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Global Existence of Weak Solutions to a Fractional Landau-Lifshitz-Gilbert Equation

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Abstract: We discuss global existence of weak solutions to a one dimensional periodical fractional Landau-Lifshitz-Gilbert equation. A Faedo-Galerkin/penalization method is employed to get approximate solutions and a fractional calculus inequality is used to deal with the convergence of nonlinear terms. We also study the asymptotic behavior of the obtained solutions when the vertical spin stiffness parameter tends to zero.

Keywords: *fractional Landau-Lifshitz-Gilbert equation; Zygmund operator; fractional calculus; global existence; weak solutions.*

Mathematics Subject Classification (2010): 35D30, 78A25, 35B40, 82D40.

1 Introduction

In the last decades the study of magnetization processes in magnetic materials has been the focus of considerable research for its application to magnetic recording technology. In fact, the design of currently widespread magnetic storage devices, such as the hard-disks, requires the knowledge of the microscopic phenomena occurring within magnetic media. In this respect, it is known that ferromagnetic materials present spontaneous magnetization which is the result of spontaneous alignment of the elementary magnetic moments that constitute the medium. The magnetic recording technology exploits the magnetization of ferromagnetic media to store information. The first example of magnetic storage device was the magnetic core memory prototype, realized by IBM in 1952. After magnetic core memories, magnetic tapes have been used, but the most widespread

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magnetic storage device is certainly the hard-disk. The progress made by research activity performed worldwide in this subject has led to exponential decay of magnetic device dimensions. For more details, we refer for example to [10, 13].

The Landau-Lifshitz (LL) equation [14] and its modification, the Landau-Lifshitz-Gilbert (LLG) equation [8], are the basic equations for studying the magnetization dynamics in ferromagnetic materials. Though these equations are equivalent from the mathematical point of view [7] (specifically, the LL equation reduces to the LLG one by a simple rescaling of the gyromagnetic ratio and damping parameter), the latter is more preferable from the physical point of view and widely used for studying the nonlinear effects in the magnetization dynamics, regimes of forced precession, magnetization switching, etc.

In this paper, we study the following one-dimensional fractional Landau-Lifshitz-Gilbert equation

$$\partial_t \boldsymbol{m} = \gamma \boldsymbol{m} \times \partial_t \boldsymbol{m} + (1+\gamma^2)\boldsymbol{m} \times \mathcal{H}_{\text{eff}}(\boldsymbol{m}). \tag{1}$$

The unknown $\boldsymbol{m}$, the magnetization vector, is an application of $Q = (0,T) \times \Omega$ ($T > 0$ and Ω is a bounded set of $\mathbb{R}$) into S^2 (the unit sphere of $\mathbb{R}^3$), $\partial_t \boldsymbol{m}$ denotes its derivative with respect to time, $\mathcal{H}_{\text{eff}}(\boldsymbol{m})$ is the effective field, "$\times$" is the three dimensional cross product and the magnitude of magnetization (which is constant in space and time) has been scaled to one

$$|\boldsymbol{m}(t,x)| = 1. \tag{2}$$

In (1), the positive constant γ is the damping coefficient, and

$$\mathcal{H}_{\text{eff}}(\boldsymbol{m}) = -\frac{\partial \mathcal{E}}{\partial \boldsymbol{m}} \tag{3}$$

is the opposite of the functional derivative of the free energy $\mathcal{E}$. Typical expressions for $\mathcal{E}$ that are usually used in practice take into account several different physical phenomena, and can be found in [10] for instance. In this work, we will focus on the case where $\mathcal{H}_{\text{eff}}(\boldsymbol{m})$ is given by

$$\mathcal{H}_{\text{eff}}(\boldsymbol{m}) = a\Lambda^{2\alpha}\boldsymbol{m} + b\, \boldsymbol{m} \times \Lambda^{2\alpha}\boldsymbol{m}, \tag{4}$$

when $\alpha \in (\frac{1}{2}, 1)$ and $a, b > 0$. The operator $\Lambda = (-\Delta)^{\frac{1}{2}}$ denotes the square root of the Laplacian and called also Zygmund operator which can be defined for example via Fourier transformation [21].

Equation (1) has broad connections with other well-known equations appearing in mathematics and physics. When $\alpha = 1$ and $b = 0$, equation (1) becomes a standard LLG equation and global existence of weak solutions and nonuniqueness is proved in [1]. When $\alpha \in (\frac{1}{2}, 1)$ and $b = 0$, the existence of weak solutions for (1) is obtained using Faedo-Galerkin/penalization (FGP) method and fractional calculus for the convergence of nonlinear terms, see [18]. When $\alpha = 1$ and $b > 0$, Eq. (1) becomes a standard LLG equation with vertical spin stiffness and global existence of weak solutions is proved in [3].

The equation (1) is subject to the periodic boundary and initial conditions

$$\boldsymbol{m}(0,.) = \boldsymbol{m}_0, \quad |\boldsymbol{m}_0| = 1 \quad \text{in} \quad \Omega. \tag{5}$$

A simplified model can be obtained by assuming that Ω is a subset of $\mathbb{R}$. Specifically, we consider one dimensional domain $\Omega = (-\pi, \pi)$ and assume periodic boundary conditions.

Throughout this paper, for $k \in \mathbb{N}^*$, $\mathbb{L}^k(\Omega) = (L^k(\Omega))^3$ and $\mathbb{H}^k(\Omega) = (H^k(\Omega))^3$ are the usual Hilbert-type Lebesgue and Sobolev spaces, respectively. $\dot{\mathbb{H}}^\alpha(\Omega)$ denotes the homogenous Sobolev-Slobodetskii space and $\mathbb{H}^\alpha(\Omega)$ denotes the inhomogeneous one.

Lemma 1.1 If $\boldsymbol{m}$ is a regular solution of the problem (1)-(5) then we have for all $t \in (0, T)$ the following energy estimate

$$\gamma \int_0^t \int_\Omega |\partial_t \boldsymbol{m}|^2 \,\mathrm{d}x\mathrm{d}t + \frac{\beta + \gamma\lambda}{2} \int_\Omega |\Lambda^\alpha \boldsymbol{m}(t)|^2 \,\mathrm{d}x \leq \frac{\beta + \gamma\lambda}{2} \int_\Omega |\Lambda^\alpha \boldsymbol{m}_0|^2 \,\mathrm{d}x,$$

where $\beta = a(1 + \gamma^2)$ at $\lambda = b(1 + \gamma^2)$.

Proof. Using the saturation constraint $|\boldsymbol{m}| = 1$, the LLG equation (1) can be written in the following form

$$\gamma\partial_t \boldsymbol{m} + \boldsymbol{m} \times \partial_t \boldsymbol{m} + \beta\Lambda^{2\alpha}\boldsymbol{m} + \lambda\boldsymbol{m} \times \Lambda^{2\alpha}\boldsymbol{m} - \beta(\boldsymbol{m} \cdot \Lambda^{2\alpha}\boldsymbol{m})\boldsymbol{m} = 0. \tag{6}$$

Taking the inner product of (6) by $\partial_t \boldsymbol{m}$ and $\Lambda^{2\alpha}\boldsymbol{m}$ respectively, we get

$$\gamma \int_\Omega |\partial_t \boldsymbol{m}|^2 \,\mathrm{d}x + \frac{\beta}{2}\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega |\Lambda^\alpha \boldsymbol{m}|^2 \,\mathrm{d}x + \lambda \int_\Omega \boldsymbol{m} \times \Lambda^{2\alpha}\boldsymbol{m} \cdot \partial_t \boldsymbol{m} \,\mathrm{d}x = 0 \tag{7}$$

and

$$\begin{aligned} \frac{\gamma}{2}\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega |\Lambda^\alpha \boldsymbol{m}|^2 \,\mathrm{d}x + \int_\Omega \boldsymbol{m} \times \partial_t \boldsymbol{m} \cdot \Lambda^{2\alpha}\boldsymbol{m} \,\mathrm{d}x + \beta \int_\Omega |\Lambda^{2\alpha}\boldsymbol{m}|^2 \,\mathrm{d}x \\ -\beta \int_\Omega (\boldsymbol{m} \cdot \Lambda^{2\alpha}\boldsymbol{m})^2 \,\mathrm{d}x = 0. \end{aligned} \tag{8}$$

Adding (7) and (8) multiplied by λ, we obtain

$$\begin{aligned} \gamma \int_\Omega |\partial_t \boldsymbol{m}|^2 \,\mathrm{d}x + \frac{\beta + \gamma\lambda}{2}\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega |\Lambda^\alpha \boldsymbol{m}|^2 \,\mathrm{d}x + \lambda\beta \int_\Omega |\Lambda^{2\alpha}\boldsymbol{m}|^2 \,\mathrm{d}x \\ = \lambda\beta \int_\Omega (\boldsymbol{m} \cdot \Lambda^{2\alpha}\boldsymbol{m})^2 \,\mathrm{d}x. \end{aligned}$$

Since

$$\int_\Omega (\boldsymbol{m} \cdot \Lambda^{2\alpha}\boldsymbol{m})^2 \,\mathrm{d}x \leq \int_\Omega |\Lambda^{2\alpha}\boldsymbol{m}|^2 \,\mathrm{d}x,$$

and integrating from 0 to t, we obtain

$$\gamma \int_0^t \int_\Omega |\partial_t \boldsymbol{m}|^2 \,\mathrm{d}x\mathrm{d}t + \frac{\beta + \gamma\lambda}{2} \int_\Omega |\Lambda^\alpha \boldsymbol{m}(t)|^2 \,\mathrm{d}x \leq \frac{\beta + \gamma\lambda}{2} \int_\Omega |\Lambda^\alpha \boldsymbol{m}_0|^2 \,\mathrm{d}x$$

for all $t \in (0, T)$. □

In this work, we are mainly interested in studying the global existence of weak solutions for (1)-(5). To this end, we first give the definition of weak solutions.

Definition 1.1 Let $\boldsymbol{m}_0 \in \mathbb{H}^\alpha(\Omega)$ with $|\boldsymbol{m}_0| = 1$ a.e., we say that a three dimensional vector $\boldsymbol{m}$ is a weak solution of the problem (1)-(5) if

- for all $T > 0$, $\boldsymbol{m} \in L^\infty(0, T, \mathbb{H}^\alpha(\Omega))$ and $\partial_t \boldsymbol{m} \in \mathbb{L}^2(Q)$ with $|\boldsymbol{m}| = 1$ a.e.;

- For all $\boldsymbol{\phi} \in \mathcal{C}^\infty(\overline{Q})$, such that $\boldsymbol{\phi}(0,.) = \boldsymbol{\phi}(T,.)$

$$\begin{aligned}&\int_Q \partial_t \boldsymbol{m} \cdot \boldsymbol{\phi} \; \mathrm{d}x\mathrm{d}t - \gamma \int_Q \boldsymbol{m} \times \partial_t \boldsymbol{m} \cdot \boldsymbol{\phi} \; \mathrm{d}x\mathrm{d}t \\ &= -\beta \int_Q \Lambda^\alpha \boldsymbol{m} \cdot \Lambda^\alpha (\boldsymbol{m} \times \boldsymbol{\phi}) \; \mathrm{d}x\mathrm{d}t - \lambda \int_Q (\boldsymbol{m} \times \Lambda^{2\alpha} \boldsymbol{m}) \cdot (\boldsymbol{m} \times \boldsymbol{\phi}) \; \mathrm{d}x\mathrm{d}t.\end{aligned} \tag{9}$$

- $\boldsymbol{m}(0,x) = \boldsymbol{m}_0(x)$ in the trace sense.

- For all $t \in (0,T)$

$$\gamma \int_0^t \int_\Omega |\partial_t \boldsymbol{m}|^2 \; \mathrm{d}x\mathrm{d}t + \frac{\beta + \gamma\lambda}{2} \int_\Omega |\Lambda^\alpha \boldsymbol{m}(t)|^2 \; \mathrm{d}x \le \frac{\beta + \gamma\lambda}{2} \int_\Omega |\Lambda^\alpha \boldsymbol{m}_0|^2 \; \mathrm{d}x. \tag{10}$$

Remark 1.1 We will show in subsect.2.2 that $\boldsymbol{m} \times \Lambda^{2\alpha} \boldsymbol{m}$ makes sense in $\mathbb{L}^2(Q)$, and for this reason, it will be clear that (9) makes sense.

The rest of the paper is organized as follows. In the next section, we prove a global existence of weak solutions result by using Faedo-Galerkin/penalization method. Section 3 is devoted to revealing the relationships between the fractional LLG equation we have studied in this paper, and the classical fractional LLG equation (i.e., in the case $b = 0$). The last section concludes the paper and provides future directions for this work.

2 Global Existence of Weak Solutions

The purpose of the present section is to prove the following result

Theorem 2.1 Let $\boldsymbol{m}_0 \in \mathbb{H}^\alpha(\Omega)$ with $|\boldsymbol{m}_0| = 1$ a.e., then there exists a weak solution of the problem (1)-(5) in the sense of Definition 1.1.

To prove Theorem 2.1, we proceed as in [1, 5, 18, 23].

2.1 The penalty problem

Let $\varepsilon > 0$. We introduce the following penalty problem. For an initial datum $\boldsymbol{m}_0 \in \mathbb{H}^\alpha(\Omega)$, and for each positive number T, find a vector field $\boldsymbol{m}_\varepsilon$ such as to satisfy the equation

$$\gamma \partial_t \boldsymbol{m}^\varepsilon + \boldsymbol{m}^\varepsilon \times \partial_t \boldsymbol{m}^\varepsilon + \beta \Lambda^{2\alpha} \boldsymbol{m}^\varepsilon + \lambda \boldsymbol{m}^\varepsilon \times \Lambda^{2\alpha} \boldsymbol{m}^\varepsilon + \frac{1}{\varepsilon}(|\boldsymbol{m}^\varepsilon|^2 - 1)\boldsymbol{m}^\varepsilon = 0. \tag{11}$$

subject to the periodic boundary and initial conditions

$$\boldsymbol{m}^\varepsilon(0,.) = \boldsymbol{m}_0, \quad |\boldsymbol{m}_0| = 1 \quad \text{in} \quad \Omega. \tag{12}$$

The last term of equation (11) was introduced at the end to represent the constraint $|\boldsymbol{m}| = 1$.

We have the following result.

Proposition 2.1 *For each fixed positive ε, there is a weak solution $\boldsymbol{m}^\varepsilon$ of problem (11)-(12) such that*

$$\gamma\int_Q \partial_t \boldsymbol{m}^\varepsilon\cdot\boldsymbol{\varphi}\ \mathrm{d}x\mathrm{d}t+\int_Q(\boldsymbol{m}^\varepsilon\times\partial_t\boldsymbol{m}^\varepsilon)\cdot\boldsymbol{\varphi}\ \mathrm{d}x\mathrm{d}t+\beta\int_Q\Lambda^\alpha\boldsymbol{m}^\varepsilon\cdot\Lambda^\alpha\boldsymbol{\varphi}\ \mathrm{d}x\mathrm{d}t$$
$$-\lambda\int_Q\Lambda^\alpha\boldsymbol{m}^\varepsilon\cdot\Lambda^\alpha(\boldsymbol{m}^\varepsilon\times\boldsymbol{\varphi})\ \mathrm{d}x\mathrm{d}t+\frac{1}{\varepsilon}\int_Q\big(|\boldsymbol{m}^\varepsilon|^2-1\big)\boldsymbol{m}^\varepsilon\cdot\boldsymbol{\varphi}\ \mathrm{d}x\mathrm{d}t=0$$

for any $\boldsymbol{\varphi}$ in $\mathbb{L}^2(0,T,\mathbb{H}^\alpha(\Omega))$. Moreover, the following energy estimate holds

$$\gamma\int_0^t\int_\Omega|\partial_t\boldsymbol{m}^\varepsilon|^2\ \mathrm{d}x\mathrm{d}t+\frac{\beta+\gamma\lambda}{2}\int_\Omega|\Lambda^\alpha\boldsymbol{m}^\varepsilon(t)|^2\ \mathrm{d}x\mathrm{d}t$$
$$+\frac{1}{4\varepsilon}(1+\frac{\gamma\lambda}{\beta})\int_\Omega(|\boldsymbol{m}^\varepsilon|^2-1)^2(t)\ \mathrm{d}x\le\frac{\beta+\gamma\lambda}{2}\int_\Omega|\Lambda^\alpha\boldsymbol{m}_0|^2\ \mathrm{d}x$$

for all $t\in(0,T)$.

Proof. We show the existence of solutions for the penalty problem by using Faedo-Galerkin method. Let $\{\chi_i\}_{i\in\mathbb{N}}$ be a complete orthonormal basis of $L^2(\Omega)$ consisting of eigenfunctions of $\Lambda^{2\alpha}$

$$\Lambda^{2\alpha}\chi_i=\lambda_i\chi_i,\quad i=1,2,\dots \tag{13}$$

under periodic boundary conditions. The existence of such a basis can be proved as in Temam [22]. For fixed $\varepsilon>0$, we seek approximate solutions $\boldsymbol{m}^{\varepsilon,N}$ for equation (11) of the form

$$\boldsymbol{m}^{\varepsilon,N}(t,x)=\sum_{i=1}^N\boldsymbol{a}_i(t)\chi_i(x),$$

where $\boldsymbol{a}_i(t)$ are $\mathbb{R}^3$-valued vectors. We obtain the following approached problem

$$\begin{aligned}\gamma\partial_t\boldsymbol{m}^{\varepsilon,N}+\boldsymbol{m}^{\varepsilon,N}\times\partial_t\boldsymbol{m}^{\varepsilon,N}+\beta\Lambda^{2\alpha}\boldsymbol{m}^{\varepsilon,N}+\lambda\boldsymbol{m}^{\varepsilon,N}\times\Lambda^{2\alpha}\boldsymbol{m}^{\varepsilon,N}\\+\frac{1}{\varepsilon}\big(|\boldsymbol{m}^{\varepsilon,N}|^2-1\big)\boldsymbol{m}^{\varepsilon,N}=0\end{aligned} \tag{14}$$

with the following initial conditions

$$\boldsymbol{m}^{\varepsilon,N}(0,.)=\boldsymbol{m}^N(0,.)\text{ in }\Omega$$

and

$$\int_\Omega\boldsymbol{m}^N(0,.)\chi_i\ \mathrm{d}x=\int_\Omega\boldsymbol{m}_0(0,.)\chi_i\ \mathrm{d}x.$$

Multiplying the equation (14) by χ_i and integrating over Ω, we get an ordinary differential system.

Note that

$$\gamma\partial_t\boldsymbol{m}^{\varepsilon,N}+\boldsymbol{m}^{\varepsilon,N}\times\partial_t\boldsymbol{m}^{\varepsilon,N}=\mathbb{A}(\boldsymbol{m}^{\varepsilon,N})\partial_t\boldsymbol{m}^{\varepsilon,N},$$

where

$$\mathbb{A}(\boldsymbol{m}^{\varepsilon,N})=\begin{pmatrix}\gamma&-m_3^{\varepsilon,N}&m_2^{\varepsilon,N}\\m_3^{\varepsilon,N}&\gamma&-m_1^{\varepsilon,N}\\-m_2^{\varepsilon,N}&m_1^{\varepsilon,N}&\gamma\end{pmatrix}.$$

We can write equation (14) in the form

$$\mathbb{A}(\boldsymbol{m}^{\varepsilon,N})\partial_t \boldsymbol{m}^{\varepsilon,N} = -\beta\Lambda^{2\alpha}\boldsymbol{m}^{\varepsilon,N} - \lambda \boldsymbol{m}^{\varepsilon,N} \times \Lambda^{2\alpha}\boldsymbol{m}^{\varepsilon,N} - \frac{1}{\varepsilon}\big(|\boldsymbol{m}^{\varepsilon,N}|^2 - 1\big)\boldsymbol{m}^{\varepsilon,N}.$$

Since $\mathbb{A}(\boldsymbol{m}^{\varepsilon,N})$ is invertible, then the resulting system is locally Lipschitz. There exists a unique local solution for the approximate problem that can extend on $[0,T]$ using a priori estimate. To get bounds on the solutions, we multiply equation (14) by $\partial_t \boldsymbol{m}^{\varepsilon,N}$ and $\Lambda^{2\alpha}\boldsymbol{m}^{\varepsilon,N}$ respectively and integrate over Ω. We obtain

$$\begin{aligned} &\gamma \int_\Omega |\partial_t \boldsymbol{m}^{\varepsilon,N}|^2 \,\mathrm{d}x + \frac{\beta}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\Lambda^\alpha \boldsymbol{m}^{\varepsilon,N}|^2 \,\mathrm{d}x \\ &+\lambda \int_\Omega \boldsymbol{m}^{\varepsilon,N} \times \Lambda^{2\alpha}\boldsymbol{m}^{\varepsilon,N} \cdot \partial_t \boldsymbol{m}^{\varepsilon,N} \,\mathrm{d}x + \frac{1}{4\varepsilon}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (|\boldsymbol{m}^{\varepsilon,N}|^2 - 1)^2 \,\mathrm{d}x = 0, \end{aligned} \tag{15}$$

and

$$\begin{aligned} &\int_\Omega \boldsymbol{m}^{\varepsilon,N} \times \partial_t \boldsymbol{m}^{\varepsilon,N} \cdot \Lambda^{2\alpha}\boldsymbol{m}^{\varepsilon,N} \,\mathrm{d}x + \beta \int_\Omega |\Lambda^{2\alpha}\boldsymbol{m}^{\varepsilon,N}|^2 \,\mathrm{d}x \\ &+\frac{\gamma}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\Lambda^\alpha \boldsymbol{m}^{\varepsilon,N}|^2 \,\mathrm{d}x + \frac{1}{\varepsilon}\int_\Omega (|\boldsymbol{m}^{\varepsilon,N}|^2 - 1)\cdot \Lambda^{2\alpha}\boldsymbol{m}^{\varepsilon,N} \,\mathrm{d}x = 0. \end{aligned} \tag{16}$$

Multiplying (16) by λ and make the sum with (15), we obtain

$$\begin{aligned} &\gamma \int_\Omega |\partial_t \boldsymbol{m}^{\varepsilon,N}|^2 \,\mathrm{d}x + \frac{\beta}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\Lambda^\alpha \boldsymbol{m}^{\varepsilon,N}|^2 \,\mathrm{d}x + \frac{1}{4\varepsilon}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (|\boldsymbol{m}^{\varepsilon,N}|^2 - 1)^2 \,\mathrm{d}x \\ &+\lambda\beta \int_\Omega |\Lambda^{2\alpha}\boldsymbol{m}^{\varepsilon,N}|^2 \,\mathrm{d}x + \frac{\lambda\gamma}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\Lambda^\alpha \boldsymbol{m}^{\varepsilon,N}|^2 \,\mathrm{d}x \\ &= -\frac{\lambda}{\varepsilon}\int_\Omega (|\boldsymbol{m}^{\varepsilon,N}|^2 - 1)\boldsymbol{m}^{\varepsilon,N} \cdot \Lambda^{2\alpha}\boldsymbol{m}^{\varepsilon,N} \,\mathrm{d}x. \end{aligned} \tag{17}$$

On the other hand, Young's inequality gives

$$\begin{aligned} &-\frac{\lambda}{\varepsilon}\int_\Omega (|\boldsymbol{m}^{\varepsilon,N}|^2 - 1)\boldsymbol{m}^{\varepsilon,N} \cdot \Lambda^{2\alpha}\boldsymbol{m}^{\varepsilon,N} \,\mathrm{d}x \\ &\le \frac{\lambda}{2d\varepsilon^2}\int_\Omega (|\boldsymbol{m}^{\varepsilon,N}|^2 - 1)^2 |\boldsymbol{m}^{\varepsilon,N}|^2 \,\mathrm{d}x + \frac{\lambda d}{2}\int_\Omega |\Lambda^{2\alpha}\boldsymbol{m}^{\varepsilon,N}|^2 \,\mathrm{d}x \end{aligned} \tag{18}$$

for any constant $d > 0$.

We multiply equation (14) by $(|\boldsymbol{m}^{\varepsilon,N}|^2 - 1)\boldsymbol{m}^{\varepsilon,N}$ and integrate over Ω, we obtain

$$\begin{aligned} &\beta \int_\Omega (|\boldsymbol{m}^{\varepsilon,N}|^2 - 1)\boldsymbol{m}^{\varepsilon,N} \cdot \Lambda^{2\alpha}\boldsymbol{m}^{\varepsilon,N} \,\mathrm{d}x + \frac{\gamma}{4}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (|\boldsymbol{m}^{\varepsilon,N}|^2 - 1)^2 \,\mathrm{d}x \\ &+\frac{1}{\varepsilon}\int_\Omega (|\boldsymbol{m}^{\varepsilon,N}|^2 - 1)^2 |\boldsymbol{m}^{\varepsilon,N}|^2 \,\mathrm{d}x = 0. \end{aligned}$$

Hence

$$\begin{aligned} &-\frac{\lambda}{\varepsilon}\int_\Omega (|\boldsymbol{m}^{\varepsilon,N}|^2 - 1)\boldsymbol{m}^{\varepsilon,N} \cdot \Lambda^{2\alpha}\boldsymbol{m}^{\varepsilon,N} \,\mathrm{d}x \\ &= \frac{\gamma\lambda}{4\beta\varepsilon}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (|\boldsymbol{m}^{\varepsilon,N}|^2 - 1)^2 \,\mathrm{d}x + \frac{\lambda}{\beta\varepsilon^2}\int_\Omega (|\boldsymbol{m}^{\varepsilon,N}|^2 - 1)^2 |\boldsymbol{m}^{\varepsilon,N}|^2 \,\mathrm{d}x. \end{aligned}$$

Therefore

$$\frac{\gamma\lambda}{4\beta\varepsilon}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega(|\boldsymbol{m}^{\varepsilon,N}|^2-1)^2\ \mathrm{d}x+\frac{\lambda}{\beta\varepsilon^2}\int_\Omega(|\boldsymbol{m}^{\varepsilon,N}|^2-1)^2|\boldsymbol{m}^{\varepsilon,N}|^2\ \mathrm{d}x$$
$$\le\frac{\lambda}{2d\varepsilon^2}\int_\Omega(|\boldsymbol{m}^{\varepsilon,N}|^2-1)^2|\boldsymbol{m}^{\varepsilon,N}|^2\ \mathrm{d}x+\frac{\lambda d}{2}\int_\Omega|\Lambda^{2\alpha}\boldsymbol{m}^{\varepsilon,N}|^2\ \mathrm{d}x.$$

That is

$$\frac{\gamma\lambda}{4\beta\varepsilon}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega(|\boldsymbol{m}^{\varepsilon,N}|^2-1)^2\ \mathrm{d}x+\frac{\lambda}{\varepsilon^2}(\frac{1}{\beta}-\frac{1}{2d})\int_\Omega(|\boldsymbol{m}^{\varepsilon,N}|^2-1)^2|\boldsymbol{m}^{\varepsilon,N}|^2\ \mathrm{d}x$$
$$\le\frac{\lambda d}{2}\int_\Omega|\Lambda^{2\alpha}\boldsymbol{m}^{\varepsilon,N}|^2\ \mathrm{d}x.$$

So for $d>\frac{\beta}{2}$

$$\frac{\lambda}{2d\beta\varepsilon^2}\int_\Omega(|\boldsymbol{m}^{\varepsilon,N}|^2-1)^2|\boldsymbol{m}^{\varepsilon,N}|^2\ \mathrm{d}x$$
$$\le\frac{\lambda d}{2(2d-\beta)}\int_\Omega|\Lambda^{2\alpha}\boldsymbol{m}^{\varepsilon,N}|^2\ \mathrm{d}x-\frac{\gamma\lambda}{4\beta\varepsilon(2d-\beta)}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega(|\boldsymbol{m}^{\varepsilon,N}|^2-1)^2\ \mathrm{d}x.$$

Therefore from (18)

$$-\frac{\lambda}{\varepsilon}\int_\Omega(|\boldsymbol{m}^{\varepsilon,N}|^2-1)\boldsymbol{m}^{\varepsilon,N}\cdot\Lambda^{2\alpha}\boldsymbol{m}^{\varepsilon,N}\ \mathrm{d}x$$
$$\le\frac{\lambda d}{2}(1+\frac{\beta}{2d-\beta})\int_\Omega|\Lambda^{2\alpha}\boldsymbol{m}^{\varepsilon,N}|^2\ \mathrm{d}x-\frac{\gamma\lambda}{4\varepsilon(2d-\beta)}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega(|\boldsymbol{m}^{\varepsilon,N}|^2-1)^2\ \mathrm{d}x.$$

Then from (17)

$$\gamma\int_\Omega|\partial_t\boldsymbol{m}^{\varepsilon,N}|^2\ \mathrm{d}x+\frac{\beta+\gamma\lambda}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega|\Lambda^\alpha\boldsymbol{m}^{\varepsilon,N}|^2\ \mathrm{d}x$$
$$+\lambda(\beta-\frac{d^2}{2d-\beta})\int_\Omega|\Lambda^{2\alpha}\boldsymbol{m}^{\varepsilon,N}|^2\ \mathrm{d}x+\frac{1}{4\varepsilon}(1+\frac{\gamma\lambda}{2d-\beta})\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega(|\boldsymbol{m}^{\varepsilon,N}|^2-1)^2\ \mathrm{d}x\le 0.$$

Choose $d=\beta$, we get $\beta-\frac{d^2}{2d-\beta}=0$ and therefore

$$\gamma\int_\Omega|\partial_t\boldsymbol{m}^{\varepsilon,N}|^2\ \mathrm{d}x+\frac{\beta+\gamma\lambda}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega|\Lambda^\alpha\boldsymbol{m}^{\varepsilon,N}|^2\ \mathrm{d}x$$
$$+\frac{1}{4\varepsilon}(1+\frac{\gamma\lambda}{\beta})\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega(|\boldsymbol{m}^{\varepsilon,N}|^2-1)^2\ \mathrm{d}x\le 0.$$

We integrate from 0 to t and we get

$$\gamma\int_0^t\int_\Omega|\partial_t\boldsymbol{m}^{\varepsilon,N}|^2\ \mathrm{d}x\mathrm{d}t+\frac{\beta+\gamma\lambda}{2}\int_\Omega|\Lambda^\alpha\boldsymbol{m}^{\varepsilon,N}(t)|^2\ \mathrm{d}x$$
$$+\frac{1}{4\varepsilon}(1+\frac{\gamma\lambda}{\beta})\int_\Omega(|\boldsymbol{m}^{\varepsilon,N}|^2-1)^2(t)\ \mathrm{d}x\le\frac{\beta+\gamma\lambda}{2}\int_\Omega|\Lambda^\alpha\boldsymbol{m}^N|^2(0)\ \mathrm{d}x \tag{19}$$
$$+\frac{1}{4\varepsilon}(1+\frac{\gamma\lambda}{\beta})\int_\Omega(|\boldsymbol{m}^N|^2-1)^2(0)\ \mathrm{d}x.$$

The right-hand side is uniformly bounded. Indeed $\mathbb{H}^\alpha(\Omega) \hookrightarrow \mathbb{L}^4(\Omega)$ with continuous embedding, therefore

$$\begin{aligned}\int_\Omega (|\boldsymbol{m}^{\varepsilon,N}|^2-1)^2(0)\ \mathrm{d}x &= \int_\Omega |\boldsymbol{m}^N(0)|^4\ \mathrm{d}x - 2\int_\Omega |\boldsymbol{m}^N(0)|^2\ \mathrm{d}x + \mathrm{meas}(\Omega)\\ &\leq \|\boldsymbol{m}^N(0)\|^4_{\mathbb{L}^4(\Omega)} + \mathrm{meas}(\Omega)\\ &\leq C_1\|\boldsymbol{m}^N(0)\|^4_{\mathbb{H}^\alpha(\Omega)} + C_2,\end{aligned}$$

where C_1 and C_2 are two constants independent of ε and N. Furthermore, note that $\boldsymbol{m}^{\varepsilon,N}(0) = \boldsymbol{m}^N(0)$, and since $\boldsymbol{m}^N(0)$ has the same components as $\boldsymbol{m}_0$ in the basis $\{\chi_i\}_{i\in\mathbb{N}}$ and $\boldsymbol{m}_0 \in \mathbb{H}^\alpha(\Omega)$, we have $\|\boldsymbol{m}_0\|_{\mathbb{H}^\alpha(\Omega)} \leq C_3$ with C_3 being a constant independent of ε and N. Hence

$$\|\boldsymbol{m}^N(0)\|_{\mathbb{H}^\alpha(\Omega)} \leq C_3.$$

Therefore,

$$\|\Lambda^\alpha \boldsymbol{m}^N(0)\|_{\mathbb{L}^2(\Omega)} \leq C_3.$$

Thus for ε fixed, we have

$$(|\boldsymbol{m}^{\varepsilon,N}|^2-1)_N \text{ is bounded in } L^\infty(0,T,\mathbb{L}^2(\Omega)),$$

$$(\Lambda^\alpha \boldsymbol{m}^{\varepsilon,N})_N \text{ is bounded in } L^\infty(0,T,\mathbb{L}^2(\Omega)).$$

By Young's inequality

$$\int_\Omega |\boldsymbol{m}^{\varepsilon,N}|^2\ \mathrm{d}x \leq C + \int_\Omega (|\boldsymbol{m}^{\varepsilon,N}|^2-1)^2\ \mathrm{d}x,$$

with C being a constant which does not depend on N. Therefore,

$$(\boldsymbol{m}^{\varepsilon,N})_N \ \text{ is bounded in }\ L^\infty(0,T,\mathbb{H}^\alpha(\Omega)),$$

$$(\partial_t \boldsymbol{m}^{\varepsilon,N})_N \ \text{ is bounded in }\ L^2(0,T,\mathbb{L}^2(\Omega)) := \mathbb{L}^2(Q),$$

and we will need a compactness lemma due to Simon [20].

Lemma 2.1 *Assume B_0, B, B_1 are three Banach spaces and satisfy $B_0 \subset B \subset B_1$ with compact embedding $B_0 \hookrightarrow B$. Let W be bounded in $L^\infty(0,T;B_0)$ and $W_t := \{w_t; w \in W\}$ be bounded in $L^q(0,T;B_1)$ where $q > 1$. Then W is relatively compact in $C([0,T];B)$.*

The proof can be found in Simon [20]. Then we have the following convergences to a subsequence further notes that $\boldsymbol{m}^{\varepsilon,N}$ for any $(1 < p < \infty)$

$$\boldsymbol{m}^{\varepsilon,N} \rightharpoonup \boldsymbol{m}^\varepsilon \ \text{ weakly in }\ L^p(0,T,\mathbb{H}^\alpha(\Omega)), \tag{20}$$

$$\boldsymbol{m}^{\varepsilon,N} \to \boldsymbol{m}^\varepsilon \ \text{ strongly in }\ C([0,T],\mathbb{H}^\delta(\Omega)) \text{ and a.e for } 0 \leq \delta < \alpha, \tag{21}$$

$$\partial_t \boldsymbol{m}^{\varepsilon,N} \rightharpoonup \partial_t \boldsymbol{m}^\varepsilon \ \text{ weakly in }\ \mathbb{L}^2(Q), \tag{22}$$

$$|\boldsymbol{m}^{\varepsilon,N}|^2 - 1 \rightharpoonup \zeta \ \text{ weakly in }\ L^p(0,T,\mathbb{L}^2(\Omega)). \tag{23}$$

The convergence (21) is a consequence of (20) and by compactness embedding of $L^2(0,T,\mathbb{H}^\alpha(\Omega))$ in $L^2(0,T,\mathbb{L}^2(\Omega))$. On the other hand $\zeta = |\boldsymbol{m}^\varepsilon|^2 - 1$. This is provided by the following lemma.

Lemma 2.2 *Let Θ be a bounded open subset of $\mathbb{R}^d_x \times \mathbb{R}_t$, h_n and h are functions of $L^q(\Theta)$ with $1 < q < \infty$ such as $\|h_n\|_{L^q(\Theta)} \leq C$, $h_n \to h$ a.e in Θ then $h_n \rightharpoonup h$ weakly in $L^q(\Theta)$.*

The proof of Lemma 2.2 can be found in [15]. In our case $\Theta = Q$, $h_N = |\boldsymbol{m}^{\varepsilon,N}|^2 - 1$, $h = |\boldsymbol{m}^{\varepsilon}|^2 - 1$ and $q = 2$ and from (21) $|\boldsymbol{m}^{\varepsilon,N}|^2 - 1 \longrightarrow |\boldsymbol{m}^{\varepsilon}|^2 - 1$ a.e, and we have in particular $|\boldsymbol{m}^{\varepsilon,N}|^2 - 1 \in L^2(\Theta)$, $|\boldsymbol{m}^{\varepsilon}|^2 - 1 \in L^2(\Theta)$ and $\big\||\boldsymbol{m}^{\varepsilon}|^2 - 1\big\|_{L^2(\Theta)} \leq C$.

Now, we pass to the limit as $N \to \infty$. Multiplying the equation (14) by $\boldsymbol{\varphi} \in \mathbf{C}^\infty(\overline{Q})$ and integrating on Q yield

$$\begin{aligned}\gamma \int_Q \partial_t \boldsymbol{m}^{\varepsilon,N} \cdot \boldsymbol{\varphi}\, \mathrm{d}x\mathrm{d}t + \int_Q \boldsymbol{m}^{\varepsilon,N} \times \partial_t \boldsymbol{m}^{\varepsilon,N} \cdot \boldsymbol{\varphi} \mathrm{d}x\mathrm{d}t + \beta \int_Q \Lambda^\alpha \boldsymbol{m}^{\varepsilon,N} \cdot \Lambda^\alpha \boldsymbol{\varphi}\, \mathrm{d}x\mathrm{d}t \\ -\lambda \int_Q \Lambda^\alpha \boldsymbol{m}^{\varepsilon,N} \cdot \Lambda^\alpha(\boldsymbol{m}^{\varepsilon,N} \times \boldsymbol{\varphi})\, \mathrm{d}x\mathrm{d}t + \frac{1}{\varepsilon}\int_Q \big(|\boldsymbol{m}^{\varepsilon,N}|^2 - 1\big)\boldsymbol{m}^{\varepsilon,N} \cdot \boldsymbol{\varphi}\, \mathrm{d}x\mathrm{d}t = 0.\end{aligned} \tag{24}$$

We have

$$\boldsymbol{m}^{\varepsilon,N} \to \boldsymbol{m}^{\varepsilon} \text{ strongly in } \mathbb{L}^2(Q).$$

Furthermore

$$\partial_t \boldsymbol{m}^{\varepsilon,N} \rightharpoonup \partial_t \boldsymbol{m}^{\varepsilon} \text{ weakly in } \mathbb{L}^2(Q).$$

Thus

$$\int_Q (\boldsymbol{m}^{\varepsilon,N} \times \partial_t \boldsymbol{m}^{\varepsilon,N}) \cdot \boldsymbol{\varphi}\, \mathrm{d}x\mathrm{d}t \to \int_Q (\boldsymbol{m}^{\varepsilon} \times \partial_t \boldsymbol{m}^{\varepsilon}) \cdot \boldsymbol{\varphi}\, \mathrm{d}x\mathrm{d}t.$$

On the other hand

$$\Lambda^\alpha \boldsymbol{m}^{\varepsilon,N} \rightharpoonup \Lambda^\alpha \boldsymbol{m}^{\varepsilon} \text{ weakly in } \mathbb{L}^2(Q).$$

Therefore

$$\int_Q \Lambda^\alpha \boldsymbol{m}^{\varepsilon,N} \cdot \Lambda^\alpha \boldsymbol{\varphi}\, \mathrm{d}x\mathrm{d}t \to \int_Q \Lambda^\alpha \boldsymbol{m}^{\varepsilon} \cdot \Lambda^\alpha \boldsymbol{\varphi}\, \mathrm{d}x\mathrm{d}t,$$

and

$$\int_Q \partial_t \boldsymbol{m}^{\varepsilon,N} \cdot \boldsymbol{\varphi}\, \mathrm{d}x\mathrm{d}t \to \int_Q \partial_t \boldsymbol{m}^{\varepsilon} \cdot \boldsymbol{\varphi}\, \mathrm{d}x\mathrm{d}t.$$

Taking into account (23), we obtain

$$\int_Q \big(|\boldsymbol{m}^{\varepsilon,N}|^2 - 1\big)\boldsymbol{m}^{\varepsilon,N} \cdot \boldsymbol{\varphi}\, \mathrm{d}x\mathrm{d}t \to \int_Q \big(|\boldsymbol{m}^{\varepsilon}|^2 - 1\big)\boldsymbol{m}^{\varepsilon} \cdot \boldsymbol{\varphi}\, \mathrm{d}x\mathrm{d}t.$$

For the third term of (24) we set

$$D_N = \int_Q (\boldsymbol{m}^{\varepsilon,N} \times \Lambda^{2\alpha} \boldsymbol{m}^{\varepsilon,N}) \cdot \boldsymbol{\varphi}\, \mathrm{d}x\mathrm{d}t \quad \text{and} \quad D = \int_Q (\boldsymbol{m}^{\varepsilon} \times \Lambda^{2\alpha} \boldsymbol{m}^{\varepsilon}) \cdot \boldsymbol{\varphi}\, \mathrm{d}x\mathrm{d}t.$$

We have

$$D_N = -\int_Q \Lambda^{2\alpha} \boldsymbol{m}^{\varepsilon,N} \cdot (\boldsymbol{m}^{\varepsilon,N} \times \boldsymbol{\varphi})\, \mathrm{d}x\mathrm{d}t = -\int_Q \Lambda^\alpha \boldsymbol{m}^{\varepsilon,N} \cdot \Lambda^\alpha(\boldsymbol{m}^{\varepsilon,N} \times \boldsymbol{\varphi})\, \mathrm{d}x\mathrm{d}t.$$

Then we define the commutator

$$[\Lambda^\alpha, \boldsymbol{\varphi}]\boldsymbol{m} := \Lambda^\alpha(\boldsymbol{\varphi} \times \boldsymbol{m}) - \boldsymbol{\varphi} \times \Lambda^\alpha \boldsymbol{m}.$$

Since Λ^α is a nonlocal operator, the following fractional calculus inequality will play a critical role in the convergence of approximate solutions, see [6] for the proof.

Lemma 2.3 *Suppose that $s > 0$ and $p \in (1, +\infty)$. Then*

$$\|\Lambda^s(fg) - f\Lambda^s g\|_{L^p} \leq C(\|\nabla f\|_{L^{p_1}}\|g\|_{\dot{H}^{s-1,p_2}} + \|f\|_{\dot{H}^{s,p_3}}\|g\|_{L^{p_4}})$$

and

$$\|\Lambda^s(fg)\|_{L^p} \leq C(\|f\|_{L^{p_1}}\|g\|_{\dot{H}^{s,p_2}} + \|f\|_{\dot{H}^{s,p_3}}\|g\|_{L^{p_4}})$$

with $p_2, p_3 \in (1, +\infty)$ such that

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4},$$

and f, g are such that the right-hand side terms make sense.

We have

$$\begin{aligned}
&\left\|[\Lambda^\alpha, \boldsymbol{\varphi}](\boldsymbol{m}^{\varepsilon,N} - \boldsymbol{m}^\varepsilon)\right\|_{\mathbb{L}^2(\Omega)} \\
&\leq C\bigg(\|\nabla\boldsymbol{\varphi}\|_{\mathbb{L}^{p_1}(\Omega)}\|\boldsymbol{m}^{\varepsilon,N} - \boldsymbol{m}^\varepsilon\|_{\dot{W}^{\alpha-1,p_2}(\Omega)} + \|\boldsymbol{\varphi}\|_{\dot{W}^{\alpha,p_3}(\Omega)}\|\boldsymbol{m}^{\varepsilon,N} - \boldsymbol{m}^\varepsilon\|_{\mathbb{L}^{p_4}(\Omega)}\bigg).
\end{aligned}$$

We choose $p_1 = \frac{1}{1-\alpha}$, $p_2 = \frac{2}{2\alpha-1}$ and $p_3, p_4 \in (2, +\infty)$. This is justified by the fact that $\dot{W}^{k,p} \hookrightarrow L^q$ for $0 \leq k < \frac{n}{p}$ and $\frac{1}{q} = \frac{1}{p} - \frac{k}{n}$, in our case $n = 1$ and $k = 1 - \alpha$ and we want $\dot{W}^{k,p} \hookrightarrow L^2$. Therefore it is sufficient that $\frac{1}{2} = \frac{1}{p} - (1-\alpha)$ that is $\frac{1}{p} = \frac{3}{2} - \alpha = \frac{1}{p_2^*}$ where $\frac{1}{p_2} + \frac{1}{p_2^*} = 1$ and therefore $\dot{W}_0^{s,p_2^*} \hookrightarrow L^2 = (L^2)' \hookrightarrow (\dot{W}_0^{k,p})' \hookrightarrow \dot{W}^{-k,p_2}$. Thus for $\delta = \frac{1}{2} - \frac{1}{p_4} < \frac{1}{2} < \alpha$

$$\begin{aligned}
&\left\|[\Lambda^\alpha, \boldsymbol{\varphi}](\boldsymbol{m}^{\varepsilon,N} - \boldsymbol{m}^\varepsilon)\right\|_{\mathbb{L}^2(\Omega)} \\
&\leq C\bigg(\|\nabla\boldsymbol{\varphi}\|_{\mathbb{L}^{p_1}(\Omega)}\|\boldsymbol{m}^{\varepsilon,N} - \boldsymbol{m}^\varepsilon\|_{\mathbb{L}^2(\Omega)} + \|\boldsymbol{\varphi}\|_{\dot{W}^{\alpha,p_3}(\Omega)}\|\boldsymbol{m}^{\varepsilon,N} - \boldsymbol{m}^\varepsilon\|_{\mathbb{H}^\delta(\Omega)}\bigg) \\
&\leq C\bigg(\|\nabla\boldsymbol{\varphi}\|^2_{\mathbb{L}^{p_1}(\Omega)}\|\boldsymbol{m}^{\varepsilon,N} - \boldsymbol{m}^\varepsilon\|^2_{\mathbb{L}^2(\Omega)} + \|\boldsymbol{\varphi}\|^2_{\dot{W}^{\alpha,p_3}(\Omega)}\|\boldsymbol{m}^{\varepsilon,N} - \boldsymbol{m}^\varepsilon\|^2_{\mathbb{H}^\delta(\Omega)}\bigg).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\left\|[\Lambda^\alpha, \boldsymbol{\varphi}](\boldsymbol{m}^{\varepsilon,N} - \boldsymbol{m}^\varepsilon)\right\|_{\mathbb{L}^2(Q)} \leq C\bigg(\|\nabla\boldsymbol{\varphi}\|^2_{L^\infty(0,T,\mathbb{L}^{p_1}(\Omega))}\|\boldsymbol{m}^{\varepsilon,N} - \boldsymbol{m}^\varepsilon\|^2_{\mathbb{L}^2(Q)} \\
&+\|\boldsymbol{\varphi}\|^2_{L^\infty(0,T,\dot{W}^{\alpha,p_3}(\Omega))}\|\boldsymbol{m}^{\varepsilon,N} - \boldsymbol{m}^\varepsilon\|^2_{L^2(0,T,\mathbb{H}^\delta(\Omega))}\bigg).
\end{aligned}$$

The right-hand side of the last inequality tends to 0 due to strong convergence of $\boldsymbol{m}^{\varepsilon,N} \to \boldsymbol{m}^\varepsilon$ in $\mathbb{L}^2(Q)$ and in $L^2(0, T, \mathbb{H}^\delta(\Omega))$. Moreover by the preceding lemma $[\Lambda^\alpha, \boldsymbol{\varphi}]\boldsymbol{m}^\varepsilon \in$

$\mathbb{L}^2(Q)$. Thus

$$\begin{aligned}
&|D_N - D| = \left| \int_Q \Lambda^\alpha \boldsymbol{m}^{\varepsilon,N} \cdot [\Lambda^\alpha, \boldsymbol{\varphi}] \boldsymbol{m}^{\varepsilon,N} \, \mathrm{d}x\mathrm{d}t - \int_Q \Lambda^\alpha \boldsymbol{m}^\varepsilon \cdot [\Lambda^\alpha, \boldsymbol{\varphi}] \boldsymbol{m}^\varepsilon \, \mathrm{d}x\mathrm{d}t \right| \\
&= \left| \int_Q \Lambda^\alpha \boldsymbol{m}^{\varepsilon,N} \cdot [\Lambda^\alpha, \boldsymbol{\varphi}] (\boldsymbol{m}^{\varepsilon,N} - \boldsymbol{m}^\varepsilon) \, \mathrm{d}x\mathrm{d}t + \int_Q \Lambda^\alpha (\boldsymbol{m}^{\varepsilon,N} - \boldsymbol{m}^\varepsilon) \cdot [\Lambda^\alpha, \boldsymbol{\varphi}] \boldsymbol{m}^\varepsilon \, \mathrm{d}x\mathrm{d}t \right| \\
&\leq \left| \int_Q \Lambda^\alpha \boldsymbol{m}^{\varepsilon,N} \cdot [\Lambda^\alpha, \boldsymbol{\varphi}] (\boldsymbol{m}^{\varepsilon,N} - \boldsymbol{m}^\varepsilon) \, \mathrm{d}x\mathrm{d}t \right| + \left| \int_Q \Lambda^\alpha (\boldsymbol{m}^{\varepsilon,N} - \boldsymbol{m}^\varepsilon) \cdot [\Lambda^\alpha, \boldsymbol{\varphi}] \boldsymbol{m}^\varepsilon \, \mathrm{d}x\mathrm{d}t \right| \\
&\leq \|\Lambda^\alpha \boldsymbol{m}^{\varepsilon,N}\|_{\mathbb{L}^2(Q)} \left\| [\Lambda^\alpha, \boldsymbol{\varphi}] (\boldsymbol{m}^{\varepsilon,N} - \boldsymbol{m}^\varepsilon) \right\|_{\mathbb{L}^2(Q)} \\
&+ \left| \int_Q \Lambda^\alpha (\boldsymbol{m}^{\varepsilon,N} - \boldsymbol{m}^\varepsilon) \cdot [\Lambda^\alpha, \boldsymbol{\varphi}] \boldsymbol{m}^\varepsilon \, \mathrm{d}x\mathrm{d}t \right| \\
&\leq \|\boldsymbol{m}^{\varepsilon,N}\|_{L^2(0,T,\mathbb{H}^\alpha(\Omega))} \left\| [\Lambda^\alpha, \boldsymbol{\varphi}] (\boldsymbol{m}^{\varepsilon,N} - \boldsymbol{m}^\varepsilon) \right\|_{\mathbb{L}^2(Q)} \\
&+ \left| \int_Q \Lambda^\alpha (\boldsymbol{m}^{\varepsilon,N} - \boldsymbol{m}^\varepsilon) \cdot [\Lambda^\alpha, \boldsymbol{\varphi}] \boldsymbol{m}^\varepsilon \, \mathrm{d}x\mathrm{d}t \right|.
\end{aligned}$$

Since $\|\boldsymbol{m}^{\varepsilon,N}\|_{L^2(0,T,\mathbb{H}^\alpha(\Omega))} \leq C$ and

$$\begin{gathered}
\left\| [\Lambda^\alpha, \boldsymbol{\varphi}] (\boldsymbol{m}^{\varepsilon,N} - \boldsymbol{m}^\varepsilon) \right\|_{\mathbb{L}^2(Q)} \to 0, \\
\left| \int_Q \Lambda^\alpha (\boldsymbol{m}^{\varepsilon,N} - \boldsymbol{m}^\varepsilon) \cdot [\Lambda^\alpha, \boldsymbol{\varphi}] \boldsymbol{m}^\varepsilon \, \mathrm{d}x\mathrm{d}t \right| \to 0,
\end{gathered}$$

this implies that

$$D_N \to D. \tag{25}$$

Using the previous convergences and passing to the limit ($N \to \infty$) in (24), we get

$$\begin{aligned}
&\gamma \int_Q \partial_t \boldsymbol{m}^\varepsilon \cdot \boldsymbol{\varphi} \, \mathrm{d}x\mathrm{d}t + \int_Q \boldsymbol{m}^\varepsilon \times \partial_t \boldsymbol{m}^\varepsilon \cdot \boldsymbol{\varphi} \, \mathrm{d}x\mathrm{d}t + \beta \int_Q \Lambda^\alpha \boldsymbol{m}^\varepsilon \cdot \Lambda^\alpha \boldsymbol{\varphi} \, \mathrm{d}x\mathrm{d}t \\
&-\lambda \int_Q \Lambda^\alpha \boldsymbol{m}^\varepsilon \cdot \Lambda^\alpha (\boldsymbol{m}^\varepsilon \times \boldsymbol{\varphi}) \, \mathrm{d}x\mathrm{d}t + \frac{1}{\varepsilon} \int_Q \big(|\boldsymbol{m}^\varepsilon|^2 - 1\big) \boldsymbol{m}^\varepsilon \cdot \boldsymbol{\varphi} \, \mathrm{d}x\mathrm{d}t = 0
\end{aligned} \tag{26}$$

for all $\boldsymbol{\varphi}$ in $L^2(0,T,\mathbb{H}^\alpha(\Omega))$ by density of $\mathbf{C}^\infty(\overline{Q})$ in $L^2(0,T,\mathbb{H}^\alpha(\Omega))$.

Now back to (19) and taking into account the previous convergences in N and using Fatou lemma, we get

$$\begin{aligned}
&\gamma \int_0^t \int_\Omega |\partial_t \boldsymbol{m}^\varepsilon|^2 \, \mathrm{d}x\mathrm{d}t + \frac{\beta + \gamma\lambda}{2} \int_\Omega |\Lambda^\alpha \boldsymbol{m}^\varepsilon(t)|^2 \, \mathrm{d}x\mathrm{d}t \\
&+ \frac{1}{4\varepsilon}(1 + \frac{\gamma\lambda}{\beta}) \int_\Omega (|\boldsymbol{m}^\varepsilon|^2 - 1)^2(t) \, \mathrm{d}x \leq \frac{\beta + \gamma\lambda}{2} \int_\Omega |\Lambda^\alpha \boldsymbol{m}_0|^2 \, \mathrm{d}x
\end{aligned} \tag{27}$$

for all $t \in (0,T)$. □

We are now in a position to prove Theorem 2.1.

2.2 Convergence of the approximate solutions

To pass to the limit in ε ($\varepsilon \to 0$), we need estimate (19) and the following result

Lemma 2.4 *If $\boldsymbol{m}^\varepsilon$ satisfies* (26) *then* $|\boldsymbol{m}^\varepsilon| \le 1$ *a.e. on* Q.

Proof. We choose $\boldsymbol{\varphi} = \psi_{\mathcal{B}}\boldsymbol{m}^\varepsilon$ with $\mathcal{B} = \{|\boldsymbol{m}^\varepsilon| > 1\}$ and $\psi_{\mathcal{B}}$ is the indicator function of the set $\mathcal{B}$. We have $\boldsymbol{\varphi}$ in $L^2(0,T,\mathbb{H}^\alpha(\Omega))$, and replacing $\boldsymbol{\varphi}$ by $\psi_{\mathcal{B}}\boldsymbol{m}^\varepsilon$ in (26), we obtain

$$\gamma\int_0^t\int_{\mathcal{B}}\partial_t\boldsymbol{m}^\varepsilon\cdot\boldsymbol{m}^\varepsilon\ \mathrm{d}x\mathrm{d}t + \beta\int_0^t\int_{\mathcal{B}}|\Lambda^\alpha\boldsymbol{m}^\varepsilon|^2\ \mathrm{d}x\mathrm{d}t + \frac{1}{\varepsilon}\int_0^t\int_{\mathcal{B}}\big(|\boldsymbol{m}^\varepsilon|^2-1\big)|\boldsymbol{m}^\varepsilon|^2\ \mathrm{d}x\mathrm{d}t = 0.$$

Then

$$\frac{\gamma}{2}\int_0^t\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{B}}\big(|\boldsymbol{m}^\varepsilon|^2-1\big)\ \mathrm{d}x\mathrm{d}t + \beta\int_0^t\int_{\mathcal{B}}|\Lambda^\alpha\boldsymbol{m}^\varepsilon|^2\ \mathrm{d}x\mathrm{d}t + \frac{1}{\varepsilon}\int_0^t\int_{\mathcal{B}}\big(|\boldsymbol{m}^\varepsilon|^2-1\big)|\boldsymbol{m}^\varepsilon|^2\ \mathrm{d}x\mathrm{d}t = 0.$$

Hence

$$\frac{\gamma}{2}\int_0^t\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{B}}\big(|\boldsymbol{m}^\varepsilon|^2-1\big)\ \mathrm{d}x\mathrm{d}t \le 0.$$

We integrate from 0 to t, we get

$$\int_{\mathcal{B}}\big(|\boldsymbol{m}^\varepsilon(t)|^2-1\big)\ \mathrm{d}x \le \int_{\mathcal{B}}\big(|\boldsymbol{m}^\varepsilon(0)|^2-1\big)\ \mathrm{d}x = 0.$$

Hence $|\boldsymbol{m}^\varepsilon| \le 1$ a.e. on Q. □

Now we will look for an estimate of the term $\boldsymbol{m}^\varepsilon \times \Lambda^{2\alpha}\boldsymbol{m}^\varepsilon$. Multiplying equation (11) by $\boldsymbol{m}^\varepsilon \times \partial_t\boldsymbol{m}^\varepsilon$ and integrating over Ω we obtain

$$\int_\Omega |\ \boldsymbol{m}^\varepsilon\times\partial_t\boldsymbol{m}^\varepsilon\ |^2\ \ \mathrm{d}x + \beta\int_\Omega\Lambda^{2\alpha}\boldsymbol{m}^\varepsilon\cdot\boldsymbol{m}^\varepsilon\times\partial_t\boldsymbol{m}^\varepsilon\ \mathrm{d}x + \lambda\int_\Omega\boldsymbol{m}^\varepsilon\times\Lambda^{2\alpha}\boldsymbol{m}^\varepsilon\cdot\boldsymbol{m}^\varepsilon\times\partial_t\boldsymbol{m}^\varepsilon\ \mathrm{d}x = 0. \tag{28}$$

Multiply this time equation (11) by $\boldsymbol{m}^\varepsilon \times \Lambda^{2\alpha}\boldsymbol{m}^\varepsilon$ and integrating over Ω, we get

$$\gamma\int_\Omega\boldsymbol{m}^\varepsilon\times\Lambda^{2\alpha}\boldsymbol{m}^\varepsilon\cdot\partial_t\boldsymbol{m}^\varepsilon\ \mathrm{d}x + \int_\Omega\boldsymbol{m}^\varepsilon\times\Lambda^{2\alpha}\boldsymbol{m}^\varepsilon\cdot\boldsymbol{m}^\varepsilon\times\partial_t\boldsymbol{m}^\varepsilon\ \mathrm{d}x + \lambda\int_\Omega|\ \boldsymbol{m}^\varepsilon\times\Lambda^{2\alpha}\boldsymbol{m}^\varepsilon\ |^2\ \ \mathrm{d}x = 0. \tag{29}$$

Multiplying equation (29) by λ and making the sum with (28), we get

$$\int_\Omega|\ \boldsymbol{m}^\varepsilon\times\partial_t\boldsymbol{m}^\varepsilon\ |^2\ \ \mathrm{d}x + (\beta+\gamma\lambda)\int_\Omega\Lambda^{2\alpha}\boldsymbol{m}^\varepsilon\cdot\boldsymbol{m}^\varepsilon\times\partial_t\boldsymbol{m}^\varepsilon\ \mathrm{d}x - \lambda^2\int_\Omega|\ \boldsymbol{m}^\varepsilon\times\Lambda^{2\alpha}\boldsymbol{m}^\varepsilon\ |^2\ \ \mathrm{d}x = 0.$$

Then

$$\lambda^2\int_\Omega|\ \boldsymbol{m}^\varepsilon\times\Lambda^{2\alpha}\boldsymbol{m}^\varepsilon\ |^2\ \ \mathrm{d}x = \int_\Omega|\ \boldsymbol{m}^\varepsilon\times\partial_t\boldsymbol{m}^\varepsilon\ |^2\ \ \mathrm{d}x + (\beta+\gamma\lambda)\int_\Omega\Lambda^{2\alpha}\boldsymbol{m}^\varepsilon\cdot\boldsymbol{m}^\varepsilon\times\partial_t\boldsymbol{m}^\varepsilon\ \mathrm{d}x. \tag{30}$$

Multiplying (11) by $\partial_t \boldsymbol{m}^\varepsilon$, integrating over Ω, replacing $\displaystyle\int_\Omega \Lambda^{2\alpha}\boldsymbol{m}^\varepsilon \cdot \boldsymbol{m}^\varepsilon \times \partial_t \boldsymbol{m}^\varepsilon \,\mathrm{d}x$ by its value in (30) and using Lemma 2.4, we obtain

$$\begin{aligned}
&\lambda^2 \int_\Omega |\boldsymbol{m}^\varepsilon \times \Lambda^{2\alpha}\boldsymbol{m}^\varepsilon|^2 \,\mathrm{d}x = \int_\Omega |\boldsymbol{m}^\varepsilon \times \partial_t \boldsymbol{m}^\varepsilon|^2 \,\mathrm{d}x + \frac{\gamma(\beta+\gamma\lambda)}{\lambda}\int_\Omega |\partial_t \boldsymbol{m}^\varepsilon|^2 \,\mathrm{d}x \\
&+\frac{\beta(\beta+\gamma\lambda)}{2\lambda}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\Lambda^\alpha \boldsymbol{m}^\varepsilon|^2 \,\mathrm{d}x + \frac{(\beta+\gamma\lambda)}{4\varepsilon\lambda}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (|\boldsymbol{m}^\varepsilon|^2-1)^2 \,\mathrm{d}x \\
&\le \int_\Omega |\boldsymbol{m}^\varepsilon|^2|\partial_t \boldsymbol{m}^\varepsilon|^2 \,\mathrm{d}x + \frac{\gamma(\beta+\gamma\lambda)}{\lambda}\int_\Omega |\partial_t \boldsymbol{m}^\varepsilon|^2 \,\mathrm{d}x + \frac{\beta(\beta+\gamma\lambda)}{2\lambda}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\Lambda^\alpha \boldsymbol{m}^\varepsilon|^2 \,\mathrm{d}x \\
&+\frac{(\beta+\gamma\lambda)}{4\varepsilon\lambda}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (|\boldsymbol{m}^\varepsilon|^2-1)^2 \,\mathrm{d}x \\
&\le (1+\frac{\gamma(\beta+\gamma\lambda)}{\lambda})\int_\Omega |\partial_t \boldsymbol{m}^\varepsilon|^2 \,\mathrm{d}x + \frac{\beta(\beta+\gamma\lambda)}{2\lambda}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\Lambda^\alpha \boldsymbol{m}^\varepsilon|^2 \,\mathrm{d}x \\
&+\frac{(\beta+\gamma\lambda)}{4\varepsilon\lambda}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (|\boldsymbol{m}^\varepsilon|^2-1)^2 \,\mathrm{d}x.
\end{aligned}$$

We integrate from 0 to t, and using the previous lemma, we get

$$\lambda^2 \int_0^t \int_\Omega |\ \boldsymbol{m}^\varepsilon \times \Lambda^{2\alpha}\boldsymbol{m}^\varepsilon\ |^2 \ \mathrm{d}x\mathrm{d}t \le C, \tag{31}$$

where C is a constant independent of ε. Hence

$$(\boldsymbol{m}^\varepsilon \times \Lambda^{2\alpha}\boldsymbol{m}^\varepsilon)_\varepsilon \ \text{ is bounded in } \ \mathbb{L}^2(Q). \tag{32}$$

Consequently,

$$\boldsymbol{m}^\varepsilon \times \Lambda^{2\alpha}\boldsymbol{m}^\varepsilon \rightharpoonup \Phi \ \text{ weakly in } \ \mathbb{L}^2(Q). \tag{33}$$

By (27), we have

$$\begin{gathered}
(\partial_t \boldsymbol{m}^\varepsilon)_\varepsilon \ \text{ is bounded in } \ \mathbb{L}^2(Q), \\
(|\ \boldsymbol{m}^\varepsilon\ |^2 -1)_\varepsilon \ \text{ is bounded in } \ L^\infty(0,T;\mathbb{L}^2(\Omega)), \\
(\boldsymbol{m}^\varepsilon)_\varepsilon \ \text{ is bounded in } \ L^\infty(0,T;\mathbb{H}^\alpha(\Omega)).
\end{gathered}$$

Then we have the following convergences to a subsequence further notes that $(\boldsymbol{m}^\varepsilon)_\varepsilon$ for $(1 < p < \infty)$:

$$\begin{gathered}
\boldsymbol{m}^\varepsilon \rightharpoonup \boldsymbol{m} \ \text{ weakly in } \ L^p(0,T;\mathbb{H}^\alpha(\Omega)), \\
\partial_t \boldsymbol{m}^\varepsilon \rightharpoonup \partial_t \boldsymbol{m} \ \text{ weakly in } \ \mathbb{L}^2(Q), \\
|\ \boldsymbol{m}^\varepsilon\ |^2 -1 \to 0 \ \text{ strongly in } \ L^2(0,T;\mathbb{L}^2(\Omega)) \ \textit{and} \ \ |\ \boldsymbol{m}\ |= 1 \text{ a.e.}
\end{gathered}$$

By compactness embedding of $\mathbb{H}^\alpha(Q)$ into $\mathbb{L}^4(Q)$, we have

$$\boldsymbol{m}^\varepsilon \to \boldsymbol{m} \ \text{ strongly in } \ \mathbb{L}^4(Q). \tag{34}$$

In the following, we show that

$$\boldsymbol{m} \times \Lambda^{2\alpha}\boldsymbol{m} = \Phi \in \mathbb{L}^2(Q). \tag{35}$$

Let $\boldsymbol{\varphi} \in \mathbb{H}^\alpha(\Omega)$. We have

$$\int_Q \boldsymbol{m}^\varepsilon \times \Lambda^{2\alpha}\boldsymbol{m}^\varepsilon \cdot \boldsymbol{\varphi} \,\mathrm{d}x\mathrm{d}t = -\int_Q \Lambda^\alpha \boldsymbol{m}^\varepsilon \cdot \Lambda^\alpha(\boldsymbol{m}^\varepsilon \times \boldsymbol{\varphi}) \,\mathrm{d}x\mathrm{d}t.$$

On the other hand, using commutator estimate together with the same reasonings that lead to (25), we have

$$\begin{aligned}\int_Q \Lambda^\alpha \boldsymbol{m}^\varepsilon \cdot \Lambda^\alpha(\boldsymbol{m}^\varepsilon \times \boldsymbol{\varphi})\ \mathrm{d}x\mathrm{d}t &\to \int_Q \Lambda^\alpha \boldsymbol{m} \cdot \Lambda^\alpha(\boldsymbol{m} \times \boldsymbol{\varphi})\ \mathrm{d}x\mathrm{d}t \\ &= -\int_Q (\boldsymbol{m} \times \Lambda^{2\alpha}\boldsymbol{m}) \cdot \boldsymbol{\varphi}\ \mathrm{d}x\mathrm{d}t,\end{aligned}$$

and therefore (35) is proved. In particular, we have

$$\boldsymbol{m}^\varepsilon \times \Lambda^{2\alpha}\boldsymbol{m}^\varepsilon \rightharpoonup \boldsymbol{m} \times \Lambda^{2\alpha}\boldsymbol{m} \ \text{ weakly in } \ \mathbb{L}^2(Q).$$

Now back to (26) and taking $\boldsymbol{\varphi} = \boldsymbol{m}^\varepsilon \times \boldsymbol{\phi}$ with $\boldsymbol{\phi} \in \mathbf{C}^\infty(\overline{Q})$, we have

$$\begin{aligned}\gamma \int_Q \partial_t \boldsymbol{m}^\varepsilon \cdot \boldsymbol{m}^\varepsilon \times \boldsymbol{\phi}\ \mathrm{d}x\mathrm{d}t + \int_Q \boldsymbol{m}^\varepsilon \times \partial_t \boldsymbol{m}^\varepsilon \cdot \boldsymbol{m}^\varepsilon \times \boldsymbol{\phi}\ \mathrm{d}x\mathrm{d}t \\ +\beta \int_Q \Lambda^\alpha \boldsymbol{m}^\varepsilon \cdot \Lambda^\alpha(\boldsymbol{m}^\varepsilon \times \boldsymbol{\phi})\ \mathrm{d}x\mathrm{d}t + \lambda \int_Q \boldsymbol{m}^\varepsilon \times \Lambda^{2\alpha}\boldsymbol{m}^\varepsilon \cdot \boldsymbol{m}^\varepsilon \times \boldsymbol{\phi}\ \mathrm{d}x\mathrm{d}t = 0.\end{aligned} \tag{36}$$

For the first term of (36), we set $\Theta_\varepsilon = \displaystyle\int_Q \boldsymbol{m}^\varepsilon \times \partial_t \boldsymbol{m}^\varepsilon \cdot \boldsymbol{m}^\varepsilon \times \boldsymbol{\phi}\ \mathrm{d}x\mathrm{d}t$.

We have

$$\Theta_\varepsilon = \int_Q |\ \boldsymbol{m}^\varepsilon\ |^2\ \partial_t \boldsymbol{m}^\varepsilon \cdot \boldsymbol{\phi}\ \mathrm{d}x\mathrm{d}t - \int_Q (\boldsymbol{m}^\varepsilon \cdot \boldsymbol{\phi})\boldsymbol{m}^\varepsilon \cdot \partial_t \boldsymbol{m}^\varepsilon\ \mathrm{d}x\mathrm{d}t.$$

On the one hand

$$\begin{aligned}\int_Q |\boldsymbol{m}^\varepsilon|^2 \partial_t \boldsymbol{m}^\varepsilon \cdot \boldsymbol{\phi}\ \mathrm{d}x\mathrm{d}t &= \int_Q (|\boldsymbol{m}^\varepsilon|^2 - 1)\partial_t \boldsymbol{m}^\varepsilon \cdot \boldsymbol{\phi}\ \mathrm{d}x\mathrm{d}t + \int_Q \partial_t \boldsymbol{m}^\varepsilon \cdot \boldsymbol{\phi}\ \mathrm{d}x\mathrm{d}t \\ &\to \int_Q \partial_t \boldsymbol{m} \cdot \boldsymbol{\phi}\ \mathrm{d}x\mathrm{d}t.\end{aligned}$$

On the other hand

$$\begin{aligned}\int_Q (\boldsymbol{m}^\varepsilon \cdot \boldsymbol{\phi})\boldsymbol{m}^\varepsilon \cdot \partial_t \boldsymbol{m}^\varepsilon\ \mathrm{d}x\mathrm{d}t &= \frac{1}{2}\int_Q \partial_t(|\ \boldsymbol{m}^\varepsilon\ |^2 - 1)\boldsymbol{m}^\varepsilon \cdot \boldsymbol{\phi}\ \mathrm{d}x\mathrm{d}t \\ &= \frac{1}{2}\left[\int_\Omega (|\ \boldsymbol{m}^\varepsilon\ |^2 - 1)\boldsymbol{m}^\varepsilon \cdot \boldsymbol{\phi}\ \mathrm{d}x\right]_0^T \\ &\quad -\frac{1}{2}\int_Q (|\ \boldsymbol{m}^\varepsilon\ |^2 - 1)\partial_t(\boldsymbol{m}^\varepsilon \cdot \boldsymbol{\phi})\ \mathrm{d}x\mathrm{d}t.\end{aligned}$$

Now choose $\boldsymbol{\phi}$ so that $\boldsymbol{\phi} = 0$ in $t = 0$ and $t = T$. Then

$$\left[\int_\Omega (|\ \boldsymbol{m}^\varepsilon\ |^2 - 1)\boldsymbol{m}^\varepsilon \cdot \boldsymbol{\phi}\ \mathrm{d}x\right]_0^T = 0.$$

Therefore,

$$\begin{aligned}\int_Q (\boldsymbol{m}^\varepsilon \cdot \boldsymbol{\phi})\boldsymbol{m}^\varepsilon \cdot \partial_t \boldsymbol{m}^\varepsilon\ \mathrm{d}x\mathrm{d}t &= -\frac{1}{2}\int_Q (|\ \boldsymbol{m}^\varepsilon\ |^2 - 1)\partial_t(\boldsymbol{m}^\varepsilon \cdot \boldsymbol{\phi})\ \mathrm{d}x\mathrm{d}t \\ &= -\frac{1}{2}\int_Q (|\ \boldsymbol{m}^\varepsilon\ |^2 - 1)\partial_t \boldsymbol{m}^\varepsilon \cdot \boldsymbol{\phi}\ \mathrm{d}x\mathrm{d}t \\ &\quad -\frac{1}{2}\int_Q (|\ \boldsymbol{m}^\varepsilon\ |^2 - 1)\boldsymbol{m}^\varepsilon \cdot \partial_t \boldsymbol{\phi}\ \mathrm{d}x\mathrm{d}t \to 0.\end{aligned}$$

Hence

$$\Theta_\varepsilon \to \int_Q \partial_t \boldsymbol{m} \cdot \boldsymbol{\phi} \, \mathrm{d}x\mathrm{d}t.$$

For the second term of (36)

$$\beta \int_Q \Lambda^\alpha \boldsymbol{m}^\varepsilon \cdot \Lambda^\alpha(\boldsymbol{m}^\varepsilon \times \boldsymbol{\phi}) \, \mathrm{d}x\mathrm{d}t \to \beta \int_Q \Lambda^\alpha \boldsymbol{m} \cdot \Lambda^\alpha(\boldsymbol{m} \times \boldsymbol{\phi}) \, \mathrm{d}x\mathrm{d}t.$$

For the third term of (36)

$$\lambda \int_Q \boldsymbol{m}^\varepsilon \times \Lambda^{2\alpha} \boldsymbol{m}^\varepsilon \cdot \boldsymbol{m}^\varepsilon \times \phi \, \mathrm{d}x\mathrm{d}t \to \lambda \int_Q \boldsymbol{m} \times \Lambda^{2\alpha} \boldsymbol{m} \cdot \boldsymbol{m} \times \boldsymbol{\phi} \, \mathrm{d}x\mathrm{d}t.$$

For the last term of (36)

$$\gamma \int_Q \partial_t \boldsymbol{m}^\varepsilon \cdot \boldsymbol{m}^\varepsilon \times \boldsymbol{\phi} \, \mathrm{d}x\mathrm{d}t \to \gamma \int_Q \partial_t \boldsymbol{m} \cdot \boldsymbol{m} \times \boldsymbol{\phi} \, \mathrm{d}x\mathrm{d}t.$$

Let ε tends to 0 in (36), we obtain

$$\begin{aligned} &\int_Q \partial_t \boldsymbol{m} \cdot \boldsymbol{\phi} \, \mathrm{d}x\mathrm{d}t - \gamma \int_Q \boldsymbol{m} \times \partial_t \boldsymbol{m} \cdot \boldsymbol{\phi} \, \mathrm{d}x\mathrm{d}t \\ &+\beta \int_Q \Lambda^\alpha \boldsymbol{m} \cdot \Lambda^\alpha(\boldsymbol{m} \times \boldsymbol{\phi}) \, \mathrm{d}x\mathrm{d}t + \lambda \int_Q \boldsymbol{m} \times \Lambda^{2\alpha} \boldsymbol{m} \cdot \boldsymbol{m} \times \boldsymbol{\phi} \, \mathrm{d}x\mathrm{d}t = 0 \end{aligned}$$

for all $\boldsymbol{\phi} \in \mathcal{C}^\infty(\overline{Q})$. Furthermore, the inequality (10) follows from (27) and we finish the proof of Theorem 2.1.

3 The Limit as $b \to 0$

The main purpose of this section is to reveal to relationships between the fractional LLG equation we have studied in this paper, and the classical fractional LLG equation (i.e., in the case $b = 0$). We will prove the following result.

Proposition 3.1 *Let $b \to 0$. The weak solution $\boldsymbol{m}^b$ obtained in section 2 weakly converges, up to a subsequence, to a solution of the classical fractional LLG equation in the following sense.*
For all $\boldsymbol{\phi} \in \mathcal{C}^\infty(\overline{Q})$ with $\boldsymbol{\phi}(0,.) = \boldsymbol{\phi}(T,.) = 0$,

$$\int_Q \partial_t \boldsymbol{m} \cdot \boldsymbol{\phi} \, \mathrm{d}x\mathrm{d}t - \gamma \int_Q \boldsymbol{m} \times \partial_t \boldsymbol{m} \cdot \boldsymbol{\phi} \, \mathrm{d}x\mathrm{d}t = -\beta \int_Q \Lambda^\alpha \boldsymbol{m} \cdot \Lambda^\alpha(\boldsymbol{m} \times \boldsymbol{\phi}) \, \mathrm{d}x\mathrm{d}t.$$

Proof. Using the fact that $|\boldsymbol{m}^b| = 1$ a.e in Q and estimate (10), we deduce that

$$(\boldsymbol{m}^b)_b \text{ is bounded in } L^\infty(0, T, \mathbb{H}^\alpha(\Omega)),$$

and

$$(\partial_t \boldsymbol{m}^b)_b \text{ is bounded in } \mathbb{L}^2(Q).$$

Hence, up to a subsequence, we have

$$\boldsymbol{m}^b \rightharpoonup \boldsymbol{m} \text{ weakly in } L^p(0,T,\mathbb{H}^\alpha(\Omega)) \text{ for } 1<p<\infty,$$
$$\boldsymbol{m}^b \to \boldsymbol{m} \text{ strongly in } C([0,T],\mathbb{H}^\delta(\Omega)) \text{ and a.e for } 0 \le \delta < \alpha,$$
$$\partial_t \boldsymbol{m}^b \rightharpoonup \partial_t \boldsymbol{m} \text{ weakly in } \mathbb{L}^2(Q).$$

Then $|\boldsymbol{m}| = 1$ a.e in Q. On the other hand, we have

$$\gamma\partial_t \boldsymbol{m}^b + \boldsymbol{m}^b \times \partial_t \boldsymbol{m}^b + \beta\Lambda^{2\alpha}\boldsymbol{m}^b + \lambda \boldsymbol{m}^b \times \Lambda^{2\alpha}\boldsymbol{m}^b - \beta(\Lambda^{2\alpha}\boldsymbol{m}^b \cdot \boldsymbol{m}^b)\boldsymbol{m}^b = 0 \text{ a.e. in } Q.$$

Multiplying this equation by $\partial_t \boldsymbol{m}^b$ and $\boldsymbol{m}^b \times \Lambda^{2\alpha}\boldsymbol{m}^b$ respectively and integrating over Ω, we get

$$\gamma \int_\Omega |\ \partial_t \boldsymbol{m}^b\ |^2 \ \mathrm{d}x + \frac{\beta}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\ \Lambda^\alpha \boldsymbol{m}^b\ |^2 \ \mathrm{d}x + \lambda \int_\Omega \boldsymbol{m}^b \times \Lambda^{2\alpha}\boldsymbol{m}^b \cdot \partial_t \boldsymbol{m}^b \,\mathrm{d}x = 0 \qquad (37)$$

and

$$\lambda \int_\Omega |\boldsymbol{m}^b \times \Lambda^{2\alpha}\boldsymbol{m}^b|^2 \,\mathrm{d}x + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\Lambda^\alpha \boldsymbol{m}^b|^2 \,\mathrm{d}x = -\gamma \int_\Omega \boldsymbol{m}^b \times \Lambda^{2\alpha}\boldsymbol{m}^b \cdot \partial_t \boldsymbol{m}^b \,\mathrm{d}x. \qquad (38)$$

The equalities (37), (38) allow to get

$$\lambda^2 \int_\Omega |\boldsymbol{m}^b \times \Lambda^{2\alpha}\boldsymbol{m}^b|^2 \,\mathrm{d}x = \gamma^2 \int_\Omega |\partial_t \boldsymbol{m}^b|^2 \,\mathrm{d}x + (\frac{\gamma\beta-\lambda}{2})\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\Lambda^\alpha \boldsymbol{m}^b|^2 \,\mathrm{d}x.$$

We integrate from 0 to t to get

$$\begin{aligned}\lambda^2 \int_0^t \int_\Omega |\boldsymbol{m}^b \times \Lambda^{2\alpha}\boldsymbol{m}^b|^2 \,\mathrm{d}x\mathrm{d}t + (\frac{\gamma\beta-\lambda}{2})\int_\Omega |\Lambda^\alpha \boldsymbol{m}_0|^2 \,\mathrm{d}x \\ = \gamma^2 \int_0^t \int_\Omega |\partial_t \boldsymbol{m}^b|^2 \,\mathrm{d}x\mathrm{d}t + (\frac{\gamma\beta-\lambda}{2})\int_\Omega |\Lambda^\alpha \boldsymbol{m}^b|^2 \,\mathrm{d}x\end{aligned} \qquad (39)$$

for all $t \in (0,T)$.

Recall that

$$\beta = a(1+\gamma^2) \text{ and } \lambda = b(1+\gamma^2).$$

Since b is small enough, we assume that $b < a\gamma$ *i.e.,* $\lambda < \gamma\beta$. Using estimate (10), we have

$$\int_\Omega |\Lambda^\alpha \boldsymbol{m}^b|^2 \,\mathrm{d}x \le \int_\Omega |\Lambda^\alpha \boldsymbol{m}_0|^2 \,\mathrm{d}x$$

and

$$\gamma^2 \int_0^t \int_\Omega |\partial_t \boldsymbol{m}^b|^2 \,\mathrm{d}x\mathrm{d}t \le \frac{\gamma\beta(1+\gamma^2)}{2}\int_\Omega |\Lambda^\alpha \boldsymbol{m}_0|^2 \,\mathrm{d}x.$$

Then, (39) implies that

$$b^2 \int_0^t \int_\Omega |\boldsymbol{m}^b \times \Lambda^{2\alpha}\boldsymbol{m}^b|^2 \,\mathrm{d}x\mathrm{d}t \le \frac{\gamma a}{2}\int_\Omega |\Lambda^\alpha \boldsymbol{m}_0|^2 \,\mathrm{d}x.$$

Hence

$$(b\, \boldsymbol{m}^b \times \Lambda^{2\alpha}\boldsymbol{m}^b)_b \text{ is bounded in } \mathbb{L}^2(Q).$$

Therefore,

$$b\, \boldsymbol{m}^b \times \Lambda^{2\alpha} \boldsymbol{m}^b \rightharpoonup \xi \text{ weakly in } \mathbb{L}^2(Q).$$

Let $\boldsymbol{\psi} \in \mathbb{H}^{\alpha}(Q)$. We have

$$\int_Q b\, \boldsymbol{m}^b \times \Lambda^{2\alpha} \boldsymbol{m}^b \cdot \boldsymbol{\psi} \, \mathrm{d}x\mathrm{d}t = -b \int_Q \Lambda^{\alpha} \boldsymbol{m}^b \cdot \Lambda^{\alpha}(\boldsymbol{m}^b \times \boldsymbol{\phi}) \, \mathrm{d}x\mathrm{d}t,$$

which tends to zero as b goes to zero. We conclude that $\xi = 0$.

Now, we can pass to the limit as $b \to 0$ in the weak formulation

$$\int_Q \partial_t \boldsymbol{m}^b \cdot \boldsymbol{\phi} \, \mathrm{d}x\mathrm{d}t - \gamma \int_Q \boldsymbol{m}^b \times \partial_t \boldsymbol{m}^b \cdot \boldsymbol{\phi} \, \mathrm{d}x\mathrm{d}t$$
$$= -\beta \int_Q \Lambda^{\alpha} \boldsymbol{m}^b \cdot \Lambda^{\alpha}(\boldsymbol{m}^b \times \boldsymbol{\phi}) \, \mathrm{d}x\mathrm{d}t - (1+\alpha^2) \int_Q b\, \boldsymbol{m}^b \times \Lambda^{2\alpha} \boldsymbol{m}^b \cdot \boldsymbol{m}^b \times \boldsymbol{\phi} \, \mathrm{d}x\mathrm{d}t.$$

We obtain

$$\int_Q \partial_t \boldsymbol{m} \cdot \boldsymbol{\phi} \, \mathrm{d}x\mathrm{d}t - \alpha \int_Q \boldsymbol{m} \times \partial_t \boldsymbol{m} \cdot \boldsymbol{\phi} \, \mathrm{d}x\mathrm{d}t = -\beta \int_Q \Lambda^{\alpha} \boldsymbol{m} \cdot \Lambda^{\alpha}(\boldsymbol{m} \times \boldsymbol{\phi}) \, \mathrm{d}x\mathrm{d}t.$$

Then Proposition 3.1 is proved. □

4 Concluding Remarks

In this paper, global existence of weak solutions to a modified fractional LLG equation is proved. The modification lies in the presence in the effective field of the term $b\, \boldsymbol{m} \times \Lambda^{2\alpha} \boldsymbol{m}$ describing fractional vertical spin stiffness. Due to nonlocal nonlinearities in the model, special structures of the equation, the commutator estimate and some calculus inequalities of fractional order are exploited to get the convergence of the approximating solutions. The relationship between the model and the classical fractional LLG equation is also revealed by discussing the limit of the obtained solutions when the vertical spin stiffness parameter b tends to zero.

Let us mention that important progress has been made in the design of schemes constructing weak solutions to classical LLG equation. Several schemes were proposed, and their convergence to weak solutions was proved (see for examples [2, 4]). An interesting direction of future research is to propose numerical scheme for the fractional LLG equation. This will be helpful to give a strategy for efficient computer implementation which may reflect the true nature of the augmentation of the LLG model considered in this paper.

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References

[1] Alouges, F. and Soyeur, A. On global weak solutions for Landau-Lifshitz equations: Existence and nonuniqueness. *Nonlinear Anal.* **18** (1992) 1071–1084.

[2] Alouges, F. and Jaisson, P. Convergence of a finite element discretization for the Landau-Lifshitz equations in micromagnetism. *Math. Models Methods Appl. Sci.* **16** (2006) 299–319.

[3] Ayouch, C., Essoufi, E. H. and Tilioua, M. On a model of magnetization dynamics with vertical spin stiffness. *Boundary Value Problems* (2016) 2016:110.

[4] Bartels, S., Ko, J. and Prohl, A. Numerical approximation of an explicit approximation scheme for the Landau-Lifshitz-Gilbert equation. *Math. Comput.* **77** (2008) 773–788.

[5] Carbou, G. and Fabrie, P. Time Average in Micromagnetism. *J. Differential Equations* **147** (1998) 383–409.

[6] Coifman, R. R. and Meyer, Y. Nonlinear harmonic analysis, operator theory and P.D.E. in Beijing Lectures in Harmonic Analysis. *Princeton University Press.* (1986) 3–45.

[7] García-Palacios, J. L. and Lázaro, F. J. Langevin-dynamics study of the dynamical properties of small magnetic particles. *Phys. Rev. B* **58** (1998) 14937.

[8] Gilbert, T. L. A lagrangian formulation of gyromagnetic equation of the magnetization field. *Phys. Rev.,* **100** (1955) 1243.

[9] Guo, B. and Hong, M. C. The Landau-Lifshitz equation of the ferromagetic spin chain and harmonic maps. *Cal. Var.* **1** (1993) 311–334.

[10] Hubert, A. and Schafer, R. *Magnetic Domains*. Springer, Berlin, 1998.

[11] Kato, T. Liapunov Functions and Monotonicity in the Navier-Stokes Equations. *Lecture Notes in Mathematics,* **1450**. Springer-Verlag, Berlin, 1990.

[12] Kenig, C., Ponce, G. and Vega, L. Well-posedness and scattering results for the generalized korteweg-de Vries equation via the contraction principle. *Commun. Pure Appl. Math.* **46**(4) (1993) 453–620.

[13] Kronmuller, H. and Parkin, S. *Handbook of Magnetism and Advanced Magnetic Materials.* Wiley, 2007.

[14] Landau, L. and Lifshitz, E. On the theory of the dispersion of magnetic permeability in ferromagnetic bodies. *Phys. Z. Sowjet.* **8** (153) (1935) 153–169.

[15] Lions, J.L. *Quelques Méthodes de Résolution des Problèmes aux Limites Non Linéaires.* Dunod & Gauthier-Villars, Paris, 1969.

[16] Podio-Guidugli, P. On dissipation mechanisms in micromagnetics. *The European Physical Journal B* **19** (2001) 417–424.

[17] Podio-Guidugli, P. and Valente, V. Existence of global-in-time weak solutions to a modified Gilbert equation. *Nonlinear Analysis: Theory, Methods and Applications* **47** (2001) 147–158.

[18] Pu, X., Guo, B. and Zhang, J. Global weak solutions to the 1-D Fractional Landau-Lifshitz Equation. *Discrete Contin. Dyn. Syst. Ser. B* **14**(1) (2010) 199–207.

[19] Shen, K., Tatara, G. and Wu, M. W. Existence of vertical spin stiffness in Landau-Lifshitz-Gilbert equation in ferromagnetic semiconductors. *Phys. Rev. B* **83** (2011) 085203.

[20] Simon, J. Compact sets in the space $L^p(0,T;B)$. *Ann. Math. Pura. Appl.* **146** (1987) 65–96.

[21] Stein, E.M. *Singular Integrals and Differentiability Properties of Functions*. Princeton University Press, Princeton, N.J., 1970.

[22] Temam, R. *Infinite-Dimensional Dynamical Systems in Mechanics and Physics.* Springer-Verlag, New York, 1997.

[23] Visintin, A. On Landau-Lifshitz equations for ferromagnetism. *Japan J. Appl. Math.* **2** (1985) 69–84.