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Relation Between Fuzzy Semigroups and Fuzzy Dynamical Systems

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Abstract: In this work we study a relation between fuzzy semigroups and fuzzy dynamical systems. Some concepts about stability are introduced to evaluate fuzzy semigroups. Several examples are given to illustrate the obtained results.

Keywords: *fuzzy strongly continuous semigroups; fuzzy dynamical systems; stability; Zadehs extension.*

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1 Introduction

Let $\pi(t,.)$ be a flow generated by solutions of autonomous differential equation. M. T. Mizukoshi et al. showed in [13] that the family of applications $\widehat{\pi}(t,.)$, indexed by $\mathbb{R}$, obtained by Zadeh's extension(see [17]) on initial condition of the flow $\pi(t,.)$, satisfies conditions that characterize $\widehat{\pi}(t,.)$ as a dynamical system in the metric space E^n. In [14], authors discuss conditions for existence of equilibrium points for $\widehat{\pi}(t,.)$ and the nature of the stability of such equilibrium points. New results about equilibrium points are presented in [2].

In [1], M. S. Cecconello discusses results obtained in [4] on invariant sets and stability of such fuzzy sets for fuzzy dynamical systems.

The fuzzy dynamical systems we consider here are obtained by Zadeh's extension of dynamical systems defined on subsets of $\mathbb{R}^n$.

In this paper we discuss relationships between fuzzy semigroups and fuzzy dynamical systems and consider results obtained in [1] on invariant sets and stability of such fuzzy sets, but in this case for fuzzy semigroups.

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2 Preliminary Notes

Let $\mathcal{P}_K(\mathbb{R}^n)$ denote the family of all nonempty compact convex subsets of $\mathbb{R}^n$ and define the addition and scalar multiplication in $\mathcal{P}_K(\mathbb{R}^n)$ as usual. Let A and B be two nonempty bounded subsets of $\mathbb{R}^n$. The distance between A and B is defined by the Hausdorf metric

$$d(A,B) = \max\Big(\rho(A,B), \rho(B,A)\Big),$$

where $\rho(A,B) = \sup\limits_{a\in A}\inf\limits_{b\in B} \|a-b\|$ and $\| \ \|$ denotes the usual Euclidean norm in $\mathbb{R}^n$. Then it is clear that $(\mathcal{P}_K(\mathbb{R}^n), d)$ becomes a complete and separable metric space (see [16]).
Denote

$$E^n = \Big\{ u \,:\, \mathbb{R}^n \longrightarrow [0,1] \ \ | \ \ \text{u satisfies (i)-(iv) below} \Big\},$$

where

(i) u is normal i.e there exists an $x_0 \in \mathbb{R}^n$ such that $u(x_0) = 1$,

(ii) u is fuzzy convex,

(iii) u is upper semicontinuous,

(iv) $[u]^0 = cl\{x \in \mathbb{R}^n : u(x) > 0\}$ is compact.

For $0 < \alpha \le 1$, denote $[u]^\alpha = \{t \in \mathbb{R}^n \,/\, u(t) \ge \alpha\}$. Then from (i)-(iv), it follows that the α-level set $[u]^\alpha \in \mathcal{P}_K(\mathbb{R}^n)$ for all $0 \le \alpha \le 1$.

According to Zadeh's extension principle, we have addition and scalar multiplication in fuzzy number space E^n as follows:

$$[u+v]^\alpha = [u]^\alpha + [v]^\alpha, \quad [ku]^\alpha = k[u]^\alpha,$$

where u, $v \in E^n$, $k \in \mathbb{R}^n$ and $0 \le \alpha \le 1$. Define $D : E^n \times E^n \to \mathbb{R}^+$ by the equation

$$D(u,v) = \sup_{0\le\alpha\le 1} d\Big([u]^\alpha \,,\, [v]^\alpha\Big),$$

where d is the Hausdorff metric for non-empty compact sets in $\mathbb{R}^n$.Then it is easy to see that D is a metric in E^n. Using the results in ([16]), we know that
(1) (E^n, D) is a complete metric space;
(2) $D(u+w, v+w) = D(u,v)$ for all u, v, $x \in E^n$;
(3) $D(k\,u, k\,v) = |k| \ D(u,v)$ for all u, $v \in E^n$ and $k \in \mathbb{R}^n$.

On E^n, we can define the substraction $\ominus$, called the H-difference (see [5]) as follows: $u \ominus v$ has sense if there exists $w \in E^n$ such that $u = v + w$.

Nguyens theorem provides an important relationship between α-levels of image of fuzzy subsets and the image of their α-levels by a function $f : X \times Y \longrightarrow Z$. According to [12], if $X \subseteq \mathbb{R}^n$, Y $\subseteq \mathbb{R}^m$ and $f : X \longrightarrow Y$ is continuous, then Zadehs extension $\widehat{f} : \mathcal{F}(X) \longrightarrow \mathcal{F}(Y)$ is well defined and

$$\Big[\widehat{f}(u)\Big]^\alpha = f\left([u]^\alpha\right), \quad \forall u \in \mathcal{F}(X), \quad \forall \alpha \in [0,1]. \tag{1}$$

Theorem 2.1 *(see [17]) Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a function. Then the following conditions are equivalents:*
(i) f is continuous;
(ii) $\widehat{f} : (E^n; D) \longrightarrow (E^n; D)$ is continuous.

2.1 Fuzzy strongly continuous semigroups.

We give here a definition of a fuzzy semigroup.

Definition 2.1 A family $\{T(t), t \geq 0\}$ of operators from E^n into itself is a fuzzy strongly continuous semigroup if

(i) $T(0) = I_{E^n}$, the identity mapping on E^n,

(ii) $T(t+s) = T(t)T(s)$ for all t, $s \geq 0$,

(iii) the function $g : [0, \infty[\to E^n$, defined by $g(t) = T(t)x$ is continuous at $t = 0$ for all $x \in E^n$ i.e
$$\lim_{t \to 0^+} T(t)x = x,$$

(iv) There exist two constants $M > 0$ and ω such that
$$D\Big(T(t)x, T(t)y\Big) \leq M\, e^{\omega t}\, D(x, y), \quad \text{for}\, t \geq 0, \quad x,\; y \in E^n.$$
In particular if $M = 1$ and $\omega = 0$, we say that $\{T(t), t \geq 0\}$ is a contraction fuzzy semigroup.

Remark 2.1 The condition (iii) implies that the function $t \longrightarrow T(t)(x)$ is continuous on $[0, \infty[$ for all $x \in E^n$.

Definition 2.2 Let $\{T(t), t \geq 0\}$ be a fuzzy strongly continuous semigroup on E^n and $x \in E^n$. If for $h > 0$ sufficiently small, the Hukuhara difference $T(h)x \ominus x$ exits, we define
$$Ax = \lim_{h \to 0^+} \frac{T(h)x \ominus x}{h}$$
whenever this limit exists in the metric space (E^n, D). Then the operator A defined on
$$D(A) = \left\{ x \in E^n : \lim_{h \to 0^+} \frac{T(h)x \ominus x}{h} \,\text{exists} \right\} \subset E^n$$
is called the infinitesimal generator of the fuzzy semigroup $\{T(t), t \geq 0\}$.

Lemma 2.1 *Let A be the generator of a fuzzy semigroup $\{T(t), t \geq 0\}$ on E^n, then for all $x \in E^n$ such that $T(t)x \in D(A)$ for all $t \geq 0$, the mapping $t \to T(t)x$ is differentiable and*
$$\frac{d}{dt}\Big(T(t)\, x\Big) = A\, T(t)\, x, \quad \forall t \geq 0.$$

2.2 Fuzzy dynamical systems.

Definition 2.3 We say that a family of continuous maps, defined on the complete metric space (X, H),
$$\begin{array}{rll} \pi : & \mathbb{R}_+ \times X & \longrightarrow X \\ & (t, x_0) & \longmapsto \pi\,(t, x_0) \end{array}$$
is a dynamical system, or semiflow, if $\pi(t, .)$ satisfies

1. $\pi\left(0, x_0\right) = x_0$;
2. $\pi\left(t, \pi(s, x_0)\right) = \pi\left(t+s, x_0\right)$

for all $t, s \in \mathbb{R}_+$ and $x_0 \in X$.
The set X is called phase space of the dynamical system. In the case of $X \subset \mathbb{R}^2$, X is said to be phase plane.

In dynamical systems context, an orbit (positive) of a point $xo \in X$ is the subset of the phase space defined by

$$\theta(x_0) = \bigcup_{t\in\mathbb{R}_+} \pi\left(t, x_0\right) = \left\{\pi(t, x_0), \quad t \in \mathbb{R}_+\right\},$$

and for each subset $B \subset X$ we have $\theta(B) = \bigcup\limits_{x_0\in B} \theta(x_0)$.

The set $\theta(x_0)$ is called periodic orbit if there exists $\tau > 0$ such that $\pi\left(t+\tau, x_0\right) = \pi\left(t, x_0\right)$. The smallest number $\tau > 0$ for which this property is satisfied is called period of the orbit [6].

The ω-limit of a subset $B \subset X$ is defined as

$$W(B) = \bigcap_{s\geq 0} \overline{\bigcup_{t\geq s} \pi\left(t, B\right)}.$$

Remark 2.2 Let $x_0 \in X$, we have

$$W(x_0) = \left\{ y, \quad \exists t_n \longrightarrow +\infty, \quad \lim_{n\to+\infty} \pi(t_n, x_0) = y \right\}.$$

A set $S \subset U$ is called invariant if $\theta\left(x_0\right) \subset S$ for all $x_0 \in S$. It follows that S is invariant if and only if $\pi(t, S) = S$ for all $t \in \mathbb{R}_+$.

Example : The orbits and ω-limit are examples of invariant sets.

Definition 2.4 We say that a set $M \subset X$ attracts a set $B \subset \mathbb{R}^n$, by flow $\pi(t, .)$, if $\rho\left(\pi(t, B), M\right) \longrightarrow 0$ when $t \to +\infty$. In other words, we say that "a set M attracts a set B" is equivalent to saying that M attracts uniformly all orbits with initial condition in B, that is,

$$\lim_{t\to+\infty} \sup\left\{\rho(\pi(t, x_0), M) : \quad x_0 \in B\right\} = 0.$$

The basin of attraction of a set M is the set $A(M)$ defined by

$$A(M) = \left\{x_0 \in U : \quad \rho\left(\pi(t, x_0), M\right) \longrightarrow 0, \quad t \to +\infty\right\}.$$

The set M is called attractor if there exists an open subset $V \supset M$ such that $V \subset A(M)$. If M is an attractor and attracts compacts subsets of $A(M)$, then M is a uniform attractor.

Definition 2.5 Let $\bar{x} \in X$, $\bar{x}$ is an equilibrium point if $\pi(t, \bar{x}) = \bar{x}$ for all $t \in \mathbb{R}_+$.

Definition of stability for invariant sets is similar to definition of stability for equilibrium points. That is, an invariant set S is stable if for every neighborhood V of S, there exists a neighborhood V' of S such that $\pi\left(t, V'\right) \subset V$ for all $t \in \mathbb{R}_+$. When S is stable and moreover there exists a neighborhood W such that S attracts points of W then S is an asymptotically stable set.

Theorem 2.2 *([1]) Let M be compact and invariant. Then M is asymptotically stable if and only if M is a uniform attractor.*

Let $\pi(t,.)$ be a dynamical system defined in a subset $U \subset \mathbb{R}^n$, that we will call deterministic dynamical system.

Theorem 2.3 *(see [13]) Let $\pi(t,.) : U \longrightarrow U$ be a deterministic dynamical system. Then $\widehat{\pi}(t,.)$ defined by Zadeh's extension applied in $\pi(t,.)$ has the following properties:*

1. $\widehat{\pi}\,(0, x_0) = x_0$, $\forall x_0 \in \mathcal{F}(U)$;
2. $\widehat{\pi}\,(t+s, x_0) = \widehat{\pi}(t, \widehat{\pi}(s, x_0))$, $\forall x_0 \in \mathcal{F}(U)$, $t, s \geq 0$.

Thus, the Zadeh's extension $\widehat{\pi}(t,.) : \mathcal{F}(U) \longrightarrow \mathcal{F}(U)$, of deterministic dynamical system $\pi(t,.) : U \longrightarrow U$, is a dynamical system in $\mathcal{F}(U)$ and we will call it fuzzy dynamical system. Then concepts of stability and asymptotic stability for invariant sets in $\mathcal{F}(U)$ follow definitions given previously to general metric spaces.

3 Main Results

3.1 Relation between fuzzy semigroups and fuzzy dynamical systems.

Let $(\pi(t,.))_{t\geq 0}$ be a dynamical system on $\mathbb{R}^n$, i.e for all $t \geq 0$

$$\begin{aligned} \pi(t,.) : \quad \mathbb{R}^n &\longrightarrow \mathbb{R}^n \\ x &\longmapsto \pi(t,x) \end{aligned}$$

satisfies

1. $\pi(0,x) = x$, $\forall x \in \mathbb{R}^n$;
2. $\pi(t+s,x) = \pi(t, \pi(s,x))$, $\forall t, s \geq 0, \quad x \in \mathbb{R}^n$;
3. $t \longrightarrow \pi(t,x)$ is continuous for all $x \in \mathbb{R}^n$.

We consider the family $\{T(t),\ t \geq 0\}$ given by

$$\begin{aligned} T : \quad \mathbb{R}_+ \times \mathbb{R}^n &\longrightarrow \mathbb{R}^n \\ (t,x) &\longmapsto T(t)x = \pi(t,x). \end{aligned}$$

Then the family of continuous maps $T(t)$ verifies

1. $T(0) = I$;
2. $T(t+s) = T(t)T(s)$, $\forall t, s \geq 0$.

Then the family $\{T(t),\ t \geq 0\}$ defines a strongly continuous semigroup on $\mathbb{R}^n$.

By Zadeh's extension, we can define a fuzzy dynamical system $\widehat{\pi}(t,.)$.

Define a mapping $\left\{\widehat{T}(t), t \geq 0\right\}$ as follows

$$\begin{aligned} \widehat{T} : \quad \mathbb{R}_+ \times \mathbb{R}^n &\longrightarrow \mathbb{R}^n \\ (t,x) &\longmapsto \widehat{T}(t)x = \widehat{\pi}(t,x). \end{aligned}$$

From Theorem 2.3, we have

1. $\widehat{T}(0)x = x$, $\forall x \in \mathcal{F}(\mathbb{R}^n)$;
2. $\widehat{T}(t+s)x = \widehat{T}(t) \circ \widehat{T}(s)x$, $\forall x \in \mathcal{F}(\mathbb{R}^n)$, $t, s \geq 0$.

Theorem 3.1 *Let $\{T(t), t \geq 0\}$ be a strongly continuous semigroup on $\mathbb{R}^n$ which satisfies*

$$\|T(t)x\| \leq Me^{wt}\|x\|, \quad \forall x \in \mathbb{R}^n, \quad t \geq 0.$$

Then

$$D\left(\widehat{T}(t)x, \widehat{T}(t)y\right) \leq Me^{wt}D(x,y), \quad \forall x, y \in E^n, \quad t \geq 0.$$

Proof. Let $x, y \in E^n$, $\alpha \in [0,1]$, we have

$$\begin{aligned}
\rho(T(t)[x]^\alpha, T(t)[y]^\alpha) &= \sup_{a\in[x]^\alpha} \inf_{b\in[y]^\alpha} \|T(t)a - T(t)b\| \\
&\leq Me^{wt} \sup_{a\in[x]^\alpha} \inf_{b\in[y]^\alpha} \|a-b\| \\
&\leq Me^{wt}\rho([x]^\alpha, [y]^\alpha) \\
&\leq Me^{wt} \max\{\rho([x]^\alpha, [y]^\alpha), \rho([y]^\alpha, [x]^\alpha)\},
\end{aligned}$$

and

$$\begin{aligned}
\rho(T(t)[y]^\alpha, T(t)[x]^\alpha) &= \sup_{a\in[y]^\alpha} \inf_{b\in[x]^\alpha} \|T(t)a - T(t)b\| \\
&\leq Me^{wt} \sup_{a\in[y]^\alpha} \inf_{b\in[x]^\alpha} \|a-b\| \\
&\leq Me^{wt}\rho([y]^\alpha, [x]^\alpha) \\
&\leq Me^{wt} \max\{\rho([x]^\alpha, [y]^\alpha), \rho([y]^\alpha, [x]^\alpha)\}.
\end{aligned}$$

This implies

$$\begin{aligned}
d\left([\widehat{T}(t)x]^\alpha, [\widehat{T}(t)y]^\alpha\right) &= d\left(T(t)[x]^\alpha, T(t)[y]^\alpha\right) \\
&= \max\{\rho(T(t)[x]^\alpha, T(t)[y]^\alpha), \rho(T(t)[y]^\alpha, T(t)[x]^\alpha)\} \\
&\leq Me^{wt} \max\{\rho([x]^\alpha, [y]^\alpha), \rho([y]^\alpha, [x]^\alpha)\} \\
&= Me^{wt} d\left([x]^\alpha, [y]^\alpha\right).
\end{aligned}$$

Hence, we conclude that

$$D\left(\widehat{T}(t)x, \widehat{T}(t)y\right) \leq Me^{wt}D(x,y).$$

□

Corollary 3.1 $\left\{\widehat{T}(t), t \geq 0\right\}$ *is a fuzzy strongly continous semigroup on E^n.*

Proof. (i) and (ii) are immediate consequences of Theorem 2.3.
Theorem 2.1 ensures (iii).
(iv) follows immediately from Theorem 3.1.

□

Now, we can conclude that from fuzzy dynamical systems, we can define fuzzy strongly continuous semigroups.

Example. We define on $\mathbb{R}$ the family of operator $(\pi(t,.))_{t\geq 0}$ by

$$\pi(t,x) = e^{at}x, \quad a \in \mathbb{R}.$$

$(\pi(t,.))_{t\geq 0}$ is a dynamical system on $\mathbb{R}$. We consider the family $\{T(t), t \geq 0\}$ given by

$$T(t)x = \pi(t,x).$$

$\{T(t), t \geq 0\}$ is a strongly continuous semigroup on $\mathbb{R}$, and the linear operator A defined by $Ax = ax$ is the infinitesimal generator of this semigroup. Then the family of continuous maps $\left\{\widehat{T}(t), t \geq 0\right\}$ defined by $\widehat{T}(t)x = \widehat{\pi}(t,x)$, where $(\widehat{\pi}(t,.)$ is the fuzzy dynamical system obtained by Zadeh's extension applied in $\pi(t,.))$ defines fuzzy strongly continuous semigroups on E^1.

3.2 Invariant and attractor sets for fuzzy strongly continuous semigroups.

In this section, we give the results obtained by M. S. Cecconello, J. Leite , R. C. Bassanezi, A. J. V. Brando (see [1]), but in this case for a fuzzy semigroups.

Let $\{T(t), t \geq 0\}$ be a strongly continuous semigroup on E^n and $\left\{\widehat{T}(t), t \geq 0\right\}$ be the fuzzy strongly continuous semigroup obtained by Zadeh's extension applied in $\{T(t), t \geq 0\}$.

Proposition 3.1 *$\bar{x}$ is an equilibrium point of $T(t)(T(t)\bar{x} = \bar{x})$ if, and only if $\chi_{\{\bar{x}\}}$ is an equilibrium point of $\widehat{T}(t)$, where $\widehat{T}(t)$ is the characteristic function of $\bar{x}$.*

Proof. We have

$$\bar{x} = T(t)\left([\chi_{\{\bar{x}\}}]^\alpha\right) \Leftrightarrow \left[\chi_{\{\bar{x}\}}\right]^\alpha = \left[\widehat{T}(t)(\chi_{\{\bar{x}\}})\right]^\alpha.$$

□

To prove the following results it is suffcient to denote $\widehat{\pi}(t,x) = \widehat{T}(t)x$ in [1].

Theorem 3.2 *Let $S \subset U \subset \mathbb{R}^n$ and consider $S_{\mathcal{F}} \in \mathcal{F}(U)$ defined by*

$$S_{\mathcal{F}} = \{x \in E^n : [x]^\alpha \subset S\}.$$

S is invariant by $T(t)$ if and only if $S_{\mathcal{F}}$ is invariant by $\widehat{T}(t)$.

The ω-limit of a subset $B \subset U$ is defined as

$$\omega(B) = \bigcap_{s\geq 0}\overline{\bigcup_{t\geq s} T(t)(B)}.$$

Consider the set $\omega_{\mathcal{F}}(B) \subset \mathbb{F}(U)$ defined by

$$\omega_{\mathcal{F}}(B) = \{x \in \mathcal{F}(U) : [x_0]^\alpha \subset \omega(B)\}.$$

Corollary 3.2 *The set $\omega(B)$ is invariant by $T(t)$ if and only if the set $\omega_{\mathcal{F}}(B)$ is invariant by $\widehat{T}(t)$.*

Theorem 3.3 *Let $C \subset U$ be an invariant set by $T(t)$ and consider the set $C_{\mathcal{F}} \subset \mathcal{F}(U)$ defined by*

$$C_{\mathcal{F}} = \left\{ x \in \mathcal{F}(U) : [x]^0 \subset C \right\}.$$

Then we have:

1. *C is stable for $T(t)$ if and only if $C_{\mathcal{F}}$ is stable for $\widehat{T}(t)$;*
2. *C is asymptotically stable for $T(t)$ if and only if $C_{\mathcal{F}}$ is asymptotically stable for $\widehat{T}(t)$.*

By the previous theorem we can establish the following result.

Corollary 3.3 *Let $\bar{x} \in U$ be an equilibrium point of $T(t)$. Then*

1. *$\bar{x}$ is stable for $T(t)$ if and only if $\chi\{\bar{x}\}$ is stable for $\widehat{T}(t)$;*
2. *$\bar{x}$ is asymptotically stable for $T(t)$ if and only if $\chi\{\bar{x}\}$ is asymptotically stable for $\widehat{T}(t)$.*

An orbit (positive) of a point x_0 is the subset of the phase space defined by

$$\theta(x_0) = \bigcup_{t \in \mathbb{R}_+} T(t)x_0 = \{T(t)x_0, \quad t \in \mathbb{R}_+\}.$$

Similarly, if θ is a periodic orbit for $T(t)$ then θ is invariant. By Theorem 4.2, the set θ is invariant for $\widehat{T}(t)$ and we have:

Corollary 3.4 *Let θ be a periodic orbit for $T(t)$ with period $\tau > 0$ and $\theta_{\mathcal{F}}$ be the fuzzy periodic set defined by*

$$\theta_{\mathcal{F}} = \left\{ x \in \mathcal{F}(U) : [x]^0 \subset \theta \right\}.$$

Then

1. *θ is stable for $T(t)$ if and only if $\theta_{\mathcal{F}}$ is stable for $\widehat{T}(t)$;*
2. *θ is asymptotically stable for $T(t)$ if and only if $\theta_{\mathcal{F}}$ is asymptotically stable for $\widehat{T}(t)$.*

Let A, $B \subset \mathbb{R}^n$ and $A_{\mathcal{F}}, B_{\mathcal{F}} \subset E^n$ be defined, respectively, by

$$A_{\mathcal{F}} = \left\{ x \in E^n : [x]^0 \subset A \right\} \quad \text{and} \quad B_{\mathcal{F}} = \left\{ x \in E^n : [x]^0 \subset B \right\}.$$

Theorem 3.4 *Set A attracts B by $T(t)$ if and only if $A_{\mathcal{F}}$ attracts $B_{\mathcal{F}}$ by $\widehat{T}(t)$.*

So we have the following result for $\omega(B)$ and the set $\omega_{\mathcal{F}}(B)$:

Corollary 3.5 *The set $\omega(B)$ attracts $B \subset U$ by $T(t)$ if and only if $\omega_{\mathcal{F}}(B)$ attracts $A_{\mathcal{F}} = \left\{ x \in \mathcal{F}(U) : [x_0]^0 \subset A \right\}$ by $\widehat{T}(t)$.*

3.3 Some examples

Example: Given the classic initial value problem

$$(2)\begin{cases} x' = -kx, \\ x(0) = x_0, \end{cases}$$

we can verify that its solution is given by $T(t)x_0 = e^{-kt}x_0$ and that the origin is an asymptotically stable equilibrium point for $k > 0$. By Proposition 4.1, we have that $\chi_{\{0\}}$ is an equilibrium point of $\widehat{T}(t)$, since $\bar{x} = 0$ is an equilibrium point of (2).
Moreover, $\chi_{\{0\}}$ is asymptotically stable.

Example: Let us consider the deterministic Verhulst model

$$(3)\begin{cases} x^{'} = ax(1-x), \\ x(0) = x_0, \end{cases}$$

whose solution is given by

$$T(t)x_0 = \frac{x_0}{x_0 + (1-x_0)e^{-at}}.$$

The equilibrium points of (3) are 0 and 1. The first one is unstable while the latter is asymptotically stable.

So, $\chi_{\{0\}}$ and $\chi_{\{1\}}$ are equilibrium points of $\widehat{T}(t)$ being the fuzzy strongly semigroup obtained by Zadeh's extension of $T(t)$ and the first one is unstable equilibrium point while the latter is asymptotically stable. Some other details for this example are presented in [1].

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