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# Existence and Uniqueness Results by Progressive Contractions for Integro-Differential Equations

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**Abstract:** In this brief note we present a simple proof of global existence and uniqueness of a solution of an integro-differential equation

$$x'(t) = g(t, x(t)) + \int_0^t A(t-s)f(s, x(s))ds,$$

where $f$ and $g$ satisfy a Lipschitz condition with constant $K = K(t)$ where $K(t)$ is allowed to tend to infinity with $t$. The proof employs the idea of progressive contractions. It is a general fixed point theorem for differential equations.

**Keywords:** *fixed points; existence; uniqueness; progressive contractions; integro-differential equations.*

**Mathematics Subject Classification (2010):** 45J05, 37C25, 47H09.

## 1 Introduction

This is the third in a series of very short notes which we are constructing to illustrate the power, flexibility, and simplicity of a technique which we call *progressive contractions* to obtain a unique global solution of various kinds of differential and integral equations. We have applied the method to integral equations [4], fractional differential equations [6] of the type considered in [2], and integral equations of the Krasnoselskii type featuring a sum of two operators [5]. Each of the problems is of an essentially different type and the title of each note is chosen to allow interested readers to detect which subject is being treated.

In most of the existing literature investigators prove existence and uniqueness of solutions of differential equations by writing them as integral equations and applying

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some type of fixed point theorem which can be tedious and challenging, often patching together solutions on short intervals after making complicated translations. Here, we make three simple short steps, two of which are actually the same. Moreover, we treat the equation directly without changing into an integral equation and we use a method which we introduced earlier and called *direct fixed point mappings*. Each of the three steps is an elementary contraction mapping on a short interval.

Examples of direct fixed point mappings can be seen in [1, 3, 7, 8]. In each case there are excellent reasons for not first converting to an integral equation. In this note there are two reasons. First, while one can prove that there is an inversion because of the fundamental properties of contractions, we see no way to actually achieve it in a workable form. The second reason is accidental. We had begun by asking a contraction condition on $g$ which had been necessary in earlier work with integral equations, but noticed that the integral in the mapping allowed us to ask only a Lipschitz condition. The result is still true when $f$ is identically zero and that means there is a simple proof of global existence in case of an ordinary differential equation with only a (possibly growing) Lipschitz condition.

The equation we treat is the scalar equation

$$x'(t) = g(t, x(t)) + \int_0^t A(t-s)f(s, x(s))ds, \quad ' = \frac{d}{dt}, \quad x(0) = a \in \Re, \tag{D}$$

although a vector system is handled in the same way. In that case, $x, g, f$ are vectors and $A$ is an $n \times n$ matrix. As we are obtaining solutions on $[0, \infty)$ and asking no sign conditions, it is clear that we will need some growth restrictions. As we are asking for uniqueness it is also clear that we will need something of a Lipschitz condition. In fact, we will ask for a Lipschitz condition on $f$ and $g$, but the Lipschitz “constant” can grow to infinity as $t$ tends to infinity.

In order to obtain an integral equation for mapping, we write the direct fixed point equation as

$$\xi(t) = g\left(t, a + \int_0^t \xi(s)ds\right) + \int_0^t A(t-s)f\left(s, a + \int_0^s \xi(u)du\right)ds \tag{1.1}$$

so that if we obtain a continuous solution of (1.1), then

$$x(t) = a + \int_0^t \xi(s)ds$$

will be a continuously differentiable solution of the original equation (D).

Specifically, we ask that

$$f, g : [0, \infty) \times \Re \to \Re \text{ are continuous,} \tag{1.2}$$

and for each $E > 0$ there is a $K = K(E) > 0$ such that

$$0 \le t \le E, \quad x, y \in \Re \implies |g(t, x) - g(t, y)| \le K|x - y|, \tag{1.3}$$

$$0 \le t \le E, \quad x, y \in \Re \implies |f(t, x) - f(t, y)| \le K|x - y|. \tag{1.4}$$

Finally, we ask that

$$A : (0, \infty) \to \Re \quad \text{be continuous,} \tag{1.5}$$

that if $\phi : [0, \infty) \to \Re$ is continuous then

$$\int_0^t A(t-s)\phi(s)ds \text{ be continuous,} \tag{1.6}$$

and that

$$\int_0^t |A(s)|ds \text{ be continuous and converge to zero as } t \downarrow 0. \tag{1.7}$$

For the $E$ and $K$ pick $\alpha \in (0,1)$ and then choose a positive $T^* < 1$ with $KT^* < \alpha$. Finally, select $T = T(K, T^*) > 0$ with $T < T^* < 1$ so that, collecting:

$$K\int_0^T |A(s)|ds < \frac{1-\alpha}{2}, \quad T^*K < \alpha, \quad 0 < T < T^* < 1. \tag{1.8}$$

We begin with a solution to (1.1) on $[0, E]$ and parlay it to $[0, \infty)$.

## 2 Existence and Uniqueness

**Theorem 2.1** *If conditions (1.2) –(1.8) hold then for each $E > 0$ and each $a \in \Re$ there is a unique solution $\xi(t)$ of (1.1) on $[0, E]$.*

***Proof.*** For the given $E > 0$ find $K > 0$ satisfying (1.3) and(1.4), while $T$ satisfies (1.8) with

$$0 < T < T^* < 1, \quad KT^* < \alpha < 1. \tag{2.1}$$

Divide $[0, E]$ into $n$ pieces of length $S < T$ and with end points $0 = T_0, T_1, ..., T_n = E$ so that

$$S = T_i - T_{i-1} < T < 1. \tag{2.2}$$

We will take two steps leading to an induction which generalizes the second step. The first step takes place in a Banach space, but the subsequent step is in a complete metric space.

Step 1. Let $(\mathcal{M}_1, |\cdot|_1)$ be the Banach space of continuous functions $\phi : [0, T_1] \to \Re$ with the supremum norm. Define $P_1 : \mathcal{M}_1 \to \mathcal{M}_1$ by $\phi \in \mathcal{M}_1$ which implies that

$$(P_1\phi)(t) = g\left(t, a + \int_0^t \phi(s)ds\right) + \int_0^t A(t-s)f\left(s, a + \int_0^s \phi(u)du\right)ds. \tag{2.3}$$

Notice that if $P_1$ has a fixed point $\xi_1$, then

$$\frac{d}{dt}\left[a + \int_0^t \xi_1(u)du\right] = \xi_1(t)$$

and

$$x(t) = a + \int_0^t \xi_1(s)ds$$

satisfies (D) with $x(0) = a$.

Let us see that we have a contraction. If $\phi, \psi \in \mathcal{M}_1$ then by (1.8)

$$\int_0^t |\phi(s) - \psi(s)|ds \le T^*|\phi - \psi|_1 \le |\phi - \psi|_1, \quad KT^* < \alpha$$

so

$$
\begin{aligned}
|(P_1\phi)(t) - (P_1\psi)(t)| &\le K\left| a + \int_0^t \phi(s)ds - a - \int_0^t \psi(s)ds \right| \\
&+ \int_0^t |A(t-s)|K\int_0^s |\phi(u) - \psi(u)|duds \\
&\le \alpha|\phi-\psi|_1 + |\phi-\psi|_1 K\int_0^t |A(s)|ds \\
&\le |\phi-\psi|_1\left[\alpha + \frac{1-\alpha}{2}\right] = \frac{1+\alpha}{2}|\phi-\psi|_1,
\end{aligned}
$$

a contraction with unique fixed point $\xi_1$ solving (2.3) on $[0, T_1]$.

Step 2. Let $(\mathcal{M}_2, |\cdot|_2)$ be the *complete metric space* of continuous functions $\phi : [T_0, T_2] \to \Re$ with the supremum metric and $\phi(t) = \xi_1(t)$ for $T_0 \le t \le T_1$. Define $P_2 : \mathcal{M}_2 \to \mathcal{M}_2$ by $\phi \in \mathcal{M}_2$ which implies

$$
(P_2\phi)(t) = g\left(t, a + \int_0^t \phi(s)ds\right) + \int_0^t A(t-s)f\left(s, a + \int_0^s \phi(u)du\right)ds. \tag{2.4}
$$

As $\xi_1$ is a fixed point of $P_1$ on $[T_0, T_1]$ for $0 \le t \le T_1$ we have for any $\phi \in M_2$ that

$$
\begin{aligned}
(P_2\phi)(t) &= g\left(t, a + \int_0^t \xi_1(s)ds\right) + \int_0^t A(t-s)f\left(s, a + \int_0^s \xi_1(u)du\right)ds \\
&= \xi_1(t) \qquad (2.5)
\end{aligned}
$$

and so $P_2$ does map $\mathcal{M}_2 \to \mathcal{M}_2$.

Let us see that $P_2$ is a contraction. If $\phi, \psi \in \mathcal{M}_2$ then

$$
\begin{aligned}
|(P_2\phi)(t) - (P_2\psi)(t)| &\le K\left| \int_0^t [\phi(s) - \psi(s)]ds \right| \\
&+ \int_0^t |A(t-s)|K\left| \int_0^s [\phi(u) - \psi(u)]du \right| ds.
\end{aligned}
$$

Let $T_1 \le t \le T_2$ and fix $s$ at any value $0 \le s \le T_1$. Then examine the last integral above. As $s \le T_1$, then $0 \le u \le T_1$ and so $\phi(u) = \psi(u)$ and that last integral is zero. This is true for every value of $s$ with $0 \le s \le T_1$. If $|\phi|^{[T_1,T_2]}$ denotes the sup then as $S = T_2 - T_1 < T^*$

$$
\int_{T_1}^{T_2} |\phi(s) - \psi(s)|ds \le T^*|\phi-\psi|^{[T_1,T_2]} \le |\phi-\psi|^{[T_1,T_2]} = |\phi-\psi|_2. \tag{2.6}
$$

Hence we may continue the above display as

$$
\begin{aligned}
&= K\left| \int_{T_1}^t [\phi(s) - \psi(s)]ds \right| + \int_{T_1}^t |A(t-s)|K\int_{T_1}^s |\phi(u) - \psi(u)|duds \\
&\le KT^*|\phi-\psi|^{[T_1,T_2]} + \int_{T_1}^t |A(t-s)|K|\phi-\psi|^{[T_1,T_2]}ds \\
&(\text{ by a change of variable and } |\phi-\psi|_2 = |\phi-\psi|^{[T_1,T_2]}) \\
&\le |\phi-\psi|_2\left[\alpha + \frac{1-\alpha}{2}\right] = \frac{1+\alpha}{2}|\phi-\psi|_2
\end{aligned}
$$

a contraction with unique fixed point $\xi_2$ on $[0, T_2]$. Note that $\xi_1 = \xi_2$ on $[0, T_1]$ because both are unique and the definition of the space demands it.

This Step 2 is the first step in the induction since it has the first complete metric space with the function $\xi_1$. We pattern the induction on $\mathcal{M}_2$ which uses $\xi_1$ from Step 1, the mapping $P_2$ which truncates the integrals using the $\xi_1$, and the fixed point $\xi_2$ which is the final product of Step 2 and upon which Step 3 relies.

Inductive hypothesis. Assume that we have a solution $\xi_{i-1}(t)$ satisfying (1.1) for $0 \leq t \leq T_{i-1}$.

From this and the assumptions (1.2)–(1.8) we will obtain a solution $\xi_i(t)$ satisfying (1.1) for $0 \leq t \leq T_i$. That will complete the induction for we can then reach $E$ with the solution $\xi_n$ satisfying (1.1) on $[0, E]$. The proof will then be complete.

Let $\xi_{i-1}$ satisfy (1.1) on $[0, T_{i-1}]$ for $i - 1 \geq 1$. Let $(\mathcal{M}_i, |\cdot|_i)$ be the complete metric space of continuous functions $\phi : [0, T_i] \to \Re$ with the supremum metric and for $0 \leq t \leq T_{i-1}$ every function satisfies $\phi(t) = \xi_{i-1}(t)$. Next, we define $P_i : \mathcal{M}_i \to \mathcal{M}_i$ by $\phi \in \mathcal{M}_i$ which implies that

$$(P_i\phi)(t) = g\left(t, a + \int_0^t \phi(s)ds\right) + \int_0^t A(t-s) f\left(s, a + \int_0^s \phi(u)du\right)ds.$$

Because $\xi_{i-1}$ is a solution on $[0, T_{i-1}]$ if $0 \leq t \leq T_{i-1}$ then $(P_i\xi_{i-1})(t) = \xi_{i-1}(t)$ and so the mapping is into $\mathcal{M}_i$.

We now show that $P_i$ is a contraction. If $\phi, \psi \in \mathcal{M}_i$ then

$$|(P_i\phi)(t) - (P_i\psi)(t)| \leq K\left|\int_0^t [\phi(s) - \psi(s)]ds\right|$$
$$+ \int_0^t |A(t-s)|K\left|\int_0^s [\phi(u) - \psi(u)]du\right|ds$$

(as in Step 2 at this same point in the display and now $T_{i-1} \leq t \leq T_i$ )

$$= K\left|\int_{T_{i-1}}^t [\phi(s) - \psi(s)]ds\right| + \int_{T_{i-1}}^t |A(t-s)|K \int_{T_{i-1}}^s |\phi(u) - \psi(u)|duds$$
$$\leq KT^*|\phi - \psi|^{[T_{i-1}, T_i]} + \int_{T_{i-1}}^t |A(t-s)|K|\phi - \psi|^{[T_{i-1}, T_i]}ds$$

(by a change of variable and $|\phi - \psi|_i = |\phi - \psi|^{[T_{i-1}, T_i]}$)

$$\leq |\phi - \psi|_i\left[\alpha + \frac{1-\alpha}{2}\right] = \frac{1+\alpha}{2}|\phi - \psi|_i,$$

a contraction with unique fixed point $\xi_i$ on $[0, T_i]$. Note that $\xi_{i-1} = \xi_i$ on $[0, T_{i-1}]$ because both are unique and the definition of the space demands it. □

**Theorem 2.2** *Under the conditions of Theorem 2.1 there is a unique solution* $\xi$ *of (1.1) on* $[0, \infty)$.

***Proof.*** Using Theorem 2.1 we construct a unique solution $\xi_n$ on every interval $[0, n]$ for every positive integer $n$. Extend each of those solutions to the interval $[0, \infty)$ by defining $\xi_n$ past $n$ by the function $\xi_n^* = \xi_n(n)$ for $t > n$. Thus we have a sequence of uniformly continuous functions on $[0, \infty)$ which converge uniformly on compact sets to a continuous function $\xi$ which is a solution of (1.1) because at every value of $t$ the function on $[0, t]$ coincides with any $\xi_n$ for $n > t$. □

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