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Hybrid Projective Synchronization of Fractional Order Chaotic Systems with Fractional Order in the Interval (1,2)

Ayub Khan and Muzaffar Ahmad Bhat *

Department of Mathematics, Jamia Millia Islamia, New Delhi–110025, India

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Abstract: A hybrid projective synchronization scheme for two identical fractional-order chaotic systems with fractional order $1 < q < 2$ has been discussed in this paper. Based on the stability theory of fractional-order systems, a controller for the synchronization of two identical fractional-order chaotic systems is designed. To illustrate the effectiveness of the proposed scheme, we discuss two examples: (i) the fractional-order Lorenz chaotic system with fractional-order $q = 1.17$, (ii) the fractional-order Lu chaotic system with fractional-order q = 1.13. The numerical simulations exhibit the validity and feasibility of the proposed scheme.

Keywords: *fractional order in the interval (1,2); chaotic systems; hybrid projective synchronization.*

Mathematics Subject Classification (2010): 37B25, 37D45, 37N30, 37N35, 70K99.

1 Introduction

The theory of derivatives of fractional order, i.e., non-integer order, goes back to Leibniz's note in his list to L'Hopital, dated 30 September 1695, in which the meaning of derivative of order one half was discussed. Fractional calculus is a 300 year old mathematical topic. Although it has a long history, the applications of fractional calculus to physics and engineering are just a recent focus of interest [1] and [2]. It was found that many systems in interdisciplinary fields can be elegantly described with the help of fractional derivatives. Many systems are known to display fractional-order dynamics, such as viscoelastic systems [3], dielectric polarization [4], electrode–electrolyte polarization [5], electromagnetic waves [6], quantitative finance [7], and quantum evolution of complex systems [8]. In recent years, chaotic phenomenon has been found in many

* Corresponding author: mailto:mzfar012@gmail.com

fractional-order nonlinear systems, such as the fractional-order Lorenz chaotic system [9], [10], Chuas fractional-order chaotic circuit system [10], the fractional-order modified Duffing chaotic system [11], the fractional-order Rossler chaotic system [12], [13], the fractional-order Chen chaotic system [10]- [12], the fractional-order memristor chaotic system [14], and so on.

In 1999, projective synchronization was first proposed by Mainieri and Rehacek [15], where the drive and response systems were synchronized up to a scaling factor. Its proportional feature can be used to extend binary digital to M-nary digital communication for achieving fast communication [16]. Both complete synchronization and anti-phase synchronization are special cases of projective synchronization. Recently, various kinds of projective synchronization for fractional order chaotic systems without time-delay have been studied, such as hybrid projective synchronization [17], generalized projective synchronization [18], function projective synchronization [19], lag projective synchronization [20] and modified projective synchronization [21].

However, many previous synchronization methods [22]- [25], [26]- [29] for fractional-order chaotic systems only focused on the fractional-order $0 < q < 1$, while in fact, there are many fractional-order systems with fractional-order $1 < q < 2$ in the real world. For example, the time fractional heat conduction equation [30], the fractional telegraph equation [31], the time fractional reaction-diffusion systems [31], the fractional diffusion-wave equation [32], the space-time fractional diffusion equation [33], the super-diffusion systems [34], etc., but the chaos phenomenon was not considered in [30]- [35]. Meanwhile, based on numerical simulation, Ge and Jhuang [31] reported some results on synchronization of the fractional order rotational mechanical system with fractional-order $q = 1.1$. Up to now, there seem to be no results on chaotic synchronization for fractional-order chaotic systems with $1 < q < 2$ through precise theorization. So, how to achieve the chaotic synchronization for fractional-order nonlinear systems with $1 < q < 2$ through precise theorization is an interesting and open question of academic significance as well as practical importance.

Motivated by the above mentioned discussion, in this paper we propose a hybrid projective synchronization approach for a class of fractional-order chaotic systems with fractional-order $1 < q < 2$ through precise theorization. To show the effectiveness of the proposed scheme, the hybrid projective synchronization for a fractional-order Lorenz chaotic system with fractional-order $q = 1.17$ and Lu fractional-order chaotic system with fractional-order $q = 1.13$ are discussed, respectively. The numerical simulations have indicated the validity and feasibility of our scheme.

2 The Review and the Approximation of a Fractional Operator

The differintegral operator, denoted by ${}_aD_t^q$, is a combined differentiation-integration operator commonly used in fractional calculus. This operator is a notation for taking both the fractional derivative and the fractional integral in a single expression and is defined by:

$$
{}_aD_t^q = \begin{cases} \frac{d^q}{dt^q}, & q > 0, \\ 0, & q = 0, \\ \int\limits_a^t (d\tau)^{-q}, & q < 0. \end{cases} \tag{1}
$$

There are some definitions for fractional derivatives [1]. The commonly used definitions are Grunwald-Letnikov, Riemann-Liouville, and Caputo definitions. The Grunwald-Letnikov definition is given by:

$$
\begin{aligned}
{}_aD_t^q f(t) &= \frac{d^q f(t)}{d(t-a)^q} \\
&= \lim[\frac{t-a}{N}]^{-q} \sum_{j=0}^{N-1} (-1)^j \binom{q}{j} f\Big(t - j\Big[\frac{t-a}{N}\Big]\Big).
\end{aligned}
\tag{2}
$$

The Riemann-Liouville definition is the simplest and easiest definition to use. This definition is given by:

$$
\begin{aligned}
{}_aD_t^q f(t) &= \frac{d^q f(t)}{d(t-a)^q} \\
&= \frac{1}{\Gamma(n-q)} \frac{d^n}{dt^n} \int_0^t (t-\tau)^{n-q-1} f(\tau) d(\tau),
\end{aligned}
\tag{3}
$$

where n is the first integer which is not less than q, i.e., $n-1 \leq q < n$ and Γ is the Gamma function defined as:

$$
\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt. \tag{4}
$$

For functions $f(t)$ having n continuous derivatives for $t \geq 0$ where $n-1 \leq q < n$, the Grunwald–Letnikov and the Riemann–Liouville definitions are equivalent. The Laplace transforms of the Riemann–Liouville fractional integral and derivative are given as follows:

$$
L\{{}_0D_t^q f(t)\} = S^q F(s), q \leq 0, \tag{5}
$$

$$
L\{{}_0D_t^q f(t)\} = S^q F(s) - \sum_{k=0}^{n-1} S_0^k D_t^q f(0), n-1 < q \leq n \in N. \tag{6}
$$

Unfortunately, the Riemann-Liouville fractional derivative appears unsuitable to be treated by the Laplace transform technique in that it requires the knowledge of the non-integer order derivatives of the function at $t = 0$. This problem does not exist in the Caputo definition that is sometimes referred to as smooth fractional derivative in literature. This definition of derivative is defined by

$$
{}_0D_t^q = \begin{cases} \frac{1}{\Gamma(m-q)} \int_0^t \frac{f^m(\tau)}{(t-\tau)^{q+1-m}} d\tau, & m-1 < q < m, \\ \frac{d^m f(t)}{dt^m}, & q = m, \end{cases} \tag{7}
$$

where m is the first integer larger than q. It is found that the equations with Riemann–Liouville operators are equivalent to those with Caputo operators by homogeneous initial conditions assumption [1].

3 Stability of Fractional Order Systems

Stability of fractional systems has been thoroughly investigated where necessary and sufficient conditions have been derived in [39]. The stability region of a linear set of fractional order equations, each of order q, such that $1 < q < 2$ is shown in Figure 1. An autonomous system is asymptotically stable iff $\mid arg\lambda \mid > \frac{q\pi}{2}$ is satisfied for all eigenvalues λ of matrix A. Also this system is stable iff $\mid arg\lambda \mid \geq \frac{q\pi}{2}$ is satisfied for all eigenvalues of a matrix A and those critical eigenvalues which satisfy $\mid arg\lambda \mid > \frac{q\pi}{2}$, and have geometric multiplicity one [38].

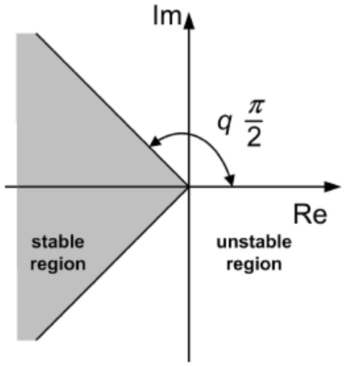

Figure 1: Stability of fractional order systems such that $1 < q < 2$.

4 System Description Problem Formulation for Hps Between Fractional Order Systems

In this section we put a glimpse of methodology and problem formulation for hybrid projective synchronization between fractional order chaotic systems via tracking control. The fractional order chaotic drive and response systems can be described as follows:

$$\frac{d^q(x)}{dt^q} = f(x) \tag{8}$$

and

$$\frac{d^q(y)}{dt^q} = g(y) + \phi(x,y), \tag{9}$$

where $x \in R^n, y \in R^m$ are state vectors of the drive system (8), and the response system (9) and $f, g : R^n \to R^n$ are continuous vector functions, respectively, $\phi(x,y)$ is a vector controller to be designed.

Definition 4.1 For the drive system (8) and the response system (9), the Hybrid Projective Synchronization (HPS) is achieved if there exists an $n \times n$ invertible matrix A such that

$$\lim_{t \to \infty} \|e(t)\| = \|Ay - x\| = 0$$

where $\| \bullet \|$ is an Euclidean norm.

Remark 4.1 If $A = \sigma I; \sigma \in R$, the HPS problem will reduce to Projective Synchronization(PS) where I is an $n \times n$ matrix with proper dimensions. In particular, if $\sigma = 1$ and $\sigma = -1$ the problem is further simplified to complete synchronization and anti-phase synchronization, respectively. If $A = diag(a_1, a_2, \ldots, a_n)$, where $a_1, a_2, \ldots, a_n$ are not all zeros and $a_i \neq a_j$ for some i and j, then the modified projective synchronization will appear. Therefore CS, AS, PS, and MPS are the special cases of hybrid projective synchronization.

In order to obtain the HPS for the fractional order chaotic system we consider that for fractional order chaotic system (8) as drive system, and construct a response system as follows

$$\frac{d^q}{dt^q}(y) = A^{-1}\Big[f(Ay) + \phi(x,y)\Big], \tag{10}$$

where A^{-1} is the inverse of the invertible matrix A, $y \in R^n$ are state vectors of the response system (10) and $\phi(x,y)$ is a controller which may be designed.

Define the HPS error between the response system (10) and the drive system (8) as

$$e = Ay - x,$$

$$e = (e_1, e_2, ..., e_n),$$

$$e_i = \Big(\sum_{j=i}^{n} a_{ij} y_j\Big) - x_i \qquad (i, j = 1, 2, ..., n).$$

Let

$$f(Ay) - f(x) = E(x,e). \tag{11}$$

Now we assume that the error vector e can be subdivided into two vectors $e_\alpha = (e_{n1}, e_{n2}, ..., e_{nk})$ and $e_\beta = (e_{n(k+1)}, e_{n(k+2)}, ..., e_{nl})$, so that $E(x,y)$ has the following form:

$$E(x,e) = \begin{pmatrix} B_\alpha e_\alpha + h_1(x, e_\alpha, e_\beta) \\ B_\beta e_\beta + h_{21}(x, e_\alpha, e_\beta) + h_{22}(x, e_\alpha, e_\beta) \end{pmatrix}, \tag{12}$$

where $h_1(x, e_\alpha, e_\beta) \in R^m, h_{21}(x, e_\alpha, e_\beta) \in R^{n-m}, h_{22}(x, e_\alpha, e_\beta) \in R^{n-m}$ and $\lim\limits_{e_\alpha \to 0} h_{21}(x, e_\alpha, e_\beta) = 0$, respectively. $B_\alpha \in R^{n\times m}$ and $B_\beta \in R^{(n-m)\times(n-m)}$ are constant matrices.

Rewrite the controller $\phi(x,y)$ as follows

$$\phi(x,y) = \mu(x,e) = \begin{pmatrix} \mu_\alpha(x,e) \\ \mu_\beta(x,e) \end{pmatrix}, \tag{13}$$

where $\mu_\alpha(x,e) \in R^m$ and $\mu_\beta(x,e) \in R^{n-m}$, respectively.

Now the following theorem is based on the stability of fractional order chaotic systems, which gives the final destination to problem formulation.

Theorem 4.1 *If the controller $\phi(x,y)$ in the response system* (10) *can be chosen as*

$$\phi(x,y) = \mu(x,e) = \begin{pmatrix} \mu_\alpha(x,e) \\ \mu_\beta(x,e) \end{pmatrix} = \begin{pmatrix} Q_\alpha e_\alpha - h_1(x, e_\alpha, e_\beta) \\ Q_\beta e_\beta - h_{22}(x, e_\alpha, e_\beta) \end{pmatrix},$$

where $Q_\alpha \in R^{m\times m}$ and $Q_\beta \in R^{(n-m)\times(n-m)}$ are suitable constant matrices respectively. If all the eigenvalues of $B_\alpha + Q_\alpha$ satisfy $\mid arg\lambda_i \mid > \frac{q\pi}{2}, (i = 1, 2, ..., m)$ and all the eigenvalues of $B_\beta + Q_\beta$ satisfy $\mid arg\lambda_i \mid > \frac{q\pi}{2}, (i = 1, 2, ..., n-m)$, then HPS between drive and response system can be achieved.

Remark 4.2 In order to use the stability theory of linear fractional-order systems [37], the controller $\phi(x,y)$ or $\mu(x,y)$ is chosen as $\begin{pmatrix} Q_\alpha e_\alpha - h_1(x, e_\alpha, e_\beta) \\ Q_\beta e_\beta - h_{22}(x, e_\alpha, e_\beta) \end{pmatrix}$. Moreover, the nonlinear term $h_{21}(x, e_\alpha, e_\beta) \in R^{n-m}$ in the error dynamic system (12) or response system (10) is preserved.

5 Illustrative Examples

In this section, to show the effectiveness of the hybrid projective synchronization approach, we apply the hybrid projective synchronization scheme for the fractional-order Lorenz chaotic system with fractional-order $1 < q < 2$ and the fractional order Lu system with fractional order $1 < q < 2$ respectively.

5.1 HPS for fractional order Lorenz chaotic system with $1 < q < 2$.

The fractional order Lorenz system is a system of three ordinary differential equations displaying a chaotic behaviour for certain values of parameters (σ, β, γ), and fractional orders $(q_1 = q_2 = q_3 = q)$. The fractional order Lorenz system with parameter values $(\sigma, \beta, \gamma) = (10, 28, 8/3)$ and fractional order q with $1 < q < 2$ is given by

$$\begin{aligned} \frac{d^q x_1}{dt^q} &= 10(x_2 - x_1), \\ \frac{d^q x_2}{dt^q} &= \frac{8}{3}x_1 - x_2 - x_1x_3, \\ \frac{d^q x_3}{dt^q} &= x_1x_2 - 28x_3. \end{aligned} \tag{14}$$

The chaotic attractor of fractional order Lorenz system for different values of q, $1 < q < 2$ is depicted in Figures 2-7.

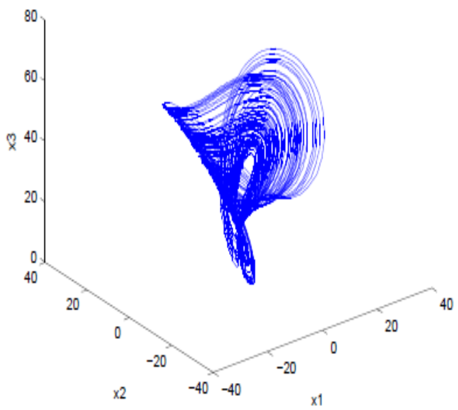

Figure 2: 3D chaotic attractor of the Lorenz system with $q_1 = q_2 = q_3 = 1.15$.

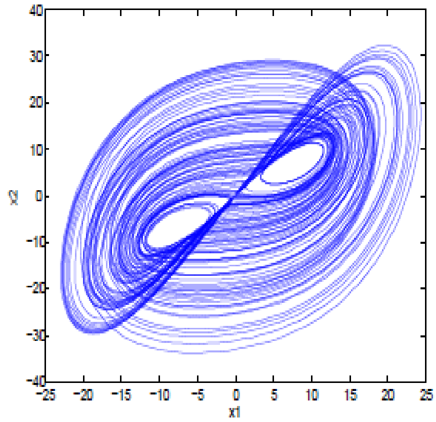

Figure 3: 2D projection of the Lorenz system with $q_1 = q_2 = q_3 = 1.15$.

According to the HPS scheme presented in the above section, the response system is described by

$$\begin{pmatrix} \frac{d^q y_1}{dt^q} \\ \frac{d^q y_2}{dt^q} \\ \frac{d^q y_3}{dt^q} \end{pmatrix} = A^{-1} \begin{pmatrix} 10(\sum_{j=1}^{3} a_{2j}y_j - \sum_{j=1}^{3} a_{1j}y_j) \\ 8/3(\sum_{j=1}^{3} a_{1j}y_j) - \sum_{j=1}^{3} a_{2j}y_j - \sum_{j=1}^{3} a_{1j}y_j \sum_{j=1}^{3} a_{3j}y_j \\ \sum_{j=1}^{3} a_{1j}y_j \sum_{j=1}^{3} a_{2j}y_j - 28 \sum_{j=1}^{3} a_{3j}y_j \end{pmatrix} + A^{-1}\phi(x,y), \tag{15}$$

where $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ is a reversible matrix and A^{-1} is its reverse matrix.

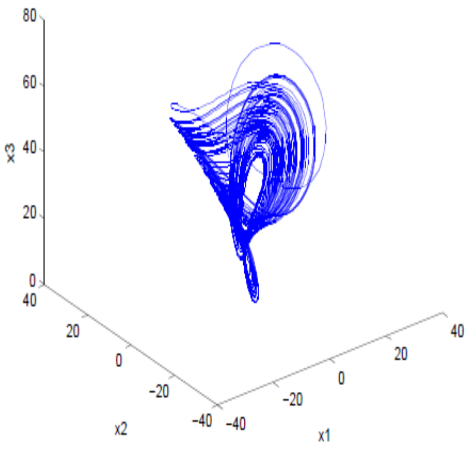

Figure 4: 3D chaotic attractor of the Lorenz system with $q_1 = q_2 = q_3 = 1.16$.

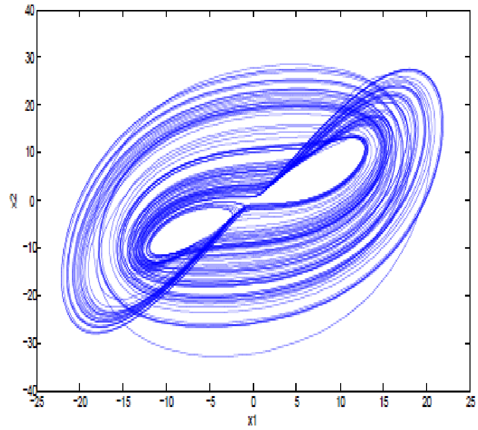

Figure 5: 2D projection of the Lorenz system with $q_1 = q_2 = q_3 = 1.16$.

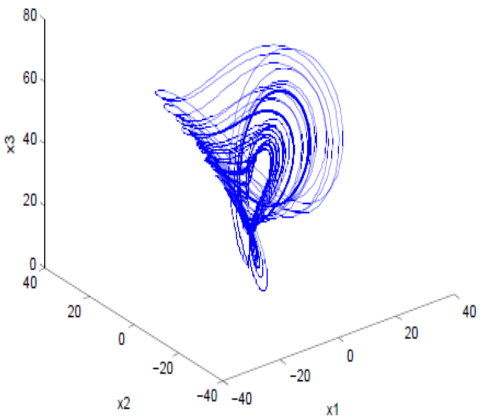

Figure 6: 3D chaotic attractor of the Lorenz system with $q_1 = q_2 = q_3 = 1.17$.

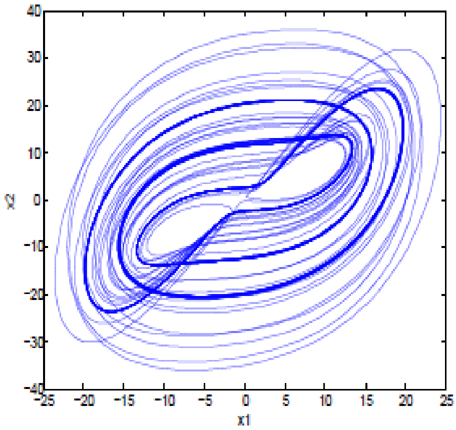

Figure 7: 2D projection of the Lorenz system with $q_1 = q_2 = q_3 = 1.17$.

According to definition of HPS error dynamics, we have

$$\begin{aligned}\frac{d^q e}{dt^q} &= A\frac{d^q y}{dt^q} - \frac{d^q x}{dt^q} \\ &= f(Ay) - f(x) + \phi(x,y). \qquad (16)\end{aligned}$$

Let

$$f(Ay) - f(x) = E(x,e). \qquad (17)$$

Therefore, from (16) we have

$$\frac{d^q e}{dt^q} = E(x,e) + \phi(x,y). \qquad (18)$$

Our goal is to find $E(x,e)$ and design a controller to achieve HPS.

Now from equation (17) we have

$$E(x,e) = \begin{pmatrix} 10(\sum_{j=1}^{3} a_{2j}y_j - \sum_{j=1}^{3} a_{1j}y_j) \\ 8/3(\sum_{j=1}^{3} a_{1j}y_j) - \sum_{j=1}^{3} a_{2j}y_j - \sum_{j=1}^{3} a_{1j}y_j \sum_{j=1}^{3} a_{3j}y_j \\ \sum_{j=1}^{3} a_{1j}y_j \sum_{j=1}^{3} a_{2j}y_j - 28 \sum_{j=1}^{3} a_{3j}y_j \end{pmatrix} - \begin{pmatrix} 10(x_2 - x_1) \\ \frac{8}{3}x_1 - x_2 - x_1x_3 \\ x_1x_2 - 28x_3 \end{pmatrix} \quad (19)$$

which gives

$$E(x,e) = \begin{pmatrix} 10e_2 - 10e_1 \\ \frac{8}{3}e_1 - e_2 - e_1x_3 - e_3x_1 - e_1e_3 \\ e_1e_2 + x_1e_2 + x_2e_1 - 28e_3 \end{pmatrix}. \quad (20)$$

We choose the following:

$e_\alpha = e_1; e_\beta = (e_2, e_3)^T; B_\alpha = -10; h_1(x, e_\alpha, e_\beta) = 10e_2, B_\beta = \begin{pmatrix} -1 & 0 \\ 0 & -28 \end{pmatrix}; h_{21}(x, e_\alpha, e_\beta) = \begin{pmatrix} \frac{8}{3}e_1 - e_1x_3 - -e_1e_3 \\ e_1e_2 + x_2e_1 \end{pmatrix}$; and $h_{21}(x, e_\alpha, e_\beta) = \begin{pmatrix} -e_3x_1 \\ e_2x_1 \end{pmatrix}$. Clearly $\lim_{e_\alpha \to 0} h_{21}(x, e_\alpha, e_\beta) = 0$.

According to Theorem 4.1, the controller $\phi(x,y)$ is now defined as

$$\phi(x,y) = \begin{pmatrix} \mu_\alpha(x,e) \\ \mu_\beta(x,e) \end{pmatrix} = \begin{pmatrix} Q_\alpha e_\alpha - h_1(x, e_\alpha, e_\beta) \\ Q_\beta e_\beta - h_{22}(x, e_\alpha, e_\beta) \end{pmatrix}. \quad (21)$$

So, from equations (20) and (21) error dynamics can be rewritten as:

$$\begin{aligned} \frac{d^q e_\alpha}{dt^q} &= (B_\alpha + Q_\alpha)e_\alpha, \\ \frac{d^q e_\beta}{dt^q} &= (B_\beta + Q_\beta)e_\beta + h_{21}(x, e_\alpha, e_\beta). \end{aligned} \quad (22)$$

Therefore, choose suitable matrices $Q_\alpha \in R^1$ and $Q_\beta \in R^{2\times 2}$ such that all the eigenvalues of $(B_\alpha + Q_\alpha)$ satisfy $|\, arg\lambda_i \,| > \frac{q\pi}{2} (i = 1)$ and all the eigenvalues of $(B_\beta + Q_\beta)$ satisfy $|\, arg\lambda_i \,| > \frac{q\pi}{2}\ (i = 1, 2)$.

Since equation (22) is asymptotically stable with equilibrium points $e_\alpha = 0, e_\beta = 0$. Obviously $\lim_{e_\alpha \to 0} h_{21}(x, e_\alpha, e_\beta) = 0$. This implies that the HPS between drive system and response system can be achieved.

5.2 Numerical simulations

Parameters of the fractional order Lorenz system are $(\sigma, \beta, \gamma) = (10, 8/3, 28)$ and fractional order is taken to be $q = 1.17$, for which the system displays a chaotic behaviour. In equation (22), we choose $Q_\alpha = 8$ and $Q_\beta = \begin{pmatrix} -1 & 0 \\ 0 & 24 \end{pmatrix}$, which gives that the stability condition of the above Theorem 4.1 is satisfied, as eigenvalue of $(B_\alpha + Q_\alpha)$ is -2 and eigenvalues of $(B_\beta + Q_\beta)$ are -2 and -4 and for all eigenvalues condition of Theorem 4.1 has been satisfied as $|\, arg\lambda_i \,| \geq \frac{q\pi}{2}$, where $q = 1.17$. The initial conditions for the master and slave systems are $(x_1(0), x_2(0), x_3(0)) = (7, 9, 6)$ and $(y_1(0), y_2(0), y_3(0)) = (6, 8, 5)$, respectively and $A = \begin{pmatrix} 1 & 0 & 0.83 \\ 1 & 0 & -0.03 \\ 1 & -1 & 0.16 \end{pmatrix}$. Then for $(e_1(0), e_2(0), e_3(0)) = (0.50, -0.80, -1)$ and $T_{sim} = 20$, diagram of convergence of errors (Figures 9-11) is the witness for achieving hybrid projective synchronization between the drive and response systems.

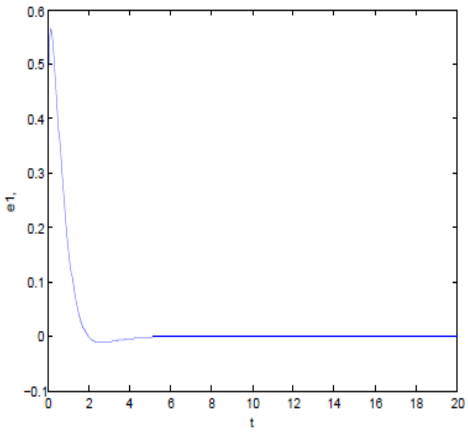

Figure 8: The synchronization error signal $e_1(t)$.

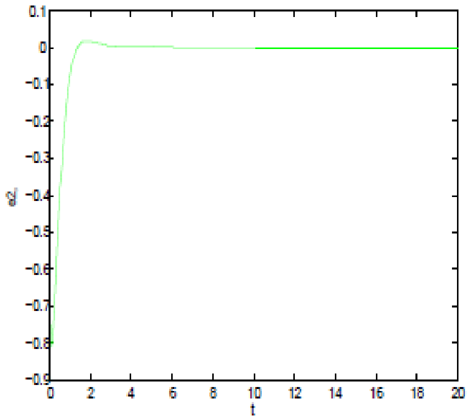

Figure 9: The synchronization error signal $e_2(t)$.

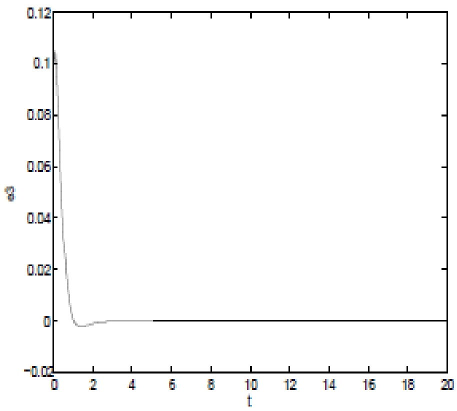

Figure 10: The synchronization error signal $e_3(t)$.

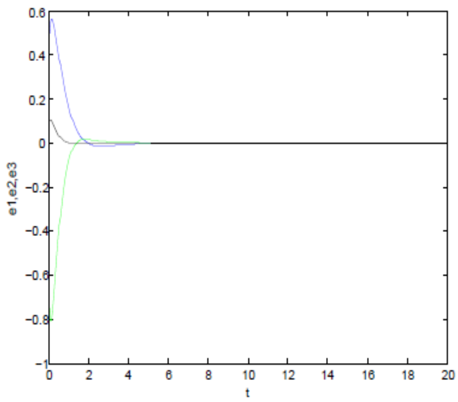

Figure 11: Error convergence diagram for HPS.

5.3 HPS for fractional order Lu chaotic system with with fractional order $1 < q < 2$

The fractional order Lu system is a system of three fractional order differential equation exhibiting chaotic behaviour for certain values of parameters. The equation of the system is:

$$\begin{aligned} \frac{d^q x_1}{dt^q} &= 36(x_2 - x_1), \\ \frac{d^q x_2}{dt^q} &= 20x_2 - x_1 x_3, \\ \frac{d^q x_3}{dt^q} &= x_1 x_2 - 3x_3. \end{aligned} \tag{23}$$

The chaotic attractor of fractional order Lu system for different values of q, $1 < q < 2$ is depicted in Figures 12-17.

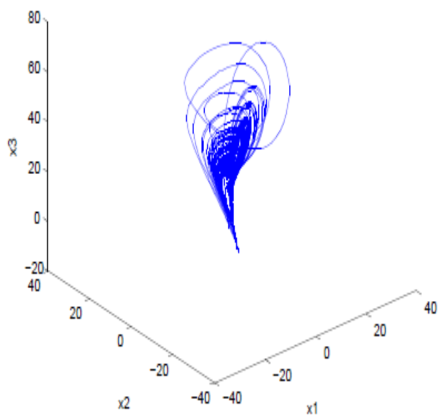

Figure 12: 3D chaotic attractor of the Lu system with $q_1 = q_2 = q_3 = 1.11$.

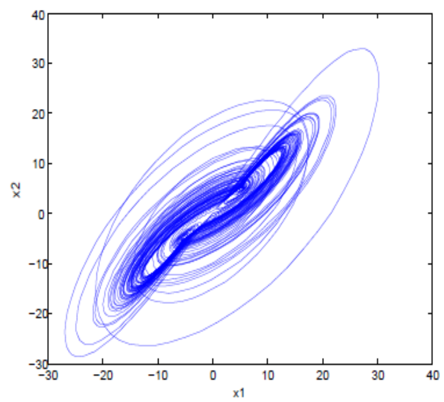

Figure 13: 2D projection of the Lu system with $q_1 = q_2 = q_3 = 1.11$.

According to the HPS scheme presented in the above section, the response system is described by

$$\begin{pmatrix} \frac{d^q y_1}{dt^q} \\ \frac{d^q y_2}{dt^q} \\ \frac{d^q y_3}{dt^q} \end{pmatrix} = A^{-1} \begin{pmatrix} 36(\sum_{j=1}^{3} a_{2j} y_j - \sum_{j=1}^{3} a_{1j} y_j) \\ 20(\sum_{j=1}^{3} a_{2j} y_j) - \sum_{j=1}^{3} a_{1j} y_j \sum_{j=1}^{3} a_{3j} y_j \\ \sum_{j=1}^{3} a_{1j} y_j \sum_{j=1}^{3} a_{2j} y_j - 3 \sum_{j=1}^{3} a_{3j} y_j \end{pmatrix} + A^{-1} \phi(x, y), \tag{24}$$

where $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ is a reversible matrix and A^{-1} is its reverse matrix.

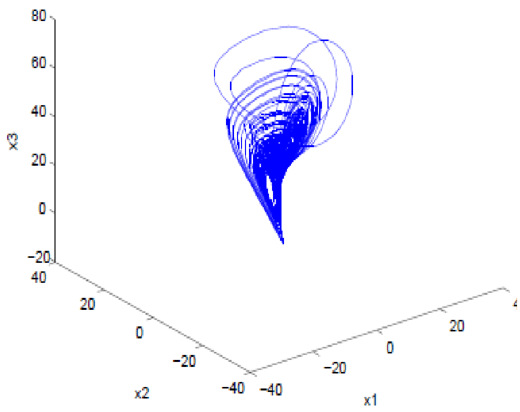

Figure 14: 3D chaotic attractor of the Lu system with $q_1 = q_2 = q_3 = 1.12$.

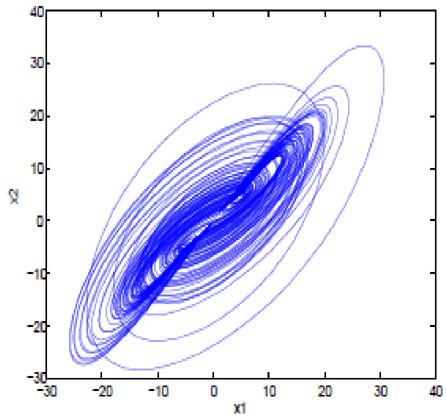

Figure 15: 2D projection of the Lu system with $q_1 = q_2 = q_3 = 1.12$.

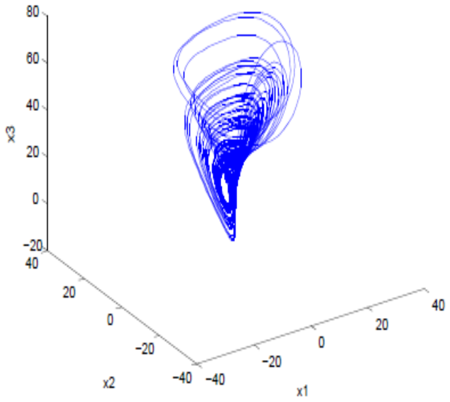

Figure 16: 3D chaotic attractor of the Lu system with $q_1 = q_2 = q_3 = 1.13$.

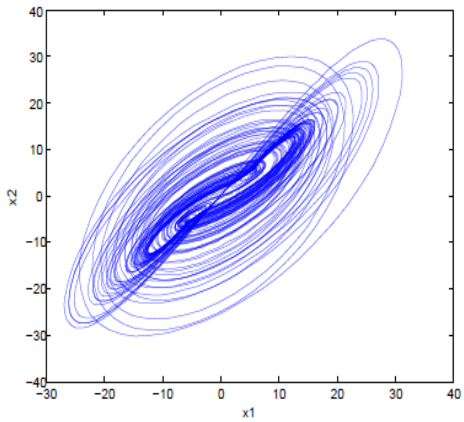

Figure 17: 2D projection of the Lu system with $q_1 = q_2 = q_3 = 1.13$.

Now, according to definition of HPS error dynamics, we have

$$\begin{aligned} \frac{d^q e}{dt^q} &= A\frac{d^q y}{dt^q} - \frac{d^q x}{dt^q} \\ &= f(Ay) - f(x) + \phi(x,y). \end{aligned} \tag{25}$$

Let

$$f(Ay) - f(x) = E(x,e). \tag{26}$$

Therefore, (25) implies that

$$\frac{d^q e}{dt^q} = E(x,e) + \phi(x,y). \tag{27}$$

Our goal is to find $E(x,e)$ and design a controller to achieve HPS. Equation (10), gives

$$E(x,e) = \begin{pmatrix} 36(\sum_{j=1}^{3} a_{2j}y_j - \sum_{j=1}^{3} a_{1j}y_j) \\ 20(\sum_{j=1}^{3} a_{2j}y_j) - \sum_{j=1}^{3} a_{1j}y_j \sum_{j=1}^{3} a_{3j}y_j \\ \sum_{j=1}^{3} a_{1j}y_j \sum_{j=1}^{3} a_{2j}y_j - 3\sum_{j=1}^{3} a_{3j}y_j \end{pmatrix} - \begin{pmatrix} 36(x_2 - x_1) \\ 20x_1 - x_2 - x_1x_3 \\ x_1x_2 - 3x_3 \end{pmatrix} \tag{28}$$

which gives

$$E(x,e) = \begin{pmatrix} 36e_2 - 36e_1 \\ 20e_2 - e_1x_3 - e_3x_1 - e_1e_3 \\ e_1e_2 + x_1e_2 + x_2e_1 - 3e_3 \end{pmatrix}. \tag{29}$$

We choose

$$e_\alpha = e_1,\ e_\beta = (e_2, e_3)^T;\ B_\alpha = -36;\ h_1(x, e_\alpha,\ e_\beta) = 36e_2,$$

$$B_\beta = \begin{pmatrix} 20 & 0 \\ 0 & -3 \end{pmatrix};\ h_{21}(x, e_\alpha,\ e_\beta) = \begin{pmatrix} -e_1x_3 - e_1e_3 \\ e_1e_2 + x_2e_1 \end{pmatrix};\ h_{22}(x, e_\alpha, e_\beta) = \begin{pmatrix} -e_3x_1 \\ e_2x_1 \end{pmatrix}.$$

Clearly, $\lim\limits_{e_\alpha \to 0} h_{21}(x, e_\alpha, e_\beta) = 0$.

According to Theorem 4.1, the controller $\phi(x,y)$ is now defined as

$$\phi(x,y) = \begin{pmatrix} \mu_\alpha(x,e) \\ \mu_\beta(x,e) \end{pmatrix} = \begin{pmatrix} Q_\alpha e_\alpha - h_1(x, e_\alpha, e_\beta) \\ Q_\beta e_\beta - h_{22}(x, e_\alpha, e_\beta) \end{pmatrix}. \tag{30}$$

So from equation (29) and (30) error dynamical system can be rewritten as:

$$\begin{aligned} \frac{d^q e_\alpha}{dt^q} &= (B_\alpha + Q_\alpha)e_\alpha, \\ \frac{d^q e_\beta}{dt^q} &= (B_\beta + Q_\beta)e_\beta + h_{21}(x, e_\alpha, e_\beta). \end{aligned} \tag{31}$$

Therefore, choose suitable matrices $Q_\alpha \in R^1$ and $Q_\beta \in R^{2\times 2}$ such that all the eigenvalues of $(B_\alpha + Q_\alpha)$ satisfy $\mid arg\lambda_i \mid > \frac{q\pi}{2}$ $(i = 1)$ and all the eigenvalues of $(B_\beta + Q_\beta)$ satisfy $\mid arg\lambda_i \mid > \frac{q\pi}{2} (i = 1, 2)$

The equilibrium points $e_\alpha = 0, e_\beta = 0$ of system (31) is asymptotically stable.

Obviously, $\lim\limits_{e_\alpha \to 0} h_{21}(x, e_\alpha, e_\beta) = 0$. This implies that the HPS between drive system and response system can be achieved.

5.4 Numerical Simulations

Parameters of the fractional order Lu system are $(a, b, c) = (36, 3, 20)$, and fractional order is taken $q = 1.13$ for which the system displays a chaotic behaviour. In equation (31), we choose

$$Q_\alpha = 34, \quad Q_\beta = \begin{pmatrix} -23 & 0 \\ 0 & -2 \end{pmatrix}.$$

This implies that the stability conditions of Theorem 4.1 are satisfied, as eigenvalue of $(B_\alpha + Q_\alpha)$ is -2 and eigenvalues of $(B_\beta + Q_\beta)$ are -3 and -5, and for all eigenvalues condition of Theorem 4.1 are satisfied as $\mid arg\lambda_i \mid \geq \frac{q\pi}{2}$, where $q = 1.13$. The initial conditions for

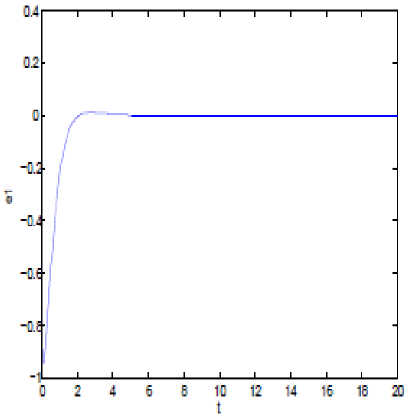

Figure 18: The synchronization error signal $e_1(t)$.

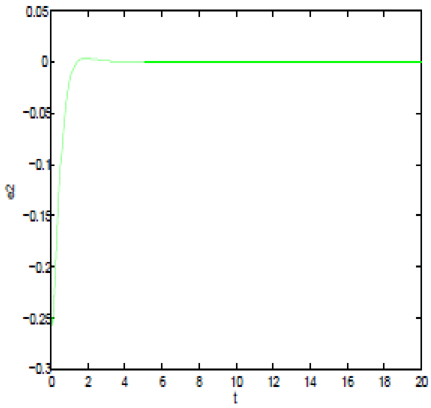

Figure 19: The synchronization error signal $e_2(t)$.

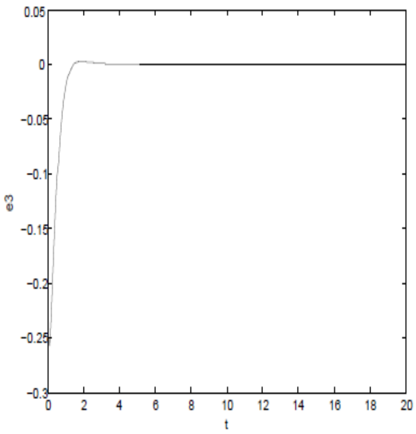

Figure 20: The synchronization error signal $e_3(t)$.

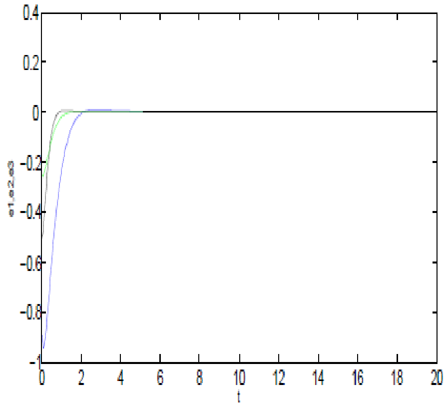

Figure 21: Error convergence diagram for HPS.

the master and slave systems are $(x_1(0), x_2(0), x_3(0)) = (2,3,6)$, $(y_1(0), y_2(0), y_3(0)) = (4,5,8)$, respectively, and

$$A = \begin{pmatrix} 1 & 0 & -0.18 \\ 0 & -0.42 & 1 \\ 1 & 1.83 & 0 \end{pmatrix}.$$

Then for $(e_1(0), e_2(0), e_3(0)) = (-0.89, -0.25, -0.50)$ and $T_{sim} = 20$, diagram of convergence of errors (Figures 18-21) is the witness of achieving hybrid projective synchronization between the drive and response systems.

6 Conclusion

In this paper, we have investigated a new synchronization scheme to achieve hybrid projective synchronization for two identical fractional order chaotic systems with fractional order q such that $1 < q < 2$ via tracking control method and stability of fractional order system. Hybrid projective synchronization (HPS) is a more general definition of projective synchronization, in which the drive system and response system could be synchronized up to a vector function factor. HPS is different from the PS and more beneficial to enhance security of communication than any other synchronization because it is obvious that the unpredictability of the vector function factor in HPS is more than that of the same scaling factor in PS. The numerical simulations exhibit the validity and feasibility of the proposed scheme. Numerical and computational results are in excellent agreement.

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