Nonlinear Dynamics and Systems Theory, **17** (1) (2017) 86–94

A Simple Approach for Q-S Synchronization of Chaotic Dynamical Systems in Continuous-Time

Adel Ouannas *

Department of Mathematics and Computer Science, University of Tebessa, 12002 Tebessa, Algeria

Received: August 31, 2015; Revised: January 31, 2017

Abstract: In this paper, the problem of Q-S synchronization for arbitrary dimensional chaotic dynamical systems in continuous-time is investigated. Based on new control scheme and Lyapunov stability theory, a simple synchronization approach is designed to achieve Q-S synchronization between n-D and m-D continuous-time chaotic systems in arbitrary dimension d. In order to verify the effectiveness of the proposed method, our approach is applied to some typical chaotic systems and numerical simulations are given to validate the derived results.

Keywords: *chaos; Q-S synchronization; continuous-time systems; control scheme; Lyapunov stability.*

Mathematics Subject Classification (2010): 37B25, 37B55, 93C10, 93C55.

1 Introduction

Since the discover of synchronization [1, 2], chaos synchronization has played important roles in sciences and enginering, due to its potential applications in secure communication and telecommunications [3–6], control theory [7, 8], biology [9, 10], lasers [11], and so on. Chaos synchronization has received increasing interest and various methods have been proposed for synchronization of chaotic dynamical systems such as adaptive control [12], backstepping design [13], sliding mode control [14], and generalized hamiltonian systems approach [15, 16] etc. Many types of chaos synchronization have been presented such as complete and anti-synchronization [17, 18], hybrid function projective synchronization [19], reduced order function projective combination synchronization [20], etc. Among all types of synchronization, Q-S synchronization is an interesting generalized-type of synchronization which has been extensively considered [21, 22]. In Q-S synchronization,

* Corresponding author: mailto:ouannas_adel@yahoo.fr

different dimensional chaotic systems can be synchronized in arbitrary dimensions due to functional relationships between the states of the master and the slave chaotic systems. Recently, Q-S synchronization has received a great deal of attention and a series of works on Q-S synchronization have been published for chaotic dynamical systems in continuous-time [23–26], and discrete-time [27–29].

The main aim of the present work, is to propose a new general control scheme to study the problem of Q-S synchronization for coupled continuous-time chaotic systems. Based on nonlinear control method, we would like to present a constructive scheme to investigate Q-S synchronization between two different dimensional chaotic systems in arbitrary dimension. The new derived synchronization result is proved using Lyapunov stability theory and numerical examples are used to show the effectiveness of the proposed control method.

The rest of this paper is arranged as follows. In Section 2, the problem of Q-S synchronization in arbitrary dimension is formulated. In Section 3, we present our approach of Q-S synchronization. In Section 4, numerical examples and simulations are used to show the effectiveness of the proposed method. Finally, conclusion is given in Section 5.

2 Problem formulation

Consider the following master chaotic system

$$\dot{X}\left(t\right) = F(X\left(t\right)), \tag{1}$$

where $X\left(t\right) = \left(x_i\left(t\right)\right)_{1\leq i\leq n}$ is the state vector of the master system (1) and $F = \left(F_i\right)_{1\leq i\leq n}$ is a differentiable vector function. As a slave system, we consider the following chaotic system

$$\dot{Y}\left(t\right) = G(Y\left(t\right)) + U, \tag{2}$$

where $Y\left(t\right) = \left(y_i\left(t\right)\right)_{1\leq i\leq m}$, is the state vector of the slave system (2), $G = \left(G_i\right)_{1\leq i\leq m}$ is a differentiable vector function and $U = \left(u_i\right)_{1\leq i\leq m}$ is a vector controller to be determined. The definition of Q-S synchronization for the master system (1) and the slave system (2) is given below.

Definition 2.1 The master system (1) and the slave system (2) are said to be Q-S synchronized, in dimension d, if there exists a controller $U = \left(u_i\right)_{1\leq i\leq m}$ and two continuously differentiable vector functions $Q\left(Y\left(t\right)\right) = \left(Q_i\left(Y\left(t\right)\right)\right)_{1\leq i\leq d}$, $S\left(X\left(t\right)\right) = \left(S_i\left(X\left(t\right)\right)\right)_{1\leq i\leq d}$, respectively, such that the synchronization error

$$e\left(t\right) = \left(e_1\left(t\right), e_2\left(t\right), ..., e_d\left(t\right)\right)^T = Q\left(Y\left(t\right)\right) - S\left(X\left(t\right)\right), \tag{3}$$

satisfies the condition $\lim_{t\longrightarrow+\infty} \left\|e\left(t\right)\right\| = 0$.

In order to study Q-S synchronization between the master and the slave systems given in equations (1) and (2), we discuss the asymptotic stability of zero solution of synchronization error system $e\left(t\right) = Q\left(Y\left(t\right)\right) - S\left(X\left(t\right)\right)$ i.e., we find the controllers u_i, $i = 1, 2, \cdots, m$, such that the solutions of the error system $e_i(t) = Q_i\left(Y\left(t\right)\right) - S_i\left(X\left(t\right)\right)$ go to 0, $i = 1, 2, \cdots, d$, as t goes to $+\infty$.

3 A New Q-S Synchronization Approach

The error system (3), between the master system (1) and the slave system (2), can be derived as

$$\dot{e}\left(t\right)=DQ\left(Y\left(t\right)\right)\times\left(G(Y\left(t\right))+U\right)-DS\left(X\left(t\right)\right)\times F(X\left(t\right)), \tag{4}$$

where $DQ\left(Y\left(t\right)\right)\in\mathbb{R}^{d\times m}$, $DS\left(X\left(t\right)\right)\in\mathbb{R}^{d\times n}$ are the Jacobian matrices of the functions Q and S, respectively,

$$DQ\left(Y\left(t\right)\right)=\begin{pmatrix}\frac{\partial Q_1}{\partial y_1} & \frac{\partial Q_1}{\partial y_2} & \cdots & \frac{\partial Q_1}{\partial y_m}\\ \frac{\partial Q_2}{\partial y_1} & \frac{\partial Q_2}{\partial y_2} & \cdots & \frac{\partial Q_2}{\partial y_m}\\ \vdots & \vdots & \ddots & \vdots\\ \frac{\partial Q_d}{\partial y_1} & \frac{\partial Q_d}{\partial y_2} & \cdots & \frac{\partial Q_d}{\partial y_m}\end{pmatrix}, \tag{5}$$

$$DS\left(X\left(t\right)\right)=\begin{pmatrix}\frac{\partial S_1}{\partial x_1} & \frac{\partial S_1}{\partial x_2} & \cdots & \frac{\partial S_1}{\partial x_n}\\ \frac{\partial S_2}{\partial x_1} & \frac{\partial S_2}{\partial x_2} & \cdots & \frac{\partial S_2}{\partial x_n}\\ \vdots & \vdots & \ddots & \vdots\\ \frac{\partial S_d}{\partial x_1} & \frac{\partial S_d}{\partial x_2} & \cdots & \frac{\partial S_d}{\partial x_n}\end{pmatrix}, \tag{6}$$

and we assume that $d\leq m$. The error system (4) can be described as follows

$$\begin{aligned}\dot{e}_i\left(t\right) &= \sum_{j=1}^{m}\left(\frac{\partial Q_i}{\partial y_j}\times\left(G_j\left(Y\left(t\right)\right)+u_j\right)\right)-\sum_{j=1}^{n}\left(\frac{\partial S_i}{\partial x_j}\times F_j\left(X\left(t\right)\right)\right)\\ &= -k_ie_i\left(t\right)+R_i+\sum_{j=1}^{d}\left(\frac{\partial Q_i}{\partial y_j}\times u_j\right)+\sum_{j=d+1}^{m}\left(\frac{\partial Q_i}{\partial y_j}\times u_j\right),\ 1\leq i\leq d, \end{aligned}\tag{7}$$

where

$$R_i=k_i\left(Q_i\left(Y\left(t\right)\right)-S_i\left(X\left(t\right)\right)\right)+\sum_{j=1}^{m}\left(\frac{\partial Q_i}{\partial y_j}\times G_j\left(Y\left(t\right)\right)\right)-\sum_{j=1}^{n}\left(\frac{\partial S_i}{\partial x_j}\times F_j\left(X\left(t\right)\right)\right), \tag{8}$$

and $k_i\in\mathbb{R}_*^+$, $(1\leq i\leq d)$ are control constants. To achieve Q-S synchronization between the systems (1) and (2), the vector controller $U=\left(u_i\right)_{1\leq i\leq m}$ is chosen as follows

$$U=\left(u_1,...,u_d,0,...,0\right)^T, \tag{9}$$

and by using equation (9) into equation (7), the error system (7) can be written as follow:

$$\dot{e}_i\left(t\right)=-k_ie_i\left(t\right)+R_i+\sum_{j=1}^{d}\left(\frac{\partial Q_i}{\partial y_j}\times u_j\right),\quad 1\leq i\leq d, \tag{10}$$

rewriting the error system (10) in the compact form

$$\dot{e}\left(t\right)=-Ke\left(t\right)+R+JV, \tag{11}$$

where $e(t) = (e_i(t))_{1\leq i\leq d}$, $K = diag(k_1, ..., k_d)$, $R = (R_i)_{1\leq i\leq d}$,

$$J = \begin{pmatrix} \frac{\partial Q_1}{\partial y_1} & \frac{\partial Q_1}{\partial y_2} & \cdots & \frac{\partial Q_1}{\partial y_d} \\ \frac{\partial Q_2}{\partial y_1} & \frac{\partial Q_2}{\partial y_2} & \cdots & \frac{\partial Q_2}{\partial y_d} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial Q_n}{\partial y_1} & \frac{\partial Q_n}{\partial y_2} & \cdots & \frac{\partial Q_d}{\partial y_d} \end{pmatrix}, \tag{12}$$

and $V = (u_1, ..., u_d)^T$. Now, we can choose V as follows

$$V = -J^{-1}R, \tag{13}$$

where J^{-1} is the inverse of (12). Substitute equation (13) into equation (11), then the error system can be written as

$$\dot{e}(t) = -Ke(t). \tag{14}$$

To study the asymptotic stability of zero solution of the error system (14), we consider the candidate Lyapunov function:

$$V(e(t)) = \frac{1}{2}e^T(t)e(t), \tag{15}$$

then the derivative of the function (15) along the solution of the system (14) is given as follows

$$\begin{aligned} \dot{V}(e(t)) &= \dot{e}^T(t)e(t) + e^T(t)\dot{e}(t) \\ &= -\frac{1}{2}Ke^T(t)e(t) - \frac{1}{2}Ke^T(t)e(t) \\ &= -Ke^T(t)e(t) \\ &= \sum_{i=1}^{d} -k_i e_i^2(t) < 0, \end{aligned}$$

and by Lyapunov stability theory, it is immediate that

$$\lim_{t\to\infty} e_i(t) = 0, \quad 1 \leq i \leq d, \tag{16}$$

and from the fact

$$\lim_{t\to\infty} \|e(t)\| = 0. \tag{17}$$

Hence, we have proved the following result.

Theorem 3.1 *The master system (1) and the slave system (2) are globally $Q - S$ synchronized under the control law (9)-(13).*

4 Numerical Examples

In order to show the effectiveness of the presented approach of synchronization, two numerical examples are used to observe Q-S synchronization in 3D between two identical dimensional (3D) chaotic systems and two different dimensional (3D and 4D) chaotic systems , respectively.

4.1 Example 1: Q-S synchronization between Rössler and Sprott-WINDMI systems

In this example, we consider the Rössler system [31] as a master system and the controlled Sprott-WINDMI system [30] as a slave system. The Rössler system and the controlled Sprott-WINDMI system can be described, respectively, as follows

$$\begin{aligned} \dot{x}_1 &= -(x_2 + x_3), \\ \dot{x}_2 &= x_1 + 0.2x_2, \\ \dot{x}_3 &= -5.7x_3 + x_1x_3 + 0.2, \end{aligned} \tag{18}$$

and

$$\begin{aligned} \dot{y}_1 &= y_2 + u_1, \\ \dot{y}_2 &= y_3 + u_2, \\ \dot{y}_3 &= -y_2 - 0.7y_3 + 2.5 - e^{y_1} + u_3, \end{aligned} \tag{19}$$

where u_1, u_2 and u_3 are synchronization controllers. In this case, we select the vector functions Q and S respectively as

$$Q\left(y_1, y_2, y_3\right) = \left(y_1, \frac{1}{3}y_2^3 + y_2, y_3\right)^T, \tag{20}$$

$$S\left(x_1, x_2, x_3\right) = \left(x_1, x_2x_3, x_1 + x_3\right)^T, \tag{21}$$

so

$$DQ\left(y_1, y_2, y_3\right) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & y_2^2 + 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{22}$$

$$DS\left(x_1, x_2, x_3\right) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & x_3 & x_2 \\ 1 & 0 & 1 \end{pmatrix}. \tag{23}$$

According to our approach presented in Section 3, and by using (20), (21), (22) and (23), the controllers u_1, u_2,and u_3 can be constructed as follows

$$\begin{aligned} u_1 &= -\left(k_1 + 1\right)y_1 + k_1x_1 - x_2 - x_3, \\ u_2 &= \frac{-1}{y_2^2 + 1}\left[k_2\left(\frac{1}{3}y_2^3 + y_2\right) + \left(y_2^2 + 1\right)y_3 - 0.2x_2 + \left(5.5 - k_2\right)x_2x_3 - x_1x_3 - x_1x_2x_3\right], \\ u_3 &= y_2 + \left(0.7 - k_3\right)y_3 + e^{y_1} + k_3x_1 - x_2 + \left(5.7 - k_3\right)x_3 + x_1x_3 - 2.3, \end{aligned} \tag{24}$$

where the control constants $\left(k_i\right)_{1\leq i\leq 3}$ are chosen as $\left(k_1, k_2, k_3\right) = (1, 2, 3)$. The error functions can be written as follows $\dot{e}_1\left(t\right) = -e_1\left(t\right)$, $\dot{e}_2\left(t\right) = -2e_2\left(t\right)$ and $\dot{e}_3\left(t\right) = -3e_3\left(t\right)$. Then, the numerical simulation of the error functions evolution is shown in Figure 2.

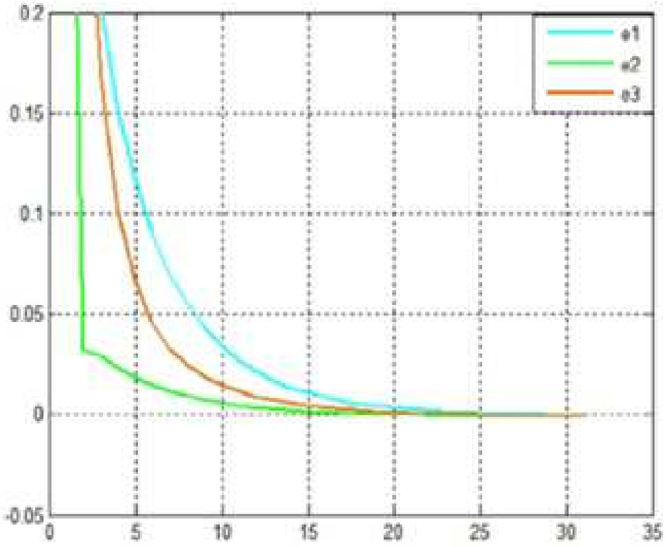

Figure 1: Time evolution of Q-S synchronization errors between the master system (18) and the slave system (19).

4.2 Example 2: Q-S synchronization between Lorenz and hyperchaotic Chen systems

In this example, we consider the Lorenz system [32] as a master system and the controlled hyperchaotic Chen system [33] as a slave system. The Lorenz system and the controlled hyperchaotic Chen system can be described, respectively, as follows

$$\begin{aligned} \dot{x}_1 &= 10(x_2 - x_1), \\ \dot{x}_2 &= 28x_1 - x_2 - x_1x_3, \\ \dot{x}_3 &= -8/3x_3 + x_1x_2, \end{aligned} \tag{25}$$

and

$$\begin{aligned} \dot{y}_1 &= -27.5y_1 + 27.5y_2 + u_1, \\ \dot{y}_2 &= 3y_1 + 19.3y_2 + y_4 - y_1y_3 + u_2, \\ \dot{y}_3 &= -2.9y_4 + u_3, \\ \dot{y}_4 &= -3.3y_1 + y_2^2 + u_4, \end{aligned} \tag{26}$$

where u_1, u_2, u_3 and u_4 are synchronization controllers. In this case, the vector functions Q and S are chosen, respectively, as follows

$$Q\left(y_1, y_2, y_3, y_4\right) = \left(y_1 + y_4, y_2 + y_4, y_4 + y_3\right)^T, \tag{27}$$

$$S\left(x_1, x_2, x_3\right) = \left(x_1x_3, x_2, x_3\right)^T, \tag{28}$$

so

$$DQ\left(y_1,y_2,y_{3,}y_4\right)=\begin{pmatrix}1&0&0&1\\0&1&0&1\\0&0&1&1\end{pmatrix},\tag{29}$$

$$DS\left(x_1,x_2,x_3\right)=\begin{pmatrix}x_3&0&x_1\\0&1&0\\0&0&1\end{pmatrix}.\tag{30}$$

According to the control law (9)-(13) proposed in Section 3, and by using (27), (28), (29) and (30), the controllers u_1,u_2, u_3 and u_4 can be designed as follows

$$\begin{aligned}u_1 &= (k_1-30.8)y_1+27.5y_2+k_1y_4+y_2^2+(\frac{38}{3}-k_1)x_1x_3 \\ &\quad -10x_3x_2-x_1^2x_2, \\ u_2 &= -0.3y_1+(k_2+19.3)y_2+(k_2+1)y_4-y_1y_3+y_2^2-28x_1 \\ &\quad +(1-k_2)x_2+x_1x_3, \\ u_3 &= -3.3y_1+k_3y_3+\left(k_3-2.9\right)y_4+y_2^2+(8/3-k_3)-x_1x_2, \\ u_4 &= 0,\end{aligned}\tag{31}$$

where the control constants $(k_i)_{1\leq i\leq 4}$ are chosen as $(k_1,k_2,k_3,k_4)=(0.1,0.2,0.3,0.4)$. The error functions can be written as follows: $\dot{e}_1\left(t\right)=-0.1e_1\left(t\right)$, $\dot{e}_2\left(t\right)=-0.2e_2\left(t\right)$, $\dot{e}_3\left(t\right)=-0.8e_3\left(t\right)$ and $\dot{e}_3\left(t\right)=-0.4e_3\left(t\right)$. Then, the numerical simulation of the error functions evolution is shown in Figure 2.

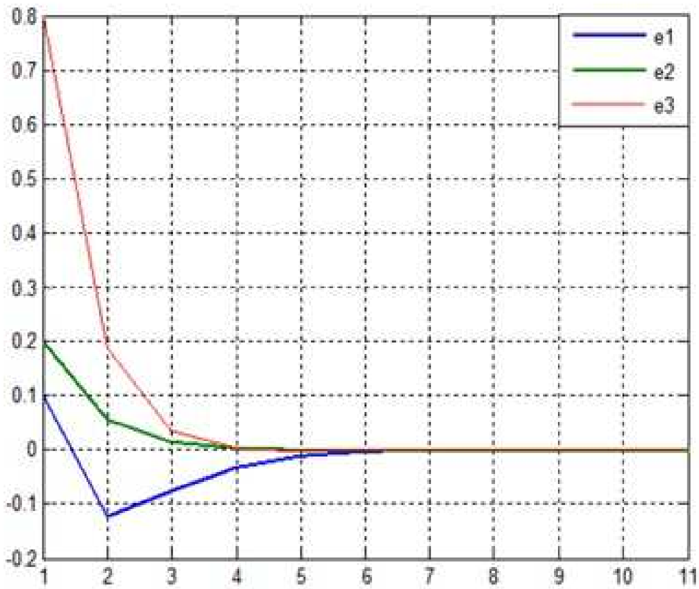

Figure 2: Time evolution of Q-S synchronization errors between the master system (25) and the slave system (26).

5 Conclusion

In this paper, we have developed a new systematic and powerful synchronization scheme, which is used to study Q-S synchronization between two n-D and m-D continuous-time chaotic dynamical systems. Numerical examples are used to verify the effectiveness of the proposed approach.

References

[1] Fujisaka, H. and Yamada, T. Stability theory of synchronized motion in coupled-oscillator systems. *Progress of Theoretical Physics* **69** (1) (1983) 32–47.

[2] Pecora, L. M. and Carroll, T. L. Synchronization in chaotic systems. *Physical Review Letter*s **64** (8) (1990) 821–824.

[3] Kocarev, L. and Parlitz, U. General approach for chaotic synchronization with applications to communication. *Physical Review Letters* **74** (25) (1995) 5028–5031.

[4] Yu, W., Cao, J., Wong, K. W. and Lü, J. New communication schemes based on adaptive synchronization. *Chaos* **17** (3) (2007) 033114-13.

[5] Aguilar Bustos, A. Y., Cruz Hernández, C., López Gutiérrez, R. M. and Posadas Castillo, Y. C. Synchronization of different hyperchaotic maps for encryption. *Nonlinear Dynamics and Systems Theory* **8** (3) (2008) 221–236.

[6] Inzunza González, E. Y. and Cruz Hernandez, C. Double hyperchaotic encryption for security in biometric systems. *Nonlinear Dynamics and Systems Theory* **13** (1) (2013) 55–68.

[7] Pyragas, K. Continuous control of chaos by self-controlling feedback. *Physics Letters A* **170** (6) (1992) 421–428.

[8] Wu, C. W. and Chua, L. O. A unified framework for synchronization and control of dynamical systems. *International Journal of Bifurcation and Chaos* **4** (4) (1994) 979–998.

[9] Han, S. K., Kurrer, C. and Kuramoto, Y. Dephasing and bursting in coupled neural oscillators. *Physical Review Letters* **75** (17) (1995) 3190–3193.

[10] Elson, R. C., Selverston, A. I., Huerta, R., Rulkov, N. F., Rabinovich, M. I. and Abarbanel, H. D. I. Synchronous behavior of two coupled biological neurons. *Physical Review Letters* **81** (25) (1998) 5692–5695.

[11] Fabiny, L., Colet, P., Roy, R. and Lenstra, D. Coherence and phase dynamics of spatially coupled solidstate lasers. *Physical Review A* **47** (5) (1993) 4287–4296.

[12] Vincent, U. E. and Guo, R. Adaptive Synchronization for Oscillators in ϕ^6 Potentials. *Nonlinear Dynamics and Systems Theory* **13** (1) (2013) 93–106.

[13] Tan, X., Zhang, J. and Yang, Y. Synchronizing chaotic systems using backstepping design. *Chaos, Solitons and Fractals* **16** (2003) 37–45.

[14] Yau, H.T. Design of adaptive sliding mode controller for chaos synchronization with uncertainties. *Chaos, Solition and Fractals* **22** (2004) 341–347.

[15] Pei, L.J. and Liu, S.H. Application of generalized Hamiltonian systems to chaotic synchronization. *Nonlinear Dynamics and Systems Theory* **9** (4) (2009) 415–432.

[16] Fu, S. H. and Pei, L. J. Synchronization of chaotic systems by the generalized hamiltonian systems approach. *Nonlinear Dynamics and Systems Theory* **10** (4) (2010) 387–396.

[17] Ouannas, A. Chaos Synchronization Approach Based on New Criterion of Stability. *Nonlinear Dynamics and Systems Theory* **14** (4) (2014) 395–401.

[18] Filali, R. L., Hammami, S., Benrejeb, M. and Borne, P. On Synchronization, Antisynchronization and Hybrid Synchronization of 3D Discrete Generalized Hénon map. *Nonlinear Dynamics and Systems Theory* **12** (1) (2012) 81-95.

[19] Khan, A. and Pal, R. Adaptive hybrid function projective synchronization of Chaotic Space-Tether System. *Nonlinear Dynamics and Systems Theory* **14** (1) (2014) 44–57.

[20] Ojo, K. S., Njah, A. N., Ogunjo, S. T. and Olusola, O. I. Reduced order function projective combination synchronization of three Josephson functions using backstepping technique. *Nonlinear Dynamics and Systems Theory* **14** (2) (2014) 119–133.

[21] Yang, Y. and Chen, Y. The generalized Q-S synchronization between the generalized Lorenz canonical form and the Rossler system. *Chaos, Solitons and Fractals* **39** (2009) 2378–2385.

[22] Wang, Z. L. and Shi, X. R. Adaptive Q–S synchronization of non-identical chaotic systems with unknown parameters. *Nonlinear Dynamics* **59** (4) (2010) 559–567.

[23] Zhao, J., Wu, Y. and Wang, Y. Adaptive function Q-S synchronization of different chaotic (hype-rchaotic) systems. *International Journal of Modern Physics B* **27** (20) (2013) 1350109-9.

[24] Zhao, J. and Zhang, K. Adaptive function Q-S synchronization of chaotic systems with unknown parameters. *International Journal of Adaptive Control and Signal Processing* **24** (8) (2009) 675-686.

[25] Zhao, J. and Zhang, K. A general scheme for Q-S synchronization of chaotic systems with unknown parameters and scaling functions. *Applied Mathematics and Computation* **216** (7) (2010) 2050–2057.

[26] Zhao, J. and Ren, T. Q–S synchronization between chaotic systems with double scaling functions. *Nonlinear Dynamics* **62** (3) (2010) 665–672.

[27] Yan, Z. Q-S synchronization in 3D Hénon-like map and generalized Hénon map via a scalar controller. *Physics Letters A* **342** (2005) 309–317.

[28] Yan, Z. Q-S (complete or anticipated) synchronization backstepping scheme in a class of discrete-time chaotic (hyperchaotic) systems. *Chaos* **16** (2006) 013119-6.

[29] Ouannas, A. A new Q-S synchronization scheme for discrete choatic systems. *Far East Journal of Applied Mathematics* **84** (2) (2013) 89–94.

[30] Sprott, J. C. Chaos and Time-Series Analysis. *Oxford University Press, New York, NY, USA*, 2003.

[31] LI Xin, L. and Yong, C. Generalized Projective Synchronization Between Rössler System and New Unified Chaotic System. *Communications in Theoretical Physics* **48** (2007) 132–136.

[32] Lorenz, E. N. Deterministic nonperiodic flows. *Journal of the Atmospheric Sciences* **20** (1963) 130–141.

[33] Xin, J. L., Hao, D. and Meng, H. A new four-dimensional hyperchaotic Chen system and its generalized synchronization. *Chinese Physics B* **19** (2010) 501–517.

[34] Olusola, O. I., Vincent, U. E., Njah, A. N. and Idowu, B. A. Global Stability and Synchronization Criteria of Linearly Coupled Gyroscope. *Nonlinear Dynamics and Systems Theory* **13** (3) (2013) 258–269.