Nonlinear Dynamics and Systems Theory, **17** (1) (2017) 5–18

Approximate Controllability of Non-densely Defined Semilinear Control System with Non Local Conditions

Urvashi Arora * and N. Sukavanam

Department of Mathematics, Indian Institute of Technology Roorkee, Roorkee-247667, India

Received: January 21, 2016; Revised: January 23, 2017

Abstract: The present paper is devoted to the study of approximate controllability of nondensely defined semilinear control system with nonlocal conditions. The approximate controllability is obtained with nonlinearity satisfying the monotone condition and integral contractor condition. Finally, an example is provided to illustrate the application of the obtained results.

Keywords: *approximate controllability; semilinear systems; nondense domain; nonlocal conditions.*

Mathematics Subject Classification (2010): 93B05, 93C10.

1 Introduction

Controllability concepts play a vital role in deterministic control theory. It is well known that controllability of deterministic equation is widely used in many fields of science and technology. Kalman [16] introduced the concept of controllability for finite dimensional deterministic linear control systems. Then Barnett [3] and Curtain [5] introduced the concepts of deterministic control theory in finite and infinite dimensional spaces. Balachandran [2] and Dauer et al. [7] studied the controllability of nonlinear systems in infinite dimensional spaces. The controllability of linear and nonlinear systems in infinite dimensional spaces has been extensively studied by many authors, when the operator A is densely defined, see [2, 7, 15, 19, 21, 23, 26]. On the other hand, we sometimes need to deal with non-densely defined operator. It is a very important case, which occurs in many practical situations. For example, the space C^1 with null values on the boundaries is not dense in the space of continuous functions, see [6]. For more examples and

* Corresponding author: mailto:urvashiaroraiitr@gmail.com

details on non-densely defined operators, one can refer [6, 8]. Xianlong [10] considered this case and studied the controllability of the semilinear system with delay, in which the nonlinear function was uniformly bounded. Recently many authors have discussed this case [17, 20].

Moreover, nonlocal conditions have a better effect on the solution and are more precise for physical measurements than classical condition $x(0) = x_0$ alone. Byszewski and Lakshmikantham [4] introduced nonlocal conditions into the initial value problems and argued that the corresponding models more accurately describe the phenomena since more information was taken into account at the oneset of the experiment, thereby reducing the ill effects incurred by a single initial measurement. Also, in controllability literature, it is common to use fixed point theory to prove the controllability of the system, which makes it necessary to assume certain inequality conditions involving system constants.(For example, see inequality (3) in [9]). In this paper, it is shown that for certain type of nonlinear functions, the non-densely defined semilinear control system with nonlocal conditions is approximately controllable without assuming any inequality conditions on the system constants.

Let U and V be two Banach spaces. $Y = L_2[0, T; U]$ and $Z = L_2[0, T; V]$ be the corresponding function spaces respectively defined on $J = [0, T]$, $0 \leq T < \infty$. Consider the semilinear control system with nonlocal conditions:

$$\left.\begin{aligned} \frac{dy(t)}{dt} &= Ay(t) + Bv(t) + f(t, y(t)) \ \text{ for } t \in (0, T], \\ y(0) &= y_0 + g(y), \end{aligned}\right\} \tag{1.1}$$

where the state $y(t)$ takes values in space V and $v : [0, T] \to U$ is the control function. B is a bounded linear operator from U into V. The map $f : [0, T] \times V \to V$ is a purely nonlinear function and $g(y)$ is a continuous function from $C(J, V) \to V$. $A : D(A) \subset V \to V$ is a closed (not necessarily bounded) linear operator whose domain $D(A)$ need not be dense in V. The linear system corresponding to (1.1) is given by

$$\left.\begin{aligned} \frac{dy(t)}{dt} &= Ay(t) + Bu(t) \ \text{ for } t \in (0, T], \\ y(0) &= y_0 + g(y), \end{aligned}\right\} \tag{1.2}$$

where $u : [0, T] \to U$ is the control function for the linear system.

2 Preliminaries

We introduce the integrated semigroup.

Definition 2.1 Let V be a Banach space. A one parameter family of bounded linear operators $\{S'(t) : t \geq 0\}$ from V into itself is said to be an integrated semigroup on V if

1. $S'(0) = 0$.
2. $t \to S'(t)$ is strongly continuous.
3. $S'(s)S'(t) = \displaystyle\int_0^s \{S'(t + r) - S'(r)\}dr = S'(t)S'(s)$; for all $t, s \geq 0$.

Definition 2.2 [10] A function $y : [0, T] \to V$ is said to be an integrated solution of the system (1.1) if the following conditions hold

1. y is continuous on $[0, T]$.

2. $\int_0^t y(s)ds \in D(A)$; for all $t \in J$.

3. $y(t) = (y_0 + g(y)) + A \int_0^t y(s)ds + \int_0^t [Bv(s) + f(s, y(s))]ds$.

Definition 2.3 [27] An operator A is called a generator of an integrated semigroup if there exists $\omega \in \mathbb{R}$ such that $(\omega, \infty) \subset \rho(A)$ (the resolvent set of A) and there exists a strongly continuous exponentially bounded family $\{S'(t) : t \geq 0\}$ of bounded linear operators such that $S'(0) = 0$ and $(\lambda I - A)^{-1} = \int_0^\infty e^{-\lambda t} S'(t)dt$ for all $\lambda > \omega$.

Let $y(T, y_0, v)$ denote the state value of the system (1.1)at time T corresponding to the control $v \in Y$ and the initial value y_0. Now, we introduce the set defined by

$$K_T(f) = \{y(T, y_0, v); v \in Y\}$$

which consists of all the possible final states and is called the reachable set of the system.

Definition 2.4 A control system is said to be approximate controllable on $[0, T]$, if $K_T(f)$ is dense in $\overline{D(A)}$, that is $\overline{K_T(f)} = \overline{D(A)}$.

Throughout this paper, the operator A is assumed to satisfy the following Hille-Yosida condition(without being densely defined), see [25]:
(H_0) there exists a constant $\overline{M} \geq 0$ and $\overline{\omega} \in N$ such that $(\overline{\omega}, \infty) \subset \rho(A)$ and

$$\sup\{(\lambda - \overline{\omega})^n ||R(\lambda, A)^n|| : n \in N \text{ and } \lambda > \overline{\omega}\} \leq \overline{M}, \tag{2.1}$$

where $R(\lambda, A) = (\lambda I - A)^{-1}$.

It is well known that the above condition is equivalent to the fact that operator A is the generator of a locally Lipschitz integrated semigroup $\{S'(t) : t \geq 0\}$ on V, see [18].

Let A_0 be the part of A defined on the domain

$$D(A_0) = \{x \in D(A) : Ax \in \overline{D(A)} \text{ and } A_0 x = Ax, \text{ for all } x \in D(A_0)\}.$$

Then $\overline{D(A_0)} = \overline{D(A)}$ and the generator A_0 generates a C_0- semigroup $\{T_0(t) : t \geq 0\}$ on $\overline{D(A)}$, see [18]. If the integral solution, as given in Definition 2.2 exists then it is given by

$$y(t) = T_0(t)(y_0 + g(y)) + \lim_{\lambda \to \infty} \int_0^t T_0(t - s)C(\lambda)[Bv(s) + f(s, y(s))]ds, \tag{2.2}$$

where $C(\lambda) = \lambda R(\lambda, A) = \lambda(\lambda I - A)^{-1}$.

Now, we define the following functions:
$F : Z \to Z$ as

$$(Fy)(t) = f(t, y(t)), \quad y \in Z,$$

and $K : Z \to Z$ as

$$(Ky)(t) = \lim_{\lambda \to \infty} \int_0^t T_0(t - s)C(\lambda)y(s)ds.$$

3 Controllability Results with Monotone Nonlinearity

In this section, we prove the controllability results of the system when the nonlinear function satisfies monotone condition. To prove the approximate controllability of the system (1.1), we assume the following conditions:

(H_1) There exists a constant $\mu > 0$ such that for all $x \in D(A)$

$$< -Ax, x >_V \geq \mu ||x||_V^2.$$

(H_2) Linear system is approximate controllable up to $\overline{D(A)}$.

(H_3) The semigroup $\{T_0(t), t \geq 0\}$ generated by A is compact on $\overline{D(A)}$ and there is a constant $M \geq 0$ such that

$$||T_0(t)|| \leq M, \text{ for all } t \in [0, T].$$

(H_4) f satisfies monotone condition, that is, there is a positive constant β such that

$$< f(t, x) - f(t, y), x - y >_V \leq -\beta ||x - y||_V^2.$$

(H_5) $||Fy||_Z \leq a + b||y||_Z$ where a and b are constants.

(H_6) $R(F) \subseteq \overline{R(B)}$.

(H_7) The function g is a continuous function and there exists a constant M_1 such that

$$||g(y)|| \leq M_1 \text{ for all } y \in D(A).$$

This section has two cases. In subsection 3.1, the controllability is proved for the case when the control operator B is an identity operator and subsection 3.2 contains the general case.

3.1 Controllability of semilinear system when $B = I$

In this subsection, it is proved that the semilinear control system (1.1) in which B is an identity operator is approximate controllable under simple sufficient conditions. Obviously, here $V = U$. In this case, the semilinear control system (1.1) becomes

$$\left.\begin{array}{rcl} y'(t) &=& Ay(t) + Bv(t) + f(t, y(t)); \quad 0 \leq t \leq T, \\ y(0) &=& y_0 + g(y), \end{array}\right\} \tag{3.1}$$

and the system (1.2) becomes

$$\left.\begin{array}{rcl} y'(t) &=& Ay(t) + Bu(t); \quad 0 \leq t \leq T, \\ y(0) &=& y_0 + g(y). \end{array}\right\} \tag{3.2}$$

Before proving the main result, we prove one lemma.

Lemma 3.1 *Under the condition (H_1), the operator K which is defined on Z satisfies the condition*

$$< Kx, x >_Z \geq \mu ||x||_Z^2 \textit{ for all } x \in Z. \tag{3.3}$$

Proof. Let $x \in Z$. Now, let us define a function ϕ as follows

$$\phi(t) = \lim_{\lambda\to\infty} \int_0^t T_0(t-s)C(\lambda)x(s)ds$$

since

$$< Kx, x >_Z = \int_0^T < Kx(t), x(t) >_V \, dt = \int_0^T < \phi(t), x(t) >_V \, dt. \tag{3.4}$$

But

$$\begin{aligned} \phi'(t) &= \lim_{\lambda\to\infty} \left[C(\lambda)x(t) + A\int_0^t T_0(t-s)C(\lambda)x(s)ds \right] \\ &= x(t) + A\phi(t) \\ \Rightarrow x(t) &= \phi'(t) - A\phi(t). \end{aligned} \tag{3.5}$$

From the equations (3.4) and (3.5), we get

$$\begin{aligned} < Kx, x >_Z &= \int_0^T < \phi(t), \phi'(t) - A\phi(t) >_V \, dt \\ &= \int_0^T < \phi(t), \phi'(t) >_V \, dt + \int_0^T < \phi(t), -A\phi(t) >_V \, dt. \end{aligned} \tag{3.6}$$

Since

$$\begin{aligned} \frac{d}{dt} < \phi(t), \phi(t) >_V &= < \phi(t), \phi'(t) >_V + < \phi'(t), \phi(t) >_V \, dt \\ &= 2 < \phi(t), \phi'(t) >_V \\ \Rightarrow \int_0^T \frac{d}{dt} < \phi(t), \phi(t) >_V \, dt &= 2\int_0^T < \phi(t), \phi'(t) >_V \, dt \\ \Rightarrow \int_0^T < \phi(t), \phi'(t) >_V \, dt &= \frac{1}{2}||\phi(T)||_V^2 \geq 0 \end{aligned} \tag{3.7}$$

and by the condition (H_1), we have

$$\int_0^T < \phi(t) - A\phi(t) >_V \, dt \geq \mu||x||^2. \tag{3.8}$$

Therefore, from the equations (3.6),(3.7) and (3.8), we get

$$< Kx, x >_Z \geq \mu||Kx||_Z^2.$$

Theorem 3.1 *Under the conditions* $(H_0) - (H_4)$*, the semilinear control system (3.1) is approximate controllable in the time interval* $[0, T]$.

Proof. Let $x(t)$ be the integral solution of the system (3.2) corresponding to the control u. Consider the following semilinear system

$$\left.\begin{aligned} y'(t) &= Ay(t) + u(t) - f(t, x(t)) + f(t, y(t)); \quad 0 \leq t \leq T, \\ y(0) &= y_0 + g(y). \end{aligned}\right\} \tag{3.9}$$

Comparing (3.1) and (3.9), it can be seen that the control function $v(t)$ is chosen that

$$v(t) = u(t) - f(t, x(t)). \tag{3.10}$$

The integral solutions of systems (3.2) and (3.9) can be written as

$$x(t) = T_0(t)(y_0 + g(y)) + \lim_{\lambda\to\infty} \int_0^t T_0(t-s)C(\lambda)u(s)ds, \tag{3.11}$$

$$y(t) = T_0(t)(y_0 + g(y)) + \lim_{\lambda\to\infty} \int_0^t T_0(t-s)C(\lambda)[u(s) - f(s, x(s)) + f(s, y(s))]ds. \tag{3.12}$$

Substracting (3.12) from (3.11), we get

$$x(t) - y(t) = \lim_{\lambda\to\infty} \int_0^t T_0(t-s)C(\lambda)[f(s, x(s)) - f(s, y(s))]ds, \tag{3.13}$$

which in operator theoretic terms can be written as

$$x - y = KFx - KFy.$$

Taking inner product on both sides in Z with $Fx - Fy$, we get

$$< x - y, Fx - Fy >_Z = < KFx - KFy, Fx - Fy >_Z . \tag{3.14}$$

Note, that the left-hand side satisfies the condition (H_4) and it is less than or equal to $-\beta||x-y||_Z^2$ and the right-hand side is nonnegative, from Lemma 3.1. This is possible only when $x = y$ in Z, which implies that $F(x) = F(y)$, where F is Nemytskii operator defined by f. Therefore, from the equation (3.13), it follows that $x(t) = y(t)$ for all $t \in [0, T]$. Thus, any mild solution of the linear system (3.2) is also a mild solution of the semilinear system (3.1), that is, $K_T(f) \supset K_T(0)$, which is dense in $\overline{D(A)}$. Hence, system (3.1) is approximate controllable on $[0, T]$.

3.2 Controllability of semilinear system when $B \neq I$

In this subsection, the approximate controllability of the system (1.1) is proved under some sufficient conditions on the operators A, B and f.

Since $R(F) \subseteq \overline{R(B)}$, (see condition (H_6)), for any given $\epsilon_1 > 0$, there exists a ω in $L_2[0, T; U]$ such that

$$||Fx - Bw||_Z \leq \epsilon_1. \tag{3.15}$$

Before proving the main result, we prove one lemma.

Lemma 3.2 *The solution of the system (1.1) and the corresponding control $v = u - w$ satisfy the following inequality*

$$\begin{aligned} ||y(t)||_V \leq\ & [M(1 + bM\overline{M}\sqrt{T})(||y_0|| + M_1) + M\overline{M}\sqrt{T}\{(1 + M\overline{M}bT)||Bu||_Z + a + \epsilon_1\} \\ & + M\overline{M}aT]e^{M\overline{M}bT}, \end{aligned}$$

where M is a positive constant such that $||T_0(t)|| \leq M$, for each $t \in [0, T]$ and $\overline{M}$ is defined in (2.1).

Proof. Let $y(t)$ be the integral solution of the system (1.1) corresponding to the control $v = u - w$. Then the integral solution of the system (1.1) can be written as

$$\begin{aligned} y(t) &= T_0(t)(y_0 + g(y)) + \lim_{\lambda\to\infty}\int_0^t T_0(t-s)C(\lambda)B(u-w)(s)ds + \\ &\quad \lim_{\lambda\to\infty}\int_0^t T_0(t-s)C(\lambda)f(s,y(s))ds. \end{aligned}$$

Taking V-norm on both sides and using the fact that $\lim_{\lambda\to\infty}||C(\lambda)|| = \overline{M}$, we get

$$\begin{aligned} ||y(t)||_V &\leq M||y_0|| + M||g(y)|| + M\overline{M}\int_0^t ||B(u-w)(s)||_V ds \\ &\quad + M\overline{M}\int_0^t ||f(s,y(s))||_V ds \\ &\leq M||y_0|| + M||g(y)|| + M\overline{M}\sqrt{T}(||Bu||_Z + ||Bw||_Z) \\ &\quad + M\overline{M}\int_0^t (a + b||y(s)||_V)ds \\ &\leq M||y_0|| + MM_1 + M\overline{M}\sqrt{T}(||Bu||_Z + ||Fx||_Z + \epsilon_1) + M\overline{M}aT \\ &\quad + M\overline{M}b\int_0^t ||y(s)||_V ds \\ &\leq M||y_0|| + MM_1 + M\overline{M}\sqrt{T}(||Bu||_Z + a + b||x||_Z + \epsilon_1) + M\overline{M}aT \\ &\quad + M\overline{M}b\int_0^t ||y(s)||_V ds \\ &\leq M||y_0|| + MM_1 + M\overline{M}\sqrt{T}(||Bu||_Z + a + bM||y_0|| + bMM_1 \\ &\quad + M\overline{M}bT||Bu||_Z + \epsilon_1) + M\overline{M}aT + M\overline{M}b\int_0^t ||y(s)||_V ds \\ &\leq M||y_0|| + MM_1 + M\overline{M}\sqrt{T}\{(1 + M\overline{M}bT)||Bu||_Z + a + bM||y_0|| + bMM_1 \\ &\quad + \epsilon_1\} + M\overline{M}aT + M\overline{M}b\int_0^t ||y(s)||_V ds \\ &\leq (1 + bM\overline{M}\sqrt{T})M||y_0|| + MM_1 + M\overline{M}\sqrt{T}\{(1 + M\overline{M}bT)||Bu||_Z + a \\ &\quad + bMM_1 + \epsilon_1\} + M\overline{M}aT + M\overline{M}b\int_0^t ||y(s)||_V ds \\ &\leq M(1 + bM\overline{M}\sqrt{T})(||y_0|| + M_1) + M\overline{M}\sqrt{T}\{(1 + M\overline{M}bT)||Bu||_Z + a + \epsilon_1\} \\ &\quad + M\overline{M}aT + M\overline{M}b\int_0^t ||y(s)||_V ds. \end{aligned}$$

Now, Gronwall's inequality implies that

$$\begin{aligned} ||y(t)||_V &\leq [M(1 + bM\overline{M}\sqrt{T})(||y_0|| + M_1) + M\overline{M}\sqrt{T}\{(1 + M\overline{M}bT)||Bu||_Z + a + \epsilon_1\} \\ &\quad + M\overline{M}aT]e^{M\overline{M}bT}, \end{aligned}$$

which completes the proof.

The main result of this chapter is given below.

Theorem 3.2 *Under the conditions* $(H_0)-(H_7)$*, the semilinear system (1.1) is approximate controllable in the time interval* $[0,T]$.

Proof. Let $x(t)$ be the integral solution of the system (1.2) corresponding to the control u, which can be written as

$$x(t) = T_0(t)(y_0+g(y)) + \lim_{\lambda\to\infty}\int_0^t T_0(t-s)C(\lambda)(Bu)(s)ds. \tag{3.16}$$

Let $y(t)$ be the integral solution of the system (1.1) corresponding to the control $v=u-w$, which can be written as

$$\begin{aligned} y(t) &= T_0(t)(y_0+g(y)) + \lim_{\lambda\to\infty}\int_0^t T_0(t-s)C(\lambda)B(u-w)(s)ds \\ &\quad + \lim_{\lambda\to\infty}\int_0^t T_0(t-s)C(\lambda)f(s,y(s))ds. \end{aligned} \tag{3.17}$$

From the equations (3.16) and (3.17), we get

$$x(t)-y(t) = \lim_{\lambda\to\infty}\int_0^t T_0(t-s)C(\lambda)Bw(s)ds - \lim_{\lambda\to\infty}\int_0^t T_0(t-s)C(\lambda)f(s,y(s))ds, \tag{3.18}$$

which in operator theoretic terms can be written as

$$\begin{aligned} x-y &= KBw-KFy \\ &= K(Bw-Fx)+(KFx-KFy). \end{aligned}$$

Taking inner products on both sides in Z with $Fx-Fy$, we get

$$\begin{aligned} <x-y, Fx-Fy>_Z &= <K(Bw-Fx)+(KFx-KFy), Fx-Fy>_Z \\ &= <K(Bw-Fx), Fx-Fy>_Z \\ &\quad + <(KFx-KFy), Fx-Fy>_Z . \end{aligned} \tag{3.19}$$

Since, by condition (H_4), the left-hand side of the equation (3.19) is less than or equal to $-\beta||x-y||^2$ and from Lemma 3.1, the second term of the right-hand side is nonnegative, if $<K(Bw-Fx^L), Fx^L-Fx>_Z$ is negligibly small, then from the equation (3.19), it follows that $||x-y||_Z$ is also arbitrary small.

Now, we show that $<K(Bw-Fx^L), Fx^L-Fx>_Z$ is arbitrarily small. For it, by using Cauchy-Schwarz inequality, we have

$$\begin{aligned} |<K(Bw-Fx), Fx-Fy>_Z| &\le ||K(Bw-Fx)||_Z||Fx-Fy||_Z \\ &\le M\overline{M}T||Bw-Fx||_Z\{||Fx||_Z+||Fy||_Z\} \\ &\le M\overline{M}T\epsilon_1\{a+b||x||_Z+a+b||y||_Z\}. \end{aligned} \tag{3.20}$$

From the equation(3.20) and Lemma 3.2, we get

$$|<K(Bw-Fx), Fx-Fy>_Z| \le M\overline{M}CT\epsilon_1, \tag{3.21}$$

where $C = 2a+b[M||y_0||+MM_1+M\overline{M}T||Bu||_Z]+b[M(1+bM\overline{M}\sqrt{T})(||y_0||+M_1)+M\overline{M}\sqrt{T}\{(1+M\overline{M}bT)||Bu||_Z+a+\epsilon_1\}+M\overline{M}aT]e^{M\overline{M}bT}$. Thus, for given u and ϵ_1, C is finite. Since ϵ_1 is arbitrarily small, it implies that $< K(Bw - Fx), Fx - Fy >_Z$ is arbitrary small.

Hence, from the equations (3.19), (3.20) and condition (H_4), it follows that $||x-y|| \leq \epsilon_2$ is arbitrary small, for some $\epsilon_2 > 0$.

Further, we prove that $||x(T) - y(T)||_V$ is arbitrary small. Now,

$$\begin{aligned} x(t) - y(t) &= KBw(t) - K(Fy)(t) \\ &= K\{Bw(t) - (Fx)(t)\} + K\{(Fx)(t) - (Fy)(t)\}. \end{aligned} \tag{3.22}$$

Taking norm on both sides in V, we have

$$\begin{aligned} ||x(t) - y(t)||_V &= ||K\{Bw(t) - (Fx)(t)\} + K\{(Fx)(t) - (Fy)(t)\}||_V \\ &\leq ||K\{Bw(t) - (Fx)(t)\}||_V + ||K\{(Fx)(t) - (Fy)(t)\}||_V. \end{aligned}$$

Now,

$$\begin{aligned} ||K\{Bw(t) - (Fx)(t)\}||_V &= \left|\left| \lim_{\lambda\to\infty} \int_0^t T_0(t-s)C(\lambda)\{Bw(s) - (Fx)(s)\}ds \right|\right|_V \\ &\leq M\overline{M} \int_0^t ||\{Bw(s) - (Fx)(s)\}ds||_V \\ &\leq M\overline{M}\sqrt{T}||Bw - Fx||_Z \\ &\leq M\overline{M}\sqrt{T}\epsilon_1 \end{aligned}$$

and

$$\begin{aligned} ||K\{(Fx)(t) - (Fy)(t)\}||_V &= \left|\left| \lim_{\lambda\to\infty} \int_0^t T_0(t-s)C(\lambda)\{(Fx)(s) - (Fy)(s)\}ds \right|\right|_V \\ &\leq M\overline{M}\sqrt{T}||Fx - Fy||_Z. \end{aligned} \tag{3.23}$$

The right-hand side of the equation (3.23) can be made arbitrarily small as $||x-y|| \leq \epsilon_2$ and F is continuous on Z. Therefore, it is concluded that for a given ϵ and x, there exists a y such that

$$||x(t) - y(t)|| \leq \epsilon \text{ for all } t \in [0, T]. \tag{3.24}$$

Thus, $||x(t) - y(t)||$ can be made arbitrarily small by choosing suitable ω. It follows that reachable set of the system (1.1) is dense in the reachable set of the system (1.2), which is dense in $\overline{D(A)}$ due to condition (H_3). Hence the theorem is proved.

4 Controllability Results with Integral Contractor Nonlinearity

In this section, approximate controllability of semilinear control system (1.1) is considered when the nonlinear function f has integral contractor.

Let C be the Banach space of all continuous functions from $[0, T]$ to Banach space V with supremum norm. The problem of controllability of infinite dimensional semilinear control systems has been studied widely by many authors, when the nonlinear function is uniformly Lipschitz continuous or monotone, see [11,21,22]. In this section, we study the approximate controllability of the system (1.1), when the operator A is not densely defined and the nonlinear function satisfies integral contractor condition, which is a weaker condition in comparison with Lipschitz condition. The concept of contractor was introduced by Altman [1] as a functional analytic tool for solving deterministic operators equations in Banach spaces and subsequently this tool was exploited by many authors for the existence and uniqueness of the solution of nonlinear evolution equations, see [24]. Govindan and Joshi [14] employed this method to investigate optimal control problem and stability problems of nonlinear stochastic control system. George [12] investigated the approximate controllability of semilinear non-autonomous system with nonlinearity satisfying integral contractor condition. Further, George et al. [13] obtained the exact controllability of the third order dispersion equation through the approach of integral contractor. In this section, our aim is to obtain a result similar to that of [28], for the non-densely defined semilinear system (1.1) by replacing Lipschitz condition with integral contractor.

Now, we define the integral contractor.

Definition 4.1 [12] Let $\Gamma : J \times V \to BC(C)$ be a bounded continuous operator and γ be a positive constant such that for any $x, y \in C$, we have

$$\left|\left| f\left(t, \left(x(t) + y(t) + \int_0^t T_0(t-s)\Gamma(s, x(s))y(s)ds\right) - f(t, x(t)) - \Gamma(t, x(t))y(t)\right|\right| \leq \gamma ||y(t)|| \tag{4.1}$$

Then, we say that f has a bounded integral contractor $\{I + \int T_0\Gamma\}$ with respect to $T_0(t)$. The constant γ will be called the contractor constant.

Remark 4.1 If $\Gamma \equiv 0$, the condition (4.1) reduces to the Lipschitz condition, as we get

$$||f(t, x(t) + y(t)) - f(t, x(t))|| \leq \gamma ||y(t)||. \tag{4.2}$$

Definition 4.2 [12] A bounded integral contractor Γ is said to be regular if the following integral equation

$$\hat{y}(t) = z(t) + \int_0^t T_0(t-s)\Gamma(s, \hat{x}(s))z(s)ds \tag{4.3}$$

has a solution z in C for any $\hat{x}, \hat{y} \in C$.

Let us assume the following conditions:

(H_8) The nonlinear function has bounded integral contractor.

(H_9) $R(F) \subseteq R(B)$.

Theorem 4.1 *Under the conditions (H_2),(H_3), (H_8) and (H_9), the semilinear system (1.1) is approximate controllable on the time interval $[0, T]$.*

Proof. Let $x(t)$ be the integral solution of the linear system (1.2) corresponding to the control u. The integral solution of (1.2) is given by the nonlinear integral equation

$$x(t) = T_0(t)(y_0 + g(y)) + \lim_{\lambda \to \infty} \int_0^t T_0(t-s)C(\lambda)Bu(s)ds. \tag{4.4}$$

Let $y(t)$ be the integral solution of the semilinear system (1.2) corresponding to the control v, which satisfies the equation (3.10). Then $y(t)$ can be written as

$$y(t) = T_0(t)(y_0 + g(y)) + \lim_{\lambda \to \infty} \int_0^t T_0(t-s)C(\lambda)[Bu(s) - f(s, x(s)) + f(s, y(s))]ds. \tag{4.5}$$

From (4.4) and (4.5) for all $t \in [0, T]$, we have

$$y(t) - x(t) = \lim_{\lambda \to \infty} \int_0^t T_0(t-s)C(\lambda)[f(s, y(s)) - f(s, x(s))]ds. \quad (4.6)$$

By the regularity condition of the integral contractor Γ with $\hat{x} = x$ and $\hat{y} = y - x$, there exists $z \in C$ such that

$$y(t) - x(t) = z(t) + \lim_{\lambda \to \infty} \int_0^t T_0(t-s)C(\lambda)\Gamma(s, x(s))z(s)ds. \quad (4.7)$$

By the equation (4.6), we have

$$\lim_{\lambda \to \infty} \int_0^t T_0(t-s)C(\lambda)[f(s, y(s)) - f(s, x(s))]ds = z(t) \quad (4.8)$$
$$+ \lim_{\lambda \to \infty} \int_0^t T_0(t-s)\Gamma(s, x(s))z(s)ds$$
$$\Rightarrow z(t) = \lim_{\lambda \to \infty} \int_0^t T_0(t-s)C(\lambda)[f(s, y(s)) - f(s, x(s)) - \Gamma(s, x(s))z(s)]ds. \quad (4.9)$$

Since

$$||C(\lambda)|| \leq \frac{\lambda \overline{M}}{\lambda - \overline{\omega}} \to \overline{M} \text{ as } \lambda \to \infty,$$

taking V-norm on both sides of (4.8), we have

$$||z(t)|| \leq M\overline{M} \int_0^t ||f(s, y(s)) - f(s, x(s)) - \Gamma(s, x(s))z(s)||ds. \quad (4.10)$$

By the condition (H_7), the nonlinear function f has a regular integral contractor, we have

$$\left|\left| f\left(t, x(t) + z(t) + \int_0^t T_0(t-s)\Gamma(s, x(s))z(s)ds\right) - f(t, x(t)) - \Gamma(t, x(t))z(t) \right|\right| \leq \gamma ||z(t)||. \quad (4.11)$$

Now, using the equation (4.6), we get

$$||f(t, y(t)) - f(t, x(t)) - \Gamma(t, x(t))z(t)|| \leq \gamma ||z(t)||. \quad (4.12)$$

From (4.10) and (4.12), we have

$$||z(t)|| \leq \gamma M\overline{M} \int_0^t ||z(s)||ds.$$

Thus, by using Gronwall's inequality, we get $||z(t)|| = 0$. Therefore by (4.7), we have that $x(t) = y(t)$ for all $t \in [0, T]$. Hence the set of the solutions of the linear system (1.2) is equal to the set of all solutions of the semilinear system (1.1). Here, $K_\tau(f) \supset K_\tau(0)$, which is dense in $\overline{D(A)}$. Therefore, system (1.1) is approximate controllable on $[0, T]$.

5 Example

Consider the following partial differential equation with nonlocal conditions of the form

$$\begin{aligned} \frac{\partial}{\partial t} y(t,x) &= y_{xx}(t,x) + Bv(t,x) + f(t, y(t,x)), \\ y(t,0) &= y(t,\pi) = 0, \\ y(0,x) &= y_0(x) + g(y), \quad 0 \le x \le \pi, 0 \le t \le T. \end{aligned} \tag{5.1}$$

In order to write system (5.1) in the abstract form (1.1), choose $V = C[0,\pi]$(with sup norm) and consider the operator A defined by

$$Aw = w'' = \frac{d^2 w}{dx^2}$$

with the domain

$$D(A) = \{w \in V : w, w' \text{ are absolutely continuous }, w'' \in V, w(0) = w(\pi) = 0\}.$$

Then, $\overline{D(A)} = \{w \in V : w(0) = w(\pi) = 0\}$. It is clear that $\overline{D(A)} \neq V$, and the resolvent set, $\rho(A) \supseteq (0, +\infty)$

$$||(\lambda I - A)^{-1}|| \le \frac{1}{\lambda}$$

for $\lambda > 0$. This implies that A satisfies the Hille-Yosida condition (H_0) on V. It is well known that A generates a C_0 semigroup $T_0(t)$ on $\overline{D(A)}$ for all $t \ge 0$, see [25] and linear system corresponding to (5.1) is approximate controllable in the space $\overline{D(A)}$(condition H_2 is satisfied). Hence, Theorem 4.1 implies the approximate controllability of system (5.1) for any nonlinear function which has integral contractor.

6 Conclusion

In this paper, we introduced a set of sufficient conditions for the approximate controllability of the semilinear control system with nonlocal conditions (1.1) with an important case in which operator A need not be densely defined. Two types of nonlinearity were considered; namely nonlinearity satisfying a monotone condition and nonlinearity having an integral contractor. Some approaches made by earlier authors led to certain inequality conditions involving various system constants. But, in this approach, there is no need of any inequality condition to prove the approximate controllability of system (1.1).

Acknowledgment

The first author is thankful to the "Ministry of Human Resource Development (MHRD)", India for financial support to carry out her research work.

References

[1] Altman, M. *Contractors and Contractor Directions. Theory and Applications.* Dekker, New York, 1977.

[2] Balachandran, K. and Dauer, J. P. Controllability of nonlinear systems in Banach spaces: a survey. *J. Optim. Theory Appl.* **115** (1) (2002) 7–28.

[3] Barnett, S. *Introduction to Mathematical Control Theory.* Clarendon Press, Oxford, 1975.

[4] Byszewski, L. and Lakshmikantham, V. Theorem about the existence and uniqueness of a solution of a nonlocal abstract Cauchy problem in a Banach space. *Appl. Anal.* **40** (1) (1991) 11–19.

[5] Curtain, R. F. and Zwart, H. *An introduction to infinite-dimensional linear systems theory.* Texts in Applied Mathematics, 21. Springer-Verlag, New York, 1995.

[6] Da Prato, G. and Sinestrari, E. Differential operators with nondense domain. *Ann. Scuola Norm. Sup. Pisa Cl. Sci.(4)* **14** (2) (1987) 285–344.

[7] Dauer, J. P. and Mahmudov, N. I. Approximate controllability of semilinear functional equations in Hilbert spaces. *J. Math. Anal. Appl.* **273** (2) (2002) 310–327.

[8] Ezzinbi, K. and Liu, J. H. Nondensely defined evolution equations with nonlocal conditions. *Math. Comput. Modelling* **36** (9-10) (2002) 1027–1038.

[9] Fu, X. Controllability of non-densely defined functional differential systems in abstract space. *Appl. Math. Lett.* **19** (4) (2006) 369–377.

[10] Fu, X. and Liu, X. Controllability of non-densely defined neutral functional differential systems in abstract space. *Chin. Ann. Math. Ser. B* **28** (2) (2007) 243–252.

[11] George, R. K. Approximate controllability of nonautonomous semilinear systems. *Nonlinear Anal.* **24** (9) (1995) 1377–1393.

[12] George, R. K. Approximate controllability of semilinear systems using integral contractors. *Numer. Funct. Anal. Optim.* **16** (1-2) (1995) 127–138.

[13] George, R. K., Chalishajar, D. N. and Nandakumaran, A. K. Exact controllability of the nonlinear third-order dispersion equation. *J. Math. Anal. Appl.* **332** (2) (2007) 1028–1044.

[14] Govindan, T. E. and Joshi, M. C. Stability and optimal control of stochastic functional-differential equations with memory. *Numer. Funct. Anal. Optim.* **13** (3-4) (1992) 249–265.

[15] Jeong, J. M., Kwun, Y. C. and Park, J. Y. Approximate controllability for semilinear retarded functional-differential equations. *J. Dynam. Control Systems* **5** (3) (1999) 329–346.

[16] Kalman, R. E. Mathematical description of linear dynamical systems. *J. SIAM Control Ser. A* **1** (1963) 152–192.

[17] Kavitha, V. and Mallika Arjunan, M. Controllability of non-densely defined impulsive neutral functional differential systems with infinite delay in Banach spaces. *Nonlinear Anal. Hybrid Syst.* **4** (3) (2010) 441–450.

[18] Kellerman, H. and Hieber, M. Integrated semigroups. *J. Funct. Anal.* **84** (1) (1989) 160–180.

[19] Kwun, Y. C., Park, J. Y. and Ryu, J. W. Approximate controllability and controllability for delay Volterra system. *Bull. Korean Math. Soc.* **28** (2) (1991) 131–145.

[20] Magal, P. and Ruan, S. On semilinear Cauchy problems with non-dense domain. *Adv. Differential Equations* **14** (11-12) (2009) 1041–1084.

[21] Naito, K. Controllability of semilinear control systems dominated by the linear part. *SIAM J. Control Optim.* **25** (3) (1987) 715–722.

[22] Naito, K. An inequality condition for approximate controllability of semilinear control systems. *J. Math. Anal. Appl.* **138** (1) (1989) 129–136.

[23] Naito, K. and Park, J. Y. Approximate controllability for trajectories of a delay Volterra control system. *J. Optim. Theory Appl.* **61** (2) (1989) 271–279.

[24] Padgett, W. J. and Rao, A. N. V. Solution of a stochastic integral equation using integral contractors. *Inform. and Control* **41** (1) (1979) 56–66.

[25] Pazy, A. *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Applied Mathematical Sciences, **44**, Springer, New York, 1983.

[26] Sukavanam, N. and Divya. Exact and approximate controllability of abstract semilinear control systems. *Indian J. Pure Appl. Math.* **33** (12) (2002) 1827–1835.

[27] Thieme, H. R. "Integrated semigroups" and integrated solutions to abstract Cauchy problems. *J. Math. Anal. Appl.* **152** (2) (1990) 416–447.

[28] Tomar, Nutan K. and Sukavanam, N. Approximate controllability of non-densely defined semilinear delayed control systems. *Nonlinear Stud.* **18** (2) (2011) 229–234.

[29] Zhou, H. X. A note on approximate controllability for semilinear one-dimensional heat equation. *Appl. Math. Optim.* **8** (3) (1982) 275–285.