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Searching Functional Exponents for Generalized Fourier Series and Construction of Oscillatory Functions Spaces

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Abstract: This paper is intended to provide a framework for further developments of the theory of generalized Fourier series of the form

$$\sum_{k=1}^{\infty} a_k \exp[i\lambda_k(t)],\ t \in\ R, \tag{1}$$

where $a_k \in \mathcal{C},\ k \geq 1,\ \lambda_k : R \to R,\ k \geq 1$. Series of the form (1) will be called, in this paper, *series representing oscillatory functions*, by the last term understanding the sum of any series of the form (1), when convergent in some sense, classical or generalized, such as summability procedure or, in respect to a certain norm on the space of series, or in the associated function space of sums or generalized sums. A basic idea we follow is to start from linear spaces of series like (1), then to organize them by introducing a norm or a kind of convergence. The connection between a space of generalized trigonometric series of the form (1) and the space of functions resulting from introducing a topology/norm is our main objective. It is also emphasized that the preceding stages of Fourier analysis, i.e., the classical trigonometric series (the first stage) or the almost periodic functions (the second stage) are also parts of the *third stage* in the development of Fourier analysis. This study is based on classical theory of Fourier Analysis and on the theory of almost periodicity, as developed since 1920's to present. It is also based on methods and results of functional analysis.

Keywords: *generalized Fourier series; oscillatory functions; trigonometric series; almost periodic functions.*

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1 Introduction

Series of the form (1) and construction of spaces of oscillatory functions, consisting of the sum or generalized 'sum', have been investigated by researches during the last 20-25 years. We shall further provide the references, adequate to the subject. It has to be emphasized that both engineering and mathematical literature contain results related to this topic, generated by the applied problems. Mathematicians have started a theory related to the series of the form (1) and their attached oscillatory function spaces. The method used consists in completing certain spaces of generalized trigonometric polynomials, with respect to uniform convergence as basic tool, or the convergence in the mean (of order 2).

Since we take the *series* as primary element in the construction of function spaces of oscillatory functions, we need to proceed with the investigation of spaces whose elements are series of the form (1), to organize them algebraically and then topologically to obtain the series spaces. After the construction of series spaces, we shall be able to obtain the function spaces, consisting of oscillatory functions.

First, let us briefly present the examples already existing in the literature, due to Osipov [15] and Zhang [17]– [20]. These mathematical constructions have been preceded by contributions coming from the engineering literature, due to several researchers, and mentioned in the references to Zhang's papers quoted above. Such applied sources have appeared, particularly, in the IEEE publications, during the last two decades, sporadically, in other journals.

It is interesting to mention the fact that the first stage of development of Fourier analysis (in its main goal of establishing the connection between series and functions), besides many other aspects, started in the 18-th century with names like Euler and continued its vigorous development in the 19-th century, when a great number of mathematicians brought very important contributions, starting with Fourier.

An example connected to the advancement of the first stage is the proof of a conjecture due to Luzin (from 1915), about the convergence almost everywhere, of the Fourier series of any functions $f : [0, 2\pi] \to R$, $f \in L^2([0, 2\pi], R)$. In such case,

$$f \simeq \frac{a_0}{2} + \sum_{j=1}^{\infty} (a_j \sin jt + b_j \cos jt),$$

where a_j, b_j are given by the classical Euler formulae. The sign $\simeq$ above can be substituted by the sign $=$, excepting a subset of $[0, 2\pi]$ of Lebesgue measure zero. This result is due to Carleson (1966).

The books by Bary [1] and Zygmund [22] are almost of encyclopedic type for the Fourier Analysis, the first stage, since its inception until the mid of the 20-th century. Needless to say that the first stage is not yet quitting the scene and new contributions are abundant.

In the 1920's, the second stage is appearing with Bohr, followed by Stepanov, Bochner and Besicovitch, to mention only a few of the great contributors to the theory of almost periodicity, a kind of oscillatory motion, more complex than periodicity.

The current mathematical literature, dedicated to the case of almost periodic functions is quite rich, the following quotations providing a rather complete source for this subject: Bohr [3], Besicovitch [2], Favard [11], Levitan [13], Fink [12], Levitan [13] and Zhikov [14], Corduneanu [4, 5].

The development of Science and Technology, especially in the 20-th century, lead to the new form, a generalized one, for Fourier series (trigonometric, when not generated by a function). This new type of series is of the form presented in formula (1) above, with the functional exponents $\lambda_k(t) : R \to R$, $k \geq 1$, subject to conditions further specified. In Zhang's papers quoted above, several spaces of oscillatory functions are constructed, starting with a class of generalized Fourier exponents, of the form

$$\lambda(t) = \sum_{k=1}^{m} c_k \exp[iq_k(t)], \ t \in R, \ c_k \in \mathcal{C}, \tag{2}$$

where $q_k(t)$ are defined by formulae like

$$q(t) = \begin{cases} \sum\limits_{i=1}^{m} \lambda_j t^{\alpha_j}, & t \geq 0, \\ -\sum\limits_{j=1}^{m} \lambda_j (-t)^{\alpha_j}, & t < 0, \end{cases} \tag{3}$$

with $\lambda_j \in R$, $j = 1, 2, ..., m$ and $\alpha_1 > \alpha_2 > \cdots > \alpha_m > 0$. The class of generalized exponents in (2), (3), is denoted by $Q(R, R)$ and, according to Zhang [21], it has been considered by Gelfand in another context.

The first space of oscillatory function, defined by Zhang [19], has been called the space of *strong limit power* functions and denoted by $SLP(R, R)$, is obtained by completing the linear space of all generalized trigonometric polynomials of the form

$$P(t) = \sum_{k=1}^{n} c_k \exp[iq_k(t)], \ t \in R, \tag{4}$$

with $q_k(t)$ as in (3) and $c_k \in \mathcal{C}$, $k = 1, 2, ..., n = n(P)$, the norm being the supremum, on R, of the polynomial $P(t)$ in (4). Of course, the topology induced by this norm is that of uniform convergence on R. Consequently, the construction of the space $SLP(R, \mathcal{C})$ is achieved by the method of completion of linear vector spaces, in this case, the norm being the $\sup\limits_{R} |\cdot|$.

Therefore, the space $SLP(R, \mathcal{C})$ is a Banach space over $\mathcal{C}$, which is also a subspace of the richer Banach space $BC(R, \mathcal{C})$, of continuous and bounded maps from R into $\mathcal{C}$, with the uniform convergence on R.

Taking the space $SLP(R, \mathcal{C})$ as a base space, new oscillatory function spaces have been constructed by Zhang [20], namely the Besicovitch type spaces, similar to the spaces $B_1(R, \mathcal{C})$, or $B_2(R, \mathcal{C})$. For the first case, one has to complete $SLP(R, \mathcal{C})$ with respect to the norm $f \to M(|f|)$, while in the second case, of the space $B_2(R, \mathcal{C})$, the norm chosen for the completion procedure will be $f \to \{M(|f|^2)\}^{1/2}$.

The interested reader can find the details in Zhang's papers, quoted above, or in the book by Corduneanu et al. [10]. Many properties are known for classical almost periodic functions, in which case the functional exponents are linear functions, of the form λt, $t \in R$, $\lambda \in R$.

In summarizing the discussion above, about the oscillatory function spaces constructed by Zhang, one can notice the following steps which are necessary in the procedure: first, one needs a set (possibly with an algebraic structure) of generalized functional exponents, say $\{f(t)\}$, such that $\exp[if(t)]$ has the following property:

$$\lim_{\ell \to \infty} (2\ell)^{-1} \int_{-\ell}^{\ell} f(t) \exp[if(t)] dt$$

exists (as a finite complex number); second, for the completion property of the space of oscillatory functions, one needs to choose a topology, or a norm, based on which we obtain the completed (or Banach) space. In the case we use a seminorm, instead of a norm, the need to work with a factor space is required. See, for instance, Corduneanu [5].

To briefly summarize the connection between the series and its sum, let us denote this connection by

$$f(t) \simeq \sum_{k=1}^{\infty} c_k \exp[iq_k(t)] \tag{5}$$

and provide the formulae ($k \geq 1$)

$$c_k = \lim_{\ell \to \infty} (2\ell)^{-1} \int_{-\ell}^{\ell} f(t) \exp[-iq_k(t)] dt. \tag{6}$$

As proved in Zhang's quoted papers, the Parseval equation

$$\sum_{k=1}^{\infty} |c_k|^2 = \lim_{\ell \to \infty} (2\ell)^{-1} \int_{-\ell}^{\ell} |f(t)|^2 dt \tag{7}$$

also holds. It has many implications, among them we mention the uniqueness of the generalized Fourier series attached to a function $f \in SLP(R, \mathcal{C})$. Or, the one to one correspondence between the elements of the space $SLP(R, \mathcal{C})$ and those of the space ℓ^2.

For more details on these matters, the reader is invited to consult the Appendix to the book of Corduneanu et al. [10]. See also the paper by Zhang [19], for the construction of the spaces of Besicovitch type, $B_1(R, \mathcal{C})$ and $B_2(R, \mathcal{C})$, by using the completion method, as specified above, by using the norms $M(|f|)$ and $\{M(|f|^2)^{1/2}$, with respect to which the space $SLP(R, \mathcal{C})$ is not complete. In the paper of Corduneanu [8], the space $B^2_\lambda(R, \mathcal{C})$ is constructed by this method, for an arbitrary set $\lambda = \{\lambda_\alpha,\ \alpha \in$ an arbitrary set of generalized Fourier exponents$\}$.

The remaining part of the Introduction will be concerned with the space constructed by Osipov [15], also pertaining to the third stage in the development of Fourier Analysis.

The Osipov space is known under the name of *Bohr-Fresnel almost periodic functions space*. Actually, these functions are oscillatory in the sense of adopted definition and a result of Osipov states: Let $f(t) : R \to \mathcal{C}$ be a Bohr-Fresnel almost periodic function. Then, there exists a Bohr almost periodic function $F(t, x) : R \times R \to \mathcal{C}$, such that $f(t) = F(t, t^2)$, $t \in R$. Of course, the result shows the close relationship between Bohr and Bohr-Fresnel almost periodic functions, but the theory of the later is much more complex, as it appears in the book of Osipov, quoted above.

Following our procedure in constructing new spaces of oscillatory functions, we shall start from the set of all formal trigonometric series, of the form

$$\sum_{k=1}^{\infty} c_k \exp(i\alpha t^2 + 2i\lambda_k t), \tag{8}$$

where $c_k \in \mathcal{C}$, $\alpha, \lambda_k \in R$, $k \geq 1$. One usually assumes that λ_k's are distinct.

It follows from Zhang's case discussed above that each term in (8) has a finite limit (Poincaré) on the whole real axis. Moreover, if the series in (8) is absolutely convergent and denotes the sum by $f(t)$, then the connections between f and series (8) are given by the formulae for coefficients, in terms of $f(t)$:

$$c_k = \lim_{\ell\to\infty} (2\ell)^{-1} \int_{-\ell}^{\ell} f(t)\exp(-i\alpha t^2 - 2i\lambda_k t)dt. \tag{9}$$

Let us note that α is a real number which is determined by the function $f(t)$. Also, the formula (9) is valid in cases when the series (8) is not necessarily absolute (hence, also uniform) convergent. The right hand side of (9) makes sense in more general situations, as we shall see. It is, again, the Poincaré mean value on R.

If one assumes the condition

$$\sum_{k=1}^{\infty} |c_k|^2 < \infty, \tag{10}$$

which is less restrictive than the condition of absolute convergence, we obtain a larger space of oscillatory functions, which is in a slighter modified form – the space of Osipov [15], consisting of oscillatory functions.

We shall list now some properties of the space of Bohr–Fresnel almost periodic functions, presented in detail in Osipov's book quoted above.

We point out the fact that the Parseval type equation

$$\lim_{\ell\to\infty} (2\ell)^{-1} \int_{-\ell}^{\ell} |f(t)|^2 dt = \sum_{k=1}^{\infty} |c_k|^2, \tag{11}$$

where the c_k's are given by (9), holds true for every B^2-almost periodic function.

Another property, following from (1) and some extra arguments, is the *uniqueness* of the generalized Fourier series, associated to a function in the Bohr-Fresnel space.

As shown in Section 2 below, to each sequence $\{c_k;\ k \geq 1\}$ satisfying (10), there corresponds a unique Bohr-Fresnel function. The *approximation* property is also valid, in the following format (different than in Osipov's text): Any function $f(t)$, in the class of Bohr-Fresnel almost periodic functions, can be approximated with any degree of accuracy by polynomials in this class, with frequencies belonging to the set of frequencies in its generalized Fourier series. Using the norm derived from the Poincaré mean value

$$M(f) = \lim_{\ell\to\infty} (2\ell)^{-1} \int_{-\ell}^{\ell} f(t)dt, \tag{12}$$

the approximation property can be stated: for each $\varepsilon > 0$, there exists $n \in N$, such that

$$M\left\{ |f(t)| - \sum_{k=1}^{n} c_k \exp(i\alpha t^2 + 2i\lambda_k t)|^2 \right\} < \varepsilon^2, \tag{13}$$

with $c_k,\ k = 1, 2, ..., n$, given by (9). We notice that, unlike in the case of Zhang's space $SLP(R, \mathcal{C})$, the 'measure' of length used is based on the Poincaré mean value, inducing a convergence in the mean (of order 2), instead of the uniform convergence, achieved by the sup-norm.

We shall conclude this introductory remarks related to the oscillatory function spaces constructed by Osipov and Zhang, mentioning the fact that, in the paper [21] by Zhang et al. the case of generalized Fourier exponents having the form of a quadratic polynomial, with real valued coefficients, has been thoroughly investigated, all possible cases (for constructing a space of oscillatory functions) being emphasized.

2 Finding Generalized Functional Fourier Exponents

From the form of formula (1), we realize that in order to attach a function to the series which we would like to represent an oscillatory function, with some basic properties encountered for classical Fourier series or the ones characterizing various types of almost periodic functions, two necessary conditions have to be satisfied:

First, we must find the generalized Fourier exponents, denoted by $\lambda_k(t),\ k \geq 1$; more precisely, we need to identify sets we shall represent by Λ, containing sequences of functions $R \to \mathcal{C}$, at least locally integrable on R. Since each sequence of $\lambda_k(t)$'s must contain distinct terms, it is obvious that Λ has to be at least countable. Moreover, in case we want to represent certain functions $R \to \mathcal{C}$ by such series, which means we have to determine the coefficients of the series like (1), we realize that, each sequence involved, must be formed from mutually 'orthogonal' elements. This condition will be imposed in the form suggested by Poincaré mean value, namely

$$\lim_{\ell\to\infty} (2\ell)^{-1} \int_{-\ell}^{\ell} \exp[i(\lambda_j(t) - \lambda_k(t))]dt = \begin{cases} 0, & j \neq k, \\ 1, & j = k. \end{cases} \tag{14}$$

This condition is also suggested by the theory of Hilbert (rather pre-Hilbert) spaces, but we are not getting into details here.

Second, one needs to make precise the kind of attaching to a given series of the type (1), a function that could be reasonably called a generalized sum. Of course, the most natural way is to have a condition assuring the convergence of the series with respect to a certain norm. Since this is a rather restrictive condition (if, for instance, we keep in mind the fact that the classical Fourier series of a continuous function is only summable to the generating function, using Euler's formulae for coefficients), we may use, when adequately, instead of a norm, a seminorm. This feature will lead to further problems when constructing the spaces of oscillatory functions, but it serves well our purpose, as we see below, in this paper.

We can obtain sequences $\{\lambda_k(t);\ k \geq 1\}$, such that (14) is satisfied, if we can construct distinct solutions of the equation/relation

$$\lim_{\ell\to\infty} (2\ell)^{-1} \int_{-\ell}^{\ell} \exp[i\lambda(t)]dt = \begin{cases} 0, & \lambda(t) \not\equiv 0, \\ 1, & \lambda(t) \equiv 0. \end{cases} \tag{15}$$

Indeed, if $\lambda_k(t),\ k \geq 1$, are distinct solutions of (15), then the sequence $\{\lambda_k(t); k \geq 1\}$ satisfies obviously the relationship (14).

Let us determine solutions of the equation/relation (15), choosing a simple procedure based on Cauchy's integral theorem.

Namely, limiting our considerations to those $\lambda(t) : R \to R$, which constitute restrictions of entire functions $\lambda = \lambda(z),\ z \in \mathcal{C}$ and applying Cauchy's theorem for a closed contour, consisting of the interval of the real axis $(-\ell, \ell)$ and the semicircle C_ℓ having $(-\ell, \ell)$ as diameter, situated in the half-plane $\operatorname{Im} z \geq 0$, one obtains for $\ell > 0$

$$\int_{-\ell}^{\ell} \exp[i\lambda(t)]dt + \int_{C_\ell} \exp[i\lambda(z)]dz = 0, \tag{16}$$

on C_ℓ being from ℓ to $-\ell$. Let $\Lambda(z)$ be a primitive of $e^{i\lambda(z)}$, which is also an entire function. Then (16) yields for $\ell > 0$,

$$\int_{-\ell}^{\ell} \exp[i\lambda(t)]dt = \Lambda(-\ell) - \Lambda(\ell). \tag{17}$$

From (15) one derives now the condition for λ:

$$\ell^{-1}[\Lambda(\ell) - \Lambda(-\ell)] = o(1),\ \ell \to \infty. \tag{18}$$

Consequently, the equation/relation (18) provides a source for obtaining $\lambda(z)$, such that $\Lambda'(z) = \exp(i\lambda(z))$ and, taking a sequence of distinct solutions of (18), we have the possibility of constructing series of the form (1).

Let us notice that the second case in (15) is obviously verified, i.e., when $\lambda(z) \equiv 0$. If one chooses $\lambda(z) = \lambda z$, $\lambda \in R$, $\lambda \neq 0$, $z \in \mathcal{C}$, then we derive from above

$$\lim (i\lambda\ell)^{-1}[e^{i\lambda\ell} - e^{-i\lambda\ell}] = 0,\ \text{as } \ell \to \infty. \tag{19}$$

Since the bracket is bounded as $\ell \to \infty$, there results the validity of (18).

Hence, the series resulting from the above considerations, namely

$$\sum_{k=1}^{\infty} a_k \exp(i\lambda_k t),\ t \in R, \tag{20}$$

with λ_k being arbitrary real numbers, are series for oscillatory functions.

But we recognize in (20) the Fourier series corresponding with the almost periodic functions. Depending on the nature of their convergence of summability, we obtain the classical Bohr almost periodic functions and its multiple generalizations (Stepanov, Besicovitch, the AP_r-almost periodic functions).

Remark 2.1 From the formula (17), we can draw the following conclusion. If $\Lambda(z)$ is a function satisfying the condition $\Lambda(\ell) = \Lambda(-\ell)$, i.e., is an even function, then (17) is verified. This is a rather special case and we invite the readers to find other solutions to the equation/relation (18), in the class of entire functions.

We shall deal now, with another condition imposed, to the function $\lambda(t)$, namely

$$\lambda(-t) = -\lambda(t),\ t \in R. \tag{21}$$

Finding generalized Fourier exponents, in the class of odd functions on R, leads to another relation/equation similar to (18). This restriction was also imposed by Zhang, when constructing the space $SLP(R, \mathcal{C})$.

Let us notice that the left hand side in (17) can be rewritten as

$$\begin{aligned}\int_{-\ell}^{\ell} \exp[i\lambda(t)]dt &= \int_{-\ell}^{0} \exp[i\lambda(t)]dt + \int_{0}^{\ell} \exp[i\lambda(t)]dt \\ &= 2\int_0^\ell \cos\lambda(t)dt,\ \ell > 0,\end{aligned} \tag{22}$$

if we take into account (21) and change t for $-t$ in the first integral. Therefore, in order to satisfy the first condition of (15) it is necessary and sufficient to satisfy the equation/relation

$$\int_0^\ell \cos\lambda(t)dt = o(\ell), \quad \text{as } \ell \to \infty. \tag{23}$$

Only odd solutions $\lambda(t)$ at least locally integrable are candidates for functional exponents in series representing oscillatory functions.

In what follows, we shall deal with finding nontrivial solutions to the equation/relation (23), as well as (18).

The relation/equation (23) has, indeed, nontrivial solutions. We notice that any function of the form $\lambda(t) = \mu t$, with $\mu = \text{const.} \in R$ and $t \in R$, is an odd function which satisfies both (18) – as seen above, and (23). Hence, we reobtain the functional exponents that characterize various classes of almost periodic functions. This remark is a confirmation of the fact that the oscillatory functions contain the classical cases of periodic and almost periodic functions. More comments on these matters will be made in forthcoming text.

Of course, it is interesting to emphasize classes of generalized exponents, using the equation/relation (23). And let us examine the case of oscillatory functions of Osipov [15] type.

Still remaining in the classical field, let us remind the Fresnel integrals, related to his theory in Optics: for $\alpha > 0$, one has

$$\int_0^\infty \cos(\alpha t^2)dt = (2\alpha)^{-1}\sqrt{\frac{\pi}{2}}\cdot \tag{24}$$

Taking (24) into account, we find out that the relation/equation (23) is verified by any function αt^2, $\alpha > 0$, $t \geq 0$. In order to obtain the odd function satisfying (21), one has to consider (on R) $\lambda(t) = \alpha t^2$ for $t \geq 0$ and $\lambda(t) = -\alpha t^2$ for $t < 0$. Then we rely on Zhang's et al. results in [21] to find that $\lambda(t)$ defined above can be used to construct generalized trigonometric polynomials, based on quadratic algebraic polynomials. This means, generalized trigonometric polynomials of the form

$$P(t) = \sum_{k=1}^{m} a_k \exp[i(\alpha t^2 + \beta_k t)], \tag{25}$$

with $\alpha, \beta_k \in R$, $1 \leq k \leq m$. There is no free term at the exponent, because it is absorbed by a_k. This approach, used by Zhang and his collaborators, does not lead to the original space constructed by Osipov. The method used by Osipov [15] requires that polynomials of the form (25), with $\lambda(t) = \alpha t^2 + \beta_k t$, $t \in R$, be used to construct the functions "sum" on the whole R. More precisely, (24) can be used only on R_+, or on the whole R. In such a way, we actually obtain two spaces of oscillatory functions, based on second degree algebraic polynomials as functional exponents. In the introduction, we have sketched the construction of the original Osipov space. The details are given in Osipov's book [15], besides a short presentation of Bohr's theory, to serve for the parallelism between two concepts of almost periodicity (actually, Bohr-Fresnel functions constitute an example of oscillatory functions, even though they can be represented by the classical Bohr almost periodic functions). Their Fourier series is representative for the third stage of Fourier Analysis.

Concerning Zhang's $SLP(R,\mathcal{C})$ functions, one sees from their construction that they are odd functions. The fact of possessing a finite Poincaré mean value is proven in the paper by Zhang [19].

Let us now consider an example corresponding to $\lambda(z) = \sin \lambda z,\ a \in R,\ z \in \mathcal{C}$. Obviously, $\lambda(z)$ is an odd function. But this $\lambda(z)$ is not a solution of (23). The associated generalized Fourier series is

$$\sum_{k=1}^{\infty} a_k \exp[i \sin \lambda_k t],\ \ t \in R, \tag{26}$$

which is characteristic for the third stage of Fourier Analysis. If we admit the condition $\{a_k;\ k \geq 1\} \in \ell^1(N,\mathcal{C})$, then (26) is absolutely and uniformly convergent on R. Since every term is a Bohr almost periodic function, the series is convergent to a function $f \in AP(R,\mathcal{C})$. In other words, this case is an example of a series whose construction is not based on the use of equation (18) or (23), but the sum is an oscillatory function, even of classical type.

Of course, if instead of the condition imposed above, $\{a_k;\ k \geq 1\} \subset \ell^1(N,\mathcal{C})$, one chooses another similar one, the result may lead to other classical spaces of almost periodic or oscillatory functions. It is also clear that the same oscillatory function can be represented by different types of generalized Fourier series. An in depth study of this fact would be welcome.

One can find many other sequences of generalized Fourier exponents, just relying on above considerations. An example, also resulting from Zhang's constructions, is given by a sequence of odd degree polynomials, say like $\mu(z) = a_1 z + a_3 z^3 + \cdots + a_{2k+1}^{2k+1}$. Indeed, these polynomials and their linear combinations are satisfying the request appearing in Zhang's construction of generalized Fourier series [19]. These exponents satisfy, starting with $k = 1$, requirements coming from applications.

We shall prove now a lemma, showing how one can get more complex generalized exponents, relying on some already found.

Lemma 2.1 *Let us assume we are given a set of generalized exponents, say* $\Lambda = \{\lambda_\alpha(t) : \alpha \in A\}$*, where* A *is a set of indices, at least countable. If* $\varphi : R \to R$ *is a locally integrable map, such that* $\lim \exp]i\varphi(t)]$ *exists when* $t \to \infty$*, while* $\{\lambda_j(t);\ j \geq 1\} \in \Lambda$ *and form an orthogonal system as shown in* (14)*, then the sequence* $\{\varphi(t) + \lambda_j(t);\ j \geq 1\} \subset \Lambda$ *is also orthogonal in the sense shown by* (14)*.*

The proof is immediate if we notice that $[\varphi(t) + \lambda_j(t)] - [\varphi(t) + \lambda_k(t)] = \lambda_j(t) - \lambda_k(t)$, and take (14) into account.

In this way, we have obtained in case of Osipov's kind of generalized Fourier series, i.e., the Bohr-Fresnel case of almost periodic functions: $\alpha t^2 + \beta_k t,\ k \geq 1$, representing the exponents of terms in the series for Osipov's oscillatory functions.

We invite the reader to investigate solutions of the form $\lambda(t) = t^\alpha$, $\alpha \in R_+$, for the equation (23). Also, for the relation/equation (18). In particular, the odd polynomials mentioned above, justified by Zhang's argument.

In concluding this section, we shall make two brief remarks/suggestions, which may be helpful in the search of new classes of generalized Fourier exponents.

First one is related to the use of the general formula for residues, instead of Cauchy's integral theorem. This formula has the form, with notations similar to those in (16),

$$\int_{-\ell}^{\ell} e^{i\lambda(t)} dt + \int_{c_\ell} e^{i\lambda(z)} dz = 2\pi i \sum \operatorname{res}(e^{i\lambda(z)}), \tag{27}$$

the Σ being extended at the poles of $\exp[i\lambda(z)]$, within the interior of the semidisc formed by c_ℓ and $(-\ell, \ell)$. The function $\exp(i\lambda(z))$ must be meromorphic, with zeros at $(z_1, z_2, ..., z_n) \in \mathcal{C}^n$, so that, for large enough ℓ, one can take the limit of both sides in (21), as $\ell \to \infty$. Apparently, this is not an easy task, but in the affirmative case it will provide other solutions for determining generalized Fourier exponents.

Second remark relates to the notation Λ for the set of generalized exponents. It is obvious that, from algebraic point of view, this set of real valued functions must form at least an additive group. This can be seen, for instance, from the formulas providing the coefficients of a generalized Fourier series, such as (6), (9), or the orthogonality conditions.

Zhang [19] required more algebraic conditions, for instance the ring structure for Λ, a necessity imposed by the fact that the product of two function in Λ, must be in Λ.

3 Construction of a Space of Oscillatory Functions

In Section 1, we have summarily presented the construction of the oscillatory function spaces, following the two authors who have brought significant contributions to the development of the third stage of Fourier Analysis. We shall present, in this section, the construction of a space of oscillatory functions, denoted by '$AP_1(R, \mathcal{C}; \Lambda)$, the AP just reminding us of the case of almost periodic functions, which functions are also *oscillatory* type (see the definition in the Abstract of the paper). It is the corresponding, more general, case of the space $AP_1(R, \mathcal{C})$, see Corduneanu [6, 7], the name of Poincaré being properly attached, since he has provided the first example of an almost periodic function (Bohr), in a rather important case: when the Fourier series attached is absolutely and uniformly convergent on R.

The first step in the construction consists in specifying the set/class of generalized Fourier/trigonometric series, of the form (1), which will be the elements of $AP_1(R, \mathcal{C}; \Lambda)$. Namely, to obtain the space $AP_1(R, \mathcal{C}; \Lambda)$, we shall assume that all series of the form (1), for which

$$\sum_{k=1}^{\infty} |a_k| < \infty, \tag{28}$$

will be the elements of $AP_1(R, \mathcal{C}; \Lambda)$, and only them.

Since the series satisfying (28) imply the absolute convergence, due to the fact $|a_k \exp[i\lambda_k(t)]| \leq |a_k|$, $k \geq 1$, $t \in R$, $\lambda_k \in \Lambda$, the norm on this space appears naturally to be the one given in (28), i.e.,

$$\left| \sum_{k=1}^{\infty} a_k \exp[i\lambda_k(t)] \right|_{AP_1} = \sum_{k=1}^{\infty} |a_k|. \tag{29}$$

Hence, the set $AP_1(R, \mathcal{C}; \Lambda)$ is a linear normed space on $\mathcal{C}$. Moreover, this space is a Banach space, i.e., complete as a linear metric space, a statement which is implied by the completeness of the space $\ell^1(R, \mathcal{C})$.

We shall try now to derive some properties of this space, particularly looking at its connections with function spaces on R. The natural approach seems to be in attaching to the series (1), the function representing its sum. This means the correspondence/map is given by

$$\sum_{k=1}^{\infty} a_k \exp[i\lambda_k(t)] \to \sum_{k=1}^{\infty} a_k \exp[i\lambda_k(t)], \ t \in R, \tag{30}$$

with the left hand side in (39) regarded as the formal series, while the right hand side is the sum of the series, i.e., a function $f : R \to \mathcal{C}$.

It is obvious that $f = f(t)$, $t \in R$, is a complex valued function, defined on R and taking values which are uniformly bounded, by the right side in (29). It is also a continuous and bounded map from R into $\mathcal{C}$, which tells us that $AP_1(R, \mathcal{C}; \Lambda) \subset BC(R, \mathcal{C})$ = the space of bounded and continuous maps from R into $\mathcal{C}$. We have admitted that Λ consists of continuous functions. When this condition does not hold for the elements of $AP_1(R, \mathcal{C}; \Lambda)$, we can obtain spaces of measurable functions (for instance), more general than $BC(R, \mathcal{C})$.

Let us summarize now the discussion above regarding the space $AP_1(R, \mathcal{C}; \Lambda)$ and its Banach space structure, over the field $\mathcal{C}$. We need to keep in mind that $AP_1(R, \mathcal{C}; \Lambda)$ can be regarded either as a series space or a function space. Their isomorphism is the motivation for using the same notation for both of them. We shall write now the formula which represents the space $AP_1(R, \mathcal{C}; \Lambda)$:

$$\begin{aligned} AP_1(R, \mathcal{C}; \Lambda) = \Bigg\{ f : R \to \mathcal{C}, f(t) = \sum_{k=1}^{\infty} a_k \exp[i\lambda_k(t)], \\ \sum_{k=1}^{\infty} |a_k| < \infty,\ \lambda_k(t) \in \Lambda,\ k \geq 1 \Bigg\}. \end{aligned} \tag{31}$$

The norm is given by formula (29). The completion of the space $AP_1(R, \mathcal{C}; \Lambda)$ follows easily from the following argument. Indeed, from our assumption (25), there follows that $AP_1(R, \mathcal{C}; \Lambda)$ is the closure of the subset of generalized trigonometric polynomials of the form $\sum\limits_{k=1}^{n} a_k \exp[i\lambda_k(t)]$, with a_k and $\lambda_k(t)$, $k \geq 1$, as considered above.

Since the completion of a linear normed space is the minimal complete Banach space, containing the given linear normed space, while any element of $AP_1(R, \mathcal{C}; \Lambda)$ can be regarded as the limit in the sense of the norm, we obtain a contradiction if we assume that there exists a complete linear space, larger than $AP_1(R, \mathcal{C}; \Lambda)$, i.e., containing at least one element outside $AP_1(R, \mathcal{C}; \Lambda)$, which can be reached by the limit process with terms from the space of trigonometric polynomials of the above shown form (sections of the series in the space $AP_1(R, \mathcal{C}; \Lambda)$).

Theorem 3.1 *The space of oscillatory functions $AP_1(R, \mathcal{C}; \Lambda)$ is constructed in the following steps:*

1) *One chooses a set Λ, at least countable, consisting of continuous functions $R \to R$, such that any sequence $\{\lambda_k(t);\ k \geq 1\} \subset \Lambda$ is orthogonal in the sense of Poincaré's mean value on R, as shown in formula* (14).

2) *See Section* 2 *for details in obtaining such a set Λ.*

3) *One considers the set of all generalized Fourier series of the form* (1), *with $\{\lambda_k(t),\ k \geq 1\} \subset \Lambda$, which can be routinely organized as a linear space over $\mathcal{C}$.*

4) *In order to introduce a topology/convergence on this linear space, we have denoted it by $AP_1(R, \mathcal{C}; \Lambda)$, we consider on it the norm defined by* (29).

5) *One derives, as shown above, that the space $AP_1(R, \mathcal{C}; \Lambda)$ is a Banach space, by proving its completeness in the norm* (29).

Remark 3.1 The isomorphism of the series space $AP_1(R, \mathcal{C}; \Lambda)$ and the function spaces of the sums of its series, in other words, the one to one correspondence between the series and functions-sums, will follow easily when we are able to prove the uniqueness theorem for Fourier generalized series in $AP_1(R, \mathcal{C}; \Lambda)$, based on Parseval's formula

$$\sum_{k=1}^{\infty} |a_k|^2 = \lim_{\ell \to \infty} (2\ell)^{-1} \int_{-\ell}^{\ell} |f(t)|^2 dt, \tag{32}$$

to be established in the sequel. There is an alternative approach, based on the formula for the coefficients, in terms of the sum of the series

$$a_k = \lim_{\ell \to \infty} (2\ell)^{-1} \int_{-\ell}^{\ell} f(t) \exp[-i\lambda_k(t)] dt. \tag{33}$$

Both approaches will be substantiated in the presentation to follow.

Remark 3.2 Since we shall deal with product of elements/series of $AP_1(R, \mathcal{C})$, we notice that this operation (Cauchy's rule of multiplication can be performed only in case when Λ is an additive group of real valued functions $\lambda = \lambda(t) : R \to R$, which we shall use to form the generalized Fourier series.

Now, let us prove the formula (32), which establishes the connection between the function $f(t) : R \to \mathcal{C}$, and its generalized Fourier series in (30). One obtains, by multiplying both sides by $\exp[-i\lambda_j(t)] \neq 0$, the following relation:

$$f(t) \exp[i\lambda_j(t)] = \sum_{k=1}^{\infty} a_k \exp[i(\lambda_k(t) - \lambda_j(t)], \tag{34}$$

which we can integrate from $-\ell$ to ℓ, both sides, the second, term by term. This follows from the condition $\{a_k;\ k \geq 1\} \subset \ell^1(N, \mathcal{C})$, taking also into account the fact that each exponential has module equal to 1. This leads to the equation

$$\begin{aligned} \int_{-\ell}^{\ell} f(t) \exp[-i\lambda_j(t)] dt &= \int_{-\ell}^{\ell} \sum_{k=1}^{\infty} a_k \exp[i(\lambda_k(t) - \lambda_j(t))] dt \\ &= \int_{-\ell}^{\ell} \sum_{k=1}^{n} a_k \exp[i(\lambda_k(t) - \lambda_j(t)] dt \\ &+ \int_{-\ell}^{\ell} \sum_{k=n+1}^{\infty} a_k \exp[i(\lambda_k(t) - \lambda_j(t)] dt, \end{aligned} \tag{35}$$

assuming $n > j$. Both sides of this equation must be multiplied by $(2\ell)^{-1}$ and then take the limit as $\ell \to \infty$. Taking into account the equations (14), one obtains from above, since

$$\left| (2\ell)^{-1} \int_{-\ell}^{\ell} \sum_{k=n+1}^{\infty} a_k \exp[i\lambda_k(t) - \lambda_j(t)] dt \right| \leq \sum_{k=n+1}^{\infty} |a_k| < \varepsilon,$$

provided $n > N(\varepsilon) \subset N$, and what remains from (33) when $\ell \to \infty$ is:

$$\lim_{\ell \to \infty} (2\ell)^{-1} \int_{-\ell}^{\ell} f(t)[-i\lambda_j(t)] dt = a_k,$$

i.e., the formula (32) for the coefficients of the function $f(t)$ = the sum of the associated Fourier series, with generalized exponents from Λ.

We can now proceed to prove the validity of the Parseval formula (32), for any $f \in AP_1(R, \mathcal{C}; \Lambda)$. Indeed, we have

$$\int_{-\ell}^{\ell} |f(t)|^2 dt = \int_{-\ell}^{\ell} f(t)\bar{f}(t) dt = \int_{-\ell}^{\ell} \Sigma\, \overline{\Sigma} dt, \tag{36}$$

with Σ from (29)-(31); but, for large n, we can also write

$$\begin{aligned} f(t)\bar{f}(t) &= \sum_{k=1}^{n} |a_k|^2 + \sum_{\substack{k,j=1 \\ k \neq j}}^{n} a_k \bar{a}_j e^{i[\lambda_k(t) - \bar{\lambda}_j(t)]} \\ &+ \left[\sum_{k=n+1}^{\infty} a_k e^{i\lambda_k t} \right] \bar{r}_n(t) + \left[\sum_{k=n+1}^{\infty} \bar{a}_k e^{-i\lambda_k(t)} \right] r_n(t) + |r_n(t)|^2, \end{aligned} \tag{37}$$

with

$$r_n(t) = \sum_{k=n+1}^{\infty} a_k e^{i\lambda_k t}.$$

Let us integrate both sides of the last equation (37) above, from $-\ell$ to ℓ, and multiply both sides by $(2\ell)^{-1}$. If one takes into account the relationships (14), n is sufficiently large, such that $|r_n(t)| < \varepsilon < 1$ for $n \geq N(\varepsilon)$, then, integrating leads to the inequality (38) below, as $\ell \to \infty$:

$$\lim_{\ell \to \infty} (2\ell)^{-1} \int_{-\ell}^{\ell} |f(t)|^2 dt - \sum_{k=1}^{n} |a_k|^2 \leq (2M + 1)\varepsilon, \tag{38}$$

where $M = \sum\limits_{k=1}^{\infty} |a_k| < \infty$, because each of the last two terms in (37) is dominated in modulus by M, while $|r_n(t)|^2 < \varepsilon^2 < \varepsilon$. From (38) one obtains the Bessel inequality, which easily leads to Parseval (32). See our book [5], for instance.

Therefore, we conclude that Parseval's formula (32) is valid for any $f \in AP_1(R, \mathcal{C}; \Lambda)$. We shall see, in the sequel, that its validity takes place in richer spaces of generalized Fourier series, containing $AP_1(R, \mathcal{C}; \Lambda)$.

To continue with the properties of the elements/functions of the space $AP_1(R, \mathcal{C}; \Lambda)$, we shall remark first that the *boundedness* on R, of each $f \in AP_1(R, \mathcal{C}; \Lambda)$, with Λ consisting of continuous generalized exponents, is a direct consequence of the norm definition in formula (31). Let us point out the fact that this property remains valid in more general spaces than $\mathcal{C}$, for example when $\mathcal{C}$ is substituted by a complex Banach space.

Another important fact following from the Parseval formula (32) is the existence of the Poincaré mean value of the square of any $f \in AP_1(R, \mathcal{C}; \Lambda)$. This property will be taken in constructing a richer space of oscillatory functions, denoted by $AP_2(R, \mathcal{C}; \Lambda)$.

We notice the property of *continuity* of the functions in $AP_1(R, \mathcal{C}; \Lambda)$, fact easily derived if we admit the continuity of elements in Λ (the generalized Fourier exponents) and we rely on the absolute and uniform convergence of the series constituting the space $AP_1(R, \mathcal{C}; \Lambda)$.

Concerning the property of uniform *continuity* of functions in $AP_1(R, \mathcal{C}; \Lambda)$, known to be valid for the special case when $\Lambda = \{\lambda t;\ \lambda, t \in R\}$, we notice that we should look closer at the set Λ of generalized exponents, the answer to the problem being certainly determined by the properties of the elements of Λ.

Let us consider the formula from (31), namely

$$f(t) = \sum_{k=1}^{\infty} a_k \exp[i\lambda_k(t)],\ t \in R, \tag{31$'$}$$

and estimate the difference $f(t+h) - f(t)$, $h > 0$. One finds, based on the absolute convergence of the series involved,

$$f(t+h) - f(t) = \sum_{k=1}^{\infty} a_k[\exp i\lambda_k(t+h) - \exp i\lambda_k(t)],\ t \in R, h > 0, \tag{39}$$

with help from the classical formula

$$\exp i\alpha = \cos\alpha + i\sin\alpha,\ \alpha \in R, \tag{40}$$

one easily derive the Lipschitz type inequality for $t \in R$, $h > 0$, $\varepsilon \geq 1$:

$$|\exp i[\lambda_k(t+h)] - \exp[i\lambda_k(t)]| \leq 2|\lambda_k(t+h) - \lambda_k(t)|.$$

Therefore, one obtains from (39)

$$|f(t+h) - f(t)| \leq 2\sum_{k=1}^{\infty} |a_k|\, |\lambda_k(t+h) - \lambda_k(t)|, \tag{41}$$

an inequality which can be discussed in regard to the properties of the set Λ of generalized exponents.

The most direct answer seems to be the following:

The sequence $\{\lambda_k(t),\ k \geq 1\} \subset \Lambda$ admits a continuity module on R, say $\omega(h)$, with $h \to 0$ implying $\omega(h) \to 0$. In other words, one obtains from (41), $f(t+h) \to f(t)$ as $h \to 0$, uniformly with respect to $t \in R$. A more stringent condition would be to have ω as a continuity module for all $\lambda(t) \in \Lambda$. This answer, in the weak form, is suggested by the case when $\Lambda = \{\lambda t;\ \lambda \in R,\ t \in R\}$, i.e., the almost periodic case for the space $AP_1(R, \mathcal{C})$ of Poincaré. In this case, with $\lambda_k(t) = \lambda_k t$, $\lambda_k \in R - \{0\}$, $t \in R$, the continuity module is $\omega_k(h) = |\lambda_k| h$.

Another formulation related to the concept of module of continuity could be phrased in terms of equicontinuity of functions in the set Λ, or some of its parts; for instance, the sequence of exponents $\{\lambda_k(t);\ k \geq 1\}$ is equicontinuous if, for each $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$, such that $|\lambda_k(t) - \lambda_k(s)| < \varepsilon$, for any $t, s \in R$, such that $|t - s| < \delta$. In particular, any sequence $\{\lambda_k(t);\ k \geq 1\} \subset \Lambda$, which is uniformly convergent on R, satisfies the conditions of equicontinuity. Also, a compact subset, countable or not, of Λ, which is compact in respect to the uniform convergence (for instance, a compact subset of the space $BC(R, \mathcal{C})$.

Obviously, from the discussion above, we can infer that the problem of uniform continuity of functions in $AP_1(R, \mathcal{C}; \Lambda)$ has more than one answer. We invite the reader to consider other cases when the uniform continuity is assured.

In the last part of this section, we will consider an example of a space in the same category as the space $AP_1(R,\mathcal{C};\Lambda)$, which presents a particular flavour and allows the illustration of several kinds of convergence. Also, this example will display a sort of classical type of space.

Namely, we shall assume that the set of generalized exponents is given by $\Lambda = AP(R,R)$, i.e., the space of real valued almost periodic functions in the sense of Bohr.

In this case, the series of the real parts of the terms in $\sum\limits_{k=1}^{\infty} a_k \exp[i\lambda_k(t)]$, which has the form $\alpha_0 + \sum\limits_{k=1}^{\infty}[\alpha_k \cos\lambda_k(t) + \beta_k \sin\lambda_k(t)]$, appears to belong to the third stage of generalized Fourier Analysis.

Let us notice that each term in the series above reminding us of the classical form of Fourier series is in $AP_r(R)$, which means that a third stage in Fourier Analysis can produce spaces of oscillatory functions also belonging to the classical heritage. Of course, the main problem in constructing spaces of oscillatory functions consists in obtaining new spaces, not pertaining to the classical category. The kind of convergence we associate with the linear space of formal series, like (1), may or may not lead to the space $AP(R,\mathcal{C})$, or to a subspace of the latter in the case $\Lambda = AP(R,R)$.

With these considerations, we end the problems/properties related to the space $AP_1(R,\mathcal{C};\Lambda)$, moving to another space of oscillatory functions, constructed in a similar manner as above and relying on the construction and the consequences for the space $AP_1(R,\mathcal{C};\Lambda)$.

4 Construction of the Space $AP_2(R,\mathcal{C};\Lambda)$

In constructing the space of oscillatory functions, denoted by $AP_2(R,\mathcal{C};\Lambda)$, we can associate the names of Besicovitch and Zhang to this type of space. In case of classical spaces of almost periodic functions, the space $AP_2(R,\mathcal{C})$ represents the Besicovitch space. In case of oscillatory functions spaces, the first examples are those described in Section 1 (Introduction) of this paper, when $\Lambda = Q(R,R)$. See formulae (3) and (4) for details. This type of space, with a special choice of Λ, is due to Zhang, who was the first to express the need of getting more comprehensive spaces of oscillatory functions, than the spaces of almost periodic functions. This need is motivated by the applications of Fourier Analysis, found in engineering literature and pertinent references are included in Zhang's papers. His pseudo almost periodic functions (1992, Ph.D. thesis), which have generated a vast literature in the last 20 years constitute a convincing example that shows the necessity of constructing new spaces of oscillatory functions. Moreover, the pseudo almost periodic functions appear as "perturbations" of the classical almost periodic functions, while their theory has many points of contact with the old theory.

In order to construct the space of oscillatory functions $AP_2(R,\mathcal{C};\Lambda)$, we will introduce in the linear (algebraic) space of generalized trigonometric series, with Λ as in the case of the space $AP_1(R,\mathcal{C};\Lambda)$ already described, the norm

$$\left|\sum_{k=1}^{\infty} a_k \exp[i\lambda_k(t)]\right|_{AP_2} = \left(\sum_{k=1}^{\infty} |a_k|^2\right)^{1/2}, \tag{42}$$

i.e., the norm of the classical space $\ell^2 = \left\{a_k,\ k \geq 1, \sum\limits_{k=1}^{\infty} |a_k|^2 < \infty\right\}$ of Hilbert.

In the space of sum functions, associated to the series space $AP_2(R, \mathcal{C}; \Lambda)$, we shall use the seminorm, compatible with (42), which looks

$$|f|_{AP_2} = \left[\lim_{\ell \to \infty} (2\ell)^{-1} \int_{-\ell}^{\ell} |f(t)|^2 dt \right]^{1/2}, \tag{43}$$

which is derived from Poincaré mean value on R and has been used by Besicovitch in the space $B^2(R, \mathcal{C})$ of his almost periodic functions (a natural generalization of Bohr's theory).

The compatibility will result from the validity of Parseval's formula (32), whose validity has been already established in AP_1. In order to obtain Parseval's formula in case $f \in AP_2(R, \mathcal{C}; \Lambda)$, we can proceed in the same way as in case of the space $AP_1(R, \mathcal{C}; \Lambda)$.

But we need, first, to look closely to the relationship/correspondence between series in AP_2 and sum-function attached. We shall show, first, that to each series in AP_2 one can attach a function belonging to the space $L^2_{\text{loc}}(R, \mathcal{C})$. Indeed, for such a series of the form $\sum\limits_{k=1}^{\infty} a_k \exp[i\lambda_k(t)]$, with $\{a_k;\ k \geq 1\} \subset \ell^2$ and $\lambda_k : R \to R$, we can write for $n, p \in N$,

$$\lim_{\ell \to \infty} (2\ell)^{-1} \int_{-\ell}^{\ell} \left| \sum_{k=n+1}^{n+p} a_k \exp(i\lambda_k(t)) \right|^2 dt = \sum_{k=n+1}^{n+p} |a_k|^2, \tag{44}$$

taking into account the orthogonality of the sequence of $\lambda_k(t)$'s and the relationship $|u|^2 = u\bar{u}$. From (44) and our assumption, we have included in defining the $AP_2(R, \mathcal{C}; \Lambda)$,

$$\sum_{k=1}^{\infty} |a_k|^2 < \infty, \tag{45}$$

we conclude that the series of $AP_2(R, \mathcal{C}; \Lambda)$ are convergent with respect to the seminorm chosen for this space. Moreover, the convergence in $AP_2(R, \mathcal{C}; \Lambda)$ is implying the convergence in $L^2_{\text{loc}}(R, \mathcal{C})$. This property of Fourier series is proven in our paper [8], in the special case $\Lambda = \{\lambda t;\ \lambda \in R,\ t \in R\}$. It remains valid in the general case, when $\lambda_k(t),\ k \geq 1$, are more general functions than in the case $\lambda_k(t) = \lambda_k t,\ k \geq 1,\ \lambda \in R$, corresponding to the almost periodic functions of all known types.

We shall write, as usual in the theory of oscillatory functions, including the classical types, in the traditional form

$$f(t) \simeq \sum_{k=1}^{\infty} a_k \exp[i\lambda_k(t)],\ t \in R, \tag{46}$$

the fact that the function $f(t)$ is constructed by means of the series in the right hand side of (46). The manner of determining the coefficients a_k, $k \geq 1$, in terms of $f(t)$, will be discussed in this section. The formulae providing the a_k's are

$$a_k = \lim_{\ell \to \infty} (2\ell)^{-1} \int_{-\ell}^{\ell} f(t) \exp[-i\lambda_k(t)] dt, \tag{47}$$

i.e., formally, the same as (32), valid for $f \in AP_1(R, \mathcal{C}; \Lambda)$.

In order to derive (47) for $f \in AP_2(R, \mathcal{C}; \Lambda)$, we shall mention the fact that the space $AP_1(R, \mathcal{C}; \Lambda)$ is everywhere dense in the space $AP_2(R, \mathcal{C}; \Lambda)$. This property follows from the fact that, taking into account the definitions of the norm/seminorm in the spaces AP_1 and AP_2, the generalized trigonometric polynomials of the form

$$P(t) = \sum_{k=1}^{n} a_k \exp[i\lambda_k(t)], \ t \in R, \tag{48}$$

constitute everywhere dense sets in both spaces AP_1 and AP_2. Of course, the exponents $\lambda_k(t)$, $1 \leq k \leq n$, are chosen from Λ, for either space.

Let us notice that (49) is elementary in case of $f(t)$ being polynomial of the form (48). We have proven its validity, above in this section, for any $f \in AP_1(R, \mathcal{C}; \Lambda)$. Since $AP_1 \subset AP_2$, due to the inclusion $\ell^1 \subset \ell^2$, we can regard the whole operations as taking place in the space $AP_2(R, \mathcal{C}; \Lambda)$.

As observed above, for each $f \in AP_2(R, \mathcal{C}; \Lambda)$, there exists a sequence in $AP_1(R, \mathcal{C}; \Lambda)$, such that for each $f \in AP_2(R, \mathcal{C}; \Lambda)$ one has $f^{(j)} \to f$ in $AP_2(R, \mathcal{C}; \Lambda)$, as $j \to \infty$. But the convergence of a sequence in either space AP_1 or AP_2, is uniform on coordinates. That means that from

$$f^{(j)} \to f \text{ in } AP_2(R, \mathcal{C}; \Lambda), \tag{49}$$

there follow the convergence relations

$$a_k^j \to a_k \text{ as } j \to \infty, \ k \geq 1, \text{ uniformly.} \tag{50}$$

There remains to prove that $a_k, \ k \geq 1$, are indeed the coefficients of $f \in AP_2(R, \mathcal{C}; \Lambda)$.

It is useful to remark the following: If one deals with a countable set of series like the set of series for $f^{(j)}, \ j \geq 1$, there is no loss of generality if we assume that all series have the same generalized Fourier exponents. This is achieved by adding terms, with zero coefficients, after having the set of all exponents, forming a sequence, hence a countable set. This operation does not influence the conditions of convergence (31) and (45).

There remains to prove that the limits $a_k, \ k \geq 1$, are given by the formulae (47), i.e.,

$$a_k^{(j)} - a_k = \lim_{\ell \to \infty} (2\ell)^{-1} \int_{-\ell}^{\ell} [f^{(j)}(t) - f(t)] \exp[-i\lambda_k(t)] dt, \tag{51}$$

tends to zero as $j \to \infty, \ k \geq 1$.

The following estimates are routine in a calculus course. Indeed, one has

$$\begin{aligned} &\left| (2\ell)^{-1} \int_{-\ell}^{\ell} [f^{(j)}(t) - f(t)] \exp[-i\lambda_k(t)] dt \right| \\ &\quad \leq (2\ell)^{-1} \int_{-\ell}^{\ell} |f^{(j)}(t) - f(t)| dt \\ &\quad \leq (2\ell)^{-1} \left[\int_{-\ell}^{\ell} |f^{(j)}(t) - f(t)|^2 dt \right]^{1/2} (2\ell)^{1/2} \\ &\quad = \left[(2\ell)^{-1} \int_{-\ell}^{\ell} |f^{(j)}(t) - f(t)|^2 dt \right]^{1/2} . \end{aligned} \tag{52}$$

The last term in (52) is as $\ell \to \infty$, exactly the norm of $f^{(j)}(t) - f(t) \in AP_2(R, \mathcal{C}; \Lambda)$, which implies it tends to zero as $j \to \infty$, by the choice of the approximating sequence $\{f^{(j)}(t);\ j \geq 1\} \subset AP_2(R, \mathcal{C}; \Lambda)$. Taking into account (51) and (52), one obtains what is required to derive that (49) is correct, it representing the connection between the Fourier series and its generalized sum, in $AP_2(R, \mathcal{C}; \Lambda)$.

Based on facts easily obtained in case of the space $AP_1(R, \mathcal{C}; \Lambda)$, which is dense in the space $AP_2(R, \mathcal{C}; \Lambda)$, we can extend results from $AP_1(R, \mathcal{C}; \Lambda)$ to the richer space $AP_2(R, \mathcal{C}; \Lambda)$, using the procedure above, when getting the formulae for the coefficients of the generalized Fourier series.

For instance, the Parseval equality (32), valid for $f \in AP_1(R, \mathcal{C})$, can be extended as proceeded above for $f \in AP_2(R, \mathcal{C}; \Lambda)$. It will look exactly as (32), which in the geometry of the Hilbert space $\ell^2 = \ell^2(N, \mathcal{C})$ means that the "length" of the limit of a convergent sequence is the limit of the sequence of lengths of the terms in the sequence. We leave to the reader the task of carrying out the details of the proof of (32), for $f \in AP_2(R, \mathcal{C}; \Lambda)$. Of course, Λ has to be the same set of generalized Fourier exponents, in AP_1 and AP_2.

Another proof of the Parseval formula (32) can be obtained based on the model we inherited from the classical period of almost periodicity. The details can be found in the author's book [5], as well as in many other sources. Instead of the exponents $\lambda_k t$, for almost periodic functions, one can substitute the general exponents $\lambda_k(t) \in \Lambda$, for oscillatory functions.

Further properties of the space $AP_2(R, \mathcal{C}; \Lambda)$ can be derived, taking into account its structure of a Banach space, whose elements are generalized Fourier series of the form (1).

We want to define the identity of two series of the form $\sum\limits_{k=1}^{\infty} a_k \exp[i\lambda_k(t)]$, with the usual significance of the data involved: one must have $\lambda_k(t) \in \Lambda =$ the set of generalized exponents, the same in both formal series and with equal coefficients for the same exponent.

When a norm or a seminorm is defined, usually implying a kind of convergence, we obtain a linear normed space which requires the completeness in order to become a Banach space. Another type of condition can be imposed, to help organizing the space of series (for instance, a kind of summability).

Once found a way of organizing the space of series like a linear metric space, the next step is to move from the series space to a function space, the series playing the role of a vehicle, or an intermediate stage, in the construction of the function space. We have illustrated this in constructing the spaces $AP_1(R, \mathcal{C}; \Lambda)$ and $AP_2(R, \mathcal{C}; \Lambda)$. In the literature, see particularly the quotation in the bibliography to this paper under the names of Osipov and Zhang, cases which we have summarily presented in the Introduction. Basically, one obtains such spaces of oscillatory functions by using the procedure of completion with respect to various norms or seminorms of simpler spaces (usually, classical ones).

5 More Spaces of Oscillatory Functions

It is clear from the preceding sections of this paper, including the Introduction, that once we succeed to find a set Λ of generalized Fourier exponents, one can construct several types of spaces of oscillatory functions. So far, we've got acquainted, to some extent, with the spaces built up by Osipov, Zhang and those in Sections 3 and 4, denoted by $AP_1(R,\mathcal{C};\Lambda)$ and $AP_2(R,\mathcal{C};\Lambda)$. In case of spaces $AP_1(R,\mathcal{C};\Lambda)$ and $AP_2(R,\mathcal{C};\Lambda)$, the set of generalized Fourier exponents Λ does not possess an algebraic structure, necessarily. The operations of multiplication of elements will imply the necessity of having the set Λ organized as an additive group of real-valued functions on R. The classical examples, periodic and almost periodic, illustrate the need and the involved groups: in the periodic case, the set Λ is given by $\Lambda = \{\lambda t;\ \lambda = k\omega,\ k \in \mathcal{Z},\ \omega > 0,\ \text{constant}\}$ = any closed subgroup of the topological group R; in the almost periodic case, $\Lambda = \{\lambda t;\ \lambda, t \in R\}$. In the Introduction, in case of the examples due to Osipov and Zhang, the generalized exponents for the Osipov type oscillatory functions have the form $\Lambda = \{at^2 + b_k t;\ a \in R,\ b_k \in R\}$, while in case of Zhang constructions, the generalized Fourier exponents belong to the set $Q(R;R)$; see formulae (3), (4), where the definition of the set $Q(R;R)$ is provided.

Section 2 is attempting to provide some tools in finding generalized exponents for series forming oscillatory spaces. The problem of finding such exponents must be investigated further. Some suggestions must come from the applicative problems. One can construct already many spaces of oscillatory functions, but their significance is depending of their area of applications.

In this closing section, we shall briefly list and describe some other spaces of oscillatory functions, constructed in several ways, always starting with a set of formal series characteristic for oscillatory functions and giving some comments on possible developments of the theory, formulating also some open problems. Of course, these ideas are directed toward the theoretic, but also deeply practical aim, to have in the future a developed theory of the spaces of oscillatory functions. This development, if achieved, will certainly constitute the third stage in the Fourier theory of vibrations and waves.

We shall start with the definition of the oscillatory function spaces we shall denote by $AP_r(R,\mathcal{C};\Lambda)$, $1 < r < 2$, the cases $r = 1, 2$ being treated in the preceding section. Taking the example from existing literature, namely Shubin [16] and Corduneanu [5], the series spaces $AP_r(R,\mathcal{C};\Lambda)$ will be formed from the generalized Fourier series like (1), i.e., $\sum_{k=1}^{\infty} a_k \exp[i\lambda(t)]$, with $a_k \in \mathcal{C}$ and $\lambda_k \in C(R,R)$, $\lambda_k \in \Lambda$, $k \geq 1$, with the following property:

$$\sum_{k=1}^{\infty} |a_k|^r < \infty. \tag{53}$$

We introduce the norm, in the linear space (over $\mathcal{C}$), of the set of formal series of the form (1), by the formula

$$\left|\sum_{k=1}^{\infty} a_k \exp[i\lambda_k(t)]\right|_{AP_1} = \left(\sum_{k=1}^{\infty} |a_k|^r\right)^{1/r}. \tag{54}$$

These norms are known as Minkowski's norms and the related inequalities are making easier the proof of the norm properties in linear normed spaces. The completion of the space of series satisfying (53) follows from the simple remark that the polynomials associated to such series (sections like $\sum_{k=1}^{n} a_k \exp[i\lambda_k(t)]$), form an everywhere dense set in $AP(R,\mathcal{C};\Lambda)$. Hence, the spaces $AP_r(R,\mathcal{C};\Lambda)$ can be organized as Banach spaces.

These spaces, with $1 < r < 2$, enjoy many properties that can be derived from the inclusions

$$AP_1 \subset AP_r \subset AP_s \subset AP_2,\ 1 < r < s < 2, \tag{55}$$

which show that they are part of AP_2, the space we have constructed above. In particular, being also AP_2-series, they have Fourier type series (generalized) of the form (1).

For a more detailed discussion of one space in the categories of AP_r-spaces, one can consult the author's paper [6]. Several applications are provided for several types of functional differential equations, including integral equations, convolution and mixed types of functional equations (integro-differential, convolutions, delay type).

Like in the special case when $\Lambda = \{\lambda t;\ \lambda, t \in R\}$, i.e., the almost periodic type of functions, the series spaces $AP_r(R, \mathcal{C}; \Lambda)$, with the same Λ, they form a scale of oscillatory functions when we regard their elements as parts of $AP_2(R, \mathcal{C}; \Lambda)$, for which space we have more accessible information (they are modeled on the Hilbert space $\ell^2(N, \mathcal{C})$). The stronger type of convergence we find in $AP_1(R, \mathcal{C}; \Lambda)$, while the weaker one corresponds to $AP_2(R, \mathcal{C}; \Lambda)$.

We point out the fact that spaces of this scale have been seldom in attention of researchers. Many problems, like convergence of their series in different meanings (say, pointwise to uniform or a.e.) still wait for detailed investigation. Also, the problem of compactness for sets in such spaces is still unsolved, excepting in case of Zhang's space $SLP(R, \mathcal{C}; \Lambda)$, for $\Lambda = Q(R; R)$. See Zhang [19] and the Appendix in Corduneanu's et al. book [10].

With regard to the space $SLP(R, \mathcal{C}; \Lambda)$, in more general cases than a specific Λ has been considered, it is worth getting in some details of the construction. This type of space is different from those in the scale $AP_r(R, \mathcal{C}; \Lambda)$, $1 \le r \le 2$, in the fact that, instead of conditions on the coefficients only, like (28), (45), from the beginning one imposes the type of convergence. Namely, the space $SLP(R, \mathcal{C}; \Lambda)$ is the function space whose elements/functions can be *uniformly* approximated on R by means of generalized trigonometric polynomials of the form (4).

Since Zhang wanted to organize the space as an algebra, which idea brought some advantages, the special type of Λ has been used. A question: are there other choices for Λ, in order to achieve new spaces in the family of $SLP(R, Q)$?

Zhang [19] relied on this space ($\Lambda = Q$) to construct two new spaces of generalized Fourier type (oscillatory functions spaces).

These new spaces generalize the Besicovitch type of almost periodicity. In very brief format, the first of these spaces is obtained by completing the space $SLP(R, \mathcal{C}; Q)$ with respect to the norm

$$f \to \{M(|f|^2)\}^{1/2}, \tag{56}$$

while the second is the completion of $SLP(R, \mathcal{C}; \Lambda)$ with respect to the norm

$$f \to M(|f|), \tag{57}$$

where M stands for the Poincaré mean value on R. It turns out that the normed spaces with either norm (56) or (57) are not complete, in general. For instance, the norm which is given by (57), derives from

$$M(g) = \lim_{\ell \to \infty} (2\ell)^{-1} \int_{-\ell}^{\ell} g(t)dt,$$

and satisfies the inequality $|M(g)| \le |g|$, where $|g|$ represents the supremum norm (as used by Zhang in constructing $SLP(R, \mathcal{C}; Q)$). But, in the case of almost periodic functions space $AP(R, \mathcal{C})$, the space itself generates the Besicovitch space $B(R, \mathcal{C})$, or $B = B^1$, which is not complete. See an example in Corduneanu et al. [10], the Appendix, or in [8]. Examples for oscillatory functions spaces await their apparition. That's depending on the possibility of getting an adequate Λ.

In the author's paper [7], one finds a reconstruction of the Bohr space $AP(R, \mathcal{C})$, starting from the set of all formal trigonometric series of the form $\sum_{k=1}^{\infty} a_k \exp[i\lambda t]$, $t \in R$, with $a_k \in \mathcal{C}$ and $\lambda_k \in R$, $k \geq 1$.

The condition which allows us to detach those that characterize those of Bohr almost periodic functions is somewhat of a different nature than conditions (28) and (45), utilized above. Instead of imposing conditions on coefficients, of a quantitative nature, we shall require that the series be summable by a linear method (for instance, the Cesaro-Fejer-Bochner method), with respect to the uniform convergence on R. The set of exponents, apparently, does not play a direct role, such method being also based on the coefficients.

Indeed, it is known, from the theory of Bohr almost periodic functions, that their series are summable by the Cesaro-Fejer-Bochner method with respect to the uniform convergence on R. Then, the "sum" is Bohr almost periodic. In other words, a trigonometric series like $\sum_{k=1}^{\infty} a_k \exp[i\lambda_k t]$, $a_k \in \mathcal{C}$, $\lambda \in R$, $k \geq 1$, is characterizing an almost periodic function in Bohr space $AP(R, \mathcal{C})$, iff it is summable with respect to the uniform convergence on RF. As we know, the uniform convergence is induced by the supremum norm.

In concluding this paper, we emphasize again the need of investigation of these spaces of series, like (1), defining the third stage of development in generalized Fourier Analysis. Of course, the Fourier Analysis has many other chapters, inspired by the investigation of classical series and the extension of such aspects appears as a future task for researchers.

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