this paper we continue our studies of a stochastic game of two players of a fully antagonistic nature initiated in [1] by the same authors. The game evolves as a mutual conflict involving two players A and B hitting each other at random and continued until one of the players is "exhausted." In short, the players attack each other in accordance with two independent marked point processes

$$\mathcal{A} := \sum_{j\geq 1} w_j \varepsilon_{s_j}, \text{ and } \mathcal{B} := \sum_{k\geq 1} z_k \varepsilon_{t_k}, s_1, t_1 > 0,$$

representing respective attacks to players A and B. Here ε_a is the Dirac point mass at point $a \in \mathbb{R}$, $\sum_{j\geq 1} \varepsilon_{s_j}$, and $\sum_{k\geq 1} \varepsilon_{t_k}$ are underlying point random measures of the times of attacks, while the marks w_j's and z_k's represent respective damages to players A and

* Corresponding author: mailto:eugene@fit.edu

B. Players A and B can sustain the attacks until their respective cumulative casualties cross thresholds M and N (positive real numbers). At a time when it takes place (at the first passage time), i.e., when one of the players loses the game, the game should formally stop. However, the game was assumed to be tracked by a third party observer upon random epochs of time $\tau_1, \tau_2, \ldots$ and consequently, the outcome of the game is unknown in real time. The first passage time is then shifted to epoch τ_ρ (called the first observed passage time) that takes place upon one of the observation epochs. Thus, the narrative of the game is delayed allowing the players to continue fighting even after one of the players lost the game thereby letting the game to proceeed in a more realistic scenario.

We further assumed in [1] that $\mathcal{A}$ and $\mathcal{B}$ are marked Poisson random measures and $\tau := \sum_{i\geq 1} \varepsilon_{\tau_i}$, $\tau_0 > 0$ was a renewal process with interrenewal times being exponentially distributed. If X_i and Y_i are increments of the casualties to players A and B on $(\tau_{i-1}, \tau_i]$ observed at time τ_i, then

$$A_k = X_0 + X_1 + \ldots + X_k, \ B_k = Y_0 + Y_1 + \ldots + Y_k$$

form the cumulative damages to players A and B by time τ_k. With the exit indices

$$\mu := inf\{j \geq 0 : A_j = X_0 + X_1 + \ \ldots + X_j > M\}$$

and

$$\nu := inf\{k \geq 0 : B_k = Y_0 + Y_1 + \ \ldots + Y_k > N\},$$

A_μ and B_ν are the respective cumulative damages to players A and B at their respective observed or virtual ruin times. In [1], the functional of interest was

$$\Phi_{\mu\nu} = \Phi_{\mu\nu}(\alpha, \beta, \theta) = Ee^{-\alpha A_\mu - \beta B_\mu - \theta\tau_\mu}\mathbf{1}_{\{\mu<\nu\}}$$

giving the joint transform of the first observed passage time τ_μ (the ruin time of player A), along with the status of the respective casualties to players A and B at $\tau_\mu = \tau_\rho$ on the confine σ-algebra $\mathcal{F}(\Omega) \cap \{\mu < \nu\}$. This functional was obtained in terms of the double Laplace-Carson and Laplace-Stieltjes transforms under the claim that it was analytically invertible. We succeeded in doing this. The inverse formulas contained various special functions but seemed to be cumbersome. We go on the further claim that the results are numerically tame.

We ended [1] with obtaining the marginal functional $Ee^{-\alpha A_\mu}\mathbf{1}_{\{\mu<\nu\}}$ in terms of modified Bessel functions and their integrals. The objective of this paper is to continue with other marginal functionals and a subsequent inversion of their Laplace-Stieltjes transforms to arrive at explicit probability distributions and then illustrate the result with computational examples. Note that either the present paper and [1] are abridged and their complete version is available in [2].

2 Further Cases of Marginal Functionals

Our next goal is to get the other marginal transforms. They are to be obtained from $\Phi_{\mu\nu}(\alpha, \beta, \theta) = Ee^{-\alpha A_\mu - \beta B_\mu - \theta\tau_\mu}\mathbf{1}_{\{\mu<\nu\}}$ in (2.27) and (3.21-3.73) of [1]. In Case 1 [1], we gave $\Phi_{\mu\nu}(\alpha, 0, 0) = Ee^{-\alpha A_\mu}\mathbf{1}_{\{\mu<\nu\}}$. We continue with the other cases.

Case 2. Setting $\alpha = \theta = 0$ in $\Phi_{\mu\nu}(\alpha, \beta, \theta)$ leads us to the marginal Laplace-Stieltjes transform of the casualties to player B at the exit from the game to be lost by player A,

$\Phi_{\mu\nu}(0,\beta,0) := Ee^{-\beta B_\mu}\mathbf{1}_{\{\mu<\nu\}}$. After setting $\alpha = \theta = 0$ in (3.70-3.71) [1], we arrive at the following.

(i) Case $\boldsymbol{\delta \neq \lambda_A}$**.** Proceeding as in Case 1 (see more details in [2]) we have

$$\begin{aligned}
\Phi^{(1)}_{\mu\nu}(0,\beta,0) = \Bigg\{ & \frac{\lambda_A\delta}{(\lambda_A+\lambda_B)(\delta+\lambda_B)} \cdot e^{-N\beta} e^{-(\frac{\lambda_B g}{\lambda_A+\lambda_B})(M-X_0)} e^{-(\frac{\lambda_A h}{\lambda_A+\lambda_B})(N-Y_0)} \\
& \times I_0\big(2\sqrt{\frac{\lambda_A\lambda_B hg(M-X_0)(N-Y_0)}{(\lambda_A+\lambda_B)^2}}\big) \\
& + \int_{z=0}^{N-Y_0} \Big[\Big(\frac{\lambda_A\delta\beta}{(\lambda_A+\lambda_B)(\delta+\lambda_B)} + \frac{\lambda_A h(\delta^2+2\lambda_B\delta)}{(\lambda_A+\lambda_B)(\delta+\lambda_B)^2} + \frac{\lambda_A\lambda_B^2 h^2\delta}{(\lambda_A+\lambda_B)(\delta+\lambda_B)^3} \\
& \times \frac{1}{\beta+\frac{h\delta}{\delta+\lambda_B}}\Big) e^{-(Y_0+z)\beta} e^{-(\frac{\lambda_B g}{\lambda_A+\lambda_B})(M-X_0)} e^{-(\frac{\lambda_A h}{\lambda_A+\lambda_B})z} \\
& \times I_0\big(2\sqrt{\frac{\lambda_A\lambda_B hg(M-X_0)z}{(\lambda_A+\lambda_B)^2}}\big)\Big] dz + \frac{-\lambda_A\lambda_B^2 h^2\delta}{(\lambda_A+\lambda_B)(\delta+\lambda_B)^3} \cdot \frac{1}{\beta+\frac{h\delta}{\delta+\lambda_B}} \\
& \times e^{-N\beta} e^{-(\frac{\lambda_B g}{\lambda_A+\lambda_B})(M-X_0)} e^{-(\frac{h\delta}{\delta+\lambda_B})(N-Y_0)} \\
& \times \int_{z=0}^{N-Y_0} e^{(\frac{\lambda_B h(\delta-\lambda_A)}{(\lambda_A+\lambda_B)(\delta+\lambda_B)})z} I_0\big(2\sqrt{\frac{\lambda_A\lambda_B hg(M-X_0)z}{(\lambda_A+\lambda_B)^2}}\big) dz \Bigg\} \\
& \times \mathbf{1}_{(X_0,\infty)}(M)\mathbf{1}_{(Y_0,\infty)}(N).
\end{aligned} \tag{2.1}$$

(ii) Case $\boldsymbol{\delta = \lambda_A}$**.** Furthermore,

$$\begin{aligned}
\Phi^{(2)}_{\mu\nu}(0,\beta,0) = \Bigg\{ & \frac{\lambda_A^2}{(\lambda_A+\lambda_B)^2} \cdot e^{-N\beta} e^{-(\frac{\lambda_B g}{\lambda_A+\lambda_B})(M-X_0)} e^{-(\frac{\lambda_A h}{\lambda_A+\lambda_B})(N-Y_0)} \\
& \times I_0\big(2\sqrt{\frac{\lambda_A\lambda_B hg(M-X_0)(N-Y_0)}{(\lambda_A+\lambda_B)^2}}\big) \\
& + \int_{z=0}^{N-Y_0} \Big[\Big(\frac{\lambda_A^2\beta}{(\lambda_A+\lambda_B)^2} + \frac{\lambda_A h(\lambda_A^2+2\lambda_A\lambda_B)}{(\lambda_A+\lambda_B)^3} + \frac{\lambda_A^2\lambda_B^2 h^2}{(\lambda_A+\lambda_B)^4} \cdot \frac{1}{\beta+\frac{\lambda_A h}{\lambda_A+\lambda_B}}\Big) \\
& \times e^{-(Y_0+z)\beta} e^{-(\frac{\lambda_B g}{\lambda_A+\lambda_B})(M-X_0)} e^{-(\frac{\lambda_A h}{\lambda_A+\lambda_B})z} I_0\big(2\sqrt{\frac{\lambda_A\lambda_B hg(M-X_0)z}{(\lambda_A+\lambda_B)^2}}\big)\Big] dz \\
& + \frac{-\lambda_A^2\lambda_B^2 h^2}{(\lambda_A+\lambda_B)^3} \sqrt{\frac{N-Y_0}{\lambda_A\lambda_B hg(M-X_0)}} \cdot \frac{1}{\beta+\frac{\lambda_A h}{\lambda_A+\lambda_B}} \cdot e^{-N\beta} e^{-(\frac{\lambda_B g}{\lambda_A+\lambda_B})(M-X_0)} \\
& \times e^{-(\frac{\lambda_A h}{\lambda_A+\lambda_B})(N-Y_0)} I_1\big(2\sqrt{\frac{\lambda_A\lambda_B hg(M-X_0)(N-Y_0)}{(\lambda_A+\lambda_B)^2}}\big) \Bigg\} \\
& \times \mathbf{1}_{(X_0,\infty)}(M)\mathbf{1}_{(Y_0,\infty)}(N).
\end{aligned} \tag{2.2}$$

Case 3. With $\alpha = \beta = 0$ we obtain the Laplace-Stieltjes transform of the exit time of the game to be lost by player A, $\Phi_{\mu\nu}(0,0,\theta) := Ee^{-\theta\tau_\mu}\mathbf{1}_{\{\mu<\nu\}}$.

(i) Case $\boldsymbol{\delta \neq \lambda_A}$.

$$\Phi^{(1)}_{\mu\nu}(0,0,\theta) = \Bigg\{ \frac{\lambda_A \delta}{\Lambda(\delta+\theta+\lambda_B)} \cdot e^{-(g-\frac{\lambda_A g}{\Lambda})(M-X_0)} e^{-(h-\frac{\lambda_B h}{\Lambda})(N-Y_0)}$$
$$\times I_0\big(2\sqrt{\frac{\lambda_A\lambda_B hg(M-X_0)(N-Y_0)}{\Lambda^2}}\big) + \frac{\lambda_A h\delta}{\Lambda(\delta+\theta)} \cdot e^{-(g-\frac{\lambda_A g}{\Lambda})(M-X_0)}$$
$$\times \int_{z=0}^{N-Y_0} e^{-(h-\frac{\lambda_B h}{\Lambda})z} I_0\big(2\sqrt{\frac{\lambda_A\lambda_B hg(M-X_0)z}{\Lambda^2}}\big)dz$$
$$+ \frac{-\lambda_A\lambda_B^2 h\delta}{\Lambda(\delta+\theta)(\delta+\theta+\lambda_B)^2} \cdot e^{-(g-\frac{\lambda_A g}{\Lambda})(M-X_0)} \cdot e^{-(h-\frac{\lambda_B h}{\delta+\theta+\lambda_B})(N-Y_0)}$$
$$\times \int_{z=0}^{N-Y_0} e^{(\frac{\lambda_B h}{\Lambda}-\frac{\lambda_B h}{\delta+\theta+\lambda_B})z} I_0\big(2\sqrt{\frac{\lambda_A\lambda_B hg(M-X_0)z}{\Lambda^2}}\big)dz \Bigg\}$$
$$\times \mathbf{1}_{(X_0,\infty)}(M)\mathbf{1}_{(Y_0,\infty)}(N). \tag{2.3}$$

(ii) Case $\boldsymbol{\delta = \lambda_A}$.

$$\Phi^{(2)}_{\mu\nu}(0,0,\theta) = \Bigg\{ \frac{\lambda_A^2}{\Lambda^2} \cdot e^{-(g-\frac{\lambda_A g}{\Lambda})(M-X_0)} e^{-(h-\frac{\lambda_B h}{\Lambda})(N-Y_0)}$$
$$\times I_0\big(2\sqrt{\frac{\lambda_A\lambda_B hg(M-X_0)(N-Y_0)}{\Lambda^2}}\big) + \frac{\lambda_A^2 h}{\Lambda(\theta+\lambda_A)} \cdot e^{-(g-\frac{\lambda_A g}{\Lambda})(M-X_0)}$$
$$\times \int_{z=0}^{N-Y_0} e^{-(h-\frac{\lambda_B h}{\Lambda})z} I_0\big(2\sqrt{\frac{\lambda_A\lambda_B hg(M-X_0)z}{\Lambda^2}}\big)dz$$
$$+ \frac{-\lambda_A^2\lambda_B^2 h}{\Lambda^2(\theta+\lambda_A)} \sqrt{\frac{N-Y_0}{\lambda_A\lambda_B hg(M-X_0)}} \cdot e^{-(g-\frac{\lambda_A g}{\Lambda})(M-X_0)} e^{-(h-\frac{\lambda_B h}{\Lambda})(N-Y_0)}$$
$$\times I_1\big(2\sqrt{\frac{\lambda_A\lambda_B hg(M-X_0)(N-Y_0)}{\Lambda^2}}\big) \Bigg\} \mathbf{1}_{(X_0,\infty)}(M)\mathbf{1}_{(Y_0,\infty)}(N). \tag{2.4}$$

Here I_j's are modified Bessel functions.

3 The Probability Distribution of the Casualties Values to Players A and B

Here we will find the probability distribution function F_{A} of the exit value of casualties to player A (special case 1) by taking the inverse Laplace transform with respect to variable α. The Laplace inverse formula that we use, along with (3.64-3.67) [1], is:

$$\mathcal{L}_y^{-1}(e^{-\alpha y} \cdot \frac{1}{(y+b)^2})(q) = (q-\alpha)e^{-b(q-\alpha)}\mathbf{1}_{(\alpha,\infty)}(q). \tag{3.1}$$

The above formula can be found in references [3,4] as well. After that, we apply the Laplace inverse to $\Phi_{\mu\nu}(\alpha,0,0) = Ee^{-\alpha A_\mu}\mathbf{1}_{\{\mu<\nu\}}$, arriving at

$$
\begin{aligned}
&F_{\mathrm{A}}\left(t\right)=\mathcal{L}_{\alpha}^{-1}\Big\{\varPhi_{\mu\nu}(\alpha,0,0)\Big\}(t)=\Big\{\frac{\lambda_A g\delta}{(\lambda_A+\lambda_B)(\delta+\lambda_A+\lambda_B)}\cdot e^{-\left(\frac{(\delta+\lambda_B)g}{\delta+\lambda_A+\lambda_B}\right)(t-M)}\\
&\times e^{-\left(\frac{\lambda_B g}{\lambda_A+\lambda_B}\right)(M-X_0)}e^{-\left(\frac{\lambda_A h}{\lambda_A+\lambda_B}\right)(N-Y_0)}I_0\big(2\sqrt{\frac{\lambda_A\lambda_B hg(M-X_0)(N-Y_0)}{(\lambda_A+\lambda_B)^2}}\big)\\
&+\frac{\lambda_A hg\delta}{(\lambda_A+\lambda_B)(\delta+\lambda_A)}e^{-\left(\frac{g\delta}{\delta+\lambda_A}\right)(t-M)}e^{-(\frac{\lambda_B g}{\lambda_A+\lambda_B})(M-X_0)}\int_{z=0}^{N-Y_0}e^{-(\frac{\lambda_A h}{\lambda_A+\lambda_B})z}\\
&\times I_0\big(2\sqrt{\frac{\lambda_A\lambda_B hg(M-X_0)z}{(\lambda_A+\lambda_B)^2}}\big)dz+\int_{z=0}^{N-Y_0}\Big[\frac{-\lambda_A\lambda_B^2 hg\delta}{(\lambda_A+\lambda_B)(\delta+\lambda_A)(\delta+\lambda_A+\lambda_B)^2}\\
&\times e^{-(\frac{(\delta+\lambda_B)g}{\delta+\lambda_A+\lambda_B})(t-M)}I_0\big(2\sqrt{\frac{\lambda_A\lambda_B hg(N-Y_0-z)(t-M)}{(\delta+\lambda_A+\lambda_B)^2}}\big)\\
&+\frac{-\lambda_A^2\lambda_B hg^2\delta}{(\lambda_A+\lambda_B)(\delta+\lambda_A)^2(\delta+\lambda_A+\lambda_B)}e^{-(\frac{g\delta}{\delta+\lambda_A})(t-M)}\int_{w=0}^{t-M}e^{-(\frac{\lambda_A\lambda_B g}{(\delta+\lambda_A)(\delta+\lambda_A+\lambda_B)})w}\\
&\times I_0\big(2\sqrt{\frac{\lambda_A\lambda_B hg(N-Y_0-z)w}{(\delta+\lambda_A+\lambda_B)^2}}\big)dw+\frac{\lambda_A^2\lambda_B hg^2\delta}{(\lambda_A+\lambda_B)(\delta+\lambda_A+\lambda_B)^2}\\
&\times\sqrt{\frac{t-M}{\lambda_A\lambda_B hg(N-Y_0-z)}}\cdot e^{-(\frac{(\delta+\lambda_B)g}{\delta+\lambda_A+\lambda_B})(t-M)}I_1\big(2\sqrt{\frac{\lambda_A\lambda_B hg(N-Y_0-z)(t-M)}{(\delta+\lambda_A+\lambda_B)^2}}\Big]\\
&\times e^{-(\frac{\lambda_B g}{\lambda_A+\lambda_B})(M-X_0)}e^{-(\frac{(\delta+\lambda_A)h}{\delta+\lambda_A+\lambda_B})(N-Y_0)}e^{(\frac{\lambda_B h\delta}{(\lambda_A+\lambda_B)(\delta+\lambda_A+\lambda_B)})z}\\
&\times I_0\big(2\sqrt{\frac{\lambda_A\lambda_B hg(M-X_0)z}{(\lambda_A+\lambda_B)^2}}\big)dz\Big\}\mathbf{1}_{(X_0,\infty)}(M)\mathbf{1}_{(Y_0,\infty)}(N)\mathbf{1}_{(M,\infty)}(t). \qquad (3.2)
\end{aligned}
$$

4 The Loss Probability

Another special case is the probability that player A loses to player B. This can be directly obtained from $\varPhi_{\mu\nu}(\alpha,\beta,\theta)=Ee^{-\alpha A_\mu-\beta B_\mu-\theta\tau_\mu}\mathbf{1}_{\{\mu<\nu\}}$ by setting $\alpha=\beta=\theta=0$:

$$\varPhi_{\mu\nu}(0,0,0):=E\mathbf{1}_{\{\mu<\nu\}}=P\{\mu<\nu\}=P\{\tau_\mu<\tau_\nu\}. \qquad (4.1)$$

With $\alpha=\beta=\theta=0$ in (3.70-3.73) [1], we have

(i) Case $\boldsymbol{\delta\neq\lambda_A}$,

$$
\begin{aligned}
&\varPhi_{\mu\nu}^{(1)}(0,0,0)=\Big\{\frac{\lambda_A\delta}{(\lambda_A+\lambda_B)(\delta+\lambda_B)}\cdot e^{-\frac{\lambda_B g(M-X_0)}{\lambda_A+\lambda_B}}e^{-\frac{\lambda_A h(N-Y_0)}{\lambda_A+\lambda_B}}\\
&\times I_0\big(2\sqrt{\frac{\lambda_A\lambda_B hg(M-X_0)(N-Y_0)}{(\lambda_A+\lambda_B)^2}}\big)+\frac{\lambda_A h}{\lambda_A+\lambda_B}\cdot e^{-\frac{\lambda_B g(M-X_0)}{\lambda_A+\lambda_B}}\\
&\times\int_{z=0}^{N-Y_0}e^{-\frac{\lambda_A h}{\lambda_A+\lambda_B}z}I_0\big(2\sqrt{\frac{\lambda_A\lambda_B hg(M-X_0)z}{(\lambda_A+\lambda_B)^2}}\big)dz\frac{-\lambda_A\lambda_B^2 h}{(\lambda_A+\lambda_B)(\delta+\lambda_B)^2}\\
&\times e^{-\frac{\lambda_B g(M-X_0)}{\lambda_A+\lambda_B}}e^{-\frac{h\delta(N-Y_0)}{\delta+\lambda_B}}\int_{z=0}^{N-Y_0}e^{(\frac{-\lambda_B h(\lambda_A-\delta)}{(\lambda_A+\lambda_B)(\delta+\lambda_B)})z}I_0\big(2\sqrt{\frac{\lambda_A\lambda_B hg(M-X_0)z}{(\lambda_A+\lambda_B)^2}}\big)dz\Big\}\\
&\times\mathbf{1}_{(X_0,\infty)}(M)\mathbf{1}_{(Y_0,\infty)}(N). \qquad (4.2)
\end{aligned}
$$

(ii) Case $\boldsymbol{\delta = \lambda_A}$,

$$\begin{aligned}
\Phi^{(2)}_{\mu\nu}(0,0,0) = \Bigg\{ & \frac{\lambda_A^2}{(\lambda_A+\lambda_B)^2} \cdot e^{-\frac{\lambda_B g(M-X_0)}{\lambda_A+\lambda_B}} e^{-\frac{\lambda_A h(N-Y_0)}{\lambda_A+\lambda_B}} \\
& \times I_0\big(2\sqrt{\frac{\lambda_A\lambda_B hg(M-X_0)(N-Y_0)}{(\lambda_A+\lambda_B)^2}}\big) + \frac{\lambda_A h}{\lambda_A+\lambda_B} \cdot e^{-\frac{\lambda_B g(M-X_0)}{\lambda_A+\lambda_B}} \\
& \times \int_{z=0}^{N-Y_0} e^{-\frac{\lambda_A h}{\lambda_A+\lambda_B} z} I_0\big(2\sqrt{\frac{\lambda_A\lambda_B hg(M-X_0)z}{(\lambda_A+\lambda_B)^2}}\big) dz \\
& + \frac{-\lambda_A\lambda_B^2 h}{(\lambda_A+\lambda_B)^2}\sqrt{\frac{N-Y_0}{\lambda_A\lambda_B hg(M-X_0)}} \cdot e^{-\frac{\lambda_B g(M-X_0)}{\lambda_A+\lambda_B}} e^{-\frac{\lambda_A h(N-Y_0)}{\lambda_A+\lambda_B}} \\
& \times I_1\big(2\sqrt{\frac{\lambda_A\lambda_B hg(M-X_0)(N-Y_0)}{(\lambda_A+\lambda_B)^2}}\big)\Bigg\} \mathbf{1}_{(X_0,\infty)}(M)\mathbf{1}_{(Y_0,\infty)}(N). \qquad (4.3)
\end{aligned}$$

5 Numerical Results

Even though the above formulas are totally explicit, they may look quite bulky. We would like to illustrate their tameness by means of simple computations. They also show how changing input parameters alters the trend of the game. For a full version of these computations including a MATLAB routine, see [2]. The program utilizes (4.2) and (4.3) with the results placed in the tables below.

λ_A	45	45	45	45	45
λ_B	45	45	45	45	45
g	18	18	18	18	18
h	18	18	18	18	18
M	35	34	33	32	31
N	33	33	33	33	33
X_0	13	13	13	13	13
Y_0	13	13	13	13	13
δ	45	45	45	45	45
Probability of A losing	0.1708	0.3106	0.4895	0.6749	0.8279

λ_A	45	45	45	45	45
λ_B	45	45	45	45	45
g	18	18	18	18	18
h	18	18	18	18	18
M	33	33	33	33	33
N	33	33	33	33	33
X_0	10	11.5	13	14.5	16
Y_0	13	13	13	13	13
δ	45	45	45	45	45
Probability of A losing	0.0811	0.2345	0.4895	0.7574	0.9268

λ_A	18	18	18	18	18
λ_B	20	20	20	20	20
g	14	14	14	14	14
h	16	12	11	10	6
M	20	20	20	20	20
N	24	24	24	24	24
X_0	7	7	7	7	7
Y_0	5	5	5	5	5
δ	100	100	100	100	100
Probability of A losing	0.9991	0.8014	0.5875	0.3324	0.0003

λ_A	18	18	18	18	18
λ_B	20	20	20	20	20
g	14	14	14	14	14
h	16	16	16	16	16
M	32	28	26	24	20
N	24	24	24	24	24
X_0	7	7	7	7	7
Y_0	5	5	5	5	5
δ	100	100	100	100	100
Probability of A losing	0.0129	0.2650	0.5910	0.8717	0.9991

λ_A	18	18	18	18	18
λ_B	20	20	20	20	20
g	14	14	14	14	14
h	16	16	16	16	16
M	20	20	20	20	20
N	24	24	24	24	24
X_0	0.0001	0.01	1	2	7
Y_0	5	5	5	5	5
δ	100	100	100	100	100
Probability of A losing	0.4191	0.4207	0.5910	0.7505	0.9991

λ_A	8	8	8	8	8
λ_B	10	10	10	10	10
g	28	28	28	28	28
h	24	32	35	38	46
M	10	10	10	10	10
N	12	12	12	12	12
X_0	2	2	2	2	2
Y_0	4	4	4	4	4
δ	50	50	50	50	50
Probability of A losing	0.0033	0.2419	0.4963	0.7431	0.9893

λ_A	8	8	8	8	8
λ_B	10	10	10	10	10
g	28	28	28	28	28
h	24	24	24	24	24
M	10	10	10	10	10
N	12	12	12	12	12
X_0	7	5	4.5	4	2
Y_0	4	4	4	4	4
δ	50	50	50	50	50
Probability of A losing	0.9996	0.7190	0.4888	0.2712	0.0033

where

$\lambda_A, \lambda_B =$ rates of strikes to player A by player B and player B to player A;

$g^{-1}, h^{-1} =$ mean magnitudes of strikes to A by B and B to A;

$M, N =$ thresholds of players A and B;

$X_0, Y_0 =$ initial casualties to players A and B;

$\delta^{-1} =$ observations frequency.

Acknowledgment

This research was supported by the US Army Grant No. W911NF-07-1-0121.

References

[1] Dshalalow, J.H., Huang, W., Ke, H-J. and Treerattrakoon, A. On tractable functionals in antagonistic games with a constant initial condition. *Nonlinear Dyanamics and System Theory* **16** (1) (2016) 1–14.

[2] Dshalalow, J.H., Huang, W., Ke, H-J., and Treerattrakoon, A. On antagonistic games with constant initial condition. A complete version. Technical Report, FIT Y15, 1-38.

[3] Bateman, H. and Erdélyi. *Tables of Integral Transforms.* Vol. I. McGraw-Hill, 1954.

[4] Bateman, H. and Erdélyi, *Higher Transcedental Functions.* Vol. 2. McGraw-Hill, 1953.

Nonlinear Dynamics and Systems Theory, **16** (3) (2016) 276–293

Capacity and Non-linear Potential in Musielak-Orlicz Spaces

M.C. Hassib [1*], Y. Akdim [2], A. Benkirane [3] and N. Aissaoui [4]

[1] *Faculty of science and technique, University Sidi Mohamed Ben Abdellah, P.O. Box 2202, road of Imouzzer Fez, Laboratory : LSI, Taza, Morocco.*
[2] *Faculty poly-disciplinary of Taza, Laboratory : LSI, Morocco.*
[3] *Faculty of Sciences Dhar El Mahraz, Laboratory LAMA, University Sidi Mohamed Ben Abdellah, P.O. Box 1796, Atlas Fez, Morocco.*
[4] *Ecole normale supérieure, P.O. Box 5206 Bensouda Fez, Morocco.*

Received: August 3, 2015; Revised: June 12, 2016

Abstract: In this paper we are going to introduce the theory of capacity in Musielak-Orlicz space. We will define the $C_{k,\varphi}$ capacity and the $D_{k,\varphi}$ capacity, prove their main properties, and establish relationship between $C_{k,\varphi}$ and $D_{k,\varphi}$. We shall introduce the theory of non-linear potential and give some of its properties.

Keywords: *Musielak-Orlicz space; Radon measures space; capacity; potential.*

Mathematics Subject Classification (2010): 31C15.

Introduction

The theory of capacity and non-linear potential in the Lebesgue space L^p studied by Maz'ya and Khavin in [10] and Meyers in [11] introduced the concept of capacity and non-linear potential in these spaces and provided very rich applications in functional analysis, harmonic analysis and the theory of partial differential equations. The previous concept was generalised by N. Aissaoui and A. Benkirane in [2] and [3], by replacing L^p by Orlicz space.

The main purpose of this paper is to study the theory of capacity and non-linear potential in Musielak-Orlicz space. Our results generalize those of N. Aissaoui and A. Benkirane in the case of Orlicz spaces [see [3] and [2]]. Let us note that this generalization was touched upon by Fumi-Yuki Maeda, Yoshihiro Mizuta, Takao Ohno and Tetsu Shimomura in [9] [see the third paragraph], but we are going to deal with another method.

* Corresponding author: mailto:cherif_hassib@yahoo.fr

The present paper is organized as follows. In the first section, we recall the main results for the Musielak-Orlicz spaces and Radon measure spaces. In the second section, we define the capacity $C_{k,\varphi}$ in the Musielak-Orlicz spaces, give some of its properties, introduce a $D_{k,\varphi}$ capacity in terms of Radon measures and give its relations with $C_{k,\varphi}$. In the third section, we introduce the theory of the non-linear potential and give some of its properties.

1 Preliminaries

1.1 Musielak-Orlicz function

Let Ω be an open set in $\mathbb{R}^N$ and let φ be a real-valued function defined in $\Omega \times \mathbb{R}^+$ and satisfying the following conditions:
a) $\varphi(x,.)$ is an N-function [convex, increasing, continuous, $\varphi(x,0)=0$, $\varphi(x,t)>0\ \forall t>0$ $\dfrac{\varphi(x,t)}{t} \to 0$ as $t \to 0$, $\dfrac{\varphi(x,t)}{t} \to \infty$ as $t \to \infty$].
b) $\varphi(.,t)$ is a measurable function.

A function $\varphi(x,t)$, which satisfies the conditions a) and b) is called a Musielak-Orlicz function. Equivalently, φ admits the representation:
$\varphi(y,t) = \displaystyle\int_0^t a(y,\tau)d\tau$, for all $y \in \Omega$ and $t \geqslant 0$, where $a(y,.) : \mathbb{R}^+ \to \mathbb{R}^+$ is non-decreasing, right continuous, for all $y \in \Omega$: $a(y,0)=0$,
$a(y,t)>0$ for t¿0 and $\lim\limits_{t\to+\infty} a(y,t) = +\infty$.

The function $a(y,.)$ is called the derivative of $\varphi(y,.)$. The Musielak-Orlicz function φ is said to satisfy the Δ_2-condition if there exists $K \geqslant 2$ such that

$$\varphi(y,2t) \leqslant K\varphi(y,t), \quad for\ all\ y \in \Omega \quad and \ \ t \geqslant 0.$$

The smallest K is called the Δ_2-constant of φ. When the last inequality holds only for $t \geqslant some\ t_0 > 0$ then φ is said to satisfy the Δ_2-condition near infinity.

1.2 Musielak-Orlicz spaces

Let φ be a Musielak-Orlicz function, we define the functional

$$\varrho_{\varphi,\Omega}(u) = \int_\Omega \varphi(x,|u(x)|)dx,$$

where $u : \Omega \mapsto \mathbb{R}$ is a Lebesgue measurable function.

In the following the measurability of a function $u : \Omega \mapsto \mathbb{R}$ means the Lebesgue measurability.

The set
$$K_\varphi(\Omega) = \{u : \Omega \mapsto \mathbb{R},\ \text{measurable}/\varrho_{\varphi,\Omega}(u) < \infty\}$$
is called the Musielak-Orlicz class.

The Musielak-Orlicz space $L_\varphi(\Omega)$ is the vector space generated by $K_\varphi(\Omega)$, that is $L_\varphi(\Omega)$ is the smallest linear space containing the set $K_\varphi(\Omega)$. Equivalently:

$$L_\varphi(\Omega) = \{u : \Omega \mapsto \mathbb{R},\ \text{measurable}/\varrho_{\varphi,\Omega}(\frac{u}{\lambda}) < +\infty \,\text{for some}\ \lambda > 0\}.$$

$K\varphi(\Omega)$ is a convex subset of $L\varphi(\Omega)$. If $\Omega = \mathbb{R}^N$ then $L_\varphi(\mathbb{R}^N)$ is denoted by L_φ.

Let

$$\psi(x,s) = sup\{st - \varphi(x,t) \ \ /t \geqslant 0\}.$$

That is, ψ is the Musielak-Orlicz function complementary to $\varphi(x,t)$ in the sense of Young with respect to the variable s. For two complementary Musielak-Orlicz functions φ and ψ the following inequality is called the Young inequality [12]

$$t.s \leqslant \varphi(x,t) + \psi(x,s) \text{ for all s, t} \geqslant 0 \ , x \in \Omega. \tag{1}$$

If $s = a(x,t)$ then

$$t.a(x,t) = \varphi(x,t) + \psi(x, a(x,t)) \text{ for all t} \geqslant 0 \ , x \in \Omega. \tag{2}$$

In the space $L_\varphi(\Omega)$ we define the following two norms:

$$||u||_{\varphi,\Omega} = \inf\{\lambda > 0 : \varrho_{\varphi,\Omega}(\frac{u}{\lambda}) \leqslant 1\}$$

called the Luxemburg norm and the so-called Orlicz norm by :

$$|||u|||_{\varphi,\Omega} = \sup_{||v||_{\psi,\Omega} \leqslant 1} \int_\Omega |u(x)v(x)|dx,$$

where ψ is the Musielak-Orlicz function complementary to φ. These two norms are equivalent [12].

For two complementary Musielak-Orlicz functions φ and ψ let $u \in L_\varphi(\Omega)$ and $v \in L_\psi(\Omega)$, we have the *Hölder* inequality [12]

$$|\int_\Omega u(x)v(x)dx| \leqslant ||u||_{\varphi,\Omega} \, |||v|||_{\psi,\Omega} \, . \tag{3}$$

In $L_\varphi(\Omega)$ we have the relation with the norm and the modular:

$$|||u|||_{\varphi,\Omega} \leqslant \varrho_{\varphi,\Omega}\,(u) + 1, \tag{4}$$

$$||u||_{\varphi,\Omega} \leqslant \varrho_{\varphi,\Omega}\,(u) \ \ , \text{if} \, ||u||_{\varphi,\Omega} > 1, \tag{5}$$

$$||u||_{\varphi,\Omega} \geqslant \varrho_{\varphi,\Omega}\,(u) \ \ , \text{if} \, ||u||_{\varphi,\Omega} \leqslant 1. \tag{6}$$

If $\Omega = \mathbb{R}^N$ then two norms $||.||_{\varphi,\mathbb{R}^N}$ and $|||.|||_{\varphi,\mathbb{R}^N}$ are denoted respectively by $||.||_\varphi$. and $|||.|||_\varphi$.

We say that a sequence of function $u_n \in L_\varphi(\Omega)$ is modular convergent to $u \in L_\varphi(\Omega)$ if there exists a constant k ¿ 0 such that

$$\lim_{n\to+\infty} \varrho_{\varphi,\Omega}\,(\frac{u_n - u}{k}) = 0.$$

If φ satisfies the $\triangle_2$ condition, then modular convergence coincides with norm convergence.

The closure in $L_\varphi(\Omega)$ of the set of bounded measurable functions with compact support in $\bar{\Omega}$ is denoted by $E_\varphi(\Omega)$ and it is a separable space. The equality $K_\varphi(\Omega) = E_\varphi(\Omega) = L_\varphi(\Omega)$ holds if and only if φ satisfies the $\triangle_2$ condition, for all t or for large t, according to whether Ω has infinite measure or not. The dual of $E_\varphi(\Omega)$ can be identified with $L_\psi(\Omega)$ by means of the pairing $\int_\Omega u(x)v(x)dx$ and the dual norm on $L_\psi(\Omega)$ is equivalent to $||.||_\psi$. The space $L_\varphi(\Omega)$ is reflexive if and only if φ and ψ satisfy the $\triangle_2$ condition, for all t or for large t according to whether Ω has infinite measure or not.

Lemma 1.1 *[8] Let φ be a Musielak-Orlicz function and f_n, f, g be measurable functions.*
(a) If $f_n \longrightarrow f$, almost everywhere , then $\varrho_{\varphi,\Omega}(f) \leqslant \liminf_{n\to+\infty} \varrho_{\varphi,\Omega}(f_n)$.
(b) If $|f_n| \nearrow |f|$, almost everywhere, then $\varrho_{\varphi,\Omega}(f) = \lim_{n\to+\infty} \varrho_{\varphi,\Omega}(f_n)$.
(c) If $f_n \longrightarrow f$, almost everywhere, $|f_n| \leqslant |g|$, almost everywhere and $\varrho_{\varphi,\Omega}(\lambda g) < \infty$ for every $\lambda > 0$, then $f_n \to f$ strongly in $L_\varphi(\Omega)$.

Theorem 1.1 *[8] Let φ be a Musielak-Orlicz function.*
(a) $||f||_{\varphi,\Omega} = ||\ |f|\ ||_{\varphi,\Omega}$ for all $f \in L_\varphi(\Omega)$.
(b) If $f \in L_\varphi(\Omega)$, g is a measurable function, and $0 \leqslant |g| \leqslant |f|$ almost everywhere, then:

$$g \in L_\varphi(\Omega) \quad and \quad ||g||_{\varphi,\Omega} \leqslant ||f||_{\varphi,\Omega}\,.$$

(c) If $f_n \to f$ almost everywhere, then: $||f||_{\varphi,\Omega} \leqslant \liminf_{n\to+\infty} ||f_n||_{\varphi,\Omega}$.
(d) If $|f_n| \nearrow |f|$ almost everywhere with $f_n \in L_\varphi(\Omega)$ and $\sup_n ||f_n||_{\varphi,\Omega} < \infty$ then:

$$f \in L_\varphi(\Omega) \quad and \quad ||f_n||_{\varphi,\Omega} \nearrow ||f||_{\varphi,\Omega}\,.$$

Theorem 1.2 *[5] Let φ and ψ be two complementary Musielak-Orlicz functions. Assume that there exists a constant $A > 0$ such that for all $x, y \in \Omega$: $|x - y| \leqslant \frac{1}{2}$ we have:*

$$\frac{\varphi(x,t)}{\varphi(y,t)} \leqslant t^{\frac{A}{log(\frac{1}{|x-y|})}} \tag{7}$$

for all $t \geq 1$. If $D \subset \Omega$ is a bounded measurable set, then $\int_D \varphi(x,1)dx < \infty$.
ψ satisfies the following condition:

$$\exists C > 0 \ : \ \psi(x,1) \leqslant C, \quad almost\ everywhere\ \ in\ \ \Omega. \tag{8}$$

Under the previous conditions, with $\Omega = \mathbb{R}^N$; $C_0^\infty(\mathbb{R}^N)$ is dense in $L_\varphi(\mathbb{R}^N)$ with respect to the modular topology.

1.3 Measures space

M designates the vector space of Radon measures. M is endowed with the weak topology for which a sequence (μ_n) converges weakly to μ, if for any continuous function f with compact support

$$\lim_{n\to+\infty} \int f d\mu_n = \int f d\mu.$$

M^+ is the cone of positive elements of M.

For all measures $\mu < \infty$, for all $X \subset \mathbb{R}^N$, the variation of μ is defined by:

$$||\mu||(X) = sup\{\sum_1^n |\mu(X_i)| : (X_i)_{i=1\ldots n} \ \ is\ \ an\ \ X\ \ partiton\}.$$

$||\mu||(\mathbb{R}^N) = ||\mu||$ is called the total variation of μ. M_1 designates the Banach space of measures, endowed with the norm total variation. M_1^+ designates the subset of M_1 consisting of positive measures.

Definition 1.1 Let $\mu \in M_1^+$. We say that μ is concentrated on X if $\mu(Y) = 0$ for all $\mu - measurable\ set\ Y$, such that $Y \subset X^c$.

2 Capacity in Musielak-Orlicz Space

2.1 $C_{k,\varphi}$-capacity

Lemma 2.1 *Let Ω be an open set in $\mathbb{R}^N$ and φ be a Musielak-Orlicz function such that*

$$\varphi(y,t) = \int_0^t a(y,\tau)d\tau, \ \forall y \in \Omega \text{ and } \ t \geqslant 0.$$

Let $u : \Omega \to \mathbb{R}$ be measurable function and $\alpha > 0$, we define a measurable function $g : \Omega \to \mathbb{R}$ so that

$$g(y) = a(y, \frac{|u(y)|}{2\alpha}), \ \forall y \in \Omega.$$

If $(\frac{u}{\alpha}) \in K_\varphi(\Omega)$ then $g \in K_\psi(\Omega)$, where ψ is the Musielak-Orlicz function complementary to φ.

Proof. For all $y \in \Omega$ and $t \geqslant O$: $\varphi(y,2t) = \displaystyle\int_0^{2t} a(y,\tau)d\tau \ \geqslant \int_t^{2t} a(y,\tau)d\tau$.
Hence $\varphi(y,2t) \geqslant ta(y,t)$, thus for all $y \in \Omega$: $\varphi(y, \frac{|u(y)|}{\alpha}) \geqslant \frac{|u(y)|}{2\alpha} a(y, \frac{|u(y)|}{2\alpha})$.
On the other hand, we have: $\frac{|u(y)|}{2\alpha} a(y, \frac{|u(y)|}{2\alpha}) - \varphi(y, \frac{|u(y)|}{2\alpha}) = \psi(y, a(y, \frac{|u(y)|}{2\alpha}))$.
Therefore, $\psi(y, a(y, \frac{|u(y)|}{2\alpha})) \leqslant \varphi(y, \frac{|u(y)|}{\alpha}) - \varphi(y, \frac{|u(y)|}{2\alpha})$, this implies that

$$\int_\Omega \psi(y, a(y, \frac{|u(y)|}{2\alpha}))dy \leqslant \int_\Omega \varphi(y, \frac{|u(y)|}{\alpha})dy - \int_\Omega \varphi(y, \frac{|u(y)|}{2\alpha})dy,$$

then

$$\varrho_{\psi,\Omega}(g) \leqslant \varrho_{\varphi,\Omega}(\frac{u}{\alpha}) - \varrho_{\varphi,\Omega}(\frac{u}{2\alpha}).$$

Since $\varrho_{\varphi,\Omega}(\frac{u}{2\alpha}) \leqslant \frac{1}{2}\varrho_{\varphi,\Omega}(\frac{u}{\alpha})$ and $\varrho_{\varphi,\Omega}(\frac{u}{\alpha}) < \infty$, the proof is complete.

Lemma 2.2 *If (f_n) is a sequence in $L_\varphi(\Omega)$ such that for all $n \in \mathbb{N}$, $f_n \geqslant 0$, then*

$$||\sup_n f_n||_{\varphi,\Omega} \leqslant ||\sum_n f_n||_{\varphi,\Omega} \leqslant \sum_n ||f_n||_{\varphi,\Omega}.$$

Proof. Since $0 \leqslant \sup_n f_n \leqslant \sum_n f_n$, thus $||\sup_n f_n||_{\varphi,\Omega} \leqslant ||\sum_n f_n||_{\varphi,\Omega}$.

Let $g_n = \sum_{k=0}^n f_k$ and $f = \sum_n f_n$, we have

$$\frac{g_n}{\sum_n ||f_n||_{\varphi,\Omega}} \nearrow \frac{f}{\sum_n ||f_n||_{\varphi,\Omega}} \ \textit{almost everywhere.}$$

By Lemma 1.1, we obtain

$$\varrho_{\varphi,\Omega}(\frac{f}{\sum_n ||f_n||_{\varphi,\Omega}}) = \lim_{n\to+\infty} \varrho_{\varphi,\Omega}(\frac{g_n}{\sum_n ||f_n||_{\varphi,\Omega}}) \leqslant \lim_{n\to+\infty} \varrho_{\varphi,\Omega}(\frac{g_n}{||g_n||_{\varphi,\Omega}}) \leqslant 1.$$

Then

$$||\frac{f}{\sum_n ||f_n||_{\varphi,\Omega}}||_{\varphi,\Omega} \leqslant 1.$$

Therefore,

$$||\sum_n f_n||_{\varphi,\Omega} \leqslant \sum_n ||f_n||_{\varphi,\Omega}\,.$$

Lemma 2.3 *Let φ be a Musielak-Orlicz function, which satisfies the $\triangle_2$ condition, and such that*

$$\varphi(y,t) = \int_0^t a(y,\tau)d\tau, \quad for\ all\ y \in \Omega \quad and \quad t \geqslant 0.$$

Let $f \in L_\varphi(\Omega)$, such that $f \geqslant 0$, and $||f||_{\varphi,\Omega} \neq 0$.

We define a measurable function $g : \Omega \to \mathbb{R}$ such that for all $y \in \Omega$; $g(y) = a(y, \frac{f(y)}{||f||_{\varphi,\Omega}})$. Then $\int f(y)g(y)dy = ||f||_{\varphi,\Omega}\,|||g|||_{\psi,\Omega}\,.$

Proof. By Lemma 2.1, we have $g \in L_\psi(\Omega)$ and by the *Hölder* inequality we have

$$\int_\Omega f(x)g(x)dx \leqslant ||f||_{\varphi,\Omega}\,|||g|||_{\psi,\Omega}\,.$$

For the opposite inequality, let $h = \frac{f}{||f||_{\varphi,\Omega}}$, and $v \in L_\varphi(\Omega)$, such that $||v||_{\varphi,\Omega} \leqslant 1$.

For all $y \in \Omega$, we have

$$g(y)h(y) = \varphi(y, h(y)) + \psi(y, g(y))$$

and

$$g(y)v(y) \leqslant \varphi(y, v(y)) + \psi(y, g(y)).$$

Hence for all $y \in \Omega$:

$$g(y)v(y) \leqslant g(y)h(y) - \varphi(y, h(y)) + \varphi(y, v(y)).$$

Then

$$\int_\Omega g(y)v(y)dy \leqslant \int_\Omega g(y)h(y)dy - \int_\Omega \varphi(y, h(y))dy + \int_\Omega \varphi(y, v(y))dy.$$

Thus,

$$\int_\Omega g(y)v(y)dy \leqslant \int_\Omega g(y)h(y)dy - \varrho_{\varphi,\Omega}(h) + \varrho_{\varphi,\Omega}(v).$$

We have $\varrho_{\varphi,\Omega}(v) \leqslant 1$. On the other hand φ satisfies the $\triangle_2$ condition, then, $\varrho_{\varphi,\Omega}$ is a continuous modular[see [8] Lemma 2.4.3]. We have $||h||_{\varphi,\Omega} = 1$, then $\varrho_{\varphi,\Omega}(h) = 1$ [see [8] Lemma 2.1.14].

Thus,

$$\int g(y)v(y)dy \leqslant \int g(y)h(y)dy$$

implies

$$\sup_{||v||_{\varphi,\Omega} \leqslant 1} \int g(y)v(y)dy \leqslant \int g(y)h(y)dy.$$

Then

$$|||g|||_{\psi,\Omega}\,||f||_{\varphi,\Omega} \leqslant \int f(y)g(y)dy.$$

Definition 2.1 Let T be a class of Borel sets in $\mathbb{R}^N$, and a function $C : T \to [0, +\infty]$.
1) C is called a capacity if the following axioms are satisfied:
i) $C(\emptyset) = 0$.
ii) $X \subset Y \Rightarrow C(X) \leqslant C(Y)$, for all X and Y in T.
iii) For all sequences $(X_n) \subset T$:

$$C(\bigcup_n X_n) \leqslant \sum_n C(X_n).$$

2) C is called an outer capacity if for all $X \in T$:

$$C(X) = \inf\{C(O) : O \supset X, \quad O \textit{ is } \ \textit{open}\}.$$

3) C is called an interior capacity if for all $X \subset T$:

$$C(X) = \sup\{C(K) : K \subset X, \quad K \ \textit{ is compact}\}.$$

4) A property, that holds true except perhaps on a set of zero capacity is said to be true C-quasi-everywhere, (C-q.e).
5) f *and* (f_n) are real-valued finite functions C-q.e. We say that (f_n) converges to f in C-capacity if:

$$\forall \varepsilon > 0, \quad \lim_{n\to+\infty} C(\{x : \ |f_n(x) - f(x)| > \varepsilon\}) = 0.$$

6) f *and* (f_n) are real-valued function finite C-q.e. We say that (f_n) converges to f C-quasi- uniformly, (C-q.u) if

$$(\forall \varepsilon > 0), (\exists\ X \in T) \ : \ C(X) < \varepsilon \ \textit{ and } (f_n) \ \textit{ converges to } f \textit{ uniformly on } X^c.$$

Remark 2.1 In the following $\Omega = \mathbb{R}^n, \varphi$ is a Musielak-Orlicz function, and $L^+_\varphi = \{\ f \in L_\varphi \ / \ f \geqslant 0\}$.

Theorem 2.1 *Let k be a positive integrable function on $\mathbb{R}^N$. For all $X \subset \mathbb{R}^N$, we put $C_{k,\varphi}(X) = \inf\{||f||_\varphi : f \in L_\varphi \quad \text{and } k * f \geqslant 1 \text{ on } \ X\}$, where $k * f$ is the convolution of k and f. $C_{k,\varphi}$ is an outer capacity.*

Remark 2.2 Let $B_{k,\varphi}(X) = \inf\{||f||_\varphi : f \in L^+_\varphi \quad \textit{and} \quad k * f \geqslant 1 \quad \textit{on } X\}$, then

$$C_{k,\varphi}(X) = B_{k,\varphi}(X).$$

Indeed, it is obvious that $C_{k,\varphi}(X) \leqslant B_{k,\varphi}(X)$. On the other hand, let $f \in L_\varphi$, then $|f| \in L^+_\varphi$ and if $k * f \geqslant 1$ on X, then $k * |f| \geqslant 1$ on X. Thus $B_{k,\varphi}(X) \leqslant ||f||_\varphi$; and therefore $B_{k,\varphi}(X) \leqslant C_{k,\varphi}(X)$.

Proof of Theorem 2.1. It is obvious that $C_{k,\varphi}(\emptyset) = 0$ and $C_{k,\varphi}(X) \leqslant C_{k,\varphi}(Y)$ if $X \subset Y$. Let $(X_n) \subset T$, so that $\sum_i C_{k,\varphi}(X_i) < +\infty$, then $(\forall i \in \mathbb{N})$ $C_{k,\varphi}(X_i) < +\infty$. Thus, $(\forall i \in \mathbb{N})(\forall \varepsilon > 0)$, $(\exists f_i \in L^+_\varphi)$ so that $k{*}f_i \geqslant 1$ *on* X_i and $||f_i||_\varphi \leqslant C_{k,\varphi}(X_i) + \dfrac{\varepsilon}{2^i}$.

Let $f = \sup_i f_i$. By Lemma 2.2, we have:

$$||f||_\varphi \leqslant \sum_i ||f_i||_\varphi.$$

We can write

$$||f||_\varphi \leqslant \sum_i C_{k,\varphi}(X_i) + \varepsilon,$$

which implies that, $f \in L_\varphi$.

Since $k * f \geqslant 1$ *on* $\bigcup_i X_i$,

$$C_{k,\varphi}(\bigcup_i X_i) \leqslant \sum_i C_{k,\varphi}(X_i) + \varepsilon, \quad \forall \varepsilon > 0.$$

Hence,

$$C_{k,\varphi}(\bigcup_i X_i) \leqslant \sum_i C_{k,\varphi}(X_i).$$

It remains to show that $C_{k,\varphi}$ is outer. Let $X \subset \mathbb{R}^N$, we have:

$$C_{k,\varphi}(X) \leqslant \inf\{C_{k,\varphi}(O) : O \supset X, \quad O \textit{ is open}\}.$$

For the reverse inequality, if $C_{k,\varphi}(X) = +\infty$ there is nothing to show.
Assume that $C_{k,\varphi}(X) < +\infty$, and let $0 < \varepsilon < 1$, then $\exists\, g \in L^+_\varphi$ so that $k{*}g \geqslant 1$ *on* X and $||g||_\varphi \leqslant C_{k,\varphi}(X) + \varepsilon$.

Let $g_\varepsilon = \dfrac{g}{1-\varepsilon}$ and $O_\varepsilon = \{x : (k * g_\varepsilon) > 1\}$, thus O_ε is open and

$$\forall x \in X; \quad (k * g_\varepsilon) \geqslant \frac{1}{1-\varepsilon} > 1.$$

Hence, $X \subset O_\varepsilon$. On the other hand, we have $C_{k,\varphi}(O_\varepsilon) \leqslant ||g_\varepsilon||_\varphi$, and we deduce that

$$C_{k,\varphi}(O_\varepsilon) \leqslant \frac{1}{1-\varepsilon}||g||_\varphi \leqslant \frac{1}{1-\varepsilon}[C_{k,\varphi}(X) + \varepsilon].$$

Therefore,

$$\inf\{C(O) : O \supset X, \quad O \textit{ isopen}\} \leqslant \frac{1}{1-\varepsilon}[C_{k,\varphi}(X) + \varepsilon], \quad \forall \varepsilon > 0.$$

Thus,

$$\inf\{C(O) : O \supset X, \quad O \textit{ isopen}\} \leqslant C_{k,\varphi}(X).$$

Theorem 2.2 *1) If there exists* $f \in L_\varphi$ *such that* $|k * f| = +\infty$ *on* X*, then* $C_{k,\varphi}(X) = 0$.
2) If $C_{k,\varphi}(X) = 0$ *then there exists* $f \in L_\varphi^+$ *such that* $k * f = +\infty$ *on* X.

Proof. 1) Let $f \in L_\varphi$ such that $|k * f| = +\infty$ on X, then $\forall \alpha > 0,\ |k * f| \geqslant \alpha$ on X. Then $C_{k,\varphi}(X) \leqslant \dfrac{||f||_\varphi}{\alpha},\ \ \forall \alpha > 0$; this means that $C_{k,\varphi}(X) = 0$.
2) If $C_{k,\varphi}(X) = 0$ then $\forall i \in \mathbb{N},\ \exists f_i \in L_\varphi^+ :\ \ k * f_i \geqslant 1\ \ on\ \ X\ \ and\ \ ||f_i||_\varphi \leqslant 2^{-i}$.
Let $f = \sum_i f_i$. By Lemma 2.2, $||f||_\varphi \leqslant \sum_i ||f_i||_\varphi$, then $||f||_\varphi < +\infty$.
We deduce that $f \in L_\varphi^+$ and $k * f = +\infty$ *on* X.

Theorem 2.3 *Consider the following propositions :*
i) $f_n \longrightarrow f$ *strongly in* L_φ.
ii) $k * f_n \longrightarrow k * f,\ C_{k,\varphi}$ *−capacity.*
iii) There is a subsequence $(f_{n_j})_j$ *such that :* $k * f_{n_j} \longrightarrow k * f\ \ C_{k,\varphi}$ *−q.u.*
iv) $k * f_{n_j} \longrightarrow k * f$ *in* $C_{k,\varphi}$ *−q.e.*
We have

$$i) \Rightarrow\ ii) \Rightarrow\ iii) \Rightarrow iv).$$

Proof. We show $i) \Rightarrow\ ii)$.
By Theorem 2.2, we have $k * f$ *and* $k * f_n$ are finite $C_{k,\varphi}$ *−q.e*, $\forall n$.
Let $\varepsilon > 0$; then

$$C_{k,\varphi}(\{x\ : |k * f_n - k * f|(x) > \varepsilon\}) \leqslant \frac{||f_n - f||_\varphi}{\varepsilon}.$$

We show $ii) \Rightarrow\ iii)$.
Let $\varepsilon > 0\ \ \exists\ f_{n_j}$ such that

$$C_{k,\varphi}(\{x\ :\ |k * f_{n_j} - k * f|(x) > 2^{-j}\}) < \varepsilon.2^{-j}.$$

We put

$$E_j = \{x\ :\ |k * f_{n_j} - k * f|(x) > 2^{-j}\} \quad \text{and} \quad G_m = \bigcup_{j \geqslant m} E_j.$$

We have $C_{k,\varphi}(G_m) \leqslant \sum_{j \geqslant m} \varepsilon.2^{-j} < \varepsilon$.
On the other hand :

$$\forall x \in (G_m)^c,\ \forall\ j \geqslant m :\ \ |k * f_{n_j} - k * f|(x) \leqslant 2^{-j}.$$

Thus $k * f_{n_j} \longrightarrow k * f\ \ C_{k,\varphi}$ *−q.u.*

We show $iii) \Rightarrow\ iv)$. We have $\forall j \in \mathbb{N}, \exists X_j\ :\ C_{k,\varphi}(X_j) \leqslant \dfrac{1}{j}$ and $k * f_{n_j} \longrightarrow k * f$ *on* $(X_j)^c$. We put $X = \bigcap_j X_j$, then $C_{k,\varphi}(X) = 0$ *and* $k * f_{n_j} \longrightarrow k * f$ *on* X^c.

Theorem 2.4 *Let* φ *be a Musielak-Orlicz function that satisfies the* $\triangle_2$ *condition, and* (f_n) *be a sequence in* L_φ *such that* $\sum_n |f_n| \in L_\varphi$*. Then,*

$$\sum_n (k * f_n) = k * (\sum_n f_n) \quad C_{k,\varphi} -q.e.$$

Proof. **First step**: Assume that $f_n \geqslant 0 \ \forall n \in \mathbb{N}$, and let $g_n = \sum_{i=1}^{n} f_i$ *and* $f = \sum_n f_n$.
We have $g_n \to f$ *almost everywhere* and $g_n \leqslant f$. On the other hand, $\varrho_\varphi(\lambda f) < \infty \ \ \forall \lambda > 0$ because $f \in L_\varphi$ and φ satisfies the $\triangle_2$ condition [see [8] paragraph 2.5].

By (c) of Lemma 1.1 we have

$$g_n \to f \textit{ strongly in } L_\varphi.$$

Theorem 2.3 implies that there is a subsequence (g_{n_i}) such that $k * g_{n_i} \to k * f$, $C_{k,\varphi}$-q.e.
Since $f_n \geqslant 0, \ \ \forall n \in \mathbb{N} \quad k * g_n \to k * f$, $C_{k,\varphi}$-q.e.

Second step: If f_n has any sign, then $\sum_n f_n^+$ and $\sum_n f_n^-$ are in L_φ because $|\sum_n f_n^+| \leqslant \sum_n |f_n|$, $|\sum_n f_n^-| \leqslant \sum_n |f_n|$ and $\sum_n |f_n| \in L_\varphi$.
By the first step the result follows.

Theorem 2.5 *Let (K_n) be a decreasing sequence of compact and $K = \bigcap_n K_n$. Then* $\lim_{n\to+\infty} C_{k,\varphi}(K_n) = C_{k,\varphi}(K)$.

Proof. First, we observe that $C_{k,\varphi}(K) \leqslant \lim_{n\to+\infty} C_{k,\varphi}(K_n)$. On the other hand, let O be an open set containing K. By the compactness of K, $K_i \subset O$ for all sufficiently large i. Therefore $\lim_{n\to+\infty} C_{k,\varphi}(K_n) \leqslant C_{k,\varphi}(O)$, and since $C_{k,\varphi}$ is an outer capacity, we obtain the claim by taking infimum over open set O containing K.

Theorem 2.6 *Let φ be a Musielak-Orlicz function, uniformly convex that satisfies the $\triangle_2$ condition. If f_n , $f \in L_\varphi$ such that $f_n \rightharpoonup f$ weakly in L_φ, then:*

$$\liminf(k * f_n) \leqslant (k * f) \leqslant \limsup(k * f_n) \quad C_{k,\varphi} - q.e.$$

Proof. $(L_\varphi \ , \ ||.||)$ is uniformly convex therefore reflexive. By the Banach-Saks theorem, there is a subsequence denoted again by (f_n) such that the sequence
$g_n = \frac{1}{n}\sum_{i=1}^{n} f_i$ converges to f strongly in L_φ. By Theorem 2.3, there is a subsequence of (g_n) denoted again by (g_n) such that

$$\lim_{n\to+\infty}(k * g_n) = (k * f) \quad C_{k,\varphi} - q.e.$$

On the other hand,

$$\liminf(k * f_n) \leqslant \lim_{n\to+\infty}(k * g_n) \ .$$

Therefore,

$$\lim_{n\to+\infty}(k * f_n) \leqslant (k * f) \quad C_{k,\varphi} - q.e.$$

For the second inequality, it suffices to replace f_n by $(-f_n)$ in the first inequality.

Theorem 2.7 *Let φ be a Musielak-Orlicz function, uniformly convex that satisfies the $\triangle_2$ condition, (X_n) be an increasing sequence of sets and $X = \bigcup_n X_n$. Then*

$$\lim_{n\to+\infty} C_{k,\varphi}(X_n) = C_{k,\varphi}(X).$$

Proof. We have $\lim_{n\to+\infty} C_{k,\varphi}(X_n) \leqslant C_{k,\varphi}(X)$. For the reverse inequality, if $C_{k,\varphi}(X) = +\infty$, there is nothing to show.

Assuming that $C_{k,\varphi}(X) < +\infty$, we have

$$\forall n \in \mathbb{N}, \ \ \exists f_n \in L_\varphi^+ \ : k * f_n \geqslant 1 \ on \ X_n \ and \ \ ||f_n||_\varphi \leqslant C_{k,\varphi}(X_n) + \frac{1}{n}.$$

Thus, (f_n) is a bounded sequence in L_φ.

On the other hand, L_φ is uniformly convex, then it is reflexive because φ is uniformly convex and satisfies the $\triangle_2$ condition, [see [8] Remark 2.4.15]. Hence there exists a subsequence which is denoted again by (f_n), and converges weakly to a function $f \in L_\varphi$. Then by Theorem 2.6,

$$\forall n \in \mathbb{N} \ : k * f \geqslant 1 \ \ on \ \ X_n, \ C_{k,\varphi} - q.e.$$

Therefore,

$$k * f \geqslant 1 \ \ on \ \ X, \ \ C_{k,\varphi} - q.e.$$

Let Y be a subset of X where $k * f \geqslant 1$, then $C_{k,\varphi}(X) = C_{k,\varphi}(Y)$. On the other hand we know that

$$\varphi(y,t) = \int_0^t a(y,\tau)d\tau, \quad \text{for all} \ \ y \in R^N \ \ and \ t \geqslant 0.$$

Let the function $g : \mathbb{R}^N \to \mathbb{R}$ be defined by $g(y) = a(y, \frac{|f(y)|}{||f||_\varphi})$ for all $y \in \mathbb{R}^N$. By Lemma 2.1, $g \in L_\psi$, and since φ satisfies the $\triangle_2$ condition, we have $L_\psi = (L_\varphi)^*$. Thus,

$$\int f_n(y)g(y)dy \to \int f(y)g(y)dy.$$

By Lemma 2.3, we have

$$\int f(y)g(y)dy = ||f||_\varphi |||g|||_\psi.$$

By the *Hölder* inequality we have:

$$\int f_n(y)g(y)dy \leq ||f_n||_\varphi |||g|||_\psi.$$

Therefore,

$$||f||_\varphi \leqslant \lim_{n\to+\infty} ||f_n||_\varphi \leqslant \lim_{n\to+\infty} (C_{k,\varphi}(X_n) + \frac{1}{n}).$$

Thus,

$$C_{k,\varphi}(X) \leqslant \lim_{n\to+\infty} C_{k,\varphi}(X_n).$$

Corollary 2.1 *Let φ be a Musielak-Orlicz function, uniformly convex, that satisfies the $\triangle_2$ condition. Let $E_n \subset \mathbb{R}^N$, then $C_{k,\varphi}(\liminf E_n) \leqslant \liminf C_{k,\varphi}(E_n)$.*

Proof. Let $E = \liminf E_n$, we have $E = \bigcup_n (\bigcap_{i \geqslant n} E_i)$.
We put $G_n = \bigcap_{i \geqslant n} E_i$. Thus a sequence (G_n) is increasing and by Theorem 2.7, $C_{k,\varphi}(E) = \lim_n C_{k,\varphi}(G_n)$. On the other hand, $C_{k,\varphi}$ is increasing, then $C_{k,\varphi}(G_n) \leqslant C_{k,\varphi}(E_n)$; therefore

$$C_{k,\varphi}(E) \leqslant \liminf C_{k,\varphi}(E_n).$$

Theorem 2.8 *Let φ be a Musielak-Orlicz function which satisfies the assumptions of Theorem 1.2. If φ satisfies the $\triangle_2$ condition, then for each $f \in L_\varphi$, there is a $C_{k,\varphi}$-quasicontinuous function $g \in L_\varphi$ such that $k * f = g$ $C_{k,\varphi}$ $-q.e.$*

Proof. Let $f \in L_\varphi$, by Theorem 1.2, there exists a sequence (f_n) in $C_0^\infty(\mathbb{R}^N)$ such that $f_n \longrightarrow f$ in L_φ. By Theorem 2.3, there exists a subsequence of (f_n) denoted again by (f_n) such that

$$k * f_n \longrightarrow k * f \quad C_\varphi - q.u.$$

Since k is integrable function and f_n is continuous $\forall n$, then $k * f_n$ is continuous. Thus, the proof is complete.

Definition 2.2 In the terminology of Choquet, C is called a capacity if it satisfies the following four properties:
i) $C(\emptyset) = 0$.
ii) C is increasing.
iii) If (E_n) is an increasing sequence of sets, then $\sup_n C(X_n) = C(\bigcup_n X_n)$.
iv) If (K_n) is a decreasing sequence of compacts, then $\inf_n C(K_n) = C(\bigcap_n K_n)$.

Remark 2.3 Let φ be a Musielak-Orlicz function, uniformly convex, that satisfies the $\triangle_2$ condition. By Theorems 2.1, 2.5 and 2.7 $C_{k,\varphi}$ is a capacity, in the sense of Choquet.

Definition 2.3 Let C be a capacity in the sense of Choquet, and $X \subset \mathbb{R}^N$.
X is called capacitable if

$$C(X) = \sup\{C(K) : K \subset X, \quad K \ iscompact\}.$$

Theorem 2.9 *Let φ be a Musielak-Orlicz function, uniformly convex that satisfies the $\triangle_2$ condition. Then all analytic sets are $C_{k,\varphi}$- capacitable .*

Proof. It is an immediate consequence of Choquet theorem [7].

2.2 Capacity in terms of measure

Theorem 2.10 *Let φ be a Musielak-Orlicz function, k be a positive integrable function on $\mathbb{R}$, and X be a μ-measurable set, for all positive measures μ. We put*
$D_{k,\varphi}(X) = \sup\{||\mu|| \ : \ \mu \in M_1^+, \mu \ is \ \ concentrated \ on \ X \ \ and \ ||k * \mu||_\psi \leqslant 1\}$
where $(k * \mu)(x) = \int k(x-y)d\mu(y)$. Then, $D_{k,\varphi}$ is an interior capacity.

Proof. It is clear that $D_{k,\varphi}(\emptyset) = 0$ and $D_{k,\varphi}(X) \leqslant D_{k,\varphi}(Y)$ if $X \subset Y$.
Let $\mu \in M_1^+$, (X_n) be a sequence of μ-measurable sets and $\mu_n = \mu|_{X_n}$ be defined by

$$\mu_n(Y) = \mu(X_n \cap Y), \quad for\ all\ \ \mu - measurable\ \ set\ \ Y.$$

First we assume that the X_n are pairwise disjoint, then

$$\mu(\bigcup_n X_n) = \sum_n \mu(X_n).$$

If μ is concentrated on $\bigcup_n X_n$ and $||k*\mu||_\psi \leqslant 1$, then $\forall n;\ \ \mu_n \in M_1^+;\ \mu_n$ is concentrated on X_n and $||k * \mu_n||_\psi \leqslant 1$.

On the other hand, we have

$$||\mu|| = \sum_n ||\mu_n|| \leqslant \sum_n D_{k,\varphi}(X_n).$$

Thus,

$$D_{k,\varphi}(\bigcup_n X_n) \leqslant \sum_n D_{k,\varphi}(X_n).$$

If the X_n are not pairwise disjoint, then by the first case and the fact that $D_{k,\varphi}$ is increasing, we have

$$D_{k,\varphi}(\bigcup_n X_n) \leqslant \sum_n D_{k,\varphi}(X_n).$$

It remains to show that $D_{k,\varphi}$ is interior.

By monotonicity we have

$$\sup\{D_{k,\varphi}(K) : K \subset X,\ K \ \ compact\} \leqslant D_{k,\varphi}(X).$$

Let $\mu \in M_1^+$ and X be a μ-measurable set such that μ is concentrated on X and $||k*\mu||_\psi \leqslant 1$.
Let a compact K be such that $K \subset X$, then $\mu|_K \in M_1^+$, $\mu|_K$ is concentrated on K and $||k * \mu|_K||_\psi \leqslant 1$. Therefore,

$$||\mu|_K||_\psi \leqslant D_{k,\varphi}(K).$$

On the other hand,

$$\sup\{||\mu\setminus_K|| : K \subset X,\ K\ is\ compact\} = ||\mu||.$$

Thus,

$$D_{k,\varphi}(X) \leqslant \sup\{D_{k,\varphi}(K) : K \subset X,\ K\ iscompact\}.$$

Theorem 2.11 *1) $D^*_{k,\varphi}$ is the outer capacity associated with $D_{k,\varphi}$, defined by:*

$$D^*_{k,\varphi}(X) = \inf\{\ D_{k,\varphi}(O)\ :\ O\ isopen\ and\ X \subset O\}.$$

Then,

$$D^*_{k,\varphi}(X) = C_{k,\varphi}(X).$$

2) If φ is a Musielak-Orlicz function, uniformly convex that satisfies the $\triangle_2$ condition, then for all analytic set X we have:

$$D_{k,\varphi}(X) = C_{k,\varphi}(X).$$

Proof. It is the same as that given in [2], Theorem 11.

Theorem 2.12 *Let φ be a Musielak-Orlicz function.*
Let K be a compact of $\mathbb{R}^N$. The following assertions are equivalents.
1)$C_{k,\varphi}(K) = \infty$.
2) $D^*_{k,\varphi}(K) = \infty$.
3) $D_{k,\varphi}(K) = \infty$.
4) There exists $x_0 \in K$ such that $k(x_0 - y) = 0$ almost everywhere.

Proof. It is the same as that given in [3], Theorem 5.

3 Non-linear Potential in Musielak-Orlicz Space

Let φ be a Musielak-Orlicz function. In this section, we propose to study the following variational problem: let X be a subset of $\mathbb{R}^N$ such that $C_{k,\varphi}(X) < \infty$. There exists $f_0 \in L^+_\varphi$ such that $k * f_0 \geqslant 1 \ \ C_{k,\varphi} -q.e$ on X, and

$$||f_0||_\varphi = inf\{||f||_\varphi : f \in L^+_\varphi \ and \ \ k * f \geqslant 1 \ C_{k,\varphi} -q.e \ on \ X\}.$$

If f_0 exists, it will be called a distribution function of X, and $k * f_0$ is called a potential of X for the $C_{k,\varphi}$ capacity.

Theorem 3.1 *Let φ be a Musielak-Orlicz function and X be a subset of $\mathbb{R}^N$ such that $C_{k,\varphi}(X) < \infty$. $\Omega_X = \{f \in L^+_\varphi : k * f \geqslant 1 \ \ C_{k,\varphi} -q.e \ on \ X\}$, and $Cl^*(\Omega_X)$ is the closure of Ω_X for the topology $\sigma(L_\varphi; E_\psi)$. Then:*
1) There exists a unique $f_0 \in L^+_\varphi$ such that:

$$||f_0||_\varphi = inf\{||f||_\varphi : \ f \in Cl^*(\Omega_X)\}.$$

2) If φ and ψ satisfy the $\triangle_2$ condition, then there exits a unique $f \in L^+_\varphi$ such that:
i) $k * f \geqslant 1$ on X and $||f||_\varphi = C_{k,\varphi}(X)$.
ii) If $C_{k,\varphi}(X) > 0$, then for all $g \in L_\varphi$ such that $k * g \geqslant 0$ on X:

$$\int a(x, \frac{f(x)}{||f||_\varphi})g(x)dx \geqslant 0,$$

where the function a(x,.) is the derivative of the function $\varphi(x, .)$.

Proof. 1) Let the function $\theta : L_\varphi \longrightarrow]-\infty; +\infty]$ be defined by $\theta(f) = ||f||_\varphi \ ; \forall f \in L_\varphi$. θ is lower semi continuous on L_φ, for topology $\sigma(L_\varphi; E_\psi)$ and coercive. Then, there exists a unique $f_0 \in L^+_\varphi$ such that

$$||f_0||_\varphi = inf\{||f||_\varphi : f \in Cl^*(\Omega_X)\}.$$

2) i) Since φ and ψ satisfy the $\triangle_2$ condition, then the space L_φ is reflexive. By Theorem 2.3, Ω_X is strongly closed in L_φ. On the other hand, Ω_X is convex, then there exists a unique $f \in L_\varphi$ such that:

$$||f||_\varphi = inf\{||g||_\varphi : g \in \Omega_X\}.$$

Let Y be a subset of X where $k * f < 1$. Then, $C_{k,\varphi}(X) = C_{k,\varphi}(X - Y)$. Since $k * f \geqslant 1$ on X-Y, $C_{k,\varphi}(X - Y) \leqslant ||f||_\varphi$.

On the other hand, we have $\{g \in L^+_\varphi : k * g \geqslant 1 \ on\ X\} \subset \Omega_X$; then $||f||_\varphi \leqslant C_{k,\varphi}(X)$.
ii) Let $g \in L_\varphi$ such that $k * g \geqslant 0$ on X. Then for all $t \geqslant 0$:

$$k * (f + tg) \geqslant 1 \ \ C_{k,\varphi} - q.e \ \ on\ X\ and \ \ (f + tg) \in \ L_\varphi.$$

Then,

$$||f + tg||_\varphi \geqslant ||f||_\varphi.$$

Therefore,

$$||\frac{1}{||f||_\varphi}(f + tg)||_\varphi \geqslant 1.$$

Thus,

$$\varrho_\varphi(\frac{1}{||f||_\varphi}(f + tg)) \geqslant 1.$$

On the other hand,

$$\varrho_\varphi(\frac{1}{||f||_\varphi} f) \leqslant 1.$$

Then, for all $t > 0$

$$\int \frac{1}{t}[\varphi(x, \frac{|f + tg|(x)}{||f||_\varphi}) - \varphi(x, \frac{|f(x)|}{||f||_\varphi})]dx \geqslant 0.$$

Let $c(x, t) = \varphi(x, \dfrac{|f + tg|(x)}{||f||_\varphi})$. Then, the function $x \longmapsto c(x, t)$ is in L^1 for all $t \in \mathbb{R}$.
On the other hand,

$$\frac{\partial c}{\partial t}(x, t) = a(x, \frac{|f + tg|(x)}{||f||_\varphi}).(\frac{g(x)}{||f||_\varphi}).sng(f + tg)(x).$$

For $0 < t < 1$ we have:

$$|\frac{\partial c}{\partial t}(x, t)| \leqslant a(x, \frac{|f + g|(x)}{||f||_\varphi}).(\frac{g(x)}{||f||_\varphi}).$$

By Lemma 2.1, the function: $x \longrightarrow a(x, \dfrac{|f + g|(x)}{||f||_\varphi})$ is in L_ψ.
Then the function: $x \mapsto a(x, \dfrac{|f + g|(x)}{||f||_\varphi}).(\dfrac{g(x)}{||f||_\varphi})$ is in L^1.
By Lebesgue's theorem

$$\lim_{t \to 0^+} \int \frac{1}{t}[\varphi(x, \frac{|f + tg|(x)}{||f||_\varphi}) - \varphi(x, \frac{|f(x)|}{||f||_\varphi})]dx = \frac{1}{||f||_\varphi} \int a(x, \frac{|f(x)|}{||f||_\varphi})dx \geqslant 0.$$

Remark 3.1 Under the assumptions of Theorem 3.1, 2) ii), if $C_{k,\varphi}(X) > 0$, then for all $g \in L_\varphi$ such that $k * g = 0$ on X:

$$\int a(x, \frac{f(x)}{||f||_\varphi})g(x)dx = 0.$$

Theorem 3.2 *Let φ be a Musielak-Orlicz function such that φ and ψ satisfy the $\triangle_2$ condition. Let $X \subset \mathbb{R}^N$ such that $0 < C_{k,\varphi}(X) < \infty$ and f be the distribution function of X for the $C_{k,\varphi}$ capacity. For all $g \in L_\varphi$*

$$|\int a(x, \frac{f(x)}{||f||_\varphi})g(x)dx| \leqslant K_\varphi \sup_{x\in X} |(k*g)(x)|.||f||_\varphi,$$

where K_φ is a constant that depends only on φ.

Proof. The inequality is obvious if $\sup\limits_{x\in X} |(k*g)(x)| = +\infty$.
On the other hand, if $\sup\limits_{x\in X} |(k*g)(x)| = 0$, then by Remark 3.1 we have

$$\int a(x, \frac{f(x)}{||f||_\varphi}).g(x)dx = 0.$$

If $0 < \alpha = \sup\limits_{x\in X} |(k*g)(x)| < +\infty$, then $k*(f - \dfrac{g}{\alpha})(x) \geqslant 0$ for all $x \in X$.
By Theorem 3.1, we have:

$$\int a(x, \frac{f(x)}{||f||_\varphi}).(f - \frac{g}{\alpha})(x)dx \geqslant 0.$$

Thus,

$$\int a(x, \frac{f(x)}{||f||_\varphi}).g(x)dx \leqslant \alpha \int a(x, \frac{f(x)}{||f||_\varphi}).f(x)dx.$$

On the other hand, we have for all $x \in \mathbb{R}^N$ and $t \geqslant 0$:

$$\varphi(x,t) = \int_0^t a(x,t)dt \geqslant \int_{\frac{t}{2}}^t a(x,t)dt \geqslant (\frac{t}{2})a(x, \frac{t}{2}).$$

Then,

$$a(x, \frac{f(x)}{||f||_\varphi}).\frac{f(x)}{||f||_\varphi} \leqslant \varphi(x, 2\frac{f(x)}{||f||_\varphi}) \leqslant K'_\varphi \varphi(x, \frac{f(x)}{||f||_\varphi})$$

because φ satisfies the $\triangle_2$ condition.
Therefore,

$$\int a(x, \frac{f(x)}{||f||_\varphi}).g(x)dx \leqslant \alpha.K'_\varphi.\varrho_\varphi(\frac{f}{||f||_\varphi}).$$

Since $\varrho_\varphi(\dfrac{f}{||f||_\varphi}) \leqslant 1$, the proof is complete.

Theorem 3.3 *Let φ be a Musielak-Orlicz function, uniformly convex which satisfies the $\triangle_2$ condition. Let $(X_i)_i \subset \mathbb{R}^N$. For each i, f_i is the distribution function of X_i for the $C_{k,\varphi}$ capacity. Let $X \subset \mathbb{R}^N$ and f be its distribution function for the $C_{k,\varphi}$ capacity. If $X \subset \liminf X_i$ and $\lim C_{k,\varphi}(X_i) = C_{k,\varphi}(X)$ then, $f_i \longrightarrow f$ in L_φ.*
We have the same result, particularly if $(X_i)_i$ is increasing and $X = \bigcup\limits_i X_i$ or $(X_i)_i$ is a decreasing sequence of compacts and $X = \bigcap\limits_i X_i$.

Proof. $(f_i)_i$ is bounded in L_φ. Since the space L_φ is reflexive, there exists a subsequence denoted again by $(f_i)_i$ which converges weakly in L_φ^+ to a function g in L_φ.
By Theorem 2.6, $k * g \geqslant 1$ on X $C_{k,\varphi}$ $-q.e.$ Therefore,

$$C_{k,\varphi}(X) \leqslant ||g||_\varphi.$$

On the other hand for all $h \in L_\psi$

$$\int f_i(x)h(x)dx \longrightarrow \int g(x)h(x)dx.$$

By*Hölder* inequality, we have:

$$\int f_i(x)h(x)dx \leqslant ||f_i||_\varphi |||h|||_\psi.$$

Thus, $\displaystyle\int g(x)h(x)dx \leqslant \liminf ||f_i||_\varphi |||h|||_\psi \leqslant ||f||_\varphi |||h|||_\psi.$
Let the function $h : x \longrightarrow a(x, \dfrac{g(x)}{||g||_\varphi})$ for all $x \in \mathbb{R}^N$.
By Lemma 2.1, $h \in L_\psi$, and by Lemma 2.3

$$||g||_\varphi |||h|||_\psi = \int g(x)h(x)dx \leqslant ||f||_\varphi |||h|||_\psi.$$

Then,

$$||g||_\varphi \leqslant C_{k,\varphi}(X).$$

Thus,

$$||g||_\varphi = C_{k,\varphi}(X) \ and \ therefore \ f = g.$$

On the other hand, f is the unique adhesion value of the sequence $(f_i)_i$ for the topology $\sigma(L_\varphi, L_\psi)$. Then, $f_i \longrightarrow f$ weakly in L_φ. Since L_φ is uniformly convex, we have $f_i \longrightarrow f$ strongly in L_φ.

Theorem 3.4 *Let φ be a Musielak-Orlicz function. Let F be a closed subset of $\mathbb{R}^N$ such that $D_{k,\varphi}(F) < \infty$. For all $r \in \mathbb{R}_+^*$: $F_r = F \cap \{x \in \mathbb{R}^N : |x| > r\}$.
If $\lim\limits_{r\to+\infty} D_{k,\varphi}(F_r) = 0$ then there exists a measure $\mu \in M_1^+$ such that μ is concentrated on F; $||k * \mu||_\psi \leqslant 1$ and $D_{k,\varphi}(F) = ||\mu||$, where μ is called a distribution measure of F for $D_{k,\varphi}$. Particularly, if K is a compact such that $D_{k,\varphi}(K) < \infty$ then K possesses a distribution measure for $D_{k,\varphi}$.*

Proof. It is the same as that given in [3], Theorem 4.

References

[1] Adams, D.R and Hedberg, L.I. Function Spaces and Potential Theory. *Springer*, 1999.

[2] Aissaoui, N. and Benkirane, A. Capacités dans les epaces d'Orlicz. *Ann. Sci. Math. Québec* **18** (1) (1994) 1–23.

[3] Aissaoui, N. and Benkirane, A. Potentiel non lineaire dans les espaces d'Orlicz. *Ann. Sci. Math. Québec* **18** (2) (1994) 105–118.

[4] Azroul, E., Benboubker, M.B. and Ouaro, S. The Obstacle Problem Associated with Nonlinear Elliptic Equations in Generalized Sobolev Spaces. *Nonlinear Dynamics and Systems Theory* **14** (3) (2014) 224–243.

[5] Benkirane, A. and M. Ould Mohamedhen Val. An approximation theorem in Musielak-Orlicz-Sobolev spaces. *Commentationes Mathematicae* (2011) 109–120.

[6] Burton, T.A. Liapunov Functionals, Convex Kernels, and Strategy. *Nonlinear Dynamics and Systems Theory* **10** (4) (2010) 325–338.

[7] Choquet, G. Forme abstraite du théoreme de capacitabitité, *Ann. Inst. Fourier* (Grenoble) **9** (1959) 83–89.

[8] Diening, L., Harjulehto, P., HÄSTÖ, P. and Rudicka, M. Lebesgue and Sobolev spaces with variable exponents. *Lecture Notes in Mathematics*, vol. 2017. Springer, Berlin, 2011.

[9] Fumi-Yuki Maeda, Yoshihiro Mizuta, Takao Ohno and Tetsu Shimomura. Capacity for potentiels of functions in Musielak-Orlicz- space. *Nonlinear Analysis* ***74*** (2011) 6231–6243.

[10] Maz'ya, V.G. and Khavin, V.P. Nonlinear potential theory. *Uspekhi Math. Nauk.* ***27*** (1972) 67–138.

[11] Meyers, N.G. A theory of capacities for potentials of functions in Lebesgue classes. *Math, Scand.* ***26*** (1970) 255–292.

[12] Musielak, J. *Modular Spaces and Orlicz Spaces.* Lecture Notes in Mathematics, vol. 1034, 1983.

Nonlinear Dynamics and Systems Theory, **16** (3) (2016) 294–299

Cubic Operators Corresponding to Graphs

U. U. Jamilov

Institute of Mathematics, National University of Uzbekistan,
29, Do'rmon Yo'li str., 100125, Tashkent, Uzbekistan.

Received: July 31, 2015; Revised: June 9, 2016

Abstract: We introduce a notion of a cubic stochastic operator corresponding to graph. We prove that each such operator has a unique fixed point. Besides, it is shown that any trajectory of such cubic stochastic operator exponentially rapidly converges to this fixed point.

Keywords: *quadratic stochastic operator; cubic stochastic operator; Volterra and non-Volterra operators.*

Mathematics Subject Classification (2010): Primary 17D92, Secondary 17D99.

1 Introduction

The history of quadratic stochastic operator (QSO) can be traced to Bernshtein's work [1]. Since then the theory of QSOs has been further developed motivated by their frequent occurrence in several problems of physical, economical and biological systems, where QSOs serve as a tool for the study of dynamical properties and modeling, see [2, 4–12, 15, 19–23]. While they were originally introduced as "evolutionary operators" to describe the dynamics of gene frequencies for given laws of heredity in mathematical population genetics, QSOs and the dynamical systems they describe have become interesting objects of study in their own right from a purely mathematical point of view. For a recent review on the theory of quadratic operators see [7].

In modern scientific investigations non-linear operators of higher order arise. Nowadays another class of nonlinear operators which are different from QSOs arises. In particular, *cubic stochastic operator* (CSO) can be obtained in gene engineering and free population with ternary production. In paper [17] the concept of *cubic stochastic operator* was introduced.

* Corresponding author: mailto:jamilovu@yandex.ru

One such subclass that arises naturally in the biological context is given by the additional restriction

$$p_{ijk,l} = 0, \quad \text{if} \quad l \notin \{i, j, k\} \quad \text{for all} \quad i, j, k, l. \tag{1}$$

These CSOs describe a reproductory behaviour where the offspring is a genetic copy of one of its parents and are called *Volterra operators*. The asymptotic behaviour of trajectories of this kind of CSOs for some particular cases were analysed in [13, 14, 17, 18].

However, in the non-Volterra case (i.e. when condition (1) is violated), many questions remain open and there seems to be no general theory available.

In all of the above-mentioned references the authors investigated trajectories of a CSO on finite dimensional unit simplex. However, it seems natural to consider the problem for an infinite dimensional CSO. This can be done, e.g., by using a method of infinite dimensional Volterra quadratic stochastic operator considered in [16].

The paper is organised as follows. In Section 2 we recall definitions and well known results from the theory of Volterra and non-Volterra CSOs . In Section 3 we define a new class of non-Volterra CSOs and show that a CSO from this class has a unique fixed point. Moreover, we prove that the trajectory of such operators has the regularity property and consequently the ergodic hypothesis is verified.

2 Preliminaries and Known Results

Let $[m] = \{1, 2, ..., m\}$. By the $(m-1)-$ simplex we mean the set

$$S^{m-1} = \{\mathbf{x} = (x_1, ..., x_m) \in R^m : x_i \geq 0, \quad \sum_{i=1}^{m} x_i = 1\}.$$

Each element $\mathbf{x} \in S^{m-1}$ is a probability measure on $[m]$ and so it may be looked upon as the state of a biological (physical and so on) system of m elements.

A cubic stochastic operator (CSO) $V : S^{m-1} \mapsto S^{m-1}$ has the form

$$V : x'_l = \sum_{i,j,k=1}^{m} p_{ijk,l} x_i x_j x_k, \quad (l = 1, \ldots, m), \tag{2}$$

where $p_{ijk,l}$ is a coefficient of heredity and

$$p_{ijk,l} \geq 0, \quad \sum_{l=1}^{m} p_{ijk,l} = 1, \quad (i, j, k, l = 1, ..., m). \tag{3}$$

More precisely $p_{ijk,l}$ is the conditional probability $P(l|i, j, k)$ with which the ith, jth and kth species interbreed successfully, when they produce an individual l. We assume that there is no difference whatever the "next" is, and in any generation the "parents" i, j, k are independent, that is $P(i, j, k) = P(i)P(j)P(k) = x_i x_j x_k$, i.e. we consider models of free population.

For a given $\mathbf{x}^{(0)} \in S^{m-1}$, the trajectory $\{\mathbf{x}^{(n)}\}$, $n = 0, 1, 2, \ldots$ of an initial point $\mathbf{x}^{(0)}$ under the action of CSO (2) with (3) is defined by $\mathbf{x}^{(n+1)} = V(\mathbf{x}^{(n)})$, where $n = 0, 1, 2, \ldots$

A point $\mathbf{x} \in S^{m-1}$ is called a fixed point of V if $V(\mathbf{x}) = \mathbf{x}$. A CSO V on S^{m-1} is called *regular* if for any initial point $\mathbf{x} \in S^{m-1}$ the limit $\lim_{n\to\infty} V^n(\mathbf{x})$ exists. The biological interpretation of the regularity of a CSO is rather clear: in the long run the distribution of species in the next generation coincides with the distribution of species in the previous one, i.e., it is stable.

For a nonlinear dynamical system, Ulam [21] suggested an analogue of a measure-theoretic ergodicity in the form of the following ergodic hypothesis: a QSO V is said to be *ergodic* if the limit $\lim\limits_{n\to\infty}\frac{1}{n}\sum\limits_{k=0}^{n-1}V^k(\mathbf{x})$ exists for any $\mathbf{x}\in S^{m-1}$.

On the basis of numerical calculations, Ulam [21] conjectured that for any QSO the ergodic hypothesis holds. In [22], Zakharevich proved that this conjecture is false in general. In [17], the authors proved that a class of Volterra CSOs has the ergodic property. The biological interpretation of non-ergodicity of a CSO is the following: in the long run the behavior of the distributions of species is unpredictable.

Evidently, any regular CSO and, more generally, any CSO for which every trajectory converges to a (not necessarily strict) periodic orbit is ergodic, but the converse is not true.

In [18] a construction of a cubic stochastic operator is given. This construction depends on a probability measure μ which is initially given on a fixed graph G. Using the construction of CSO for μ defined as product of measures given on components of G a wide class of non-Volterra CSOs is described. It is proved that the non-Volterra CSOs can be reduced to N number of Volterra CSOs defined on the components, where N is the number of components.

In [3] a class of non-Volterra cubic operators is given and the dynamical systems generated by these CSOs are studied.

3 Asymptotic Behaviour of CSOCGs

Recall the notion of infinite dimensional simplex following [16]. Denote by S the following set:

$$S=\{\mathbf{x}=(x_i): x_i\geq 0, i\in\mathbb{N},\ \ \sum_{i=1}^{\infty}x_i=1\}.$$

Clearly, S is the closed convex hull of vectors of the form $\mathbf{e}_k=(0,0,...,1,0,0,...)$, where the unite is the $k-$th position, and precisely these vectors are the extreme elements of S.

We define an operator $V:S\mapsto S$ as follows

$$(V(\mathbf{x}))_l=\sum_{i,j,k=1}^{\infty}p_{ijk,l}x_ix_jx_k,\quad l\in\mathbb{N},\quad \mathbf{x}=(x_i)\in S, \tag{4}$$

where

$$p_{ijk,l}\geq 0,\quad \sum_{l=1}^{\infty}p_{ijk,l}=1,\quad i,j,k,l\in\mathbb{N}. \tag{5}$$

and the values $p_{ijk,l}$ do not change for any permutation of i,j, and k.

Definition 3.1 An operator defined by conditions (4) and (5) is called an infinite dimensional cubic stochastic operator.

Let $G = (\Lambda, L)$ be a graph without multiple edges, where Λ is the set of vertices which is at most a countable set, L is the set of edges of the graph G. Enumerate the vertices of the graph G by elements of $[m]_0 = \{0\} \cup \mathbb{N}$. For the vertices $i, j \in \Lambda$ define

$$\delta_{ij} := \begin{cases} 1, & \text{if } \{i,j\} \subset L; \\ 0, & \text{otherwise}; \end{cases}$$

and we denote $\langle i,j,k \rangle$ if $\delta_{ij} + \delta_{jk} + \delta_{ki} > 1$ and by $\rangle i,j,k \langle$ we denote the case $\delta_{ij} + \delta_{jk} + \delta_{ki} \leq 1$.

We define the coefficients of heredity as follows:

$$p_{ijk,l} := \begin{cases} 1, & \text{if } l = 0, \ \ \rangle i,j,k\langle, \ \ i,j,k \in [m]_0 \ \text{ or } \ \langle i,j,k\rangle, \ \ 0 \in \{i,j,k\}; \\ 0, & \text{if } l \neq 0, \ \ \rangle i,j,k\langle, \ \ i,j,k \in [m]_0 \ \text{ or } \ \langle i,j,k\rangle, \ \ 0 \in \{i,j,k\}; \\ \geq 0, & \text{if } \langle i,j,k\rangle, \ \ i,j,k \in \mathbb{N}. \end{cases} \tag{6}$$

The biological interpretation of the coefficients (6) is obvious: the individuals i, j and k might produce the offspring $l \neq 0$ if they are neighboring points of a graph.

Definition 3.2 For any fixed graph G, CSO satisfying conditions (4), (5) and (6) is called the cubic stochastic operator corresponding to the graph (CSOCG).

Remark 3.1 Any CSOCG is non-Volterra, because $p_{ijk,0} \neq 0$ if $\rangle i,j,k\langle$ and $ijk \neq 0$.

Arbitrary CSOCG has the form

$$V: \begin{cases} x_0' = \sum\limits_{i\in[m]_0} x_i^3 + 3x_0^2 \sum\limits_{i\in\mathbb{N}} x_i + 6x_0 \sum\limits_{i,j\in\mathbb{N}} x_i x_j + 6 \sum\limits_{\substack{i,j,k\in\mathbb{N}: \\ \rangle i,j,k\langle}} x_i x_j x_k + 6 \sum\limits_{\substack{i,j,k\in\mathbb{N}: \\ \langle i,j,k\rangle}} p_{ijk,0} x_i x_j x_k \\ x_l' = 6 \sum\limits_{\substack{i,j,k\in\mathbb{N}: \\ \langle i,j,k\rangle}} p_{ijk,l} x_i x_j x_k, \quad l \in \mathbb{N}. \end{cases} \tag{7}$$

Denote int$S = \{\mathbf{x} \in S : x_i > 0, \ \ i \in \mathbb{N}\}$. Let $\omega(\mathbf{x}^0)$ be the set of limit points of a trajectory $\{V^k(\mathbf{x}^0) \in S : k = 0, 1, 2, \dots\}$. Using Lyapunov functions, one can handle the set of limit points. Recall the definition of a Lyapunov function.

Definition 3.3 A continuous function $\varphi : intS \to \mathbb{R}$ is called a Lyapunov function for the operator (4) if $\varphi(V(\mathbf{x})) \geq \varphi(\mathbf{x})$ for all $\mathbf{x}$ (or $\varphi(V(\mathbf{x})) \leq \varphi(\mathbf{x})$ for all $\mathbf{x}$).

Theorem 3.1 *Any CSOCG* (7) *has a unique fixed point (1,0,0,...). Moreover for an initial* $\mathbf{x}^{(0)} \in S$*, the trajectory of operator* (7) *tends to this fixed point exponentially rapidly.*

Proof. It is easy to verify that $\mathbf{e}_0 = (1, 0, 0, ...)$ is a fixed point. We consider the function

$$\varphi(\mathbf{x}) = \sum_{k\in\mathbb{N}} x_k. \tag{8}$$

The function (8) will be a Lyapunov function for the operator (7). Indeed,

$$\begin{aligned} \varphi(V(\mathbf{x})) &= \sum_{l\in\mathbb{N}} x_l' = \sum_{l\in\mathbb{N}} \sum_{\substack{i,j,k\in\mathbb{N}: \\ \langle i,j,k\rangle}} p_{ijk,l} x_i x_j x_k = \sum_{\substack{i,j,k\in\mathbb{N}: \\ \langle i,j,k\rangle}} \sum_{l\in\mathbb{N}} p_{ijk,l} x_i x_j x_k \\ &\leq \sum_{\substack{i,j,k\in\mathbb{N}: \\ \langle i,j,k\rangle}} x_i x_j x_k \leq \left(\sum_{l\in\mathbb{N}} x_l \right)^3 \leq \sum_{l\in\mathbb{N}} x_l = \varphi(\mathbf{x}). \end{aligned} \tag{9}$$

It is evident, that $\varphi(\mathbf{x}^{(n+1)}) \leq \varphi(\mathbf{x}^{(n)}), n = 0, 1, \ldots$ implies that $\varphi(\mathbf{x})$ is a Lyapunov function, that is $\{\varphi(\mathbf{x}^{(n)})\}_{n=0}^{\infty}$ is a decreasing sequence and converges to some limit ξ. We claim that $\xi = 0$. Indeed, from (9) one has

$$\varphi(\mathbf{x}^{(n+1)}) \leq (\varphi(\mathbf{x}^{(n)}))^3 \leq \left(\varphi\big(\mathbf{x}^{(0)}\big)\right)^{3^n}. \tag{10}$$

If $x_0^{(0)} \neq 0$, then from (10) using $\varphi\big(\mathbf{x}^{(0)}\big) = \sum\limits_{k\in\mathbb{N}} x_k^{(0)} = 1 - x_0^{(0)}$ we obtain

$$\lim_{n\to\infty} \varphi(\mathbf{x}^{(n)}) = 0. \tag{11}$$

If for an initial point it holds that $x_0^{(0)} = 0$, then from (7) it is easy to see that

$$V(\mathbf{x}^{(0)}) \in \mathrm{int}S = \{\mathbf{x} \in S : x_i > 0, \sum_{k\in\mathbb{N}} x_k = 1\}, \tag{12}$$

that is $x_0' \neq 0$.

Thus from (11) and (12) it should be

$$\lim_{n\to\infty} x_k^{(n)} = 0, \quad \text{for} \quad \text{any} \quad k \in \mathbb{N},$$

consequently

$$\lim_{n\to\infty} \mathbf{x}^{(n)} = \mathbf{e}_0, \quad \text{for} \quad \text{any} \quad \mathbf{x}^{(0)} \in S.$$

Since the limit is obtained for any $\mathbf{x}^{(0)} \in S$, we conclude that $(1, 0, 0, ...)$ is unique fixed point. This completes the proof.

If an operator has the regularity property then it satisfies the ergodic hypothesis. By Theorem 3.1, a CSOCG is a regular transformation, so as a corollary we have the following theorem.

Theorem 3.2 *Any CSOCG* (7) *is an ergodic transformation.*

References

[1] Bernstein, S.N. The solution of a mathematical problem related to the theory of heredity. *Uchn. Zapiski. NI Kaf. Ukr. Otd. Mat.* **1** (1924) 83–115.

[2] Blath, J., Jamilov (Zhamilov), U.U. and Scheutzow, M. (G, μ)-quadratic stochastic operators. *J. Difference Equ. & Appl.* **20** (8) (2014) 1258–1267.

[3] Davronov, R. R., Jamilov, U.U. and Ladra, M. Conditional cubic stochastic operator. *J. Difference Equ. & Appl.* **21** (12) (2015) 1163–1170.

[4] Ganikhodzhaev, N. N. An application of the theory of Gibbs distributions to mathematical genetics. *Doklady Math.* **61** (3) (2000) 321–323.

[5] Ganikhodjaev, N., Ganikhodjaev, R. and Jamilov, U. Quadratic stochastic operators and zero-sum game dynamics. *Ergod. Th. and Dynam. Sys.* **35** (2015) 1443–1473.

[6] Ganikhodzhaev, N.N., Zhamilov, U.U. and Mukhitdinov, R.T. Nonergodic quadratic operators for a two-sex population. *Ukrainian Math. Jour.* **65** (8) (2014) 1282–1291.

[7] Ganikhodzhaev, R., Mukhamedov, F. and Rozikov, U. Quadratic stochastic operators and processes: results and open problems. *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* **14** (2) (2011) 279–335.

[8] Ganikhodzhaev, R.N. Quadratic stochastic operators, Lyapunov functions and tournaments. *Acad. Sci. Sb.Math.* **76** (2) (1993) 489–506.

[9] Ganikhodzhaev, R.N. A chart of fixed points and Lyapunov functions for a class of discrete dynamical systems. *Math. Notes* **56** (5-6) (1994) 1125–1131.

[10] Ganikhodzhaev, R.N. and Eshmamatova, D.B. Quadratic automorphisms of a simplex and the asymptotic behavior of their trajectories. *Vladikavkaz. Mat. Zh.* **8** (2) (2006) 12–28.[Russian]

[11] Kesten, H. Quadratic transformations: A model for population growth. I. *Advances in Appl. Probability.* **2** (1970) 1–82.

[12] Kesten, H. Quadratic transformations: A model for population growth. II. *Advances in Appl. Probability.* **2** (1970) 179–228.

[13] Khamraev, A.Yu. On a Volterra type cubic operators. *Uzbek. Math. Zh.* **3** (2009) 65–71.

[14] Khamraev, A.Yu. On cubic operators of Volterra type. *Uzbek. Math. Zh.* **2** (2004) 79–84.

[15] Lyubich, Y.I. *Mathematical Structures in Population Genetics.* Springer-Verlag, Berlin, 1992.

[16] Mukhamedov, F.M. Infinite-dimensional quadratic Volterra operators. *Russ. Math. Surv.* **55** (6) (2000) 1161–1163.

[17] Rozikov, U.A. and Khamraev, A.Yu. On cubic operators, defined on the finite-dimensional simplexes. *Ukrainian Math. Jour.* **56** (10) (2004) 1418–1427.

[18] Rozikov, U.A. and Khamraev, A.Yu. On construction and a class of non-Volterra cubic stochastic operators. *Nonlinear Dynamics and Systems Theory.* **14** (1) (2014) 92–100.

[19] Rozikov, U.A. and Zhamilov, U.U. F-quadratic stochastic operators. *Math. Notes* **83** (4) (2008) 554–559.

[20] Rozikov, U.A. and Zhamilov, U.U. Volterra quadratic stochastic operators of a two-sex population. *Ukrainian Math. Jour.* **63** (7) (2011) 1136–1153.

[21] Ulam, S.M. *A collection of Mathematical Problems.* Interscience Publishers, New York-London, 1960.

[22] Zakharevich, M.I. On the behaviour of trajectories and the ergodic hypothesis for quadratic mappings of a simplex. *Russ. Math. Surv.* **33** (6) (1978) 265–266.

[23] Zhamilov, U.U. and Rozikov, U.A. On the dynamics of strictly non-Volterra quadratic stochastic operators on a two-dimensional simplex. *Sb. Math.* **200** (9) (2009) 1339–1351.

Nonlinear Dynamics and Systems Theory, **16** (3) (2016) 300–311

Extremal Mild Solutions for Nonlocal Semilinear Differential Equations with Finite Delay in an Ordered Banach Space

Kamaljeet *, D. Bahuguna

Department of Mathematics & Statistics
Indian Institute of Technology Kanpur, Kanpur - 208016, India.

Received: September 23, 2015; Revised: June 9, 2016

Abstract: This paper is concerned with the existence and uniqueness of extremal mild solutions for nonlocal semilinear differential equations with finite delay in an ordered Banach space with the help of the monotone iterative technique based on lower and upper solutions. We use the theory of semigroup and measures of noncompactness to obtain the main results. The existence results are proved by assuming compact or non compact semigroup. An example is provided to illustrate the applicability of the main results.

Keywords: *initial value problem; finite delay; semigroup theory; monotone iterative technique; lower and upper solutions; Kuratowskii measure of noncompactness.*

Mathematics Subject Classification (2010): 34G20, 34K30.

1 Introduction

In this paper, we consider the following nonlocal semilinear differential equations with finite delay in an ordered Banach space:

$$\begin{cases} \frac{d}{dt}x(t) = & Ax(t) + f(t, x_t, Bx(t)), \quad t \in J = [0, b], \\ x(t) = & \phi(t) + g(x)(t), \quad t \in [-a, 0], \end{cases} \tag{1}$$

where the state $x(\cdot)$ takes values in the Banach space X endowed with norm $\|\cdot\|$; $A\colon D(A) \subset X \to X$ is a closed linear densely defined operator and an infinitesimal generator of strongly continuous semigroup $\{T(t)\}_{t\geq 0}$ of bounded linear operator in X;

* Corresponding author: mailto:kamaljeetp2@gmail.com

the nonlinear function $f\colon [0,b] \times \mathcal{D} \times X \to X$ is continuous, here $\mathcal{D} = C([-a,0],X)$; the term $Bx(t)$ is given by $Bx(t) = \int_0^t K(t,s)x(s)ds$, here $K \in C(\Sigma, \mathbb{R}^+)$ is the set of all positive functions which are continuous on $\Sigma = \{(t,s)|0 \le s \le t \le T\}$; $\phi(\cdot) \in \mathcal{D}$ and $g\colon C([-a,b],X) \to \mathcal{D}$ is a continuous operator. If $x\colon [-a,b] \to X$ is a continuous function, then x_t denotes the function in $\mathcal{D}$ defined as $x_t(\nu) = x(t+\nu)$ for $\nu \in [-a,0]$, here $x_t(\cdot)$ represents the time history of the state from the time $t-a$ up to the present time t.

It is well known that time delays are frequently encountered in various industrial and practical systems, such as chemical processing, bio engineering, fuzzy systems, automatic control, neural networks, circuits, vehicle suspension systems and so on. Hence, in recent years, the researchers have paid more attention to delay differential equations (see [1–7]). Some authors have studied differential equations with nonlocal initial conditions, see for instance, [7–13]. Nonlocal initial condition, in many cases, is more suitable and produces better results in applications of physical problems than the classical initial value of the type $x(0) = x_0$.

The monotone iterative technique based on lower and upper solutions provides an effective way to investigate the existence of solutions for the nonlinear differential equations (fractional or non-fractional ordered), see for instance, [6, 14–18]. It constructs monotone sequences of lower and upper solutions that converge uniformly to the extremal mild solutions between the lower and upper solutions.

This paper is motivated by recent works [6, 7, 16]. We extend a monotone iterative technique for nonlocal semilinear differential equations with finite delay (1) to study the existence and uniqueness of extremal mild solutions in an ordered Banach space. We use the semigroup theory and measures of noncompactness to obtain the results. The existence results are discussed by assuming compact or non compact semigroup. To the best of our knowledge, up to now, no work has been reported on nonlocal semilinear differential equations with finite delay by using the monotone iterative technique.

The rest of the paper is organized as follows: In the next section, we introduce some basic definitions, notations and preliminary results. In Section 3, we prove the existence and uniqueness of extremal mild solutions of the delay system (1) by using monotone iterative technique. Finally, in Section 4, we present an example to show the application of the main result.

2 Preliminaries

Throughout this paper, we assume that X is a Banach space with the norm $\|\cdot\|$ and $P = \{y \in X : y \ge \theta\}$ (θ is a zero element of X) is a positive cone in X which defines a partial ordering in X by $x \le y$ if and only if $y - x \in P$. If $x \le y$ and $x \ne y$, we write $x < y$. The cone P is said to be normal if there exists a positive constant N such that $\theta \le x \le y$ implies $\|x\| \le N\|y\|$. We also assume that $A\colon D(A) \subset X \to X$ is a closed linear densely defined operator that generates a strongly continuous semigroup $\{T(t), t \ge 0\}$. By Pazy [19], there exists a constant $M \ge 1$ such that $\sup_{t\in J} \|T(t)\| \le M$. For the sake of convenience, we write $B^* = \sup_{t\in J} \int_0^t K(t,s)ds$.

$C([-a,b],X)$ is the Banach space of all continuous X-valued functions on interval $[-a,b]$ with norm $\|\cdot\|_C = \sup_{t\in[-a,b]} \|x(t)\|$. Then $C([-a,b],X)$ is an ordered Banach space whose partial ordering $\le$ is induced by positive cone $P_C = \{x \in C([-a,b],X) \mid x(t) \ge \theta,\ t \in [-a,b]\}$. Similarly D is also an ordered Banach space with norm $\|\cdot\|_D = \sup_{t\in[-a,0]} \|x(t)\|$ and partial ordering $\le$ induced by $P_D = \{x \in$

$C([-a,0],X) \mid x(t) \geq \theta, t \in [-a,0]\}$. If the cone P is normal with a normal constant N, then P_C and P_D are also normal cones with the same normal constant N. For $x, y \in C([-a,b],X)$ with $x \leq y$, denote the ordered interval $[x,y] = \{z \in C([-a,b],X),\ x \leq z \leq y\}$ in $C([-a,b],X)$, and $[x(t),y(t)] = \{u \in X\colon x(t) \leq u \leq y(t)\}$ $(t \in [-a,b])$ in X.

Let us recall some basic definitions and lemmas which are used to prove our main results.

Definition 2.1 A C_0-semigroup $\{T(t)\}_{t\geq 0}$ is called a positive semigroup, if $T(t)x \geq \theta$ for all $x \geq \theta$ and $t \geq 0$.

Lemma 2.1 (see [19]) *If $h \in C^1(J,X)$, then for every $x_0 \in D(A)$ the following initial value problem*

$$\begin{cases} \frac{d}{dt}x(t) = Ax(t) + h(t), \quad t \in J, \\ x(0) = x_0, \end{cases} \tag{2}$$

has a unique solution x on J given by

$$x(t) = T(t)x_0 + \int_0^t T(t-s)h(s))\, ds, \quad t \in J.$$

Definition 2.2 (see [19]) A continuous function $x\colon [-a,b] \to X$ is said to be a mild solution of the system (1) if $x(t) = \phi(t) + g(x)(t)$ on $[-a,0]$ and the following integral equation is satisfied:

$$x(t) = T(t)(\phi(0) + g(x)(0)) + \int_0^t T(t-s)f(s,x_s,Bx(s))\, ds, \quad t \in J.$$

Lemma 2.2 (see [19]) *If $h \in L^1((0,b),X)$, then for every $x_0 \in X$ the initial value problem* (2) *has a unique mild solution.*

Let $C_1([-a,b],X) = \{u \in C([-a,b],X)\colon u'$ exists on J, $u'|_J \in C(J,X)$ and $u(t) \in D(A)$ for $t \geq 0\}$. An abstract function $u \in C_1([-a,b],X)$ is called a solution of (1) if $u(t)$ satisfies the equation (1).

Definition 2.3 (see [16]) The function $x \in C_1([-a,b],X)$ is called a lower solution of the system (1) if it satisfies the following inequalities

$$\begin{cases} \frac{d}{dt}x(t) \leq Ax(t) + f(t,x_t,Bx(t)), \quad t \in J, \\ x(\nu) \leq \phi(\nu) + g(x)(\nu), \quad \nu \in [-a,0]. \end{cases} \tag{3}$$

If all inequalities of (3) are reversed, we call x an upper solution of the system (1).

Now we recall the definition of Kuratowski's measure of noncompactness and its properties.

Definition 2.4 (see [20, 21]) Let X be a Banach space and $\mathcal{B}(X)$ be a family of bounded subset of X. Then $\mu : \mathcal{B}(X) \to \mathbb{R}^+$, defined by

$$\mu(S) = \inf\{\delta > 0\colon S \text{ admits a finite cover by sets of diameter } \leq \delta\},$$

where $S \in \mathcal{B}(X)$, is called the Kuratowski measure of noncompactness. Clearly $0 \leq \mu(S) < \infty$.

Lemma 2.3 (see [20, 21]) *Let S, S_1 and S_2 be bounded sets of a Banach space X. Then*

(i) $\mu(S) = 0$ *if and only if S is a relatively compact set in X.*

(ii) $\mu(S_1) \leq \mu(S_2)$ *if* $S_1 \subset S_2$.

(iii) $\mu(S_1 + S_2) \leq \mu(S_1) + \mu(S_2)$.

(iv) $\mu(\lambda S) \leq |\lambda|\mu(S)$ *for any* $\lambda \in \mathbb{R}$.

Lemma 2.4 (see [20, 21]) *If $S \subset C([c, d], X)$ is bounded and equicontinuous on $[c, d]$, then $\mu(S(t))$ is continuous for $t \in [c, d]$ and*

$$\mu(S) = \sup\{\mu(S(t)),\ t \in [c, d]\}, \quad \textit{where } S(t) = \{x(t) \colon x \in S\} \subseteq X.$$

Remark 2.1 (see [20,21]) If S is a bounded set in $C([c, d], X)$, then $S(t)$ is bounded in X, and $\mu(S(t)) \leq \mu(S)$.

Lemma 2.5 (see [20, 21]) *Let $S = \{u_n\} \subset C([c, d], X)(n = 1, 2, \dots)$ be a bounded and countable set. Then $\mu(S(t))$ is Lebesgue integrable on $[c, d]$, and*

$$\mu\left(\left\{\int_c^d u_n(t)\, dt \mid n = 1, 2, \dots\right\}\right) \leq 2 \int_c^d \mu(S(t))\, dt. \tag{4}$$

3 Main Result

In this section, we prove the existence and the uniqueness of extremal mild solutions of the system (1).

Theorem 3.1 *Let X be an ordered Banach space, whose positive cone P is normal with a normal constant N. Also assume that A is the infinitesimal generator of a positive and compact C_0-semigroup $\{T(t)\}_{t\geq 0}$ on X. If the system (1) has a lower solution $x^{(0)} \in C([-a, b], X)$ and an upper solution $y^{(0)} \in C([-a, b], X)$ with $x^{(0)} \leq y^{(0)}$ and satisfies the following assumptions:*

(H1) *The function $f \colon J \times \mathcal{D} \times X \to X$ satisfies that $f(t, \cdot, \cdot) \colon \mathcal{D} \times X \to X$ is continuous for $t \in J$, and $f(\cdot, \varphi, x)$ is strongly measurable for all $(\varphi, x) \in \mathcal{D} \times X$.*

(H2) *For any $t \in J$, the function $f(t, \cdot, \cdot) \colon \mathcal{D} \times X \to X$ satisfies the following*

$$f(t, \varphi_1, u_1) \leq f(t, \varphi_2, u_2),$$

where $u_1, u_2 \in X$ with $Bx^{(0)}(t) \leq u_1 \leq u_2 \leq By^{(0)}(t)$ and $\varphi_1, \varphi_2 \in \mathcal{D}$ with $x_t^{(0)} \leq \varphi_1 \leq \varphi_2 \leq y_t^{(0)}$.

(H3) *The function $g \colon C([-a, b], X) \to \mathcal{D}$ is increasing, continuous and compact.*

Then the delay system (1) has minimal and maximal mild solutions between $x^{(0)}$ and $y^{(0)}$.

Proof. Let $B = [x^{(0)}, y^{(0)}] = \{x \in C([-a,b], X) \mid x^{(0)} \le x \le y^{(0)}\}$. Define $Q\colon B \to C([-a,b], X)$ by

$$Qx(t) = \begin{cases} T(t)(\phi(0) + g(x)(0)) + \int_0^t T(t-s) f(s, x_s, Bx(s))\, ds, & t \in [0,b], \\ \phi(t) + g(x)(t), & t \in [-a, 0]. \end{cases} \tag{5}$$

For any $x \in B$ and in view of (H2), we have

$$\begin{aligned} f(t, x_t^{(0)}, Bx^{(0)}(t)) &\le f(t, x_t, Bx(t)) \\ &\le f(t, y_t^{(0)}, By^{(0)}(t)). \end{aligned}$$

By the normality of the positive cone P, there exists a constant $k > 0$ such that

$$\|f(t, x_t, Bx(t))\| \le k, \quad x \in B. \tag{6}$$

Firstly we prove that Q is a continuous and monotonically increasing operator from B to B. Let $x, y \in B$ with $x \le y$, then $x(t) \le y(t),\ t \in [-a, b]$. Therefore $x_t \le y_t$ in $\mathcal{D}$ for all $t \in [0, b]$. By the positivity of the semigroup $T(t)$ and the assumptions (H2) and (H3), we get

$$Qx \le Qy. \tag{7}$$

Let $\frac{d}{dt} x^{(0)}(t) = Ax^{(0)}(t) + h(t),\ t \in J$. In view of Lemma 2.2 and Definition 2.3, we get

$$\begin{aligned} x^{(0)}(t) =& T(t)x^{(0)}(0) + \int_0^t T(t-s)h(s)ds \\ \le& T(t)(\phi(0) + g(x^{(0)})(0)) + \int_0^t T(t-s) f(s, x_s^{(0)}, Bx^{(0)}(s))ds \\ =& Qx^{(0)}(t), \quad t \in J. \end{aligned}$$

Also $x^{(0)}(t) \le \phi(t) + g(x^{(0)})(t) = Qx^{(0)}(t),\ t \in [-a, 0]$. Thus $x^{(0)}(t) \le Qx^{(0)}(t),\ t \in [-a, b]$. Similarly we can show that $Qy^{(0)}(t) \le y^{(0)}(t),\ t \in [-a, b]$. Now let $\{x^{(n)}\} \subset B$ with $x^{(n)} \to x \in B$ as $n \to \infty$. By (6), (H1) and (H3) for any $t \in J$, we have

(i) $f(t, x_t^{(n)}, Bx^{(n)}(t)) \to f(t, x_t, Bx(t))$.

(ii) $g(x^{(n)}) \to g(x)$.

(iii) $\|f(t, x_t^{(n)}, Bx^{(n)}(t)) - f(t, x_t, Bx(t))\| \le 2k$.

These, together with Lebesgue's dominated convergence theorem, imply that

$$\begin{aligned} \|Qx^{(n)}(t) - Qx(t)\| \le& M\|g(x^{(n)})(0) - g(x)(0)\| + M \int_0^t \|f(s, x_s^{(n)}, Bx(s)) \\ & - f(s, x_s, Bx(s))\|\, ds \\ & \to 0 \text{ as } n \to \infty. \end{aligned}$$

In view of (H3), for any $t \in [-a, 0]$, we have $\|Qx^{(n)}(t) - Qx(t)\| = \|g(x^{(n)})(t) - g(x)(t)\| \to 0$ as $n \to 0$. Therefore $Q : B \to B$ is a monotonically increasing and continuous operator.

Next we show that $Q(B)$ is equicontinuous on $[-a,b]$. Since semigroup $T(t)$ is compact for $t>0$, $T(t)$ is continuous in uniform operator topology for $t>0$. For any $x\in B$ and $t_1,t_2\in J$ with $t_1<t_2$, we have that

$$\begin{aligned}
\|Qx(t_2)-Qx(t_1)\| \leq & \|T(t_2)(\phi(0)+g(x)(0))-T(t_1)(\phi(0)+g(x)(0))\| \\
& + \Big\| \int_0^{t_1} [T(t_2-s)-T(t_1-s)]\, f(s,x_s,Bx(s))\, ds \Big\| \\
& + \Big\| \int_{t_1}^{t_2} T(t_2-s) f(s,x_s,Bx(s))\, ds \Big\| \\
\leq & \|T(t_2)(\phi(0)+g(x)(0))-T(t_1)(\phi(0)+g(x)(0))\| \\
& + k\int_0^{t_1-\epsilon} \|T(t_2-s)-T(t_1-s)\|\, ds \\
& + k\int_{t_1-\epsilon}^{t_1} \|T(t_2-s)-T(t_1-s)\|\, ds + Mk(t_2-t_1) \\
\leq & \|T(t_2)(\phi(0)+g(x)(0))-T(t_1)(\phi(0)+g(x)(0))\| \\
& + k(t_1-\epsilon) \sup_{s\in[0,t_1-\epsilon]} \|T(t_2-s)-T(t_1-s)\| \\
& + 2Mk\epsilon + Mk(t_2-t_1),
\end{aligned}$$

where $\epsilon\in(0,t_1)$ is arbitrary. Therefore $\|Qx(t_2)-Qx(t_1)\|\to 0$ as $t_1\to t_2$ and $\epsilon\to 0$ independently of $x\in B$. Thus $Q(B)$ is equicontinuous on J. Since $g: C([-a,b],X)\to\mathcal{D}$ is continuously compact operator and $\phi\in\mathcal{D}$, $Q(B)$ is equicontinuous on $[-a,0]$. Hence $Q(B)$ is equicontinuous on $[-a,b]$.

Further we show that for each $t\in[-a,b]$, the set $G(t)=\{Qx(t):\ x\in B\}$ is relatively compact in X. Let $t\in(0,b]$ be a fixed real number and κ be a given real number satisfying $0<\kappa<t$. For $x\in B$, we define

$$\begin{aligned}
Q^\kappa x(t) =& T(t)\big(\phi(0)+g(x)(0)\big) + \int_0^{t-\kappa} T((t-s) f(s,x_s,Bx(s))\, ds \\
=& T(\kappa)\left[T(t-\kappa)\big(\phi(0)+g(x)(0)\big) + \int_0^{t-\kappa} T(t-\kappa-s) f(s,x_s,Bx(s))\, ds \right].
\end{aligned}$$

By (6), (H3) and the compactness of $T(\kappa)$, the set $\{Q^\kappa x(t):\ x\in B\}$ is relatively compact in X for each $t\in(0,b]$. Also

$$\begin{aligned}
\|Qx(t)-Q^\kappa x(t)\| \leq & \Big\| \int_{t-\kappa}^{t} T(t-s) f(s,x_s,Bx(s))\, ds \Big\| \\
\leq & Mk\kappa \to 0 \text{ as } \kappa\to 0^+.
\end{aligned}$$

Thus there are relatively compact sets $\{(Q^\kappa x)(t)\colon x\in B\}$ arbitrary close to the set $G(t)$ for each $t\in(0,b]$. Also $G(t),\ t\in[-a,0]$, is relatively compact in X as $g\colon C([-a,b],X)\to\mathcal{D}$ is a continuously compact operator and $\phi(\cdot)\in\mathcal{D}$. Hence the set $G(t)$ is relatively compact in X for all $t\in[-a,b]$.

In view of Ascoli-Arzela theorem, we conclude that $Q(B)$ is relatively compact. Now we define the sequences as

$$x^{(n)} = Qx^{(n-1)} \text{ and } y^{(n)} = Qy^{(n-1)}, \quad n=1,2,\dots, \tag{8}$$

and from (7), we have

$$x^{(0)} \leq x^{(1)} \leq \ldots x^{(n)} \leq \ldots \leq y^{(n)} \leq \ldots \leq y^{(1)} \leq y^{(0)}. \tag{9}$$

Since $Q(B)$ is relatively compact, the sequence $\{x^{(n)}\}$ has a convergent subsequence $\{x^{(n_j)}\}$. Let x^* be its limit. Then for each $\varepsilon > 0$ there exists an n_j (depending upon ε) such that

$$\|x^{(n_j)} - x^*\|_C < \frac{\varepsilon}{1+N}.$$

To show that the sequence $\{x^{(n)}\}$ converges to x^*, take any $n \geq n_j$ and in view of (9), we have

$$x^{(n_j)} \leq x^{(n)} \leq x^*,$$

that is

$$0 \leq x^{(n)} - x^{(n_j)} \leq x^* - x^{(n_j)}.$$

By normality of cone P of X, we have

$$\|x^{(n)} - x^{(n_j)}\|_C \leq N\|x^* - x^{(n_j)}\|_C.$$

This implies

$$\begin{aligned}\|x^{(n)} - x^*\|_C \leq& \|x^{(n)} - x^{(n_j)}\|_C + N\|x^{(n_j)} - x^*\|_C\\ \leq&(N+1)\|x^{(n_j)} - x^*\|_C\\ \leq&\varepsilon.\end{aligned}$$

Hence the sequence $\{x^{(n)}\}$ converges to x^*. By (5) and (8), we have that

$$x^{(n)}(t) = \begin{cases} T(t)(\phi(0) + g(x^{(n-1)})(0)) \\ \quad + \int_0^t T(t-s)f(s, x_s^{(n-1)}, Bx^{(n-1)}(s))\,ds, \quad t \in [0,b], \\ \phi(t) + g(x^{(n-1)})(t), \quad t \in [-a,0]. \end{cases}$$

In view of Lebesgue's dominated convergence theorem and taking $n \to \infty$, we get

$$x^*(t) = \begin{cases} T(t)(\phi(0) + g(x^*)(0)) + \int_0^t T(t-s)f(s, x_s^*, Bx^*(s))\,ds, \quad t \in [0,b], \\ \phi(t) + g(x^*)(t), \quad t \in [-a,0]. \end{cases}$$

Thus $x^* \in C([-a,b],X)$ and $x^* = Qx^*$. It means that x^* is a mild solution of (1). Similarly we can prove that there exists $y^* \in C([-a,b],X)$ such that $y^{(n)} \to y^*$ as $n \to \infty$ and $y^* = Qy^*$. Let $x \in B$ be any fixed point of Q, then by (7), $x^{(1)} = Qx^{(0)} \leq Qx = x \leq Qy^{(0)} = y^{(1)}$. By induction, $x^{(n)} \leq x \leq y^{(n)}$. Using (9) and taking the limit as $n \to \infty$, we conclude that $x^{(0)} \leq x^* \leq x \leq y^* \leq y^{(0)}$. Hence $x^*,\ y^*$ are the minimal and maximal mild solutions of the nonlocal semilinear differential equations with finite delay (1) on $[x^{(0)}, y^{(0)}]$ respectively.

In the next theorem, we again discuss the existence of extremal mild solution of (1) with the help of the measure of noncompactness and the monotone iterative procedure. In this result, semigroup $\{T(t)\}_{t\geq 0}$ does not have to be compact.

Theorem 3.2 *Let X be an ordered Banach space whose positive cone P is normal with a normal constant N and A be the infinitesimal generator of a positive C_0-semigroup $\{T(t)\}_{t\geq 0}$ on X. Also suppose that the delay system* (1) *has a lower solution $x^{(0)} \in C([-a,b],X)$ and an upper solution $y^{(0)} \in C([-a,b],X)$ with $x^{(0)} \leq y^{(0)}$ and the assumptions (H1)-(H3) hold. If the following hypotheses are satisfied*

(H4) The operator $T(t)$ is continuous in the sense of uniform operator topology for $t > 0$.

(H5) There exists a constant $L \geq 0$ such that

$$\mu(f(t,E,S)) \leq L\Big[\sup_{-a\leq\nu\leq 0}\mu(E(\nu)) + \mu(S)\Big],$$

for $t \in J$ and $E \subset \mathcal{D}$, $S \subset X$, where $E(\nu) = \{\varphi(\nu)\colon \varphi \in E\}$,

and $2MLb(1+2B^*) < 1$, then the delay system (1) *has minimal and maximal mild solutions between $x^{(0)}$ and $y^{(0)}$.*

Proof. Let $B = [x^{(0)}, y^{(0)}] = \{x \in C([-a,b],X) \mid x^{(0)} \leq x \leq y^{(0)}\}$. We define a map $Q : B \to C([-a,b],X)$ as defined in Theorem 3.1. Proceeding as in the proof of Theorem 3.1 and in view of (H4), we get that the operator $Q : B \to B$ is monotonically increasing and continuous, and $Q(B)$ is equicontinuous on $[-a,b]$. Also we define the sequences $x^{(n)}$ and $y^{(n)}$ as defined by (8) in Theorem 3.1. Since $x^{(0)} \leq Qx^{(0)}$, $Qy^{(0)} \leq y^{(0)}$ and the map Q is increasing, the equation (9) holds.

Let $S = \{x^{(n)}\}_{n=1}^{\infty}$. By (9) and the normality of positive cone P_C , the set S is bounded. As g is a continuously compact operator, we get

$$\begin{aligned}\mu(\{S(t)\}) &= \mu(\{\phi(t) + g(x^{(n-1)})(t)\}_{n=1}^{\infty}) \\ &\leq \mu(\{\phi(t)\}) + \mu(\{g(x^{(n-1)})(t)\}_{n=1}^{\infty}) = 0 \text{ for } t \in [-a,0].\end{aligned}$$

Since $S(t) = \{x^{(1)}(t)\} \cup \{Q(S)(t)\}$ for any $t \in J$, $\mu(S(t)) = \mu(Q(S)(t))$, $t \in J$. From (H3), (H5), (5) and (8), we get for $t \in J$ that

$$\begin{aligned}\mu(S(t)) =&\mu\Big(\Big\{T(t)[\phi(0) + g(x^{(n)})(0)] + \int_0^t T(t-s)f(s, x_s^{(n)}, Bx^{(n)}(s))\,ds\Big\}\Big) \\ \leq& 2M\int_0^t \mu\Big(\Big\{f(s, x_s^{(n)}, Bx^{(n)}(s))\,ds\Big\}\Big) \\ \leq& 2ML\int_0^t \left[\sup_{-a\leq\nu\leq 0}\mu\left(\left\{x^{(n)}(s+\nu)\right\}\right) + \mu\left(\left\{\int_0^s K(s,r)x^{(n)}(r)\,dr\right\}\right)\right] ds \\ \leq& 2ML\int_0^t \left[\sup_{0\leq r\leq s}\mu\left(\left\{x^{(n)}(r)\right\}\right) + 2\int_0^s K(s,r)\mu\left(\left\{x^{(n)}(r)\right\}\right)\,dr\right] ds \\ \leq& 2ML(1+2B^*)\int_0^t \sup_{0\leq r\leq s}\mu\left(\left\{x^{(n)}(r)\right\}\right)\,ds \\ \leq& 2MLb(1+2B^*)\sup_{-a\leq r\leq b}\mu\left(\{S(r)\}\right).\end{aligned}$$

Since $\left\{Qx^{(n)}\right\}_{n=0}^{\infty}$, i.e. $\left\{x^{(n)}\right\}_{n=1}^{\infty}$, is equicontinuous on $[-a,b]$ and by Lemma 2.4, we get

$$\mu(S) \leq 2MLb(1+2B^*)\mu(S).$$

Since $2MLb(1+2B^*) < 1$, this implies that $\mu(S) = 0$, i.e. $\mu(\{x^{(n)}\}_{n=1}^{\infty}) = 0$. Therefore the set $\{x^{(n)} : n \geq 1\}$ is relatively compact in B. So the sequence $\{x^{(n)}\}$ has a convergent subsequence in B. By the proof of Theorem 3.1, the sequence $\{x^{(n)}\}$ is itself convergent sequence. So there exists $x^* \in B$ such that $x^{(n)} \to x^*$ as $n \to \infty$. Similarly there exists $y^* \in B$ such that $y^{(n)} \to y^*$ as $n \to \infty$. Again by Theorem 3.1, x^* and y^* become the minimal and maximal mild solutions of the nonlocal semilinear differential equations with finite delay (1) in B respectively.

In the next theorem, we shall prove the uniqueness of the solution of the system (1) by using monotone iterative procedure. For this purpose, we make the following assumptions:

(H6) The function $f : J \times \mathcal{D} \times X \to X$ is continuous and there exists a constant $\eta \geq 0$ such that for some $\nu \in [-a, 0]$,

$$f(t, \varphi_2, u_2) - f(t, \varphi_1, u_1) \leq \eta[(\varphi_2(\nu) - \varphi_1(\nu)) + (u_2 - u_1)],$$

for any $t \in J$, $u_1, u_2 \in X$ with $Bx^{(0)}(t) \leq u_1 \leq u_2 \leq By^{(0)}(t)$ and $\varphi_1, \varphi_2 \in \mathcal{D}$ with $x_t^{(0)} \leq \varphi_1 \leq \varphi_2 \leq y_t^{(0)}$.

(H7) For any $t \in [-a, 0]$ and $x, y \in B$ with $x \leq y$, there exists a constant $\gamma(0 \leq \gamma < \frac{1}{N})$ such that

$$g(y)(t) - g(x)(t) \leq \gamma(y(t) - x(t)).$$

Theorem 3.3 *Let X be an ordered Banach space whose positive cone P is normal with a normal constant N and A be the infinitesimal generator of a positive C_0-semigroup $\{T(t)\}_{t\geq 0}$ on X. Also suppose that the system* (1) *has a lower solution $x^{(0)} \in C([-a, b], X)$ and an upper solution $y^{(0)} \in C([-a, b], X)$ with $x^{(0)} \leq y^{(0)}$. If the assumptions (H2), (H3), (H4), (H6) and (H7) hold, and $2MLb(1+2B^*) < 1$, where $L = N\eta$, then the delay system* (1) *has a unique mild solution between $x^{(0)}$ and $y^{(0)}$.*

Proof. Let $\{\varphi_n\} \subset \mathcal{D}$ and $\{u_n\} \subset X$ be two monotone increasing sequences. Take any $m, n = 1, 2, \ldots,$ with $m > n$. By (H2), (H3) and (H6), we get for some $\nu \in [-a, 0]$ that

$$\theta \leq f(t, \varphi_m, u_m) - f(t, \varphi_n, u_n) \leq \eta\Big[(\varphi_m(\nu) - \varphi_n(\nu)) + (u_m - u_n)\Big].$$

Using the normality of the positive cone P, we get

$$\|f(t, \varphi_m, u_m) - f(t, \varphi_n, u_n)\| \leq N\eta\Big[\|\varphi_m(\nu) - \varphi_n(\nu)\| + \|u_m - u_n\|\Big]. \tag{10}$$

By the definition of measure of noncompactness, we get

$$\begin{aligned}\mu\left(\{f\left(s, \varphi_n\right)\}\right) \leq & L\left[\mu\left(\{\varphi_n(\nu)\}\right) + \mu\left(\{u_n\}\right)\right] \\ \leq & L\left[\sup_{-a\leq\nu\leq 0} \mu\left(\{\varphi_n(\nu)\}\right) + \mu\left(\{u_n\}\right)\right],\end{aligned}$$

where $L = N\eta$. Clearly the assumption (H5) is satisfied. The assumption (H1) is satisfied by the inequality (10). Thus the assumptions (H1)-(H5) hold and $2MLb(1+2B^*) < 1$. So by Theorem 3.2, the delay system (1) has minimal and maximal mild solutions between $x^{(0)}$ and $y^{(0)}$.

Let $x^*(t)$ and $y^*(t)$ be the minimal and maximal solutions of the delay system (1) respectively on the ordered interval $B = [x^{(0)}, y^{(0)}]$. By (5) and H(7) for any $t \in [-a, 0]$, we have

$$\begin{aligned}\theta &\le y^*(t) - x^*(t) = Qy^*(t) - Qx^*(t)\\ &= g(y^*)(t) - g(x^*)(t)\\ &\le \gamma(y^*(t) - x^*(t))\end{aligned}$$

By using the normality of positive cone P, we get $\|y^*(t) - x^*(t)\| \le N\gamma\|y^*(t) - x^*(t)\|$ for all $t \in [-a, 0]$. This implies that $y^*(t) = x^*(t)$ for all $t \in [-a, 0]$ as $N\gamma < 1$. Let $t \in [0, b]$. In view of (5) and (H6), we have

$$\begin{aligned}\theta \le& y^*(t) - x^*(t) = Qy^*(t) - Qx^*(t)\\ =& \int_0^t T(t-s)\left[f(s, y_s^*, By^*(s)) - f(s, x_s^*, Bx^*(s))\right] ds\\ \le& \eta\int_0^t T(t-s)\left[(y_s^*(\nu) - x_s^*(\nu)) + \int_0^s K(s,r)(y^*(r) - x^*(r))\,dr\right] ds\end{aligned}$$

where $\nu \in [-a, 0]$. By applying the normality of the positive cone P, we get

$$\begin{aligned}\|y^*(t) - x^*(t)\| &\le N\eta\Big\|\int_0^t T(t-s)\Big[(y_s^*(\nu) - x_s^*(\nu))\\ &\quad + \int_0^s K(s,r)(y^*(r) - x^*(r))\,dr\Big]\,ds\Big\|\\ &\le MN\eta\int_0^t \Big[\|y^*(s+\nu) - x^*(s+\nu)\|\\ &\quad + \int_0^s K(s,r)\|y^*(r) - x^*(r)\|\,dr\Big]\,ds\\ &\le MN\eta b(1 + B^*)\|y^* - x^*\|_C.\end{aligned} \tag{11}$$

Since $y^*(t) = x^*(t)$ for $t \in [-a, 0]$ and due to the inequality (11), we get that $\|y^* - x^*\|_C \le MN\eta b(1 + B^*)\|y^* - x^*\|_C$. But $MLb(1 + 2B^*) < \frac{1}{2}$, so $\|y^* - x^*\|_C = 0$, i.e., $y^*(t) = x^*(t)$, $t \in [-a, b]$. Hence $y^* = x^*$ is the unique mild solution of the delay system (1) between $x^{(0)}$ and $y^{(0)}$.

4 Example

Consider the following nonlocal semilinear partial differential equations with finite delay of the form:

$$\begin{cases} \frac{\partial z(t,\xi)}{\partial t} = & \frac{\partial^2}{\partial \xi^2} z(t,\xi) + \int_{-a}^0 (a+\nu)^{\frac{-1}{2}}(-\nu)^{\frac{-1}{2}} z(t+\nu,\xi)\,d\nu\\ & \qquad + \int_0^t z(s,\xi)\,ds, \quad \xi \in [0,\pi],\ t \in [0,b],\\ z(t,0) = & z(t,\pi) = 0, \quad t \in [0,b],\\ z(\nu,\xi) = & \phi(\nu,\xi) + \int_0^b \rho(s,\nu)\log(1 + |z(s,\xi)|)ds, \quad -a \le \nu \le 0, \end{cases} \tag{12}$$

where $\phi \in \mathcal{D} = C([-a,0] \times [0,\pi]\colon \mathbb{R}^+)$, the operator $\rho(s,\nu)\colon [0,b] \times [-a,0] \to \mathbb{R}^+$ is continuous.

Let $X = L^2([0,\pi],\mathbb{R})$ and $P = \{v \in X\colon v(\xi) \geq 0,\ \xi \in [0,\pi]\}$. Then P is a normal cone in Banach space X. We define an operator $A : X \to X$ by $Av = v''$ with domain

$$D(A) = \{v \in X\colon v, v' \text{ is absolutely continuous } v'' \in X, v(0) = v(\pi) = 0\}.$$

It is well known that A is an infinitesimal generator of a strongly continuous semigroup $\{T(t), t \geq 0\}$ of uniformly bounded linear operators in X. Now we define $z(t)(\xi) = z(t,\xi)$, $z_t(\nu,\xi) = z(t+\nu,\xi)$, $\phi(\nu)(\xi) = \phi(\nu,\xi)$, $Bz(t)(\xi) = \int_0^t z(s,\xi)\,ds$, $f(t,\varphi,u)(\xi) = \int_{-a}^0 (a+\nu)^{\frac{-1}{2}}(-\nu)^{\frac{-1}{2}}\varphi(\nu,\xi)\,d\nu + u(\xi)$ and $g(z)(\nu)(\xi) = g(z(\nu,\xi)) = \int_0^b \rho(s,\nu)\log(1 + |z(s,\xi)|)ds$. Therefore, the above nonlocal semilinear partial differential equations with finite delay (12) can be written as the abstract form (1).

Since $T(t)$ is continuous in the sense of uniform operator topology for $t > 0$, the assumption (H4) is satisfied. We can also easily see that function f satisfies the assumptions (H1) and (H2). For $t \in [0,b]$, $\varphi_1, \varphi_2 \in C([-a,0],X)$ with $0 \leq \varphi_1 \leq \varphi_2$ and $u_1, u_2 \in X$ with $0 \leq u_1 \leq u_2$, then

$$\begin{aligned} 0 \leq & f(t,\varphi_2,u_2)(\xi) - f(t,\varphi_1,u_1)(\xi) \\ \leq & \int_{-a}^0 (a+\nu)^{\frac{-1}{2}}(-\nu)^{\frac{-1}{2}}[\varphi_2(\nu)(\xi) - \varphi_1(\nu)(\xi)]\,d\nu + [u_2(\xi) - u_1(\xi)]. \end{aligned}$$

By normality of cone P, we have

$$\|f(t,\varphi_2,u_2) - f(t,\varphi_1,u_1)\| \leq \int_{-a}^0 (a+\nu)^{\frac{-1}{2}}(-\nu)^{\frac{-1}{2}}\|\varphi_2(\nu) - \varphi_1(\nu)\|\,d\nu + \|u_2 - u_1\|.$$

Hence, for any bounded set $E \subset C([-a,0],X)$ and $S \subset X$, we have

$$\mu(f(t,E,S)) \leq \left[\pi \sup_{-a \leq \nu \leq 0} \mu(E(\nu)) + \mu(S)\right].$$

Thus f satisfies the assumption H(5). Clearly the function $g : PC([0,b],X) \to X$ is increasing, continuous and compact. Thus g satisfies the assumption (H3).

Let $v(t,\xi) = 0$, $(t,\xi) \in [-a,b] \times [0,\pi]$. Then $f(t,v_t,Bv(t)) = 0$ for $t \in [0,b]$ and $v(\nu,\xi) \leq \phi(\nu,\xi) + g(v(\nu,\xi))$ for $\nu \in [-a,0]$. Now we assume that there is a function $w(t,\xi) \geq 0$ such that $w(t,0) = w(t,\pi) = 0$,

$$\frac{\partial w(t,\xi)}{\partial t} \geq \frac{\partial^2}{\partial y^2} w(t,\xi) + f(t,w_t,Bw(t)),$$

and $w(\nu,\xi) \geq \phi(\nu,\xi) + g(w(\nu,\xi))$ for $\nu \in [-a,0]$. Thus v, w become lower and upper solutions of the system (12) respectively and $v \leq w$. If $2Mb(\pi + 2b) < 1$, then all the conditions of Theorem 3.2 are satisfied. Hence, by Theorem 3.2, the system (12) has the minimal and maximal mild solutions lying between the lower solution 0 and the upper solution w.

Acknowledgment

The first author would like to acknowledge the financial assistance provided by University Grant Commission (UGC) of India for carrying out this work. The second author would like to acknowledge that this work has been carried out under the research project SR/S4/MS:796/12 of DST, New Delhi.

References

[1] Ye, R. Existence of solutions for impulsive partial neutral functional differential equation with infinite delay. *Nonlinear Analysis* **73** (9) (2010) 155–162.

[2] Li, K. and Jia, J. Existence and uniqueness of mild solutions for abstract delay fractional differential equations. *Comput. Math. Appl.* **62** (3) (2011) 1398–1404.

[3] Bellen, A., Guglielmi, N. and Ruehli, A.E. Methods for linear systems of circuit delay differential equations of neutral type. *IEEE Trans. Circuits Systems* **46** (1) (1999) 212–216.

[4] Kuang, Y. Delay Differential Equations with Applications in Population Dynamics. *Mathematics in Science and Engineering* **191** Academic Press, Boston, 1993.

[5] Han, Q. On robust stability of neutral systems with time-varying discrete delay and norm-bounded uncertainty. *Automatica J. IFAC* **40** (6) (2004) 1087–1092.

[6] Kamaljeet and Bahuguna, D. Monotone iterative technique for nonlocal fractional differential equations with finite delay in Banach space. *Electron. J. Qual. Theory Differ. Equ.* **2015** (9) (2015) 1–16.

[7] Machado, J., Ravichandran, C., Rivero, M. and Trujillo, J. Controllability results for impulsive mixed–type functional integro–differential evolution equations with nonlocal conditions. *Fixed Point Theory and Applications* **2013** (66) (2013) 1–16.

[8] Yang, H. Existence of mild solutions for fractional evolution equations with nonlocal conditions. *Boundary Value Problems* **2012** (113) (2012) 1–12.

[9] Balachandran, K. and Park, J. Nonlocal Cauchy problem for abstract fractional semilinear evolution equations. *Nonlinear Analysis* **71** (10) (2009) 4471–4475.

[10] Byszewski, L. Theorems about the existence and uniqueness of solutions of a semilinear evolution nonlocal Cauchy problem. *J. Math. Anal. Appl.* **162** (2) (1991) 494–505.

[11] Liang, J., Liu, J. and Xiao, T. Nonlocal impulsive problems for nonlinear differential equations in Banach spaces. *Math. Comput. Modelling* **49** (3-4) (2009) 798–804.

[12] Wang, J. and Wei, W. A class of nonlocal impulsive problems for integrodifferential equations in Banach spaces. *Results Math.* **58** (3-4) (2010) 379–397.

[13] Benchohra, M., Gatsori, E. and Ntouyas, S. Controllability results for semilinear evolution inclusions with nonlocal conditions. *J. Optim. Theory Appl.* **118** (3) (2003) 493–513.

[14] Li, Y. and Liu, Z. Monotone iterative technique for addressing impulsive integro-differential equations in Banach spaces. *Nonlinear Analysis* **66** (1) (2007) 83–92.

[15] Kamaljeet and Bahuguna, D. Extremal mild solutions for finite delay differential equations of fractional order in Banach spaces. *Nonlinear Dyn. Syst. Theory* **14** (4) (2014) 371–382.

[16] Bhaskar, T.G., Lakshmikantham, V. and Devi, J.V. Monotone iterative technique for functional differential equations with retardation and anticipation. *Nonlinear Analysis* **66** (10) (2007) 2237–2242.

[17] Liu, X. Monotone iterative technique for impulsive differential equations in a Banach space. *J. Math. Phys. Sci.* **24** (3) (1990) 183–191.

[18] Chen, P. and Mu, J. Monotone iterative method for semilinear impulsive evolution equations of mixed type in Banach spaces. *Electron. J. Diff. Equ.* **2010** (149) (2010) 1–13.

[19] Pazy, A. Semigroup of Linear Operators and Applications to Partial Differential Equations. *Applied Mathematical Sciences* **44** Springer-Verlag, New York, 1983.

[20] Deimling, K. *Nonlinear Functional Analysis.* Springer-Verlag, Berlin, 1985.

[21] Heinz, H. On the behaviour of measures of noncompactness with respect to differentiation and integration of vector valued functions. *Nonlinear Analysis* **7** (12) (1983) 1351–1371.

Nonlinear Dynamics and Systems Theory, **16** (3) (2016) 312–321

Co-existence of Various Types of Synchronization Between Hyper-chaotic Maps

Adel Ouannas *

Department of Mathematics and Computer Science,
Constantine University, Algeria;
Laboratory of Mathematics, Informatics and Systems (LAMIS), University of Larbi Tebessi,
Tebessa, 12002 Algeria.

Received: January 13, 2015; Revised: December 15, 2015

Abstract: In this paper, we propose a new type of hybrid synchronization combining projective synchronization (PS), full state hybrid projective synchronization (FSHPS) and generalized synchronization (GS). We present, based on nonlinear controllers, a new control scheme to study the co-existence of (PS), (FSHPS) and (GS) between general 3D hyperchaotic maps. The capability of the proposed approach is illustrated by numerical example.

Keywords: *hyperchaotic maps; synchronization; co-existence; Lyapunov stability.*

Mathematics Subject Classification (2010): 93C10, 93C55, 93D05.

1 Introduction

Historically, hyperchaos in discrete-time systems was firstly reported by Rössler [1]. A hyperchaotic system is usually defined as a chaotic system with more than one positive Lyapunov exponent. The occurrence of hyperchaotic behavior has been found in an electronic circuit [2], NMR laser [3], in a semi-conductor system [4] and in a chemical reaction system [5]. Some interesting hyperchaotic systems in discrete-time were presented in the past two decades such as Baier-Klain system [6], Hitzl-Zele map [7], Stefanski map [8], Wang map [9], Rössler discrete-time system [10] and Grassi-Miller map [11] etc. Since hyperchaotic maps are more complex than chaotic maps, their dynamics have been investigated extensively owing to their useful potential applications in

* Corresponding author: mailto:ouannas_adel@yahoo.fr

secure communications [12–17]. Thus it is a more important subject to study hyperchaos synchronization.

Recently, more and more attention has been paid to the synchronization of chaos (hyperchaos) in discrete-time dynamical systems [18–22]. Different synchronization types have been proposed for discrete-time chaotic and hyperchaotic maps such as projective synchronization [23], adaptive function projective synchronization [24, 25], function cascade synchronization [26], generalized synchronization [27, 28], lag synchronization [29], impulsive synchronization [30], hybrid synchronization [31], Q-S synchronization [32] and full state hybrid projective synchronization [33, 34]. Among all synchronization types, projective synchronization (PS), full-state hybrid projective synchronization (FSHPS) and generalized synchronization (GS) are effective approaches for achieving the synchronization of chaotic and hyperchaotic discrete-time systems. (PS) means that the drive chaotic system and the response chaotic system synchronize up to scaling constant, FSHPS means that each drive system state synchronizes with a linear combination of response system states and (GS) appears when there exists functional relationship between the states of the drive and the response chaotic systems.

In this paper, a new general scheme of synchronization which includes (PS), (FSHPS) and (GS) between coupled 3D hyperchaotic maps is constructed. Based on stability theory of linear discrete-time systems, Lyapunov stability theory and using nonlinear controllers, a new criterion of co-existence of (PS), (FSHPS) and (GS) is derived. The derived synchronization results can have an important effect in the application due to complexity of the proposed scheme and the difficulty of the prediction of the scaling factors. To validate the proposed approach numerically, we apply it to two hyperchaotic maps: the hyperchoatic Wang map and the hyperchoatic Stefanski map.

This paper is organized as follows. In Section 2, the problem of co-existence of synchronization types is introduced. Our approach of synchronization is described in Section 3. In Section 4, numerical example is used to show the effectiveness of the proposed synchronization method. In Section 5, conclusion is made.

2 Problem Statement

We consider the following drive and response chaotic systems

$$x_i(k+1) = f_i(X(k)), \quad 1 \leq i \leq 3, \tag{1}$$

$$y_i(k+1) = g_i(Y(k)) + u_i, \quad 1 \leq i \leq 3, \tag{2}$$

where $(x_1(k), x_2(k), x_3(k))^T$, $(y_1(k), y_2(k), y_3(k))^T$ are the states of the drive and the response systems, respectively, f_i, $g_i : \mathbf{R}^3 \to \mathbf{R}$, $1 \leq i \leq 3$, and u_i, $1 \leq i \leq 3$, are controllers to be determined.

The error system between the drive system (1) and the response system (2) is defined as

$$\begin{aligned} e_1(k) &= y_1(k) - \theta x_1(k), \\ e_2(k) &= y_2(k) - \sum_{j=1}^{3} \lambda_j x_j(k), \\ e_3(k) &= y_3(k) - \phi(x_1, x_2, x_3)(k), \end{aligned} \tag{3}$$

where $\theta \in \mathbf{R}^*$, $\lambda_j \in \mathbf{R}^*$ $j = 1, 2, 3$, and $\phi : \mathbf{R}^3 \to \mathbf{R}$ is a continuously bounded function.

We said that projective synchronization (PS), full-state hybrid projective synchronization (FSHPS) and generalized synchronization (GS) co-exist in the synchronization of the systems (1) and (2), if there exist controllers u_i, $1 \leq i \leq 3$, such that the synchronization errors (3) satisfy

$$\lim_{k \longrightarrow +\infty} e_i(k) = 0, \quad i = 1, 2, 3. \tag{4}$$

3 Synchronization Approach

As the drive system, we consider the following hyperchaotic map

$$x_i(k+1) = f_i(X(k)), \quad 1 \leq i \leq 3, \tag{5}$$

where $X(k) = (x_1(k), x_2(k), x_3(k))^T$ is the state vector of the drive system, $f_i : \mathbf{R}^3 \longrightarrow \mathbf{R}$, $1 \leq i \leq 3$. As the response, we consider the following chaotic system

$$y_i(k+1) = \sum_{j=1}^{3} b_{ij} y_j(k) + g_i(Y(k)) + u_i, \quad 1 \leq i \leq 3, \tag{6}$$

where $Y(k) = (y_1(k), y_2(k), y_3(k))^T$ is the state vector of the response systems, $(b_{ij}) \in \mathbf{R}^{3\times 3}$ is the linear part of the response system, $g_i : \mathbf{R}^3 \longrightarrow \mathbf{R}$, $1 \leq i \leq 3$, are nonlinear functions and u_i, $1 \leq i \leq 3$, are controllers to be designed.

The error system, according to (3), between the drive system (5) and the response system (6) can be derived as

$$\begin{aligned} e_1(k+1) &= y_1(k+1) - \theta x_1(k+1), \\ e_2(k+1) &= y_2(k+1) - \sum_{j=1}^{3} \lambda_j x_j(k+1), \\ e_3(k+1) &= y_3(k+1) - \phi(X(k+1)). \end{aligned} \tag{7}$$

Then, the error system (7) can be written as

$$\begin{aligned} e_1(k+1) &= \sum_{j=1}^{3} b_{1j} y_j(k) + g_1(Y(k)) + u_1 - \theta f_1(X(k)), \\ e_2(k+1) &= \sum_{j=1}^{3} b_{2j} y_j(k) + g_2(Y(k)) + u_2 - \sum_{j=1}^{3} \lambda_j f_j(X(k)), \\ e_3(k+1) &= \sum_{j=1}^{3} b_{3j} y_j(k) + g_3(Y(k)) + u_2 - \phi(f_1(X(k)),\ f_2(X(k)),\ f_3(X(k))). \end{aligned} \tag{8}$$

To achieve synchronization between the drive system (5) and the response system

(6), we propose the following synchronization controllers

$$\begin{aligned} u_1 &= N_1 - b_{11}\theta x_1(k) - b_{12}\left(\sum_{j=1}^{3}\lambda_j x_j(k)\right) - \sum_{j=1}^{3} l_{1j}e_j\left(k\right), \\ u_2 &= N_2 - b_{21}\theta x_1(k) - b_{22}\left(\sum_{j=1}^{3}\lambda_j x_j(k)\right) - \sum_{j=1}^{3} l_{2j}e_j\left(k\right), \\ u_3 &= N_3 - b_{31}\theta x_1(k) - b_{32}\left(\sum_{j=1}^{3}\lambda_j x_j(k)\right) - \sum_{j=1}^{3} l_{3j}e_j\left(k\right), \end{aligned} \tag{9}$$

where

$$\begin{aligned} N_1 &= \theta f_1(X(k)) - b_{13}\phi\left(X(k)\right) - g_1(Y(k)), \\ N_2 &= \sum_{j=1}^{3}\lambda_j f_j(X(k)) - b_{23}\phi\left(X(k)\right) - g_2(Y(k)), \\ N_3 &= \phi\left(f_1(X(k)),\ f_2(X(k)),\ f_3(X(k))\right) - b_{33}\phi\left(X(k)\right) - g_3(Y(k)), \end{aligned} \tag{10}$$

and $(l_{ij}) \in \mathbf{R}^{3\times 3}$ are control constants to be determined later.

By substituting the control law (9) into (8), the error system can be described as

$$\begin{aligned} e_1\left(k+1\right) &= \sum_{j=1}^{3}\left(b_{1j} - l_{1j}\right)e_j(k), \\ e_2\left(k+1\right) &= \sum_{j=1}^{3}\left(b_{2j} - l_{2j}\right)e_j(k), \\ e_3\left(k+1\right) &= \sum_{j=1}^{3}\left(b_{3j} - l_{3j}\right)e_j(k). \end{aligned} \tag{11}$$

Now, rewrite the error system described in (11) in the compact form

$$e\left(k+1\right) = \left(B - L\right)e(k), \tag{12}$$

where $e(k) = \left(e_1(k), e_2(k), e_3(k)\right)^T$, $B = \left(b_{ij}\right)_{3\times 3}$ and $L = \left(l_{ij}\right)_{3\times 3}$.

Hence, we have the following result.

Theorem 3.1 *If the control matrix L is chosen such that one of the following conditions is satisfied:*

(i) All eigenvalues of $B - L$ are strictly inside the unit disk.

(ii) $(B-L)^T(B-L) - I$ is negative definite matrix.

(iii) $\left(l_{ij}\right)_{1\le i,\ j\le 3}$ are chosen such that

$$\begin{aligned} &\sum_{i=1}^{3}\left(b_{ip} - l_{ip}\right)\left(b_{iq} - l_{iq}\right) = 0, \quad p, q = 1, 2, 3, \ p \neq q, \\ &\sum_{i=1}^{3}\left(b_{ij} - l_{ij}\right)^2 < 1, \quad j = 1, 2, 3. \end{aligned} \tag{13}$$

Then, (PS), (FSHPS) and (GS) co-exist between the drive system (5) and the response system (6).

Proof. Firstly, according to stability theory of linear discrete-time systems, we can conclude that if condition (i) is satisfied it is immediate that $\lim_{k\longrightarrow+\infty} e_i(k)=0,\ i=1,2,3.$ Therefore, systems (5) and (6) are globally synchronized.

Secondly, we construct the Lyapunov function in the form $V\left(e(k)\right)=e^T(k)e(k)$, we obtain

$$\begin{aligned}
\Delta V\left(e(k)\right) &= e^T(k+1)e(k+1)-e^T(k)e(k) \\
&= e^T(k)(B-L)^T(B-L)e(k)-e^T(k)e(k) \\
&= e^T(k)\left[(B-L)^T(B-L)-I\right]e(k),
\end{aligned}$$

and by using condition (ii) we get $\Delta V\left(e(k)\right)<0$. Thus, from the Lyapunov stability theory, it is immediate that $\lim_{k\longrightarrow+\infty} e_i(k)=0\ \ (i=1,2,3)$ then the synchronization is achieved between systems (5) and (6).

Finally, consider the candidate Lyapunov function: $V\left(e(k)\right)=\sum_{i=1}^{3} e_i^2\left(k\right)$, we get

$$\begin{aligned}
\Delta V\left(e(k)\right) &= \sum_{i=1}^{3} e_i^2\left(k+1\right)-\sum_{i=1}^{3} e_i^2\left(k\right) \\
&= \sum_{j=1}^{3}\left(\sum_{i=1}^{3}\left(b_{ij}-l_{ij}\right)^2-1\right)e_j^2(k) \\
&+ \sum_{\substack{p,\ q=1\\ p\neq q}}^{3}\left(\sum_{i=1}^{3}\left(b_{ip}-l_{ip}\right)\left(b_{iq}-l_{iq}\right)\right)e_p(k)e_q(k),
\end{aligned}$$

and by using conditions (iii), we obtain $\Delta V\left(e(k)\right)<0$. Then, it is immediate that $\lim_{k\longrightarrow+\infty} e_i(k)=0\ (i=1,2,3)$, and we conclude that the systems (4) and (5) are globally synchronized.

4 Numerical Example

We consider hyperchaotic Stefanski map as the drive system and the controlled hyperchaotic Wang map as the response system. The drive system is described as

$$\begin{aligned}
x_1\left(k+1\right) &= 1+x_3\left(k\right)-\alpha x_2^2\left(k\right), \\
x_2\left(k+1\right) &= 1+\beta x_2\left(k\right)-\alpha x_1^2\left(k\right), \\
x_3\left(k+1\right) &= \beta x_1\left(k\right),
\end{aligned} \tag{14}$$

which has a chaotic attractor, when $(\alpha,\ \beta)=(1.4,\ 0.2)$ [36]. The hyperchaotic attractor of Stefanski map is shown in Figure 1. The response system can be defined as

$$\begin{aligned}
y_1\left(k+1\right) &= a_3\delta y_2\left(k\right)+\left(a_4\delta+1\right)y_1\left(k\right)+u_1, \\
y_2\left(k+1\right) &= a_1\delta y_1\left(k\right)+y_2\left(k\right)+a_2\delta y_3\left(k\right)+u_2, \\
y_3\left(k+1\right) &= \left(a_7\delta+1\right)y_3\left(k\right)+a_6\delta y_2\left(k\right)y_3\left(k\right)+a_5\delta+u_3,
\end{aligned} \tag{15}$$

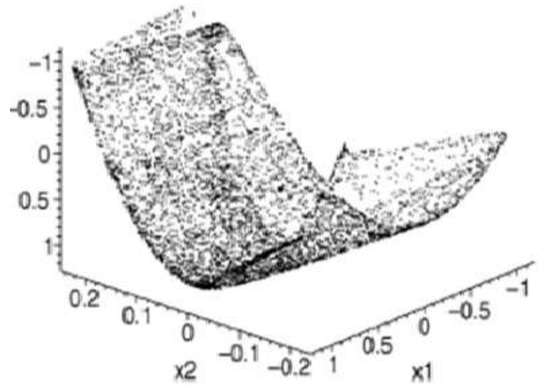

Figure 1: Hyperchaotic attractor of Stefanski map.

where $U = (u_1, u_2, u_3)^T$ is the vector controller. The hyperchaotic Wang map has a chaotic attractor, when $(a_1, a_2, a_3, a_4, a_5, a_6, a_7, \delta) = (-1.9, 0.2, 0.5, -2.3, 2, -0.6, -1.9, 1)$ [35]. The hyperchaotic attractor of Wang map is shown in Figure 2. According to our control scheme proposed in the previous section

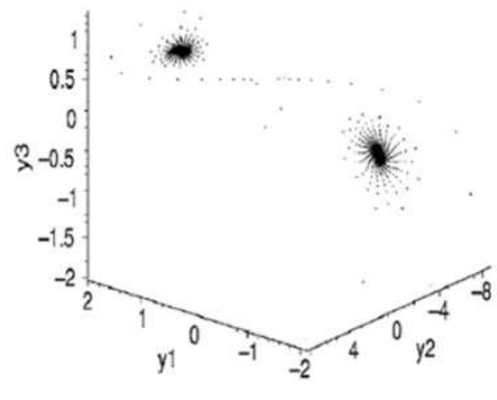

Figure 2: Hyperchaotic attractor of Wang map.

the synchronization errors between the drive system (14) and the response system (15) are defined as follows

$$\begin{aligned} e_1(k+1) &= y_1(k+1) - \theta x_1(k+1), \\ e_2(k+1) &= y_2(k+1) - \sum_{j=1}^{3} \lambda_j x_j(k+1), \\ e_3(k+1) &= y_3(k+1) - \phi(x_1(k+1), x_2(k+1), x_3(k+1)). \end{aligned} \tag{16}$$

In this example, the scaling constants $\theta, \lambda_1, \lambda_2$ and λ_3 are chosen as

$$\begin{cases} \theta = 2, \\ \lambda_1 = 1, \\ \lambda_2 = 2, \\ \lambda_3 = 3, \end{cases} \tag{17}$$

and the map $\phi : \mathbf{R}^3 \to \mathbf{R}$ is selected as

$$\phi(x_1(k), x_2(k), x_3(k)) = x_1(k) - x_2(k)x_3(k). \tag{18}$$

Then, the errors system (16) can be described as

$$\begin{aligned} e_1(k+1) &= (a_4\delta + 1) e_1(k) + R_1 + u_1,, \\ e_2(k+1) &= e_2(k) + R_2 + u_2, \\ e_3(k+1) &= (a_7\delta + 1) e_3(k) + u_3, \end{aligned} \tag{19}$$

where

$$\begin{aligned} R_1 &= a_3\delta y_2(k) + \sum_{j=1}^{3} \mu_{1j} x_j(k) + 2\alpha x_2^2(k) - 2, \\ R_2 &= a_1\delta y_1(k) + a_2\delta y_3(k) + \sum_{j=1}^{3} \mu_{2j} x_j(k) + \alpha x_2^2(k) + 2\alpha x_1^2(k) - 3, \\ R_3 &= a_6\delta y_2(k)y_3(k) + \sum_{j=1}^{3} \mu_{3j} x_j(k) - (a_7\delta + 1) x_2(k)x_3(k) + \beta x_1(k)x_2(k) \\ &- \alpha\beta x_1^3(k) + \alpha x_2^2(k) + a_5\delta - 1, \end{aligned} \tag{20}$$

where $\mu_{11} = 2(a_4\delta + 1)$, $\mu_{12} = 0$, $\mu_{13} = -2$, $\mu_{21} = -3\beta + 1$, $\mu_{22} = 2(1 - \beta)$, $\mu_{23} = 2$, $\mu_{31} = a_7\delta + 1 - \beta$, $\mu_{31} = a_7\delta + 1 + \beta$, $\mu_{32} = 0$, and $\mu_{33} = -1$.

To achieve synchronization between systems (14) and (15), we choose the synchronization controllers u_i $(i = 1, 2, 3)$, as

$$u_i = -R_i - l_i e_i, \quad i = 1, 2, 3, \tag{21}$$

where the control constants $(l_i)_{1 \le i \le 3}$ are selected as follows

$$\begin{aligned} l_1 &= a_4\delta, \\ |l_2| &< 1, \\ l_3 &= a_7\delta. \end{aligned} \tag{22}$$

Theorem 4.1 *The hyperchaotic Stefanski map (14) and the controlled hyperchaotic Wang map (15) are globally synchronized under the controllers (21).*

Proof. By substituting (21) into (19), the synchronization errors can be written as

$$\begin{aligned} e_1(k+1) &= e_1(k), \\ e_2(k+1) &= (1-l_2)\, e_2(k), \\ e_3(k+1) &= e_3(k). \end{aligned} \tag{23}$$

To prove the zero-stability of synchronization errors (23), we consider the quadratic Lyapunov function $V(e(k)) = \sum_{i=1}^{3} e_i^2(k)$, then we obtain

$$\begin{aligned} \Delta V(e(k)) &= \sum_{i=1}^{3} e_i^2(k+1) - \sum_{i=1}^{3} e_i^2(k) \\ &= e_1^2(k) + (1-l_2)^2 e_2^2(k) + e_3^2(k) - e_1^2(k) - e_2^2(k) - e_3^2(k) \\ &= (1-l_2)^2 e_2^2(k) < 0. \end{aligned}$$

Thus, by Lyapunov stability it is immediate that $\lim_{k\to\infty} e_i(k) = 0 \quad (i = 1, 2, 3)$. Finally, we get the numeric results that are shown in Figure 3.

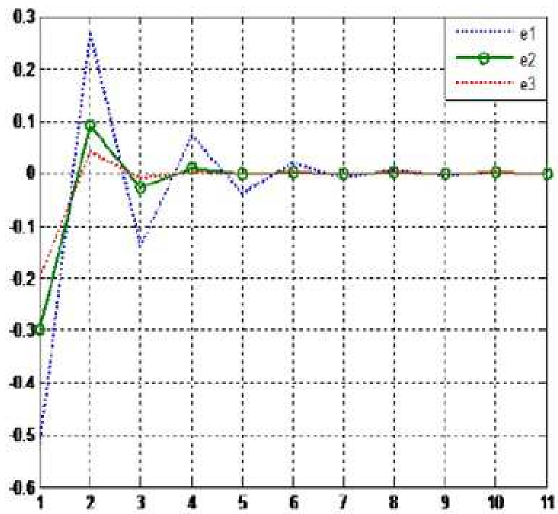

Figure 3: Time evolution of errors between systems (14) and (15).

5 Conclusion

In this paper, the co-existence of some synchronization types in general 3D coupled hyperchaotic maps has been investigated. Sufficient conditions have been derived for achieving a new synchronization scheme of co-existence of (PS), (FSHPS) and (GS) between hyperchaotic maps. The new synchronization criterion has been demonstrated using nonlinear controllers, stability theory of linear discrete-time systems and Lyapunov stability theory. An example of application and numerical simulations have been used to show the effectiveness of the derived result.

References

[1] Rössler, O.E. An equation for hyperchaos. *Physics Letters A* **8** (2008) 35–42.

[2] Matsumoto, T., Chua, L.O. and Kobayashi, K. Hyperchaos: laboratory experiment and numerical confirmation. *IEEE Transactions on Circuits and Systems* **33** (11) (1986) 1143–1147.

[3] Stoop, R. and Meier, P.F. Evaluation of Lyapunov exponents and scaling functions from time series. *Journal of the Optical Society of America B* **5** (5) (1988) 1037–1045.

[4] Stoop, R., Peinke, J., Parisi, J., Röhricht, B. and Hübener, R.P. A p-Ge semiconductor experiment showing chaos and hyperchaos. *Physica D* **35** (1989) 425–435.

[5] Eiswirth, M., Kruel, Th.-M., Ertl, G. and Schneider, F. W. Hyperchaos in a chemical reaction. *Chemical Physics Letters* **193** (4) (1992) 305–310.

[6] Baier, G. and Klein, M. Maximum hyperchaos in generalized Hénon maps. *Physics Letters A* **51** (1990) 281–284.

[7] Hitzl, D.L. and Zele, F. An exploration of the Hénon quadratic map. *Physica D* **14** (3) (1985) 305–326.

[8] Stefanski, K. Modelling chaos and hyperchaos with 3D maps. *Chaos, Solitons and Fractals* **9** (1-2) (1998) 83–93.

[9] Wang, X.Y. *Chaos in Complex Nonlinear Systems. Publishing House of Electronics Industry.* Beijing, 2003.

[10] Itoh, M., Yang, T. and Chua, L.O. Conditions for impulsive synchronization of chaotic and hyperchaotic systems. *International Journal of Bifurcation and Chaos* **11** (2001) 551–560.

[11] Grassi, G. and Miller, D. A. Dead-beat full state hybrid projective synchronization for chaotic maps using a scalar synchronizing signal. *Chinese Physics B* **17** (4) (2012) 1824–1830.

[12] Aguilar Bustos, A.Y., Cruz Hernández, C., López Gutiérrez, R.M. and Posadas Castillo, Y.C. Synchronization of different hyperchaotic maps for encryption. *Nonlinear Dynamics and Systems Theory* **8** (3) (2008) 221–236.

[13] Aguilar Bustos, A.Y. and Cruz Hernández, C. Synchronization of discrete-time hyperchaotic systems: An apllication in communications. *Chaos, Solitons and Fractals* **41** (3) (2009) 1301–1310.

[14] Cruz Hernández, C., Lopez Gutierrez, R. M., Aguilar Bustos, A. Y. and Posadas Castillo, Y.C. Communicating encrypted information based on synchronized hyperchaotic maps. *Communications in Nonlinear Sciences and Numerical Simulation* **11** (5) (2010) 337–349.

[15] Liu, W., Wang, Z.M. and Zhang, W.D. Controlled synchronization of discrete-time chaotic systems under communication constraints. *Nonlinear Dynamics* **69** (2012) 223–230.

[16] Inzunza González, E.Y. and Cruz Hernandez, C. Double hyperchaotic encryption for security in biometric systems. *Nonlinear Dynamics and Systems Theory* **13** (1) (2013) 55–68.

[17] Filali, R.L., Benrejeb, M. and Borne, P. On observer-based secure communication design using discrete-time hyperchaotic systems. *Communications in Nonlinear Science and Numerical Simulation* **19** (5) 2014 1424–1432.

[18] Ouannas, A. A synchronization criterion for a class of sinusoidal chaotic maps via linear controller. *International Journal of Contemporary Mathematical Sciences* **9** (14) (2014) 677–683.

[19] Ouannas, A. Nonlinear control method of chaos synchronization for arbitrary 2D quadratic dynamical systems in discrete-time. *International Journal of Mathematical Analysis* **8** (53) (2014) 2611–2617.

[20] Ouannas, A. A new chaos synchronization criterion for discrete dynamical systems. *Applied Mathematical Sciences* **8** (41) (2014) 2025–2034.

[21] Ouannas, A. Chaos synchronization approach for coupled of arbitrary 3-D quadratic dynamical systems in discrete-time. *Far East Journal of Applied Mathematics* **86** (3) (2014) 225–232.

[22] Ouannas, A. Some synchronization criteria for N-dimensional chaotic dynamical systems in discrete-time. *Journal of Advanced Research in Applied Mathematics* **6** (4) (2014) 1-9.

[23] Jin Y., L, Xin, Chen, Y. Function projective synchronization of discrete-time chaotic and hyperchaotic systems using backstepping method. *Communications in Theoritical Physics* **50** (2008) 111–116.

[24] Li, Y., Chen, Y., and Li, B. Adaptive control and function projective synchronization in 2D discrete-time chaotic systems. *Communications in Theoritical Physics* **51** (2009) 270–278.

[25] Li, Y., Chen, Y., and Li, B. Adaptive function projective synchronization of discrete-time chaotic systems. *Chinese Physics Lettres* **26** (4) (2009) 040504–4.

[26] Hong-Li An, Yong Chen. The function cascade synchronization scheme for discrete-time hyperchaotic systems. *Communications in Nonlinear Science and Numerical Simulation* **14** (2009) 1494–1501.

[27] Ma, Z., Liu, Z. and Zhang, G. Generalized synchronization of discrete systems. *Applied Mathematics and Mechanics* **28** 5 (2007) 609–614.

[28] Grassi, G. Generalized synchronization between different chaotic maps via dead-beat control. *Chinese Physics B* **21** (5) (2012) 050505.

[29] Chai, Y., Lü, L., and Zhao, H.Y. Lag synchronization between discrete chaotic systems with diverse structure. *Applied Mathematics and Mechanics* **31** (6) (2010) 733–738.

[30] Yanbo, G., Xiaomei, Z., Guoping, L. and Yufan, Z. Impulsive synchronization of discrete-time chaotic systems under communication constraints. *Communications in Nonlinear Science and Numerical Simulation* **16** (2011) 1580–1588.

[31] Filali, R.L., Hammami, S., Benrejeb, M. and Borne, P. On synchronization, anti-synchronization and hybrid synchronization of 3D discrete generalized Hénon map. *Nonlinear Dynamics and Systems Theory* **12** (1) (2012) 81–95.

[32] Ouannas, A. A new Q-S synchronization scheme for discrete chaotic systems. *Far East Journal of Applied Mathematics* **84** (2) (2013) 89–94.

[33] Grassi, G. Arbitrary full-state hybrid projective synchronization for chaotic discrete-time systems via a scalar signal. *Chinese Physics B* **21** (2012) 060504–5.

[34] Ouannas, A. On full-state hybrid projective synchronization of general discrete chaotic systems. *Journal of Nonlinear Dynamics* **2014** 1–6.

[35] Yan, Z.Y. Q-S (complete or anticipated) synchronization backstepping scheme in a class of discrete-time chaotic (hyperchaotic) systems: A symbolic-numeric computation approach. *Chaos* **16** (2006) 013119–11.

[36] Yan, Z.Y. Q-S synchronization in 3D Hénon-like map and generalized Hénon map via a scalar controller. *Physics Letters A* **342** (2005) 309–317.

Nonlinear Dynamics and Systems Theory, **16** (3) (2016) 322–334

Periodic Solutions for a Class of Superquadratic Damped Vibration Problems

M. Timoumi *

Department of Mathematics, Faculty of Sciences, 5000 Monastir, Tunisia

Received: June 2, 2015; Revised: June 10, 2016

Abstract: In the present paper, the following damped vibration problems

$$\begin{cases} \ddot{u}(t) + q(t)\dot{u}(t) - L(t)u(t) + \nabla W(t, u(t)) = 0, \\ u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0, \end{cases}$$

are studied, where $T > 0$, $q \in C(\mathcal{R}, \mathcal{R})$ is $T-$periodic with $\int_0^T q(t)dt = 0$, $L(t)$ is a continuous $T-$periodic and symmetric $N \times N$ matrix-valued function and $W \in C^1(\mathcal{R} \times \mathcal{R}^N, \mathcal{R})$ is $T-$periodic in the first variable. We use a new kind of superquadratic condition instead of the global Ambrosetti-Rabinowitz superquadratic condidition and we obtain a nontrivial $T-$periodic solution for the above system. The main idea here lies in the application of a variant of generalized weak linking theorem for strongly indefinite problem developed by Schechter and Zou.

Keywords: *periodic solutions; damped vibration problems; superquadradicity; weak linking theorem.*

Mathematics Subject Classification (2010): 34C25, 34B15.

1 Introduction

Consider the following damped vibration problems

$$(\mathcal{DV}) \qquad \begin{cases} \ddot{u}(t) + q(t)\dot{u}(t) - L(t)u(t) + \nabla W(t, u(t)) = 0, \\ u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0, \end{cases}$$

where $T > 0$, $q : \mathcal{R} \longrightarrow \mathcal{R}$ is a continuous $T-$periodic function with $\int_0^T q(t)dt = 0$, $Q(t) = \int_0^t q(s)ds$, $L(t)$ is a continuous $T-$periodic and symmetric $N \times N$ matrix-valued

Corresponding author: mailto:m_timoumi@yahoo.com

function and $W : \mathcal{R}\times\mathcal{R}^N \longrightarrow \mathcal{R}$ is a continuous function, $T-$periodic in the first variable and differentiable in the second variable with continuous derivative $\nabla W(t,x) = \frac{\partial W}{\partial x}(t,x)$. Equation $(\mathcal{DV})$ is a basic mathematical model for the representation of damped nonlinear oscillatory phenomena.

When $q(t) = 0$ for all $t \in \mathcal{R}$, $(\mathcal{DV})$ is just the following second-order Hamiltonian system

$$(\mathcal{HS}) \qquad \ddot{u}(t) - L(t)u(t) + \nabla W(t,u(t)) = 0,$$

which is a classical equation describing many mechanical systems, such as a pendulum. The system $(\mathcal{HS})$ has been thoroughly studied and a lot of existence results have been obtained, for example see [1-6] and references therein.

As far as the case $q(t) \neq 0$ is concerned, to our best knowledge, there are few research about the existence of periodic solutions for $(\mathcal{DV})$, see [7-9]. Recently, the existence of periodic solutions for $(\mathcal{DV})$ has been studied in [9] when W has a superquadratic growth at infinity satisfying the global Ambrosetti-Rabinowitz superquadratic condition: there exist constants $\mu > 2$ and $R > 0$ such that

$$(\mathcal{AR}) \qquad 0 < \mu W(t,x) \leq \nabla W(t,x).x$$

for all $t \in \mathcal{R}$ and $|x| \geq R$, where $x.y$ denotes the Euclidean inner product of $x, y \in \mathcal{R}^N$ and $|.|$ denotes the corresponding Euclidean norm. Our paper is motivated by the following reason: when dealing with superlinear differential equations, one often meets functionals which do not satisfy $(\mathcal{AR})$-condition. Without $(\mathcal{AR})$-condition, we do not know whether a Palais-Smale sequence is bounded. In the present paper, we shall study the existence of periodic solutions for $(\mathcal{DV})$ under a new kind of superquadratic condition given in [10] by Ding and Luan for Schrödinger's equation. Our approach is based on an application of a variant of generalized weak linking theorem for strongly indefinite problem developed by Schechter and Zou [11], where the authors developed the idea of monotonicity tric for strongly indefinite problems; the original idea is due to Struwe [12].

Our main result reads as follows:

Theorem 1.1 *Assume the following assumptions hold:*
$(\mathcal{L})$ *Zero is not an eigenvalue of* $\mathcal{L} = -\frac{d^2}{dt^2} + L(t)$;
(W_1) $\nabla W(t,x) = o(|x|)$ *as* $|x| \longrightarrow 0$, *uniformly on* $t \in [0,T]$;
(W_2) $\frac{W(t,x)}{|x|^2} \longrightarrow +\infty$, *as* $|x| \longrightarrow \infty$, $\forall t \in [0,T]$;
(W_3) $W(t,x) \geq 0$ *and* $\tilde{W}(t,x) = \frac{1}{2}\nabla W(t,x).x - W(t,x) > 0$, $\forall t \in [0,T], x \in \mathcal{R}^N - \{0\}$;
(W_4) *There exist constants* $c, r > 0$ *and* $\sigma > 1$ *such that*

$$\Big(\frac{|\nabla W(t,x)|}{|x|}\Big)^{\sigma} \leq c\tilde{W}(t,x), \ \forall\, |x| \geq r, \ \forall t \in [0,T].$$

Then $(\mathcal{DV})$ *has at least one nontrivial* $T-$ *periodic solution.*

Example 1.1 [10] Let $W(t,x) = a(t)(|x|^{\mu} + (\mu-2)\,|x|^{\mu-\epsilon}\, sin^2(\frac{|x|^{\epsilon}}{\epsilon}))$, where $a : \mathcal{R} \longrightarrow \mathcal{R}^*_+$ is a continuous $T-$ periodic function, $\mu > 2$ and $0 < \epsilon < \mu - 2$. A straigborhood calculation shows that W satisfies the conditions of Theorem 1.1, but does not satisfy the $(\mathcal{AR})$ condition.

2 Abstract Critical Point Theorem [11]

For the existence of periodic solutions for $(\mathcal{DV})$, we appeal to the following abstract critical point theorem. Let E be a Hilbert space with norm $\|.\|$ and have an orthogonal decomposition $E = N \oplus N^{\perp}$, $N \subset E$ is a closed and separable subspace. Since N is separable, there exists a norm $|.|_{\omega}$ that satisfies $|v|_{\omega} \leq \|v\|$ for all $v \in N$ and induces a topology equivalent to the weak topology of N on bounded subset of N. For $u = v + z \in N \oplus N^{\perp}$ with $v \in N$, $z \in N^{\perp}$, we define $|u|_{\omega}^2 = |v|_{\omega}^2 + |z|_{\omega}^2$, then $|u|_{\omega} \leq \|u\|$, $\forall u \in E$. Particularly, if $(u_n = v_n + z_n)$ is $\|.\|$ $-$bounded and $u_n \longrightarrow^{|\cdot|_{\omega}} u$, then $v_n \rightharpoonup v$ weakly in N, $z_n \longrightarrow z$ strongly in $N^{\perp}$, $u_n \rightharpoonup v + z$ weakly in E. Next, let us recall some definitions:
(i) A functional $f : E \longrightarrow \mathcal{R}$ is said to be $|.|_{\omega}$ $-$upper semi-continuous, i.e., $u_n \longrightarrow^{|\cdot|_{\omega}} u$ in E implies $\limsup_{n\longrightarrow\infty} f(u_n) \leq f(u)$.
(ii) Let $f \in C^1(E, \mathcal{R})$. f' is said to be weakly sequentially continuously, i.e., $u_n \longrightarrow u$ in E implies $\lim_{n\longrightarrow\infty} f'(u_n)w = f'(u)w$ for all $w \in E$.
Let $E = E^+ \oplus E^-$, $z_0 \in E^+$ with $\|z_0\| = 1$. Let $N = E^- \oplus \mathcal{R}z_0$ and $E_1^+ = N^{\perp} = (E^- \oplus \mathcal{R}z_0)^{\perp}$. For $R > 0$, let

$$M = \left\{u = u^- + sz_0 / s \in \mathcal{R}^+, u^- \in E^-, \|u\| < R\right\}$$

with $P_0 = s_0 z_0 \in M$, $s_0 > 0$. We define

$$D = \left\{u = sz_0 + z^+ / s \in \mathcal{R}, z^+ \in E_1^+, \left\|sz_0 + z^+\right\| = s_0\right\}.$$

For $f \in C^1(E, \mathcal{R})$, let Γ be the set of $\gamma : [0,1] \times \bar{M} \longrightarrow E$ satisfying

$$\begin{cases} \gamma \ is \ |.|_{\omega} - continuous, \\ \gamma(0,u) = u \ and \ f(\gamma(s,u)) \leq f(u) \ for \ all \ u \in \bar{M}, \\ for \ any \ (s_0, u_0) \in [0,1] \times \bar{M}, \ there \ is \ a \ |.|_{\omega} - neighborhood \\ U_{(s_0,u_0)} \ s.t. \ \left\{U - \gamma(s,u)/(t,u) \in U_{(s_0,u_0)} \cap ([0,1] \cap \bar{M})\right\} \subset E_{fin}, \end{cases}$$

where E_{fin} denotes various finite dimensional subspaces of E, Γ is not empty since $id \in \Gamma$.

Theorem 2.1 *Let (f_λ) be a family of C^1-functionals having the form*

$$f_\lambda(u) = g(u) - \lambda h(u), \ u \in E, \ \lambda \in [1,2].$$

a)
$$h(u) \geq 0, \ \forall u \in E, \ f_1 = f;$$

b)
$$g(u) \longrightarrow +\infty \ or \ h(u) \longrightarrow +\infty \ as \ \|u\| \longrightarrow \infty;$$

c) f_λ *is* $|.|_{\omega}$ $-$ *upper semi* $-$ *continuous,* f_λ' *is weakly sequentially continuous on* E. *Moreover,* f_λ *maps bounded sets into bounded sets;*

d)
$$\sup_{\partial M} f_\lambda < \inf_D f_\lambda, \ \forall \lambda \in [1,2].$$

Then for almost all $\lambda \in [1,2]$, there exists a sequence (u_n) such that

$$\sup_n \|u_n\| < \infty, \ f_\lambda(u_n) \longrightarrow c_\lambda, \ f_\lambda'(u_n) \longrightarrow 0 \ as \ n \longrightarrow \infty,$$

where

$$c_\lambda = \inf_{\gamma\in\Gamma} \sup_{u\in M} f_\lambda(\gamma(1,u)) \in [\inf_D f_\lambda, \sup_{\bar{M}} f].$$

As usual, we say $f \in C^1(E, \mathcal{R})$ satisfies the Palais-Smale condition ((PS) in short) if any sequence $(u_n) \subset E$ for which $(f(u_n))$ is bounded and $f'(u_n) \longrightarrow 0$ as $n \longrightarrow \infty$, possesses a convergent subsequence.

3 Proof of Theorem 1.1

For $1 \leq s < \infty$, let $L^s_Q(0, T; \mathcal{R}^N)$ be the Banach space of measurable functions u defined on $[0, T]$ with values in $\mathcal{R}^N$ satisfying $\int_0^T e^{Q(t)} |u(t)|^s \, dt < \infty$, with the norm

$$\|u\|_{L^s_Q} = (\int_0^T e^{Q(t)} |u(t)|^s \, dt)^{\frac{1}{s}}$$

and $L^\infty_Q(0, T; \mathcal{R}^N)$ denote the Banach space of measurable functions u defined on $[0, T]$ with values in $\mathcal{R}^N$ under the norm

$$\|u\|_{L^\infty_Q} = esssup_{t\in[0,T]} e^{\frac{Q(t)}{2}} |u(t)| \, .$$

The space $L^2_Q(0, T; \mathcal{R}^N)$ provided with the inner product

$$< u, v >_{L^2_Q} = \int_0^T e^{Q(t)} u(t).v(t) dt, \ u, v \in L^2_Q(0, T; \mathcal{R}^N)$$

is a Hilbert space. Let E be the space defined by

$$E = \left\{ u \in L^2_Q(0, T; \mathcal{R}^N) : \dot{u} \in L^2_Q(0, T; \mathcal{R}^N), \ u(0) = u(T) \right\} .$$

The space E provided with the inner product

$$< u, v >_0 = \int_0^T e^{Q(t)} [u(t).v(t) + \dot{u}(t).\dot{v}(t)] dt, \ u, v \in E$$

and the associated norm

$$\|u\|_0 = (\int_0^T e^{Q(t)} [|u(t)|^2 + |\dot{u}(t)|^2] dt)^{\frac{1}{2}}, \ u \in E$$

is a Hilbert space. Define an operator $K : E \longrightarrow E$ by

$$< Ku, v >_0 = \int_0^T e^{Q(t)} (I_{N\times N} - L(t)) u(t).v(t) dt$$

for all $u, v \in E$, where $I_{N\times N}$ is the $N \times N$ identity matrix. Then it is easy to check that K is a bounded self-adjoint linear operator. By the assumption $(\mathcal{L})$ and the classical spectral theory, we can decompose E into the orthogonal sum of invariant subspaces for $I - K$: $E = E^- \oplus E^+$, where E^- (respectively E^+) is the subspace of E on which $I - K$ is negative (respectively positive) definite. Here, I denotes the identity operator. Besides, E^- is finite dimensional since K is compact. Furthermore, we introduce on E the equivalent new inner product

$$< u, v > = < (I - K) u^+, v^+ >_0 - < (I - K) u^-, v^- >_0$$

for $u = u^- + u^+$ and $v = v^- + v^+ \in E$ and the equivalent norm $\|.\| =< .,. >^{\frac{1}{2}}$. It is well known that E is compactly embedded in $L^s_Q(0,T;\mathcal{R}^N)$ for all $s \in [1,\infty]$ and as a consequence for all $s \in [1,\infty]$, there exists a constant $\mu_s > 0$ such that

$$\|u\|_{L^s_Q} \le \mu_s \|u\|, \ \forall u \in E. \tag{3.1}$$

By definition of $< .,. >$, E^- and E^+ we have

$$< (I-K)u, u >_0 = \pm \|u\|^2, \ \forall u \in E^{\pm}.$$

For $(\mathcal{DV})$, we consider the functional $f(u) = \chi(u) - g(u)$ defined on the space E, where χ is the quadratic form

$$\chi(u) = \frac{1}{2}\int_0^T e^{Q(t)}[|\dot{u}(t)|^2 + L(t)u(t).u(t)]dt$$

and

$$g(u) = \int_0^T e^{Q(t)} W(t,u)dt.$$

By the definition of K, the functional f can be rewritten as

$$f(u) = \frac{1}{2} < (I-K)u, u >_0 - g(u) = \frac{1}{2}(\|u^+\|^2 - \|u^-\|^2) - g(u), \ u \in E.$$

By (W_4), for $|x| \ge r$ and $t \in [0,T]$, we have

$$|\nabla W(t,x)|^\sigma \le c\tilde{W}(t,x)|x|^\sigma \le \frac{c}{2}|\nabla W(t,x)||x|^{\sigma+1},$$

thus

$$|\nabla W(t,x)| \le (\frac{c}{2})^{\frac{1}{\sigma-1}}|x|^{p-1},$$

where $p = \frac{2\sigma}{\sigma-1}$. Let $c_1 = \max_{t\in[0,T],|x|\le r}|\nabla W(t,x)|$, then

$$|\nabla W(t,x)| \le c_1 + (\frac{c}{2})^{\frac{1}{\sigma-1}}|x|^{p-1}, \ \forall t \in [0,T], \ x \in \mathcal{R}^N. \tag{3.2}$$

By (W_1), for all $\epsilon > 0$, there exists $r_\epsilon > 0$ such that

$$|\nabla W(t,x)| \le 2\epsilon|x|, \ \forall t \in [0,T], \ |x| \le r_\epsilon. \tag{3.3}$$

For $|x| \ge r_\epsilon$, we have by (3.2), $|\nabla W(t,x)| \le pC_\epsilon|x|^{p-1}$, where $C_\epsilon = \frac{1}{p}(\frac{c_1}{r_\epsilon^{p-1}} + (\frac{c}{2})^{\frac{1}{\sigma-1}})$. So

$$|\nabla W(t,x)| \le 2\epsilon|x| + pC_\epsilon|x|^{p-1}, \ \forall t \in [0,T], \ x \in \mathcal{R}^N. \tag{3.4}$$

Hence, for all $t \in [0,T]$ and $x \in \mathcal{R}^N$

$$W(t,x) = \int_0^1 \nabla W(t,sx).xds \le \epsilon|x|^2 + C_\epsilon|x|^p, \ \forall t \in [0,T], \ x \in \mathcal{R}^N. \tag{3.5}$$

By Proposition B.37 in [13], the inequality (3.4) implies that the functional g is continuously differentiable on E and for all $u, v \in E$

$$g'(u)v = \int_0^T e^{Q(t)}\nabla W(t,u).vdt.$$

It is easy to see that the quadratic form χ is continuously differentiable and for all $u, v \in E$, we have

$$\chi'(u)v = \int_0^T e^{Q(t)}[\dot{u}.\dot{v} + L(t)u.v]dt.$$

Therefore the functional f is continuously differentiable on E and for all $u, v \in E$

$$f'(u)v = \int_0^T e^{Q(t)}[\dot{u}.\dot{v} + L(t)u.v - \nabla W(t,u).v]dt$$

$$=< u^+, v^+ > - < u^-, v^- > - \int_0^T e^{Q(t)}\nabla W(t,u).vdt.$$

Lemma 3.1 *If u is a $T-$periodic solution of the Euler equation $f'(u) = 0$, then u is a solution of problem $(\mathcal{DV})$.*

Proof. Since $f'(u) = 0$, then for all $v \in E$

$$0 = f'(u)v = \int_0^T e^{Q(t)}\dot{u}.\dot{v}dt + \int_0^T e^{Q(t)}[L(t)u - \nabla W(t,u)].vdt.$$

By the fundamental lemma and remarks in ([14], pages 6,9), we know that $e^Q\dot{u}$ has a weak derivative and

$$\frac{d}{dt}(e^Q\dot{u}) = e^Q(L(t)u - \nabla W(t,u))\ a.e.\ t \in [0,T], \tag{3.6}$$

$$e^{Q(t)}\dot{u}(t) = \int_0^t e^{Q^{(s)}}[L(s)u(s) - \nabla W(s,u(s))]ds + c\ a.e.\ t \in [0,T], \tag{3.7}$$

$$\int_0^T e^{Q(s)}[L(s)u(s) - \nabla W(s,u(s))]ds = 0, \tag{3.8}$$

where c is a constant. We identify the equivalence class $e^{Q(t)}\dot{u}(t)$ and its continuous representation $\int_0^t e^{Q(s)}[L(s)u(s) - \nabla W(s,u(s))]ds + c$. Thus by (3.7), (3.8) and the existence of $\dot{u}$, one has

$$\dot{u}(0) - \dot{u}(T) = u(0) - u(T) = 0.$$

In order to apply Theorem 2.1, we consider the family of functionals

$$f_\lambda(u) = \frac{1}{2}\left\|u^+\right\|^2 - \lambda(\frac{1}{2}\left\|u^+\right\|^2 + \int_0^T e^{Q(t)}W(t,u)dt),$$

$\lambda \in [1,2]$. It is easy to see that f_λ satisfies conditions $a), b)$ in Theorem 2.1. To verify condition $c)$, let $u_n \longrightarrow^{|.|_\omega} u$, then $u_n^+ \longrightarrow u^+$ and $u_n^- \rightharpoonup u^-$ in E. Taking a subsequence if necessary, we have $u_n \longrightarrow u$ a.e. on $[0,T]$. By (W_3), Fatou's lemma and the weak lower semi-continuity of the norm, we have

$$\limsup_{n\longrightarrow\infty} f_\lambda(u_n) \le f_\lambda(u),$$

which means that f_λ is $|.|_\omega$ −upper semi-continuous. $f_\lambda^{'}$ is weakly sequentially continuous on E is due to [15].

To continue the discussion, it remains to verify condition d) in Theorem 2.1.

Lemma 3.2 *Under assumptions* $(\mathcal{L})$, $(W_1)-(W_4)$, *we have*
(i) *There exists* $\rho>0$ *independent of* $\lambda\in[1,2]$ *such that* $m=\inf f_\lambda(S_\rho^+)>0$, *where*

$$S_\rho^+=\left\{u\in E^+/\ \|u\|=\rho\right\}.$$

(ii) *For fixed* $z_0\in E^+$ *with* $\|z_0\|=1$ *and any* $\lambda\in[1,2]$, *there is* $R>\rho>0$ *such that* $\sup f_\lambda(\partial M)\le 0$, *where*

$$M=\left\{u=u^-+sz_0/s\in\mathcal{R}^+,\ u^-\in E^-,\ \|u\|<R\right\}.$$

Proof. (i) By (3.5) and (2.1), for any $u\in E^+$, we have

$$\begin{aligned}f_\lambda(u)&\ge\frac{1}{2}\|u\|^2-\lambda\epsilon\|u\|^2_{L^2_Q}-\lambda C_\epsilon\|u\|^p_{L^p_Q}\\&\ge\frac{1}{2}\|u\|^2-2\epsilon\mu_2^2\|u\|^2-2C_\epsilon\mu_p^p\|u\|^p.\end{aligned}$$

Taking $\epsilon=\frac{1}{8\mu_2^2}$, we get

$$f_\lambda(u)\ge\frac{1}{4}\|u\|^2-2C_\epsilon\mu_p^p\|u\|^p.$$

Since $p>2$, there exists a constant $\rho>0$ independent of $\lambda\in[1,2]$ satisfying $\inf f_\lambda(S_\rho^+)>0$.
(ii) Assume by contradiction that there exists $u_n\in E^-\oplus\mathcal{R}^+z_0$ such that $f_\lambda(u_n)>0$ for all n and $\|u_n\|\longrightarrow\infty$ as $n\longrightarrow\infty$. Let $v_n=\frac{u_n}{\|u_n\|}=s_nz_0+v_n^-$, then

$$0<\frac{f_\lambda(u_n)}{\|u_n\|}=\frac{1}{2}(s_n^2-\lambda\left\|v_n^-\right\|^2)-\lambda\int_0^T e^{Q(t)}\frac{W(t,u_n)}{|u_n|^2}|v_n|^2\,dt. \tag{3.9}$$

It follows from (W_3) that

$$\left\|v_n^-\right\|^2\le\lambda\left\|v_n^-\right\|^2<s_n^2=1-\left\|v_n^-\right\|^2,$$

therefore $\left\|v_n^-\right\|^2\le\frac{1}{\sqrt{2}}$ and $1-\frac{1}{\sqrt{2}}\le s_n\le 1$. Taking a subsequence if necessary, we can assume that $s_n\longrightarrow s\neq 0$, $v_n\rightharpoonup v$ and $v_n\longrightarrow v$ almost everywhere on $[0,T]$. Hence $v=sz_0+v^-\neq 0$, and since $|u_n|\longrightarrow\infty$ almost everywhere on $[0,T]$, it follows from (W_2) and Fatou's lemma that

$$\int_0^T e^{Q(t)}\frac{W(t,u_n)}{|u_n|^2}|v_n|^2\,dt\longrightarrow\infty\ as\ n\longrightarrow\infty$$

which contradicts (3.9). The proof is finished.

Under assumptions $(\mathcal{L})$ and $(W_1)-(W_4)$, we obtain by applying Theorem 2.1, that for all $\lambda\in[1,2]$, there exists a sequence (u_n) such that

$$\sup_n\|u_n\|<\infty,\ f_\lambda'(u_n)=0,\ f_\lambda(u_n)\longrightarrow c_\lambda\in[m,\sup_{\bar M}f]. \tag{3.10}$$

Lemma 3.3 *Under assumptions* $(\mathcal{L})$ *and* $(W_1)-(W_4)$, *for all* $\lambda\in[1,2]$, *there exists* $u_\lambda\in E-\{0\}$ *such that*

$$f_\lambda'(u_\lambda)=0,\ f_\lambda(u_\lambda)\le\sup_{\bar M}f. \tag{3.11}$$

Proof. Let (u_n) be the sequence obtained in (3.10), write $u_n = u_n^- + u_n^+$ with $u_n^\pm \in E^\pm$. Since (u_n) is bounded, then (u_n^+) is bounded, so $u_n \rightharpoonup u_\lambda$ and $u_n^+ \rightharpoonup u_\lambda^+$ in E, after going to a subsequence.
We claim that $u_\lambda^+ \neq 0$. If not, then after going to a subsequence, we can assume that $u_n^+ \longrightarrow 0$ in $L^s(\mathcal{R}, \mathcal{R}^N)$ for all $s \in [1, \infty]$ since E is compactly embedded in $L^s(\mathcal{R}, \mathcal{R}^N)$. It follows from inequality (3.4) and Hölder's inequality that

$$0 \leq \int_0^T e^{Q(t)} \left|\nabla W(t,u).u_n^+\right| dt \leq 2\epsilon \int_0^T |u_n| \left|u_n^+\right| dt + \rho C_\epsilon \int_0^T e^{Q(t)} |u_n|^{p-1} \left|u_n^+\right| dt$$

$$\leq 2\epsilon \left\|u_n\right\|_{L_Q^2} \left\|u_n^+\right\|_{L_Q^2} + \left\|u_n\right\|_{L_Q^p}^{p-1} \left\|u_n^+\right\|_{L_Q^p} \longrightarrow 0$$

as $n \longrightarrow \infty$. Hence by (3.10), we get

$$f_\lambda(u_n) \leq \left\|u_n^+\right\|^2 = f_\lambda^{'}(u_n)u_n^+ + \lambda \int_0^T e^{Q(t)} \nabla W(t,u).u_n^+ dt \longrightarrow 0$$

as $n \longrightarrow \infty$, which contradicts the fact that $f_\lambda(u_n) \geq m > 0$. Therefore $u_\lambda^+ \neq 0$ and thus $u_\lambda \neq 0$. Note that f_λ is weakly sequentially continuous on E, thus

$$f_\lambda^{'}(u_\lambda)w = \lim_{n \longrightarrow \infty} f_\lambda^{'}(u_n)w = 0, \ \forall w \in E,$$

which implies that $f_\lambda^{'}(u_\lambda) = 0$. By (3.10), (W_3) and Fatou's lemma, we have

$$\sup_{\bar{M}} f \geq c_\lambda = \lim_{n \longrightarrow \infty} (f_\lambda(u_n) - \frac{1}{2} f_\lambda^{'}(u_n)u_n)$$

$$= \lim_{n \longrightarrow \infty} \lambda \int_0^T e^{Q(t)} (\frac{1}{2} \nabla W(t,u_n).u_n - W(t,u_n))dt$$

$$\geq \lambda \int_0^T e^{Q(t)} (\frac{1}{2} \nabla W(t,u_\lambda).u_\lambda - W(t,u_\lambda))dt = f_\lambda(u_\lambda).$$

Thus we get $f_\lambda(u_\lambda) \leq \sup_{\bar{M}} f$.

Lemma 3.4 *Assume* $(\mathcal{L})$ *and* $(W_1) - (W_4)$ *hold, then there exist a sequence* (λ_n) *of* $[1,2]$ *converging to* 1 *and a bounded sequence* (u_{λ_n}) *on* E *such that*

$$f_{\lambda_n}^{'}(u_{\lambda_n}) = 0, \ f_{\lambda_n}(u_{\lambda_n}) \leq \sup_{\bar{M}} f.$$

Proof. Let $(\lambda_n) \subset [1,2]$ be a sequence such that $\lambda_n \longrightarrow 1$. By Lemma 3.3, there exists a sequence (u_{λ_n}) such that

$$f_{\lambda_n}^{'}(u_{\lambda_n}) = 0, \ f_{\lambda_n}(u_{\lambda_n}) \leq \sup_{\bar{M}} f.$$

It remains to prove the boundedness of (u_{λ_n}). Arguing by contradiction, suppose that $\|u_{\lambda_n}\| \longrightarrow \infty$ as $n \longrightarrow \infty$. Let $v_{\lambda_n} = \frac{u_{\lambda_n}}{\|u_{\lambda_n}\|}$, then $\|v_{\lambda_n}\| = 1$. By going to a subsequence if necessary, we can assume that $v_{\lambda_n} \rightharpoonup v$ in E and $v_{\lambda_n} \longrightarrow v$ almost everywhere on $[0,T]$. Since $f'_{\lambda_n}(u_{\lambda_n}) = 0$, then for any $w \in E$, we have

$$< u^+_{\lambda_n}, w > -\lambda_n < u^-_{\lambda_n}, w >= \lambda_n \int_0^T e^{Q(t)} \nabla W(t, u_{\lambda_n}).wdt. \tag{3.12}$$

Consequently, (v_{λ_n}) satisfies

$$< v^+_{\lambda_n}, w > -\lambda_n < v^-_{\lambda_n}, w >= \lambda_n \int_0^T e^{Q(t)} \frac{\nabla W(t, u_{\lambda_n}).w}{\|u_{\lambda_n}\|} dt. \tag{3.13}$$

Let $w = v^{\pm}_{\lambda_n}$ in (3.13) respectively. Then we have

$$\left\|v^+_{\lambda_n}\right\|^2 = \lambda_n \int_0^T e^{Q(t)} \frac{\nabla W(t, u_{\lambda_n}).v^+_{\lambda_n}}{\|u_{\lambda_n}\|} dt,$$

$$\left\|v^-_{\lambda_n}\right\|^2 = -\int_0^T e^{Q(t)} \frac{\nabla W(t, u_{\lambda_n}).v^-_{\lambda_n}}{\|u_{\lambda_n}\|} dt.$$

Since $1 = \|v_{\lambda_n}\|^2 = \left\|v^+_{\lambda_n}\right\|^2 + \left\|v^-_{\lambda_n}\right\|^2$, we have

$$1 = \int_0^T e^{Q(t)} \frac{\nabla W(t, u_{\lambda_n}).(\lambda_n v^+_{\lambda_n} - v^-_{\lambda_n})}{\|u_{\lambda_n}\|} dt. \tag{3.14}$$

For $s \geq 0$, let

$$\varphi(s) = \inf\left\{\tilde{W}(t,x)/t \in [0,T],\ x \in \mathcal{R}^N,\ |x| \geq s\right\}.$$

By (W_3), we have $\varphi(s) > 0$ for all $s > 0$. By (W_3) and (W_4), we have for $t \in [0,T]$ and $|x| \geq r$

$$\tilde{W}(t,x) \geq \frac{1}{c}\Big(\frac{|\nabla W(t,x)|}{|x|}\Big)^{\sigma} \geq \frac{2^{\sigma}}{c}\Big(\frac{|W(t,x)|}{|x|^2}\Big)^{\sigma},$$

so by (W_2) we have $\varphi(s) \longrightarrow +\infty$ as $s \longrightarrow \infty$. For $0 \leq a < b$, let

$$A_n(a,b) = \{t \in [0,T]/a \leq |u_{\lambda_n}(t)| \leq b\},$$

$$k_{a,b} = \inf\left\{\frac{\tilde{W}(t,x)}{|x|^2}/t \in [0,T],\ x \in \mathcal{R}^N,\ a \leq |x| \leq b\right\}.$$

Since $W(t,x)$ depends periodically on t, then by (W_3), we have $k_{a,b} > 0$ for $a > 0$ and

$$\tilde{W}(t, u_{\lambda_n}(t)) \geq k_{a,b}\,|u_{\lambda_n}(t)|^2 \ \ for\ all\ t \in A_n(a,b).$$

Since $f'_{\lambda_n}(u_{\lambda_n}) = 0$ and $f_{\lambda_n}(u_{\lambda_n}) \leq \sup_{\bar{M}} f$, there exists a constant $c_0 > 0$ such that for all $n \in \mathcal{N}$

$$c_0 \geq f_{\lambda_n}(u_{\lambda_n}) - \frac{1}{2} f'_{\lambda_n}(u_{\lambda_n}) u_{\lambda_n} = \int_0^T e^{Q(t)} \tilde{W}(t, u_{\lambda_n}) dt$$

$$= \int_{A_n(0,a)} e^{Q(t)} \tilde{W}(t, u_{\lambda_n}) dt + \int_{A_n(a,b)} e^{Q(t)} \tilde{W}(t, u_{\lambda_n}) dt + \int_{A_n(b,\infty)} e^{Q(t)} \tilde{W}(t, u_{\lambda_n}) dt$$

$$\geq \int_{A_n(0,a)} e^{Q(t)}\tilde{W}(t,u_{\lambda_n})dt + k_{a,b}\int_{A_n(a,b)} e^{Q(t)}\left|u_{\lambda_n}\right|^2 dt + \varphi(b)\int_{A_n(b,\infty)} e^{Q(t)}dt. \quad (3.15)$$

Combining (3.15) with the fact that $\varphi(s) \longrightarrow \infty$ as $s \longrightarrow \infty$, yields

$$\int_{A_n(b,\infty)} e^{Q(t)}dt \longrightarrow 0 \text{ as } b \longrightarrow \infty, \text{ uniformly in } n. \quad (3.16)$$

Let $\gamma \in]p,\infty[$. By Hölder's inequality and (2.1), we have

$$\int_{A_n(b,\infty)} e^{Q(t)}\left|v_{\lambda_n}\right|^p dt \leq (\int_0^T e^{Q(t)}\left|v_{\lambda_n}\right|^\gamma dt)^{\frac{p}{\gamma}}(\int_{A_n(b,\infty)} e^{Q(t)}dt)^{1-\frac{p}{\gamma}}$$

$$\leq \mu_\gamma^p(\int_{A_n(b,\infty)} e^{Q(t)}dt)^{1-\frac{p}{\gamma}} \longrightarrow 0 \text{ as } b \longrightarrow \infty, \text{ uniformly in } n. \quad (3.17)$$

By (3.15), we have

$$\int_{A_n(a,b)} e^{Q(t)}\left|v_{\lambda_n}\right|^2 dt = \frac{1}{\left\|u_{\lambda_n}\right\|^2}\int_{A_n(a,b)} e^{Q(t)}\left|u_{\lambda_n}\right|^2 dt \leq \frac{c_0}{k_{a,b}\left\|u_{\lambda_n}\right\|^2} \longrightarrow 0 \quad (3.18)$$

as $n \longrightarrow \infty$.

Let $0 < \epsilon < \frac{1}{3}$. By (W_1) there exists $a_\epsilon > 0$ such that $|\nabla W(t,x)| \leq \frac{\epsilon}{2\mu_2^2}|x|$ for all $|x| \leq a_\epsilon$. Consequently, by Hölder's inequality and (2.1)

$$\int_{A_n(0,a_\epsilon)} e^{Q(t)}\frac{\nabla W(t,u_{\lambda_n}).(\lambda_n v_{\lambda_n}^+ - v_{\lambda_n}^-)}{\left\|u_{\lambda_n}\right\|}dt$$

$$\leq \int_{A_n(0,a_\epsilon)} e^{Q(t)}\frac{\left|\nabla W(t,u_{\lambda_n})\right|}{\left|u_{\lambda_n}\right|}\left|v_{\lambda_n}\right|\left|\lambda_n v_{\lambda_n}^+ - v_{\lambda_n}^-\right|dt$$

$$\leq \frac{\epsilon}{2\mu_2^2}\int_{A_n(0,a_\epsilon)} e^{Q(t)}\left|v_{\lambda_n}\right|\left|\lambda_n v_{\lambda_n}^+ - v_{\lambda_n}^-\right|dt$$

$$\leq \frac{\epsilon}{2\mu_2^2}(\int_{A_n(0,a_\epsilon)} e^{Q(t)}\left|v_{\lambda_n}\right|^2 dt)^{\frac{1}{2}}(\int_{A_n(0,a_\epsilon)} e^{Q(t)}\left|\lambda_n v_{\lambda_n}^+ - v_{\lambda_n}^-\right|^2 dt)^{\frac{1}{2}}$$

$$\leq \frac{\epsilon}{2\mu_2^2}\lambda_n\left\|v_{\lambda_n}\right\|_{L_Q^2}^2 \leq \epsilon,\ \forall n \in \mathcal{N}. \quad (3.19)$$

Now, by Hölder's inequality, (W_4) and (3.17), we can take $b_\epsilon \geq r$ large enough so that

$$\int_{A_n(b_\epsilon,\infty)} e^{Q(t)}\frac{\nabla W(t,u_{\lambda_n}).(\lambda_n v_{\lambda_n}^+ - v_{\lambda_n}^-)}{\left\|u_{\lambda_n}\right\|}dt$$

$$\leq \int_{A_n(b_\epsilon,\infty)} e^{Q(t)}\frac{\left|\nabla W(t,u_{\lambda_n})\right|}{\left|u_{\lambda_n}\right|}\left|v_{\lambda_n}\right|\left|\lambda_n v_{\lambda_n}^+ - v_{\lambda_n}^-\right|dt$$

$$\leq (\int_{A_n(b_\epsilon,\infty)} e^{Q(t)}\Big(\frac{\left|\nabla W(t,u_{\lambda_n})\right|}{\left|u_{\lambda_n}\right|}\Big)^\sigma dt^{\frac{1}{\sigma}}(\int_{A_n(b_\epsilon,\infty)} e^{Q(t)}(\left|v_{\lambda_n}\right|\left|\lambda_n v_{\lambda_n}^+ - v_{\lambda_n}^-\right|)^{\sigma'}dt)^{\frac{1}{\sigma'}}$$

$$\leq (\int_{A_n(b_\epsilon,\infty)} e^{Q(t)}c\tilde{W}(t,u_{\lambda_n})dt)^{\frac{1}{\sigma}}(\int_{A_n(b_\epsilon,\infty)} e^{Q(t)}\left|v_{\lambda_n}\right|^{2\sigma'}dt)^{\frac{1}{2\sigma'}}$$

$$\cdot\left(\int_{A_n(b_\epsilon,\infty)} e^{Q(t)} \left|\lambda_n v^+_{\lambda_n} - v^-_{\lambda_n}\right|^{2\sigma'} dt\right)^{\frac{1}{2\sigma'}}$$

$$\leq (cc_0)^{\frac{1}{\sigma}}\left(\int_{A_n(b_\epsilon,\infty)} e^{Q(t)} \left|v_{\lambda_n}\right|^p dt\right)^{\frac{2}{p}} < \epsilon \tag{3.20}$$

for all integer n, where $\frac{1}{\sigma}+\frac{1}{\sigma'}=1$. Since ∇W is continuous, there exists $d=d(\epsilon)$ such that $|\nabla W(t,x)| \leq d\,|x|$ for all $t\in[0,T]$ and $x\in[a_\epsilon,b_\epsilon]$. So, for all $t\in A_n(a_\epsilon,b_\epsilon)$, we have $|\nabla W(t,u_{\lambda_n})| \leq d\,|u_{\lambda_n}|$. Hence by Hölder's inequalitty and (3.18), there exists an integer n_0 such that

$$\int_{A_n(a_\epsilon,b_\epsilon)} e^{Q(t)} \frac{\nabla W(t,u_{\lambda_n}).(\lambda_n v^+_{\lambda_n} - v^-_{\lambda_n})}{\|u_{\lambda_n}\|} dt$$

$$\leq \int_{A_n(a_\epsilon,b_\epsilon)} e^{Q(t)} \frac{|\nabla W(t,u_{\lambda_n})|}{|u_{\lambda_n}|} \left|v_{\lambda_n}\right| \left|\lambda_n v^+_{\lambda_n} - v^-_{\lambda_n}\right| dt$$

$$\leq d\int_{A_n(a_\epsilon,b_\epsilon)} e^{Q(t)} \left|v_{\lambda_n}\right| \left|\lambda_n v^+_{\lambda_n} - v^-_{\lambda_n}\right| dt$$

$$\leq d\left(\int_{A_n(a_\epsilon,b_\epsilon)} e^{Q(t)} \left|v_{\lambda_n}\right|^2 dt\right)^{\frac{1}{2}}\left(\int_{A_n(a_\epsilon,b_\epsilon)} e^{Q(t)} \left|\lambda_n v^+_{\lambda_n} - v^-_{\lambda_n}\right|^2 dt\right)^{\frac{1}{2}}$$

$$\leq 2d\int_{A_n(a_\epsilon,b_\epsilon)} e^{Q(t)} \left|v_{\lambda_n}\right|^2 dt < \epsilon \tag{3.21}$$

for all integer $n\geq n_0$. Therefore, combining $(3.19)-(3.21)$ yields for $n\geq n_0$

$$\int_0^T e^{Q(t)} \frac{\nabla W(t,u_{\lambda_n}).(\lambda_n v^+_{\lambda_n} - v^-_{\lambda_n})}{\|u_{\lambda_n}\|} dt \leq 3\epsilon < 1,$$

which contradicts (3.14). Hence (u_{λ_n}) is bounded.

Lemma 3.5 *Let (u_{λ_n}) be the sequence obtained in Lemma 3.4, then it is a (PS) sequence of f satisfying*

$$\lim_{n\longrightarrow\infty} f'(u_{\lambda_n}) = 0,\ \lim_{n\longrightarrow\infty} f(u_{\lambda_n}) \leq \sup_{\bar{M}} f.$$

Proof. We have

$$\lim_{n\longrightarrow\infty} f(u_{\lambda_n}) = \lim_{n\longrightarrow\infty}[f_{\lambda_n}(u_{\lambda_n}) + (\lambda_n-1)(\frac{1}{2}\left\|u^-_{\lambda_n}\right\|^2 + \int_0^T e^{Q(t)}W(t,u_{\lambda_n})dt)]. \tag{3.22}$$

By (3.5) and (2.1), we have

$$\int_0^T e^{Q(t)}W(t,u_{\lambda_n})dt \leq \epsilon\mu_2^2 \left\|u_{\lambda_n}\right\|^2 + C_\epsilon\mu_p^p \left\|u_{\lambda_n}\right\|^p. \tag{3.23}$$

It follows from (3.22), (3.23) and the boundedness of (u_{λ_n}) that

$$\lim_{n\longrightarrow\infty} f(u_{\lambda_n}) = \lim_{n\longrightarrow\infty} f_{\lambda_n}(u_{\lambda_n}) \leq \sup_{\bar{M}} f.$$

Similarly, for all $w \in E$, we have

$$\lim_{n\longrightarrow\infty} f'(u_{\lambda_n})w = \lim_{n\longrightarrow\infty}[f'_{\lambda_n}(u_{\lambda_n})w+(\lambda_n-1)(\frac{1}{2} < u^-_{\lambda_n}, w > + \int_0^T e^{Q(t)}\nabla W(t,u_{\lambda_n}).wdt)]$$

$$= \lim_{n\longrightarrow\infty} f'_{\lambda_n}(u_{\lambda_n})w = 0,$$

for all $w \in E$. The proof is complete.

Now, let (u_{λ_n}) be the bounded sequence obtained in Lemma 3.4. Taking a subsequence if necessary, we can assume that $u_{\lambda_n} \rightharpoonup u$ in E and $u_{\lambda_n} \longrightarrow u$ in $L^s_Q(0,T)$ for all $s \in [1,\infty]$ since E is compactly embedded in $L^s_Q(0,T)$. By $f'_{\lambda_n}(u_{\lambda_n}) = 0$, (3.4), Hölder's inequality and (2.1), we obtain

$$\left\|u^+_{\lambda_n}\right\|^2 = \lambda_n \int_0^T e^{Q(t)}\nabla W(t,u_{\lambda_n}).u^+_{\lambda_n}dt$$

$$\begin{aligned}
&\leq 4\epsilon \int_0^T e^{Q(t)}\left|u_{\lambda_n}\right|\left|u^+_{\lambda_n}\right|dt + 2pC_\epsilon \int_0^T e^{Q(t)}\left|u_{\lambda_n}\right|^{p-1}\left|u^+_{\lambda_n}\right|dt \\
&\leq 4\epsilon \left\|u_{\lambda_n}\right\|_{L^2_Q}\left\|u^+_{\lambda_n}\right\|_{L^2_Q} + 2pC_\epsilon \left\|u_{\lambda_n}\right\|^{p-1}_{L^p_Q}\left\|u^+_{\lambda_n}\right\|_{L^p_Q} \\
&\leq 4\epsilon \left\|u_{\lambda_n}\right\|_{L^2_Q}\left\|u^+_{\lambda_n}\right\|_{L^2_Q} + 2pC_\epsilon \left\|u_{\lambda_n}\right\|^{p-1}_{L^p_Q}\left\|u^+_{\lambda_n}\right\|_{L^p_Q} \\
&\leq 4\epsilon\mu_2^2\left\|u_{\lambda_n}\right\|^2 + 2pC_\epsilon\mu_p^2\left\|u_{\lambda_n}\right\|^{p-2}_{L^p_Q}\left\|u_{\lambda_n}\right\|^2. \qquad (2.24)
\end{aligned}$$

Similarly, we have

$$\left\|u^-_{\lambda_n}\right\|^2 \leq 4\epsilon\mu_2^2\left\|u_{\lambda_n}\right\|^2 + 2pC_\epsilon\mu_p^2\left\|u_{\lambda_n}\right\|^{p-2}_{L^p_Q}\left\|u_{\lambda_n}\right\|^2. \qquad (2.25)$$

Combining (3.24) and (3.25) yields

$$\left\|u_{\lambda_n}\right\|^2 \leq 8\epsilon\mu_2^2\left\|u_{\lambda_n}\right\|^2 + 4pC_\epsilon\mu_p^2\left\|u_{\lambda_n}\right\|^{p-2}_{L^p_Q}\left\|u_{\lambda_n}\right\|^2. \qquad (2.26)$$

Combining Lemma 3.3 and (3.26) yields

$$1 - 8\epsilon\mu_2^2 \leq 4pC_\epsilon\mu_p^2\left\|u_{\lambda_n}\right\|^{p-2}_{L^p_Q}. \qquad (3.27)$$

Taking $\epsilon = \frac{1}{16\mu_p^2}$, we get $\left\|u_{\lambda_n}\right\|^{p-2}_{L^p_Q} \geq (8p\mu_p^2C_\epsilon)^{-1} > 0$, for all n. Since $u_{\lambda_n} \longrightarrow u$ in $L^p_Q([0,T])$ then $u \neq 0$. The fact that f' is weakly sequentially continuous on E and $u_{\lambda_n} \rightharpoonup u$ in E imply $f'(u) = 0$.
Let $K = \{u \in E/f'(u) = 0\}$ be the critical set of f and $m_0 = \inf\{f(u)/u \in K - \{0\}\}$. For any critical point u of f, assumption (W_3) implies that

$$f(u) = f(u) - \frac{1}{2}f'(u)u = \int_0^T e^{Q(t)}[\frac{1}{2}\nabla W(t,u).u - W(t,u)]dt \geq 0.$$

Therefore, $m_0 \geq 0$. Let $(u_j) \subset K - \{0\}$ be such that $f(u_j) \longrightarrow m_0$. Arguing as in the proof of Lemma 3.4, we can prove that (u_j) is bounded and by going to a subsequence if necessary, we can assume that $u_j \rightharpoonup u$ in E and $u_j \longrightarrow u$ almost everywhere on $[0,T]$, and as above $u \neq 0$. Thus by (W_3) and Fatou's lemma

$$m_0 = \lim_{j\longrightarrow\infty} f(u_j) = \lim_{j\longrightarrow\infty} \int_0^T e^{Q(t)}[\frac{1}{2}\nabla W(t,u_j).u_j - W(t,u_j)]dt$$

$$\geq \int_0^T e^{Q(t)}[\frac{1}{2}\nabla W(t,u).u - W(t,u)]dt = f(u) \geq m_0.$$

So $m_0 = f(u)$ and $m_0 > 0$ because $u \neq 0$.

References

[1] Fei, G. On periodic solutions of superquadratic Hamiltonian systems. *Elect. J. Diff. Eq.* **2012** (8) (2012) 1–12.

[2] He X. and Wu, X. Periodic solutions for a class of nonautonomous second order Hamiltonian systems. *J. Math. Anal. Appl.* **341** (2008) 1354–1364.

[3] Li, S. and Willem, M. Applications of local linking to critical point theory. *J. Math. Anal. Appl.* **189** (1995) 6–32.

[4] Long, Y.M. Multiple solutions of perturbed superquadratic second order Hamiltonian systems. *Trans. Amer. Math. Soc.* **311** (1989) 749–780.

[5] Tang, X.H. and Zhang, Q. New existence of periodic solutions for second-order nonautonomous Hamiltonian systems. *J. Math. Anal. Appl.* **369** (2010) 357–367.

[6] Tang, C.L. and Tao, Z.L. Periodic and subharmonic solutions of second-order Hamiltonian systems. *J. Math. Anal. Appl.* **293** (2004) 435–445.

[7] Chen, J. and Wu, X. Existence theorems of periodic solutions for a class of damped vibration problems. *Applied Mathematics and Computation.* **207** (2009) 230–235.

[8] Chen, S., Teng, K. and Wu, X. On variational methods for a class of damped vibration problems. *Nonlinear Analysis.* **68** (2008) 1432–1441.

[9] Wang, S. and Wu, X. On a class of damped vibration problems with obstacles. *Nonlinear Analysis* **11** (2010) 2973–2988.

[10] Ding, Y. and Luan, S. Multiple solutions for a class of nonlinear Schrödinger equations. *J. Differential Equations.* **207** (2004) 423–457.

[11] Schechter, M. and Zou, W. Weak linking theorems and Schrödinger equations with critical Sobolev exponent. *ESAIM: Contol Optim. Calc. Var.* **9** (2003) 601–619.

[12] Struwe, M. Variational Methods, Applications to Nonlinear Partial Differential Equations and Hamiltonian Systems. *Springer-Verlag*, Berlin,(2000).

[13] Rabinowitz, P.H. Minimax methods in critical point theory with applications to differential equations. *CBMS Regional Conf. Ser. in Math.* No. 65 AMs (1986).

[14] Mawhin, J. and Willem, M. Critical point theory and Hamiltonian systems. *Appl. Math. Sci.* Vol. 74, Springer, New York, (1989).

[15] Willem, M. Minimax Theorems. *Birkhäuser,* 1996.

[16] Khachnaoui, K. Existence of Even Homoclinic Solutions for a Class of Dynamical Systems. *Nonlinear Dynamics and Systems Theory* **15** (3) (2015) 298–312.