Nonlinear Dynamics and Systems Theory, **17** (1) (2017) 19–28

Integro-differential Equations, Compact Maps, Positive Kernels, and Schaefer's Fixed Point Theorem

T. A. Burton

Northwest Research Institute, 732 Caroline St., Port Angeles, WA 98362, USA

Received: December 18, 2016; Revised: January 23, 2017

Abstract: Integral equations offer a natural fixed point mapping, while an integro-differential equation

$$x'(t) = -g(t, x(t)) - \int_0^t A(t-s)f(x(s))ds$$

often prompts us to write it as an integral equation. This can be a mistake. We can convert it to an integral equation by a direct fixed point substitution which yields a very fixed point friendly equation. It is a natural sum of a continuous and compact map. Our first contribution here is to note that the direct fixed point process can change the continuous map to a compact map so that we have the sum of two compact maps and it is ready for Schauder's theorem instead of the much more complicated Krasnoselskii theorem which we usually expect to need. Schauder's theorem still requires that we find a self mapping set and that can be difficult. So we continue and combine the direct fixed point process with positive kernel theory so that we have an automatic *a priori* bound on all possible solutions of a homotopy equation. This gives us existence and boundedness of solutions of a wide class of problems from applied mathematics and it solves a classical problem which has been raised several times since 1960. See "Main note" following (3.1). That first contribution is a continuation of an earlier brief note in which we discovered that the direct fixed point process changed a Lipschitz map of an integro-differential equation into a contraction map, while here it changes a continuous map into a compact map.

Keywords: *integro-differential equations; compact maps; positive kernels; Schaefer's theorem; existence; uniqueness; fixed points.*

Mathematics Subject Classification (2010): 34A08, 34A12, 45D05, 45G05, 47H09.

* Corresponding author: mailto:taburton@olypen.com

1 Introduction

One of the two main goals of this paper is to illustrate a new compactness technique in preparing fixed point mappings for integro-differential equations. It allows us to establish a combination Schauder-Krasnoselskii fixed point theorem especially suited to integro-differential equations. We obtain a fixed point when the mapping is the sum of a compact mapping and a continuous mapping. The result is parallel to one which we presented in [2] in which we showed how to reduce a Lipschitz property to a contraction.

In the classical theory of scalar integro-differential equations

$$y'(t) = -g(t, y(t)) - \int_0^t A(t-s)f(y(s))ds$$

as treated in MacCamy and Wong [13] there is the nagging question of existence and uniqueness of solutions. It is explicitly raised at the bottom of page 6 of their treatment and again by the reviewer in the Mathematical Reviews MR0293355 (45 #2432). Moreover, the question traces back to statements by Levin [11, p. 534]. That paper was the one generating work by Halanay [8] on the use of strongly positive kernels to obtain asymptotic stability properties. In turn, an error in Halanay's paper noted by the reviewer in MR 0176304 (31 #579) generated the work by MacCamy and Wong. This brings us back, then, to Levin's paper with $g(t, x) = 0$, in which he states (but neither proves nor references) that it is possible to use successive approximations, together with a Lipschitz condition on f in every region $|x| \leq X < \infty$ to obtain a unique solution. Our "Main note" following (3.1) will point out that their problem does have a solution on any interval $[0, E]$ and the Lipschitz condition is not needed except for uniqueness.

The papers by Levin, Levin and Nohel, Halanay, and MacCamy and Wong generated a flurry of activity which largely abated by 1990 and as far as we know the existence and uniqueness question was never settled. In any case, we treat the problem of existence as an example of the method we present here as outlined in items 1 through 4 after this paragraph. Much of the work through 1989 is reported in the book by Gripenberg, Londen, and Staffans [7]. As this equation plays a fairly prominent role in some areas of applied mathematics (see for example Levin [11, p. 535] referring to Levin and Nohel [12] on work with reactor dynamics) and as it occupies a position in the general theory of integro-differential equations, it seems to be in order to offer a solution to the question using a fixed point theorem of Schaefer which adds yet another application of the work of MacCamy and Wong. There are four steps and each one seems interesting.

1. Cast the problem in the form of a "direct fixed point mapping", to be explained later.

2. Introduce a Liapunov function yielding an *a priori* bound on all possible solutions including the solution whose existence we prove.

3. Invoke a result that the kernel will generate a compact map.

4. Show that the idea of a direct fixed point mapping allows us to show that continuity of the function $g(t, x)$ generates a compact map. We believe that this idea is new and is a counterpart to a recent result in [2] that if g is Lipschitz then the direct fixed point mapping allows us to show that it generates a contraction mapping.

These properties will fulfill the conditions of Schaefer's fixed point theorem and yield at least one solution on any interval $[0, E]$, $0 < E < \infty$. A review of standard treatments, such as Miller [14, pp. 93-98], shows that investigators generally are willing to accept this as implying that there is a solution on $[0, \infty)$. A look at the figure of Hartman [10, p.

19] raises questions on just how we continue and get a solution on $[0, \infty)$. In fact, there is reason for unease. A close look at (e.g. [9, p. 42]) tells us that this is a deep problem, solved by invocation of Zorn's lemma.

There is a way out if we ask conditions implying uniqueness such as [3, p. 11]. In a sense, this is a return to the old idea of uniqueness implying existence. For with uniqueness we can present a very clear proof of the existence of a unique solution on $[0, \infty)$ and that appears as the last paragraph of this paper.

2 The Mappings and Continuity

We begin with a scalar integro-differential equation

$$y'(t) = -g(t, y(t)) - \int_0^t A(t-s)f(y(s))ds, \quad y(0) = a \in \Re, \quad ' = d/dt, \tag{2.1}$$

in which $f : \Re \to \Re, g : [0, \infty) \times \Re \to \Re$ are both continuous, $A : (0, \infty) \to \Re$ is continuous, and if $\phi : [0, \infty) \to \Re$ is continuous so is $\int_0^t A(t-s)\phi(s)ds$.

Moreover, because we ask no growth condition on either f or g and we seek a global solution it is clear that we must ask sign conditions such as

$$yf(y) \geq 0 \text{ and } yg(t, y) \geq 0. \tag{2.1a}$$

Thus, (2.1) includes both

$$y'(t) = -g(t, y) \text{ and } y'(t) = -\int_0^t A(t-s)f(y(s))ds \tag{2.1b}$$

which are an elementary differential equation and an equation of the type considered by Levin [11]. If we review the conditions of Levin's problem we see that the properties of the kernel which allowed his analysis would be destroyed if we integrate that last equation to get an integral equation which would define a natural fixed point mapping. And that is the case in so many integro-differential equations: they come to us with a very useful kernel and we strive to keep that kernel. In our problems here we start with a positive kernel. If we were to integrate (2.1) and interchange the order of integration then it would no longer be a positive kernel and all the theory of MacCamy and Wong would fail. The following has proven to be a very useful technique.

Under the above conditions, if (2.1) has a continuous solution $y(t)$ then $y'(t)$ is also continuous so we designate that derivative as $\phi(t)$ and write

$$y'(t) = \phi(t), \quad y(t) = y(0) + \int_0^t \phi(s)ds = a + \int_0^t \phi(s)ds. \tag{2.2}$$

Hence, (2.1) is written as

$$\phi(t) = -g\left(t, a + \int_0^t \phi(s)ds\right) - \int_0^t A(t-s)f\left(a + \int_0^s \phi(u)du\right)ds \tag{2.3}$$

which is a standard integral equation.

Given y solving (2.1), then $y(0) = a$ and $y'(t) = \phi(t)$ are both uniquely determined. Letting $y = a + \int_0^t \phi(s)ds$ in (2.1) yields exactly (2.3) so (2.3) is uniquely determined from y and (2.1). On the other hand, if ϕ satisfies (2.3) and if we set $y = a + \int_0^t \phi(s)ds$ in (2.3) we have $y'(t) = \phi(t)$ so (2.1) is uniquely determined.

If $(\mathcal{B}, \|\cdot\|)$ is the Banach space of continuous functions with $\phi \in \mathcal{B}$ if $\phi : [0, E] \to \Re$ with the supremum norm then the natural mapping defined by (2.3) is $\phi \in \mathcal{B}$ implies that

$$(P\phi)(t) = -g\left(t, a + \int_0^t \phi(s))ds\right) - \int_0^t A(t-s)f\left(a + \int_0^s \phi(u)du\right)ds. \tag{2.4}$$

Thus, if $\phi \in \mathcal{B}$ is a fixed point of P, then $\phi : [0, E] \to \Re$ is continuous and satisfies (2.3). The process is reversible and

$$y(t) = a + \int_0^t \phi(s)ds \tag{2.5}$$

is a solution of (2.1).

Continuity A proof on p. 443 of [1] will show that P is continuous when the space is restricted to $[0, E]$.

Remark 2.1 Here is a tour of the paper. In the next section, Theorem 3.1, we show that the first term on the right-hand side of (2.3) defines a compact map. In Section 4 we mention three papers giving conditions ensuring that the integral in (2.3) involving A defines a compact map. Thus, P will define a compact map and is continuous. That much is sufficient to satisfy Schauder's theorem and give a fixed point of P provided that we can find a self mapping set, M. That must be handled in a case-by-case way and it is never simple. But for the vast majority of problems like this occurring in applied mathematics the kernel satisfies a condition which makes the problem simple so we branch at this point and show how that is done, thereby obtaining a bounded solution for a wide class of problems in applied mathematics. The branch starts with the fixed point theorem of Schaefer which requires an *a priori* bound on all possible solutions of a homotopy equation. We use (2.4) to establish continuity and compactness of the mapping. We will need (2.1) to prove boundedness of all possible solutions. The method is important because we do not know how to establish the result without both equations unless we were to ask for a Lipschitz condition on both f and g. $\square$

For reference, here is Schauder's theorem [15, p. 25].

Theorem 2.1 *(Schauder's second theorem) Let $\mathcal{M}$ be a non-empty convex subset of a normed space $\mathcal{B}$. Let T be a continuous mapping of $\mathcal{M}$ into a compact set $\mathcal{K} \subset \mathcal{M}$. Then T has a fixed point.*

3 Compactness of the Maps from g and A

There are three recent papers showing that if $A(t)$ satisfies certain very general conditions then for any bounded set of function $M \subset \mathcal{B}$ the mapping $Q : M \to \mathcal{B}$ defined by $\phi \in M$ implies that

$$(Q\phi)(t) = \int_0^t A(t-s)\phi(s)ds \tag{3.1}$$

maps M into a bounded equicontinuous subset of $\mathcal{B}$ on $[0, E]$ and, hence, into a compact subset of $\mathcal{B}$.

1. The first condition [4, Theorem 5.1] says that both $A(t) = t^{q-1}, 0 < q < 1$, and its resolvent generate the required equicontinuity (Corollary 4.1).

2. There is an interesting condition for equicontinuity given by Dwiggins [6, pp. 48, 51] which asks too many technical details to be safely mentioned here, but we can give a sketch. It is assumed that for decreasing $A(t) > 0$ and

$$W(t) = \int_0^t A(s)ds \quad \& \quad H(t) = \int_0^t A(t-s)|f(s, \phi(s))|ds,$$

then

$$|H(t_2) - H(t_1)| \leq 2KW(t_2 - t_1),$$

where K is the bound on the functions in M. If $W(r)$ is continuous and tends to zero as $r \downarrow 0$ then equicontinuity follows.

3. A necessary and sufficient condition on a nonconvolution case, $C(t, s)$, is found in [5]. The function $C : (0, \infty) \times (0, \infty)$ is continuous for $t > s > 0$. It is of the fading memory type by which we mean

$$0 < s < t_2 < t_1 \implies C(t_2, s) \geq C(t_1, s).$$

For each $\epsilon > 0$ there exists $\delta > 0$ such that

$$0 \leq t_2 \leq t_1, \quad t_1 - t_2 < \delta \implies \int_{t_2}^{t_1} C(t_1, s)ds < \epsilon.$$

If $\int_0^t C(t, s)ds$ is uniformly continuous on an interval $[0, E]$ then QM of (3.1) is equicontinuous.

Reminder of our original question: The MacCamy-Wong conditions in Theorem 4.1 do satisfy the condition in item 2 above [6] to ensure that A will satisfy the compactness of (3.1). This is the main addition to Theorem 5.2 needed to see that the conclusion of our Theorem 5.3 includes the fact that the MacCamy-Wong problem does have a solution on any interval $[0, E]$ when (1.5) holds.

It is known that integrals smooth so the compactness of (3.1) is not unexpected. But the compactness of g is quite new.

Theorem 3.1 *Let $g : [0, E] \times \Re \to \Re$ be continuous and let M be a closed bounded nonempty subset of $\mathcal{B}$ so that there is a $K > 0$ and if $\phi \in M$ then $\|\phi\| \leq K$. If $L : M \to \mathcal{B}$ is defined by $\phi \in M$ implies*

$$(L\phi)(t) = g\left(t, a + \int_0^t \phi(s)ds\right),$$

then LM is equicontinuous.

Proof. Let $\epsilon > 0$ be given. We must find $\delta > 0$ such that $\phi \in M$ and $0 \leq t_1 < t_2 \leq E$ with $|t_1 - t_2| < \delta$ then

$$\left|g\left(t_1, a + \int_0^{t_1} \phi(s)ds\right) - g\left(t_2, a + \int_0^{t_2} \phi(s)ds\right)\right| < \epsilon. \tag{3.2}$$

For $\phi \in M$ and $0 \leq t \leq E$ the coordinates in g satisfy $0 \leq t \leq E$ and $\left|a + \int_0^t \phi(s)ds\right| \leq |a| + EK =: J$.

For the given $\epsilon > 0$ by the uniform continuity of g on $[0, E] \times [-J, J]$ there is a $\mu > 0$ such that

$$|t_1 - t_2| < \mu \text{ and } \left| \int_0^{t_1} \phi(s)ds - \int_0^{t_2} \phi(s)ds \right| < \mu$$

imply (3.2).

Now

$$\left| \int_0^{t_1} \phi(s)ds - \int_0^{t_2} \phi(s)ds \right| \le \int_{t_1}^{t_2} |\phi(s)|ds \le K|t_2 - t_1| < \mu$$

if $|t_2 - t_1| < \mu/K$ so we choose $\delta = \min[\mu, \mu/K]$ to satisfy (3.2). □

Remark 3.1 Now consider putting the mapping Q of (3.1) and its properties together with the mapping L of Theorem 3.1 into the mapping (2.4). We would have the sum of two compact maps and that is a compact map and it is continuous. Suppose that this sum maps the non-empty closed bounded convex set of continuous functions M on $[0, E]$ into itself. By Schauder's theorem there is a fixed point. We will state that below. But to understand its place in the literature, consider Krasnoselskii's theorem [15, p. 31].

Theorem 3.2 (Krasnoselskii) *Let $\mathcal{M}$ be a closed convex non-empty subset of a Banach space $\mathcal{S}$. Suppose that A and B map $\mathcal{M}$ into $\mathcal{S}$ and that*
(i) $Ax + By \in \mathcal{M}$ for each $x, y \in \mathcal{M}$.
(ii) A is compact and continuous.
(iii) B is a contraction mapping.
Then there exists $y \in \mathcal{M}$ such that $Ay + By = y$.

Notice that in Remark 3.1 we have not required L to be a contraction, but only continuous. Krasnoselskii's theorem is fundamental, but it has proven very hard to satisfy. We would ask only that P be self mapping and would state it as follows invoking Schauder's second theorem [15, p. 25] (stated in Section 2). Recall that continuity of P was mentioned with (2.4).

Theorem 3.3 *Suppose that Q in (3.1) maps bounded subsets of $\mathcal{B}$ on $[0, E]$ into bounded equicontinuous sets. Let the conditions of Theorem 3.1 hold, and suppose that M is a closed bounded convex nonempty set with P defined in (2.4) being continuous and mapping M into M. Then P has a fixed point in M.*

4 The Class of Kernels

Our work branches at this point. We have gotten past the continuous mapping by making it compact and we are going to look at one of many possible ways to proceed from here. It can be challenging to find a self mapping set and now we turn away from Schauder's theorem and, instead, look at Schaefer's theorem which asks for an *a priori* bound on all possible solutions of an equation related to (2.1). When the kernel satisfies conditions (B_1)–(B_4) of Theorem 4.1 then a Liapunov function will give that bound in a very simple way.

On p. 209 Miller [14] defines a large class of kernels which occur frequently in applied mathematics and for which there exist resolvent kernels satisfying fundamental properties given on pp. 212–213. The common kernel $A(t) = t^{q-1}, 0 < q < 1$, satisfies those conditions and it occurs in all fractional differential equations of Riemann-Liouville and

Caputo type as well as many forms of heat problems with $q = 1/2$. Moreover, it allows us to make certain transformations which we have found to be fundamental in papers since 2009. The conditions are

(A1) $A \in C(0, \infty) \cap L^1(0, 1)$,

(A2) $A(t)$ is positive and non-increasing for $t > 0$,

(A3) for each $T > 0$ the function $A(t)/A(t + T)$ is non-increasing in t for $0 < t < \infty$.

We will need a bit more than this and it leads to the origin of this study. Focus on (A2). First, if we ask that $A(t)$ is non-increasing, then we approximate this with the simple requirement that $A'(t) \leq 0$. It is more difficult to specify a condition keeping $A(t)$ positive, but for a start we could ask that $A''(t) \geq 0$. In trying to simplify these three conditions we have unwittingly almost asked for a classical condition that $A(t)$ be a strongly positive kernel. Following MacCamy and Wong [13], $A(t)$ is a positive kernel if whenever $h : [0, \infty) \to \Re$ is continuous then for $T > 0$ we have

$$\int_0^T h(t) \int_0^t A(t-s)h(s)dsdt \geq 0 \tag{4.1}$$

and that will follow under a number of conditions. It is more than sufficient [13, p. 2] if

$$A(t) > 0, A'(t) \leq 0, A''(t) \geq 0, \quad A'(t) \text{ not identically zero,}$$

and we discuss more below.

Remark 4.1 The boundedness which we mentioned in Remark 2.1 will follow from (4.1), but only by using (2.1) or (2.1λ), not the integral equation (2.3). The idea is based on a Liapunov function and works only on an integro-differential equation. Once we get the boundedness of the solution of (2.1) we will transfer it to boundedness of fixed points of the mapping in Schaefer's theorem which requires a λ on the right-hand side. We will denote that equation by (2.4λ) written in Section 5. $\square$

A standard condition for (4.1) to hold is given by a Laplace transform condition [13, p. 2, equation (1.5)], namely its Laplace transform $A^*(s) > 0$ in $\Re s > 0$, which is challenging to verify. Consequently, any claim that it holds needs to be carefully checked. Instead of using that, we would quote a result of MacCamy and Wong [13, p. 16] concerning strongly positive kernels which are positive kernels [13, p. 5, (2.9) and the line below].

In our subsequent theorems we could use the following result, but (4.1) is our basic assumption.

Theorem 4.1 *Let $A(t)$ satisfy the following conditions:*

(B_1) $A(t) \in C^1(0, \infty) \cap L^1(0, 1)$,

(B_2) $A(t) \geq 0, A'(t) \leq 0$,

(B_3) $A'(t)$ *is nondecreasing,*

(B_4) $A(t)$ *is not identically constant.*

Then $A(t)$ defines a strongly positive kernel.

The following corollary provides us examples of strongly positive kernels found widely in applied mathematics. The resolvent mentioned is found in Miller [14, p. 212].

Corollary 4.1 *Both t^{q-1} for $0 < q < 1$ and $R(t)$, the resolvent, define strongly positive kernels as both are completely monotone.*

5 Schaefer's Theorem

The following result by Schaefer is conveniently found in Smart [15, p. 29]

Theorem 5.1 (Schaefer) *Let $(\mathcal{B}, \|\cdot\|)$ be a normed space, P be a continuous mapping of $\mathcal{B}$ into $\mathcal{B}$ which is compact on each bounded subset X of $\mathcal{B}$. Then either*

(i) *the equation $x = \lambda Px$ has a solution for $\lambda = 1$, or*
(ii) *the set of all such solutions x, for $0 < \lambda < 1$, is unbounded.*

This brings us to (2.4) in which we insert the parameter $\lambda \in (0,1)$ and write

$$\lambda(P\phi)(t) = \lambda\Bigg[- g\Bigg(t, a + \int_0^t \phi(s))ds\Bigg) - \int_0^t A(t-s) f\Bigg(a + \int_0^s \phi(u)du\Bigg)ds\Bigg]. \quad (2.4\lambda)$$

Here is the array for our work in applying Schaefer's theorem. At the end of the boundedness proof we gather all of this together and state as a formal theorem.

1. The compactness follows from one of our three choices for A in (3.1) which are listed below (3.1) and the compactness of g given in Theorem 3.1.

2. The continuity of P on finite intervals is parallel to that of p. 443 of [1].

3. A fixed point for $\lambda = 1$ is a solution of our beginning equation (2.1). We have established all the conditions of Schaefer's theorem for our mapping P except the *a priori* bound which we will give now as a separate theorem. It contains an interesting pair of steps. To get a clean result we assume that

$$|y| \to \infty \implies \int_0^y f(s)ds \to \infty. \quad (5.1)$$

There are lengthy arguments yielding boundedness if the integral is infinite for either $y > 0$ or $y < 0$.

Theorem 5.2 *Consider (2.4λ) with f and g continuous, (5.1) satisfied, and let A satisfy (4.1). For each $E > 0$ there is a $K > 0$ such that if ξ is a fixed point of* (2.4λ) *on* $[0, E]$ *then $\|\xi\| \leq K$ and for y in (2.5) and (2.1) we have $L > 0$ with $\|y\| \leq L$.*

Proof. Equation (2.4λ) is our mapping equation for Schaefer's theorem. We are mapping continuous functions into continuous functions and a fixed point will be a continuous function ϕ. When we do get a fixed point ϕ for $\lambda = 1$ we will return to the paragraph after (2.3) and we will let $y = a + \int_0^t \phi(s)ds$ as a solution of (2.1), finishing this problem. However, that equivalence between (2.1) and (2.3) which we proved in that paragraph does not hold here for $\lambda \neq 1$.

We are applying Schaefer's theorem and we start afresh with ϕ the assumed fixed point of (2.4λ) for some $\lambda \in (0,1)$ so that we are considering

$$\phi(t) = \lambda\Bigg[- g(t, a + \int_0^t \phi(s))ds\Bigg) - \int_0^t A(t-s) f\Bigg(a + \int_0^s \phi(u)du\Bigg)ds\Bigg]. \quad (2.4\lambda)$$

The goal is an *a priori* bound on $\phi(t)$ for any $\lambda \in (0,1)$. Working exclusively with this equation we make the substitution $y(t) = a + \int_0^t \phi(s)ds$ so that $y(0) = a$. Thus $y'(t) = \phi(t)$ and we have

$$y'(t) = \lambda\Bigg[- g(t, y(t)) - \int_0^t A(t-s) f(y(s))ds\Bigg]. \quad (2.1\lambda)$$

We will get an *a priori* bound on any solution of that equation which is identically (2.4λ) under the substitution $y(t) = a + \int_0^t \phi(s)ds$. A bound on y will be used in (2.1λ) to get a bound on y' which is $\phi(t)$, the assumed fixed point of (2.4λ). That bound will complete Schaefer's requirements giving a solution of (2.4λ) for $\lambda = 1$. We then go back to the paragraph after (2.3) and use that solution of (2.3) to get y in (2.1).

Boundedness of y (but not yet of y') will now follow from the Liapunov function

$$V(y(t)) = \int_0^{y(t)} f(s)ds \tag{5.2}$$

which is positive definite and radially unbounded. In the following differentiation, recall that y' is that continuous function ϕ.

Using the chain rule on (5.2) and (2.1λ) we have

$$\frac{dV(y(t))}{dt} = f(y(t))\frac{dy}{dt} = \lambda f(y(t))\Big[- g(t, y(t)) - \int_0^t A(t-s)f(y(s))ds\Big]$$

so that for any $T \in (0, E]$ we have

$$\begin{aligned} &V(y(T)) - V(y(0)) \\ &= -\lambda\Big[\int_0^T f(y(t))g(t, y(t))dt + \int_0^T f(y(t))\int_0^t A(t-s)f(y(s))dsdt\Big] \le 0 \end{aligned}$$

by (2.1a), (2.1λ), and (4.1). Hence, $V(y(T)) \le V(y(0))$ or

$$\int_0^{y(T)} f(s)ds \le \int_0^{y(0)} f(s)ds \le \max \int_0^{\pm y(0)} f(s)ds =: L^* \tag{5.3}$$

so by (5.1) there is an $L > 0$ with

$$|y(T)| \le L.$$

This is true also for $\lambda = 1$ which means we have the result for (2.1) *provided the solution does exist which, of course, is the objective of this entire theorem.*

To finish the proof, let

$$G = \max_{0\le t\le E, |x|\le L} |g(t, x)|$$

and

$$F = \max_{|x|\le L} |f(x)|.$$

Then the bound on y' of any solution of (2.1λ) is given by

$$K = G + \int_0^E A(t)dtF.$$

Let us consider what this proof has given us. We have an *a priori* bound on ϕ. Thus, we now have all the conditions of Schaefer's theorem and a fixed point of P is assured. Taking that back to the paragraph following (2.3) tells us that the existence of $y(t)$ satisfying (2.1) is assured and it is bounded by L. □

We formally collect all this as the following result.

Theorem 5.3 *Let the conditions with (2.1) hold, together with (2.1a) and the conditions of Theorem 3.1. If, in addition, the conditions of Theorem 5.2 hold, then (2.1) has a solution on any interval* $[0, E]$ *and it is bounded by the L following (5.3).*

If we ask conditions to ensure uniqueness, then we can obtain a solution on $[0, \infty)$ as follows. For each positive integer n construct a solution on $[0, n]$ and continue it to all of $[0, \infty)$ by defining it to be the value at $t = n$. This gives a sequence of functions converging uniformly on compact sets to a continuous function which is a solution on all of $[0, \infty)$ because at any $t > 0$ it agrees with one of the members of the sequence on $[0, t + 1]$. With this we are willing to say that uniqueness implies existence on $[0, \infty)$.

References

[1] Becker, L. C., Burton, T. A., and Purnaras, I. K. Integral and fractional equations, positive solutions, and Schaefer's fixed point theorem. *Opuscula Math.* **36** (4) (2016) 431–458.

[2] Burton, T. A. Existence and uniqueness results by progressive contractions for integro-differential equations. *Nonlinear Dynamics and Systems Theory* **16**(4) (2016) 366–371.

[3] Burton, T. A. Integral equations, transformations and a Krasnoselskii-Schaefer type fixed point theorem. *Electronic J. Qualitative Theory Differential Eq.* **66** (2016) 1–13.

[4] Burton, T. A. and Zhang, Bo. Fixed points and fractional differential equations:Examples. *Fixed Point theory* **14** (2) (2013) 313–326.

[5] Burton, T. A. and Zhang, Bo. A NASC for equicontinuous maps for integral equations, submitted for publication.

[6] Dwiggins, D. P. Fixed point theory and integral equations. *Dynamics of Continuous, Discrete and Impulsive Systems Series A: Mathematical Analysis* **23** (2016) 47–57.

[7] Gripenberg, G., Londen, S-O, and Staffans, O. *Volterra Integral and Functional Equations.* Cambridge, 1990.

[8] Halanay, A. On the asymptotic behavior of the solutions of an integro-differential equation. *J. Math. Anal. Appl.* **10** (1965) 319–324.

[9] Hale, Jack. *Theory of Functional Differential Equations.* Springer, New York, 1977.

[10] Hartman, Philip. *Ordinary Differential Equations.* Wiley, New York, 1964.

[11] Levin, J. J. The asymptotic behavior of the solution of a Volterra equation. *Proc. Amer. Math. Soc.* **14** (1963) 534–541.

[12] Levin, J. J. and Nohel, J. A. On a system of integrodifferential equations occurring in reactor dynamics. *J. Math. Mech.* **9**(1960) 347–368.

[13] MacCamy, R. C. and Wong, J. S. W. Stability theorems for some functional equations. *Trans. Amer. Math. Soc.***164** (1972) 1–37.

[14] Miller, R. K. *Nonlinear Volterra Integral Equations.* Benjamin, Menlo Park, CA, 1971.

[15] Smart, D. R. *Fixed Point Theorems.* Cambridge, New York, 1980.