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Exponential Domination and Bondage Numbers in Some Graceful Cyclic Structure

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Abstract: The domination number is an important vulnerability parameter that it has become one of the most widely studied topics in graph theory, and also the bondage number which is related by domination number the most often studied property of vulnerability of communication networks. Recently, Dankelmann et al. defined the exponential domination number denoted by $\gamma_e(G)$ in [17]. In 2016, the exponential bondage number, denoted by $b_{exp}(G)$, is defined by $b_{exp}(G) = min\{|B_e| : B_e \subseteq E(G), \gamma_e(G - B_e) > \gamma_e(G)\}$, where $\gamma_e(G)$ is the exponential domination number of G [24]. In this paper, the above mentioned parameters is has been examined. Then exact formulas are obtained for the families of cyclic structures tend to have graceful subfamilies such as helm graph, windmill graph, circular necklace and friendship graph.

Keywords: *graph vulnerability; connectivity; domination number; bondage number; exponential domination number; exponential bondage number.*

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1 Introduction

Graph theory plays vital role in various fields. One of the important areas in graph theory is graph labeling. Interest in graph labeling began in mid-1960s with a conjecture by Kotzig-Ringel and a paper by Rosa [5]. In 1967, Rosa published a pioneering paper on graph labeling problems. Graph labeling is powerful tool that makes things ease in various fields of networking. Graph labeling is very important major areas of computer science like data mining image processing, cryptography, software testing, information

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security, communication network etc. Also, there are many applications of graph labelling in the literature such as coding theory, radar, astronomy, circuit design, missile guidance, communication network addressing, xray crystallography, data base management [5,13].

We begin by recalling some standard definitions that we need throughout this paper. Let $G = (V, E)$ be a simple undirected graph of order n. For any vertex $v \in V$, the *open neighborhood* of v is $N_G(v) = \{u \in V | uv \in E\}$ and *closed neighborhood* of v is $N_G[v] = N_G(v) \cup \{v\}$. The degree of v in G denoted by $deg(v)$, is the size of its open neighborhood. A vertex v is said to be pendant vertex if $deg(v) = 1$ [7,18]. A vertex u is called support vertex if u is adjacent to a pendant vertex. The graph G is called r-regular graph if $deg(v) = r$ for every vertex $v \in V$. The *distance* $d(u, v)$ between two vertices u and v in G is the length of a shortest path between them [7,18].

Given a graph $G = (V, E)$, the set N of non-negative integers and a commutative binary operation $* : N \times N \to N$, every vertex $f : V \to N$ induces an edge function $f* : E \to N$such that $f * (uv) = |f(u) - f(v)|$, for all $uv \in E$. A function f is called graceful labeling of a graph G if $f : V \to 0, 1, 2, ..., q$ is injective and the induced function $f* : E \to 1, 2, ..., q$ is bijective. A graph which admits graceful labeling is called graceful graph.

A set $S \subseteq V$ is a *dominating set* if every vertex in $V(G) - S$ is adjacent to at least one vertex in S. The minimum cardinality taken over all dominating sets of G is called the *domination number* of G is denoted by $\gamma(G)$ [7,18]. There are different application of domination problems. For instance, dominating sets in graphs are natural models for facility location problems in operations research [18] or domination number is the one of the most important vulnerability parameter for networks [18,23]. When investigating the domination number of a given graph G, one may want to learn the answer of the following question: How does the domination number increases in a graph G? or How many edges need to be added to decrease the domination number of the original graph? One of the vulnerability parameters known as *bondage number* in a graph G answers the former question. The bondage number $b(G)$ was introduced by Fink et al. [12] and is defined as follows:

$$b(G) = \min\{|B| : B \subseteq E, \gamma(G - B) > \gamma(G)\}.$$

We call such an edge set B that $\gamma(G - B) > \gamma(G)$ the *bondage set* and the minimum one the *minimum bondage set*. If $b(G)$ does not exist, for example empty graphs, then $b(G) = \infty$ is defined.

In 2009, Dankelmann introduced the concept of *exponential domination* [17]. This new parameter is closely in relation with distance of each pair of vertices. The exponential domination number is the theoretical vulnerability parameters for a network that is represented by a graph [1, 17]. An exponential dominating set of graph G is a kind of distance domination subset $S \subseteq V(G)$ such that $\sum_{v \in S} (1/2)^{\overline{d}(u,v)-1} \geq 1, \forall v \in V$, where $\overline{d}(u, v)$ is the length of a shortest path in $\langle V - (S - \{u\}) \rangle$ if such a path exist, and ∞ otherwise. The minimum exponential domination number, $\gamma_e(G)$ is the smallest cardinality of an exponential dominating set. We call such an edge set is a minimum exponential set which is denoted by γ_e-set.

Aytac et al. has defined exponential bondage number [24]. It is defined as follows:

$$b_{exp}(G) = \min\{|B_e| : B_e \subseteq E, \gamma_e(G - B_e) > \gamma_e(G)\},$$

where $\gamma_e(G)$ is the exponential domination number of the graph G. We call such an edge set B_e that $\gamma_e(G - B_e) > \gamma_e(G)$ the *exponential bondage set* and the minimum one the *minimum exponential bondage set*.

There are many advantages of creating a communications network that is analogous a graceful graph. One advantage is that if a link goes out, a simple algorithm could detect which two centers are no longer linked, since each connection is labeled with the difference between the two communication centers. Another advantage is that this network also would have all the same properties as a graceful graph; such as having cyclic decompositions [5,13]. Many structures that have been studied in recent years are structures that involve cycles. One reason for this is that Rosa proved that all cycles that are of lengths $n \equiv 0, 3(mod4)$ are graceful. Hence, many families of cyclic structures tend to have graceful subfamilies. We will now investigate some of these structures such as: helm graph, windmill graph, circular necklace and friendship graph.

Calculation of exponential domination and bondage numbers for simple cyclic graph types is important because if one can break a more complex network into smaller networks, then under some conditions the solutions for the optimization problem on the smaller networks can be combined to a solution for the optimization problem on the larger network.

In Section 2, some well-known basic results are given for exponential domination and bondage numbers. In Section 3, examples of the exponential dominating and the exponential bondage sets of a graph are are given. In Section 4, the exponential domination numbers have been computed for helm graph, windmill graph, circular necklace and friendship graph. In Section 5, the exponential bondage numbers have been calculated for same structures.

2 Basic Results

In this section some well-known basic results are given with regard to exponential domination number and bondage number.

Theorem 2.1 *[17] The exponential domination number of*

a) the path graph P_n of order $n \geq 2$ is $\gamma_e(P_n) = \lceil \frac{n+1}{4} \rceil$.

b) the cycle graph C_n of order $n \geq 4$ is $\gamma_e(C_n) = \left\{ \begin{array}{ll} 2 & , \text{if } n = 4; \\ \lceil \frac{n}{4} \rceil & , \text{if } n \neq 4. \end{array} \right\}$

Theorem 2.2 *[17] For every graph G, $\gamma_e(G) \leq \gamma(G)$, and also $\gamma_e(G) = 1$ if and only if $\gamma(G) = 1$.*

Theorem 2.3 *Let G be any connected graph with n vertices and $\exists v \in V(G)$ such that $deg(v) = n - 1$. Then $\gamma_e(G) = 1$.*

Theorem 2.4 *[12] If G is a connected graph of order $n \geq 2$, then $b(G) \leq n - \gamma(G) + 1$.*

Theorem 2.5 *[12] The bondage number of*

a) the path graph P_n of order $n \geq 2$ is $b(P_n) = \left\{ \begin{array}{ll} 2, & \text{if } n \equiv 1(mod\ 3); \\ 1, & \text{otherwise.} \end{array} \right.$

b) the cycle graph C_n of order $n \geq 3$ is $b(C_n) = \left\{ \begin{array}{ll} 3, & \text{if } n \equiv 1(mod\ 3); \\ 2, & \text{otherwise.} \end{array} \right.$

c) the complete graph K_n *of order* $n \geq 2$ *is* $b(K_n) = \lceil \frac{n}{2} \rceil$.

d) the star graph S_n *of order* $n \geq 3$ *is* $b(S_n) = 1$.

Theorem 2.6 *[22] If G is a nonempty graph with a unique minimum dominating set, then* $b(G) = 1$.

Theorem 2.7 *[24] Let G be a connected graph of order n. If G includes only one pendant vertex, then* $b_{exp}(G) = 1$.

3 Example

a) Let's find the exponential dominating sets of the given graph in Figure 1.

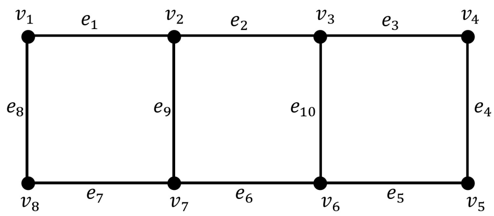

Figure 1: Graph G.

- For the set $S_1 = \{v_1,\ v_3,\ v_7,\ v_5\} \subseteq V(G)$, Table 1 is obtained.

Table 1: The weight values of S_1 at v.

v	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8
$w_{S_1}(v)$	2	3	2	2	2	3	2	2

From Table 1, it is easy to see that $w_{S_1}(v) \geq 1$. Hence, the set $S_1 \subseteq V(G)$ is an exponential dominating set of the graph G.

- For the set $S_2 = \{v_2, v_6, v_8\} \subseteq V(G)$, Table 2 is obtained.

Table 2: The weight values of S_2 at v.

v	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8
$w_{S_2}(v)$	2	2	2	1	5/4	2	3	2

From Table 2, it is easy to see that $w_{S_2}(v) \geq 1$. Hence, the set $S_2 \subseteq V(G)$ is an exponential dominating set of the graph G.

- For the set $S_3 = \{v_1, v_5\} \subseteq V(G)$, Table 3 is obtained.

  From Table 3, it is easy to see that $w_{S_3}(v) \geq 1$. Hence, the set $S_3 \subseteq V(G)$ is an exponential dominating set of the graph G.

Table 3: The weight values of S_3 at v.

v	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8
$w_{S_3}(v)$	2	5/4	1	5/4	2	5/4	1	5/4

Among some of the exponential dominating sets discussed above, the set having minimum element is the set S_3. There is not a set that is exponential dominating and $|S| < |S_3|$ of the graph G. Namely $\exists S \subseteq V(G)$ can not be found. In this case, exponential domination number of the graph G is $\gamma_e(G) = |S_3| = 2$.

b) Let's find the exponential bondage sets of the given graph in Figure 1.

- Let's consider the set $B_e^1 = \{e_1\} \subseteq E(G)$. In this case, we examine exponential domination number of the $E(G) - B_e^1$ graph. Here, it is easy to see that $S = \{v_1, v_8\} \subseteq E(G) - B_e^1$ is a member of any minimum exponential dominating set. B_e^1 is not an exponential bondage set because $\gamma_e\left(E(G) - B_e^1\right) = \gamma_e(G) = 2$.
- Let's consider the set $B_e^2 = \{e_3, e_6\} \subseteq E(G)$. In this way, we examine exponential domination number of the $E(G) - B_e^2$ graph. Here, it can be easily seen that the set $S = \{v_1, v_3, v_5\} \subseteq E(G) - B_e^2$ is a minimum exponential dominating set. B_e^2 is an exponential bondage set because $\gamma_e\left(E(G) - B_e^2\right) = 3 > \gamma_e(G) = 2$.
- Let's consider the set $B_e^3 = \{e_2, e_6\} \subseteq E(G)$. The $E(G) - B_e^3$ graph consists of two components. In this case, we examine exponential domination number of the $E(G) - B_e^3$ graph. Here, it can be easily seen that the set $S = \{v_1, v_3, v_5, v_7\} \subseteq E(G) - B_e^3$ is a member of any minimum exponential dominating set. B_e^3 is an exponential bondage set because $\gamma_e\left(E(G) - B_e^3\right) = 4 > \gamma_e(G) = 2$.
- Let's consider the set $B_e^4 = \{e_3, e_5, e_{10}\} \subseteq E(G)$. The $E(G) - B_e^4$ graph consists of two components. In this case, we examine exponential domination number of the $E(G) - B_e^4$ graph. Here, it can be easily seen that the set $S = \{v_1, v_7, v_4\} \subseteq E(G) - B_e^4$ is a member of any minimum exponential dominating set. B_e^4 is an exponential bondage set because $\gamma_e\left(E(G) - B_e^4\right) = 3 > \gamma_e(G) = 2$.

Among some of the exponential bondage sets discussed above, the set having minimum element is the set B_e^2. There is not a set that is exponential bondage and $|B_e| < \left|B_e^2\right|$ of the graph G. Namely $\exists B_e \subseteq E(G)$ can not be found. In this case, exponential bondage number of the graph G is $b_{exp}(G) = \left|B_e^2\right| = 2$.

4 The Exponential Domination Number of Some Graceful Cyclic Structure

In this section, we give definition of well-known graceful cyclic structure. Then we calculate the exponential domination number of them.

Definition 4.1 [15] A helm graph is denoted by H_n is a graph obtained by attaching a single edge and vertex of the outer circuit of a wheel graph W_n. The number of vertices of H_n is $2n + 1$ and the number of edges is $3n$. We display the graph H_4 in Figure 2.

Theorem 4.1 *If H_n is a helm graph, then $\gamma_e(H_n) = 4$.*

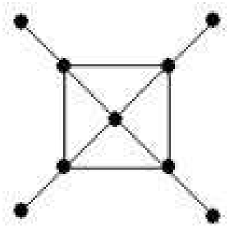

Figure 2: The Helm Graph H_4.

Proof. The Helm H_n consist of the vertex set $V(H_n) = \{v_i | 0 \leq i \leq n-1\} \cup \{a_i | 0 \leq i \leq n-1\} \cup \{c\}$. Let c be the central vertex of H_n. The degree of central vertex is n. The vertices of $H_n \setminus \{c\}$ are two kinds: vertices of degree four and one, respectively. Clearly, $deg(v_i) = 4$ and $deg(a_i) = 1$.

Let S be γ_e-set of H_n. If S consists of only one central vertex c, then this vertex is exponentially dominated all vertices except that the pendant vertices a_i. Therefore, the vertices v_i must be added to S.

If $c \in S$ and v_i is not adjacent a_i, then $\overline{d}(v_i, a_i) \geq 2$. If $c \notin S$ and v_i is not adjacent a_i, then $\overline{d}(v_i, a_i) = 2$ or $\overline{d}(v_i, a_i) = 3$.

Due to distance between a_i and v_i and because S is γ_e-set, S must not contain the central vertex c. In this case, the set S must consist only of the vertices v_i. The geodesic(shortest) distances from the vertices v_i to the other vertices of H_n are as follows: $\overline{d}(v_i, a_i) \leq 3$, $\overline{d}(v_i, v_i) \leq 3$ and $\overline{d}(v_i, c) = 1$.

Accordingly, any vertex $x \in V(H_n)$ is at most 3 distance away from the vertex $v_i \in S$.

Initially, let's assume that S is only one vertex v_i. Let x be the vertex in $V(H_n) \setminus S$ such that $\overline{d}(v_i, x) = 3$. To dominate the exponentially the vertex x by set S, the number of vertices that must be in S is

$$w_s(x) = \sum_{v_i \in S} \frac{1}{2^{\overline{d}(v_i, x)}} \geq 1,$$

$$\frac{m}{2^2} \geq 1 \Rightarrow m \geq 4,$$

where $m = |S|$.

Thus, there must be at least 4 for vertices v_i in the set S. Consequently, the exponential domination of H_n is $\gamma_e(H_n) = 4$. The proof is completed. □

Definition 4.2 [11] The windmill graph $Wd(k, n)$ can be constructed by joining n copies of the complete graph K_k with a common vertex. It has $(k-1)n + 1$ vertices and $nk(k-1)/2$ edges. We display the graph $Wd(5, 4)$ in Figure 3.

Theorem 4.2 *If $Wd(k, n)$ is a windmill graph, then $\gamma_e(Wd(k, n)) = 1$.*

Proof. By the Theorem 2.3, the proof is clear. □

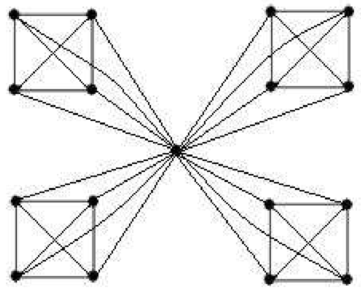

Figure 3: The Windmill graph $Wd(5,4)$.

Definition 4.3 [11] Let K_m and K_{t_i} be complete graphs on m(say $v_1, v_2, ..., v_m$) and t_i vertices, respectively. Let $t_i = 2^{r_i}$, $1 \leq i \leq m$, and $r_1 = r_2$, $r_{i+1} = r_i + 1$ for all $2 \leq i \leq m-1$ such that $K_m \uplus K_{t_i}$ has just v_i as a cut vertex, where r_i is an integer and $1 \leq i \leq m$. The resultant graph $K_m \uplus (\cup_{i=1}^{m} K_{t_i})$ is a circular necklace denoted by $CN(K_m; K_{t_1}, K_{t_2}, ..., K_{t_m})$. We display the graph $CN(K_m; K_{t_1}, K_{t_2}, ..., K_{t_m})$ in Figure 4.

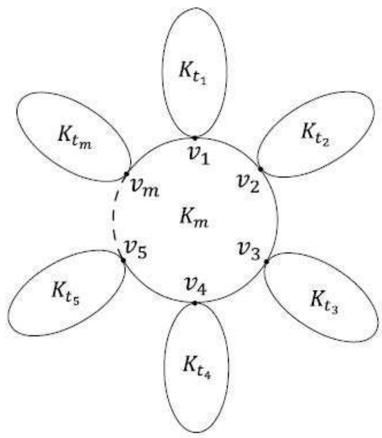

Figure 4: The Circular Necklace $CN(K_m; K_{t_1}, K_{t_2}, ..., K_{t_m})$.

Theorem 4.3 *If G is a circular necklace graph, then $\gamma_e(G) = 2$.*

Proof. By the definition of circular necklace graph, both K_m and K_{t_i} are complete graphs. Any vertex exponentially dominates all the remaining vertices in complete graph. Let v_1, v_2,..., v_m be vertices of K_m. Let S be γ_e- set of the graph G. If S consists of exactly one vertex v_x of K_m, where $1 \leq x \leq m$. Then all vertices of K_m and K_{t_x} in G are exponentially dominated. For the all remaining vertices $u \in V(G - V(K_m) - V(K_{t_x}))$, we get $\overline{d}(v_x, u) = 2$. Thus, the vertex v_x contributes 1/2 to $w_s(u)$. To exponentially dominate all the remaining vertices u, only one vertex v_i of K_m, also must be added to S. Hence, we get $\gamma_e(G) = 2$. The proof is completed. □

Definition 4.4 [15] The friendship graph F_n can be constructed by joining n copies of the cycle graph C_3 with a common vertex. We display the graph F_4 in Figure 5.

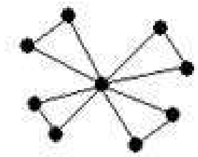

Figure 5: The Friendship graph F_4.

Theorem 4.4 *If F_n is a friendship graph, then $\gamma_e(F_n) = 1$.*

Proof. By the Theorem 2.3, the proof is clear. □

5 The Exponential Bondage Number of Some Graceful Cyclic Structure

In this section, we calculate the exponential bondage number of well-known graceful cyclic structure.

Theorem 5.1 *If H_n is a helm graph, then $b_{exp}(H_n) = 1$.*

Proof. The proof is easy to see by the Theorem 2.7. □

Theorem 5.2 *If $Wd(k, n)$ is a windmill graph, then $b_{exp}(Wd(k, n)) = 1$.*

Proof. Let c be the central vertex of $Wd(k, n)$. Clearly, $deg(c) = n(k-1)$. The removal of an edge e which is incident to c leaves a graph H. The graph H is connected graph with $(k-1)n + 1$- vertices. It is easy to see that $|V(Wd(k, n))| = |V(H)|$ and $deg(c) = n(k-1) - 1$ in the graph H. Now, we determine the exponential domination number of H. Let D be a γ_e- set of the graph H. If $D = \{c\}$, then D exponentially dominates $(k-1)n$ vertices. Thus, there remains only one vertex v exponentially dominated by D. The vertex v is the end vertex of removed edge. The vertex c contributes 1/2 to $w_D(v)$. Therefore, the vertex v or any vertex at 1/2 distance to the vertex v must be in D. Then we get $\gamma_e(H) = 2$.

Since $\gamma_e(H) > \gamma_e(Wd(k, n))$, the exponential bondage number of the windmill graph is $b_{exp}(Wd(k, n)) = 1$. The proof is completed. □

Theorem 5.3 *If G is a circular necklace graph, then*

$$b_{exp}(G) = \begin{cases} 2^{r_1} - 1, & \text{if } m > 2^{r_1}; \\ m - 1, & \text{otherwise.} \end{cases}$$

Proof. By the definition of a circular necklace graph, K_m and K_{t_i} are complete graphs and $r_1 = r_2$, where $1 \leq i \leq m$. It is the graph K_{t_1} or K_{t_2} which has the least vertices on the graph G. Let $r_1 = r_2$ be an integer value of r. Thus, $|V(K_{t_1})| = |V(K_{t_2})| = 2^r$ and $|V(K_m)| = m$. Let v_1, v_2,..., v_m and $v_i = u_{i1}, u_{i2}, ..., u_{i2^r}$ be vertices of graphs K_m and K_{t_i}, where $1 \leq i \leq m$, respectively. For every $v \in V(K_m)$, we have $deg(v) = m - 1$ in the graph K_m. Similarly, for every $u_{1j} \in V(K_{t_1})$, we have $deg(u_{1j}) = 2^r - 1$ in the graph K_{t_1}, where $1 \leq j \leq 2^r$. There are two cases depending on the degrees of the vertices of v and u_{1j}.

Case 1. $deg_{K_m}(v) > deg_{K_{t_i}}(u_{1j}) \Rightarrow m > 2^r$.
The removal of all edge incident to the vertex u_{1j} in G leaves a graph H consisting of two components. One of these is an isolated vertex and the other is connected graph $CN(K_m; K_{t_1-1}, K_{t_2}, ..., K_m)$. Thus by the Theorem 4.3 we get

$$\gamma_e(H) = \gamma_e(CN(K_m; K_{t_1-1}, K_{t_2}, ..., K_m) + 1 = 2 + 1 > \gamma_e(G).$$

Since $\gamma_e(H) > \gamma_e(G)$ is obtained, we have $b_{exp}(G) = 2^r - 1$.

Case 2. $deg_{K_m}(v) < deg_{K_{t_i}}(u_{1j}) \Rightarrow m < 2^r$.
The removal of all edge incident to vertex v in G leaves a graph H consisting of K_{t_1} and $CN(K_{m-1}; K_{t_1}, K_{t_2}, ..., K_{t_m})$. Thus by the Theorem 4.3 and 2.3 we get

$$\gamma_e(H) = \gamma_e(CN(K_{m-1}; K_{t_1}, K_{t_2}, ..., K_{t_m}) + \gamma_e(K_{t_1}) = 2 + 1 > \gamma_e(G).$$

Since $\gamma_e(H) > \gamma_e(G)$ is obtained, we have $b_{exp}(G) = m - 1$.

By combining these two cases, the exponential domination number of the circular necklace graph is

$$b_{exp}(G) = \begin{cases} 2^{r_1} - 1, & \text{if } m > 2^{r_1}; \\ m - 1, & \text{otherwise.} \end{cases}$$

The proof is completed. □

Theorem 5.4 *If F_n is a friendship graph, then $b_{exp}(F_n) = 1$.*

Proof. The vertices of F_n are two kinds. Let u and v_i be vertices of F_n, where $i \in \{1, ..., 2n\}$. Since $deg(u) = 2n$ in F_n, the vertex u is the central vertex of F_n. Furthermore, $deg(v_i) = 2$ for every $v_i \in V(F_n)$. If we remove the only one edge e_{uv_i} incident with the vertex u, then remaining graph is H.

Now we determine the exponential domination number of H. In the graph H, $deg_H(u) = 2n - 1$. Let D be a γ_e- set of the graph H. If $D = \{u\}$, then the set D exponentially dominates $(2n - 1)$- vertices. Thus, the remains only one vertex exponentially dominated by D. The vertex v_i is the end vertex of removed edge e_{uv_i}. The vertex u contributes $1/2$ to $w_D(v_i)$. Therefore, the vertex v_i or the vertex in $N(v_i) - \{u\}$ must be in D. Then we get $\gamma_e(H) = 2$.

Since $\gamma_e(H) > \gamma_e(F_n)$ is obtained, the exponential bondage number of the friendship graph is $b_{exp}(F_n) = 1$. The proof is completed. □

6 Conclusion

In this paper we determine the exact values of exponential domination and bondage numbers of a wheel helm graph, windmill graph, circular necklace and friendship graph. The problem of finding the exponential domination and bondage numbers of architecture such as Pyramid networks, Circulant networks are under investigation.

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