Nonlinear Dynamics and Systems Theory, **17** (2) (2017) 205–216

Global Dynamics of a Cooperative and Supportive Network System with Subnetwork Deactivation

P. Raja Sekhara Rao [1], K. Venkata Ratnam [2*], P. Lalitha [3] and Dipak Kumar Satpathi [4]

[1] *P. Raja Sekhara Rao, Department of Mathematics, Government Polytechnic, Addanki - 523 201, Prakasam District, A.P., India*
[2] *K. Venkata Ratnam, Department of Mathematics, Birla Institute of Technology and Science-Pilani, Hyderabad Campus, Jawahar Nagar, Hyderabad - 500078, India*
[3] *P. Lalitha, Department of Mathematics, St. Francis College for Women, Begumpet, Hyderabad, India.*
[4] *Dipak Kumar Satpathi, Department of Mathematics, Birla Institute of Technology and Science-Pilani, Hyderabad campus, Jawahar Nagar, Hyderabad - 500078, India*

Received: April 4, 2016; Revised: April 11, 2017

Abstract: In this paper a cooperative and supportive neural network, in which each neuron of main network is supported by a subnetwork of neurons, is considered. The dynamics of supportive subnetwork are subjected to some deactivation with transfer of data to the main network. Results are obtained on influence of this deactivation on global asymptotic behavior of the solutions. Numerical examples are provided to illustrate the results. The results are compared with known results.

Keywords: *neural networks; cooperative and supportive systems; deactivation; global asymptotic stability;*

Mathematics Subject Classification (2010): 34D23, 34K20, 92B20, 93D20.

* Corresponding author: mailto:vrkota@yahoo.com

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1 Introduction

Neural networks is an exciting area for research with a broad spectrum of applications [1-12, 16, 22-24]. Networks of neurons have taken many shapes as mathematical models basing on the application which they are designed for. Models of Hopfield, Cohen-Grossberg, cellular networks, recurrent networks, cooperative modular networks and spiking neural networks are such popular models to quote [1-9, 13, 20]. Along the cite, a new class of neural networks, designated as co-operative and supportive neural network (CSNN, for short) is introduced in [26]. It consists of a network of neurons called main components each of which is connected to and supported by another network of neurons called subnetwork components. The model is aimed at explaining the dynamics of systems exhibiting hierarchy that takes into account the collective capabilities of components for better performance of the system. Such systems are useful in understanding industrial information management, financial and economic systems which involve distribution and monitoring of various tasks. They are utilized to decompose complex classification tasks into simpler subtasks and puzzle them out. In particular, the network of [26] was utilized for estimation of key parameters in an infectious disease model [25]. For different models of cooperative neural networks and their applications, readers are referred to [9, 14, 15, 17, 21]. It was also claimed that the CSNN model presented in [26] was entirely new and different from all the above neural models in terms of formulation and application. Hereunder, we explain briefly the CSNN model introduced in [26], which we are going to modify and analyze further in the present study.

The model comprises two neuronal fields, say, F_x and F_y. Each neuron in F_x is denoted by x_i, $i = 1, 2, ..., n$ and is connected to other neurons x_j, $j = 1, 2, ...n$ in the same field F_x. Also each x_i is connected to r_i number of neurons in the neuronal field F_y. These are denoted by y_{i_k}, $k = 1, 2, ...r_i$, $1 \le r_i \le n$. These y_{i_k}'s support x_i in the sense that they coordinate and cooperate with it so that any task assigned to them by x_i will be attended to. The dynamics of the model are described by the following system of equations

$$x_i' = -a_i x_i + \sum_{j=1}^{n} b_{ij} f_j(x_j) + \sum_{k=1}^{r_i} c_{ii_k} g_{i_k}(x_i, y_{i_k}) + I_i, \ i = 1, 2, ..., n,$$

$$y_{i_k}' = -c_{i_k} y_{i_k} + \sum_{l=1}^{r_i} d_{i_l} h_{i_l}(y_{i_l}) + J_{i_k}, \quad k = 1, 2, ..., r_i, \ 1 \le r_i \le n. \tag{1}$$

In (1), a_i and c_{i_k} are positive constants known as decay rates, I_i, J_{i_k} are exogenous inputs and b_{ij}, d_{i_l} are the synaptic connection weights which may be real or complex constants. c_{ii_k} is the rate of distribution of information between x_i and y_{i_k}. The functions f_i, g_{i_k} and h_{i_k} are the neuronal output response functions and are more commonly known as the signal functions.

Besides a study of qualitative behavior of the system, several modifications of the CSNN model (1) are suggested and left as open problems in [26] for enthusiastic readers. Present authors have studied two such modified models [18, 19] of (1) that increase its applicability. Extending this view point, we shall address one more modification of (1) in our present study. Motivation for this stems from the following observations.

The second equation of (1) contains no term that includes x_i. That means, the subcomponents y_{i_k} work independently of x_i, supply information to x_i and do not bother whether their contribution is fully utilized or are contributing more than what is required. At the same time for the main component x_i there is a need to check this contribution from y_{i_k} either in terms of money or in terms of easing out y_{i_k} from unnecessary production beyond what is required. Thus, there is a need to check the activity of sub-components. Also when y_{i_k} depends mainly on x_i for its survival or activity there should be a term that reflects the interaction between x_i and y_{i_k}. Such a term in second equation of (1) represents: (i) physical transfer of subcomponents in case of ancillary manufacturing units; (ii) removal of data/information after transferring it to main component ('cut and paste' instead of 'copy and paste') or (iii) deactivation of subcomponents as soon as the required data is supplied.

Another argument runs as follows. System (1) reflects how x_i receives information from y_{i_k} represented by $g_{i_k}(x_i, y_{i_k})$ but not how it is sent from y_{i_k} - term at receiver's end but not at giver's end. It may also be understood as that y_{i_k} keeps a copy of what ever information/data sent to x_i. This may not be possible in all cases. We can not keep copies of physical quantities such as spare parts, components, etc., of the main item in a manufacturing unit. Even in data processing systems, retention of data at too many places may raise security problems. Absence of a term involving x_i may also infer that the requirements of x_i are insignificant when compared to the quantum of work done by y_{i_k} for all its purposes.

In order to incorporate these, we introduce a term which may be utilized for deactivating or resting of y_{i_k} once its task is done. Introduction of such term into the second equation modifies (1) to

$$\begin{aligned} x_i' &= -a_i x_i + \sum_{j=1}^{n} b_{ij} f_j(x_j) + \sum_{k=1}^{r_i} c_{ii_k} g_{i_k}(x_i, y_{i_k}) + I_i, \\ y_{i_k}' &= -c_{i_k} y_{i_k} + \sum_{l=1}^{r_i} d_{i_l} h_{i_l}(y_{i_l}) - \overline{c}_{ii_k} \overline{g}_{i_k}(x_i, y_{i_k}) + J_{i_k}. \end{aligned} \tag{2}$$

In (2), the term $\bar{c}_{ii_k}\bar{g}_{i_k}(x_i, y_{i_k})$ denotes the resting or deactivating component for the subsystem. Here each $\bar{c}_{ii_k} > 0$ may be called the rate of de-activation of y_{i_k} by x_i. The functional term $\bar{g}_{i_k}(x_i, y_{i_k})$ denotes how the deactivation takes place. System (2) is Model I in [26] which is left open for exploration. Our task in this paper shall be to study the influence of this new term on the dynamics of the system (1). Is this term going to pacify sub-components or influence the entire network will be a question of utmost importance. How to manage its influence using the system parameters may be reasonable task to take up. This we study in terms of stability of equilibrium patterns of the system (2) in the light of existing results on (1).

The paper is organized as follows. In Section 2, we provide conditions for existence and uniqueness of solutions, equilibria for system (2) — basic properties of any such dynamical system. Results on global asymptotic stability of equilibria are obtained in Section 3. The results are compared with earlier results on (1). Examples are provided for illustration of results. Finally a discussion follows in Section 4.

2 Basic Properties

In this section, we explain basic properties of (2) such as existence of solutions along with equilibria. This is to be done with appropriate assumptions or restrictions on system parameters and nonlinear functions. To begin with we assume that the response functions f_j, g_{ii_k}, h_{i_k} and $\overline{g}_{ii_k}$ satisfy local Lipschitz conditions given by

$$\begin{aligned}
\| g_{i_k}(x_i, y_{i_k}) - g_{i_k}(\overline{x}_i, \overline{y}_{i_k}) \| &\leq M_{1i_k}|y_{i_k} - \overline{y}_{i_k}| + M_{2i_k}|x_i - \overline{x}_i|, &(3)\\
\| \overline{g}_{i_k}(x_i, y_{i_k}) - \overline{g}_{i_k}(\overline{x}_i, \overline{y}_{i_k}) \| &\leq \overline{M}_{1i_k}|y_{i_k} - \overline{y}_{i_k}| + \overline{M}_{2i_k}|x_i - \overline{x}_i|, &(4)\\
|f_j(x_j) - f_j(\overline{x}_j)| &\leq p_j|x_j - \overline{x}_j|, &(5)\\
|h_{i_l}(y_{i_l}) - h_{i_l}(\overline{y}_{i_l})| &\leq q_{i_l}|y_{i_l} - \overline{y}_{i_l}|, &(6)
\end{aligned}$$

where M_{1i_k}, M_{2i_k}, p_j and q_{i_l} are positive constants. Then from the theory of differential equations, it is evident that solutions for (2) do exist, are unique and continuable in their maximal intervals of existence.

Since the stability of a system is understood in terms of the stability of its equilibria, we verify whether (2) provides scope for equilibrium patterns to exist. The following result provides one such set of sufficient conditions.

Theorem 2.1. *Let a_i and c_{i_k} be positive numbers such that*

$$\begin{aligned}
\frac{1}{a_i}\sum_{j=1}^{n}|b_{ij}|p_j + \frac{1}{a_i}\sum_{k=1}^{r_i}|c_{ii_k}|M_{2i_k} + \frac{1}{c_{i_k}}|\overline{c}_{ii_k}|\overline{M}_{2i_k} &< 1, \ i = 1, 2, ..., n.\\
\frac{1}{c_{i_k}}\sum_{l=1}^{r_i}|d_{i_l}|q_{i_l} + \frac{1}{a_i}\sum_{k=1}^{r_i}|c_{ii_k}|M_{1i_k} + \frac{1}{c_{i_k}}|\overline{c}_{ii_k}|\overline{M}_{1i_k} &< 1, \ 1 \leq r_i \leq n. &(7)
\end{aligned}$$

Then the system (2) has a unique equilibrium solution $(x_i^*, y_{i_k}^*)$ for each i, k.

Since several results are available in literature on similar systems, we omit the proof of the above result here and refer the interested readers to [3],[26] for a line of proof based on contraction mapping principle.

Since $(x_i^*, y_{i_k}^*)$ is a constant solution of (2), we have

$$\begin{aligned}
x_i^{*'} = 0 &= -a_i x_i^* + \sum_{j=1}^{n} b_{ij} f_j(x_i^*) + \sum_{k=1}^{r_i} c_{ii_k} g_{i_k}(x_i^*, y_{i_k}^*) + I_i,\\
y_{i_k}^{*'} = 0 &= -c_{i_k} y_{i_k}^* + \sum_{l=1}^{r_i} d_{i_l} h_{i_l}(y_{i_l}^*) - \overline{c}_{ii_k}\overline{g}_{i_k}(x_i^*, y_{i_k}^*) + J_{i_k}. &(8)
\end{aligned}$$

We shall now take up the aspect of stability of equilibrium pattern of (2), assuming its existence tacitly.

3 Global Stability Results

In this section we study the influence of deactivation term on the stability of the system. Whether its presence will increase strain on parameters or reduce it when compared to (1) — is the main concern.

Before we present our results, we rearrange system (2) as follows. Utilizing (8) in (2), we get

$$
\begin{aligned}
(x_i - x_i^*)' &= -a_i(x_i - x_i^*) + \sum_{j=1}^{n} b_{ij}[f_j(x_j) - f_j(x_j^*)] + \sum_{k=1}^{r_i} c_{ii_k}[g_{i_k}(x_i, y_{i_k}) - g_{i_k}(x_i^*, y_{i_k}^*)], \\
(y_{i_k} - y_{i_k}^*)' &= -c_{i_k}(y_{i_k} - y_{i_k}^*) + \sum_{l=1}^{r_i} d_{i_l}[h_{i_l}(y_{i_l}) - h_{i_l}(y_{i_l}^*)] - \overline{c}_{ii_k}[\overline{g}_{i_k}(x_i, y_{i_k}) - \overline{g}_{i_k}(x_i^*, y_{i_k}^*)].
\end{aligned} \tag{9}
$$

We shall establish our first result now.

Theorem 3.1. *Assume that the parameters of the system (2) satisfy the following conditions:*

$$
\begin{aligned}
a_i &> \sum_{j=1}^{n} |b_{ji}| p_i + \sum_{k=1}^{r_i} |c_{ii_k}| M_{2i_k} + \sum_{k=1}^{r_i} |\overline{c}_{ii_k}| \overline{M}_{2i_k}, \\
c_{i_k} &> \sum_{l=1}^{r_i} |d_{i_l}| q_{i_l} + |c_{ii_k}| M_{1i_k} + |\overline{c}_{ii_k}| \overline{M}_{1i_k}.
\end{aligned}
$$

Assume further that conditions (3) - (6) on response functions hold. Then the equilibrium $(x_i^*, y_{i_k}^*)$ is globally asymptotically stable in the sense that all solutions of (2) satisfy the convergence requirement

$$\lim_{t\to\infty} y_{i_k} \to y_{i_k}^*, \quad \lim_{t\to\infty} x_i \to x_i^*.$$

Proof. We consider the functional

$$V(t) = \sum_{i=1}^{n} \Big\{ |x_i - x_i^*| + \sum_{k=1}^{r_i} |y_{i_k} - y_{i_k}^*| \Big\}. \tag{10}$$

The upper right derivative of V along the solutions of (2) utilizing (9) may be given by

$$
\begin{aligned}
D^+V(t) \leq \; & \sum_{i=1}^{n} \Big\{ -a_i|x_i - x_i^*| + \sum_{j=1}^{n} |b_{ij}||f_j(x_j) - f_j(x_j^*)| \\
& + \sum_{k=1}^{r_i} |c_{ii_k}||g_{i_k}(x_i, y_{i_k}) - g_{i_k}(x_i^*, y_{i_k}^*)| \\
& + \sum_{k=1}^{r_i} \Big[-c_{i_k}|y_{i_k} - y_{i_k}^*| + \sum_{l=1}^{r_i} |d_{i_l}||h_{i_l}(y_{i_l}) - h_{i_l}(y_{i_l}^*)| \Big] \\
& - \sum_{k=1}^{r_i} |\overline{c}_{ii_k}||\overline{g}_{i_k}(x_i, y_{i_k}) - \overline{g}_{i_k}(x_i^*, y_{i_k}^*)| \Big\}.
\end{aligned}
$$

Then,

$$\begin{aligned} D^+V(t) \;\le\; & \sum_{i=1}^{n}\Big\{ -a_i|x_i - x_i^*| + \sum_{j=1}^{n}|b_{ij}|p_j|x_j - x_j^*| \\ & + \sum_{k=1}^{r_i}|c_{ii_k}|M_{2i_k}|x_i - x_i| + \sum_{k=1}^{r_i}|c_{ii_k}|M_{1i_k}|y_{i_k} - y_{i_k}^*| \\ & + \sum_{k=1}^{r_i}\Big[-c_{i_k}|y_{i_k} - y_{i_k}^*| + \sum_{l=1}^{r_i}|d_{i_l}|q_{i_l}|y_{i_l} - y_{i_l}^*|\Big] \\ & + \sum_{k=1}^{r_i}|\overline{c}_{ii_k}|[\overline{M}_{1i_k}|y_{i_k} - y_{i_k}^*| + \overline{M}_{2i_k}|x_i - x_i^*|]\Big\}, \end{aligned}$$

using (3) - (6) on the response functions. Thus,

$$\begin{aligned} D^+V(t) \;\le\; & -\sum_{i=1}^{n}\Big\{[a_i - \sum_{j=1}^{n}|b_{ji}|p_i - \sum_{k=1}^{r_i}|c_{ii_k}|M_{2i_k} - \sum_{k=1}^{r_i}|\overline{c}_{ii_k}|\overline{M}_{2i_k}]\;|x_i - x_i^*| \\ & + \sum_{k=1}^{r_i}[c_{i_k} - \sum_{l=1}^{r_i}|d_{i_l}|q_{i_l} - |c_{ii_k}|M_{1i_k} - |\overline{c}_{ii_k}|\overline{M}_{1i_k}]\;|y_{i_k} - y_{i_k}^*|\Big\} \\ \le\; & -\widetilde{A}V < 0, \qquad \text{by hypotheses,} \end{aligned}$$

where $\widetilde{A} = \min\big\{\overline{A}, \overline{B}\big\}$, and

$$\begin{aligned} \overline{A} &= \left\{\min\left[a_i - \sum_{j=1}^{n}|b_{ji}|p_j - \sum_{k=1}^{r_i}|c_{ii_k}|M_{2i_k} - \sum_{k=1}^{r_i}|\overline{c}_{ii_k}|\overline{M}_{2i_k}\right] > 0,\ 1 \le i \le n.\right\} \\ \overline{B} &= \left\{\min\left[c_{i_k} - \sum_{l=1}^{r_i}|d_{i_l}|q_{i_l} - |c_{ii_k}|M_{1i_k} - |\overline{c}_{ii_k}|\overline{M}_{1i_k}\right] > 0,\ 1 \le k \le r_i,\ 1 \le i \le n.\right\} \end{aligned}$$

Thus, $D^+V(t) + \widetilde{A}V(t) < 0$. Integrating on both sides with respect to t from 0 to t, we have $V(t) < V(0)e^{-\widetilde{A}t} \to 0$ for large t. The conclusion follows from the definition of V.

We shall present yet another result on global asymptotic stability of equilibrium pattern of (2) using a different Lyapunov functional providing one more set of sufficient conditions on parameters of the system.

Theorem 3.2. *Assume that the conditions (3)–(6) on response functions hold. Furthermore the parameters satisfy the following inequalities*

$$\begin{aligned} a_i \;>\; & \frac{1}{2}\sum_{j=1}^{n}|b_{ij}|p_j + \frac{1}{2}\sum_{j=1}^{n}|b_{ji}|p_i + \frac{1}{2}\sum_{k=1}^{r_i}|c_{ii_k}|M_{2i_k} \\ & + \frac{1}{2}\sum_{k=1}^{r_i}|c_{ii_k}|M_{1i_k} + \frac{1}{2}\sum_{k=1}^{r_i}|\overline{c}_{ii_k}|\overline{M}_{2i_k}, \\ c_{i_k} \;>\; & \sum_{l=1}^{r_i}|d_{i_l}|q_{i_l} + \frac{1}{2}|c_{ii_k}|M_{1i_k} + \frac{1}{2}|\overline{c}_{ii_k}|\overline{M}_{2i_k} + |\overline{c}_{ii_k}|\overline{M}_{1i_k}, \end{aligned} \tag{11}$$

for all i and i_k. Then the equilibrium pattern of (2) is globally asymptotically stable.

Proof. We consider the functional

$$V(t) = \sum_{i=1}^{n} \left\{ \frac{(x_i(t) - x_i^*)^2}{2} + \sum_{k=1}^{r_i} \frac{(y_{i_k} - y_{i_k}^*)^2}{2} \right\}.$$

The derivative of V along the solutions of (1.2), using (3.1), is given by

$$\begin{aligned}
V'(t) &= \sum_{i=1}^{n} \Big[(x_i(t) - x_i^*)(x_i'(t) - x_i^{*'}) + \sum_{k=1}^{r_i} (y_{i_k}(t) - y_{i_k}^*)(y_{i_k}'(t) - y_{i_k}^{*'}) \Big] \\
&= \sum_{i=1}^{n} \Big[\Big[-a_i(x_i(t) - x_i^*)^2 + (x_i(t) - x_i^*) \sum_{j=1}^{n} b_{ij}(f_j(x_j) - f_j(x_j^*)) \\
&\quad + (x_i(t) - x_i^*) \sum_{k=1}^{r_i} c_{ii_k}(g_{i_k}(x_i, y_{i_k}) - g_{i_k}(x_i^*, y_{i_k}^*)) \Big] \\
&\quad + \sum_{k=1}^{r_i} \Big[-c_{i_k}(y_{i_k}(t) - y_{i_k}^*)^2 + (y_{i_k}(t) - y_{i_k}^*) \sum_{l=1}^{r_i} d_{i_l}[h_{i_l}(y_{i_l}) - h_{i_l}(y_{i_l}^*)] \\
&\quad - (y_{i_k}(t) - y_{i_k}^*)\overline{c}_{ii_k}\Big(\overline{g}_{i_k}(x_i, y_{i_k}) - \overline{g}_{i_k}(x_i^*, y_{i_k}^*)\Big)\Big]\Big] \\
&\le \sum_{i=1}^{n} \Big[-a_i(x_i(t) - x_i^*)^2 + |x_i(t) - x_i^*| \sum_{j=1}^{n} |b_{ij}| p_j |x_j(t) - x_j^*| \\
&\quad + |x_i(t) - x_i^*| \sum_{k=1}^{r_i} |c_{ii_k}| \Big[M_{2i_k}|x_i - x_i^*| + M_{1i_k}|y_{i_k} - y_{i_k}^*| \Big] \\
&\quad + \sum_{k=1}^{r_i} \Big[-c_{i_k}(y_{i_k}(t) - y_{i_k}^*)^2 + |y_{i_k}(t) - y_{i_k}^*| \sum_{l=1}^{r_i} |d_{i_l}| q_{i_l} |y_{i_l} - y_{i_l}^*| \\
&\quad + |y_{i_k} - y_{i_k}^*||\overline{c}_{ii_k}|\Big(\overline{M}_{2i_k}|x_i - x_i^*| + \overline{M}_{1i_k}|y_{i_k} - y_{i_k}^*|\Big)\Big]\Big],
\end{aligned}$$

utilizing (3)-(6). Employing the inequality $ab \le \frac{a^2+b^2}{2}$ and rearranging the terms we get

$$\begin{aligned}
V'(t) &\le \sum_{i=1}^{n} \Big[-a_i(x_i(t) - x_i^*)^2 + \frac{1}{2} \sum_{j=1}^{n} |b_{ij}| p_j \Big[(x_i(t) - x_i^*)^2 + (x_j - x_j^*)^2 \Big] \\
&\quad + \frac{1}{2} \sum_{k=1}^{r_i} |c_{ii_k}| M_{2i_k}(x_i - x_i^*)^2 \\
&\quad + \frac{1}{2} \sum_{k=1}^{r_i} |c_{ii_k}| M_{1i_k} \Big[(y_{i_k} - y_{i_k}^*)^2 + (x_i - x_i^*)^2 \Big]\Big] \\
&\quad - \sum_{k=1}^{r_i} \Big[c_{i_k} - \frac{1}{2} \sum_{l=1}^{r_i} |d_{i_l}| q_{i_l} - \frac{1}{2} \sum_{k=1}^{r_i} |d_{i_k}| q_{i_k} \Big] (y_{i_k} - y_{i_k}^*)^2 \\
&\quad + \frac{1}{2} \sum_{k=1}^{r_i} |\overline{c}_{ii_k}| \overline{M}_{2i_k}(x_i - x_i^*)^2 + \frac{1}{2} |\overline{c}_{ii_k}| \overline{M}_{2i_k}(y_{i_k} - y_{i_k}^*)^2 \\
&\quad + |\overline{c}_{ii_k}| \overline{M}_{1i_k}(y_{i_k} - y_{i_k}^*)^2 \Big].
\end{aligned}$$

Thus,

$$\begin{aligned} V'(t) \leq & -\sum_{i=1}^{n}\Big[a_i - \frac{1}{2}\sum_{j=1}^{n}|b_{ij}|p_j - \frac{1}{2}\sum_{j=1}^{n}|b_{ji}|p_i - \frac{1}{2}\sum_{k=1}^{r_i}|c_{ii_k}|M_{2i_k} \\ & -\frac{1}{2}\sum_{k=1}^{r_i}|c_{ii_k}|M_{1i_k} - \frac{1}{2}\sum_{k=1}^{r_i}\overline{c}_{ii_k}\overline{M}_{2i_k}\Big](x_i - x_i^*)^2 \\ & -\sum_{i=1}^{n}\sum_{k=1}^{r_i}\Big[c_{i_k} - \frac{1}{2}\sum_{l=1}^{r_i}|d_{i_l}|q_{i_l} - \frac{1}{2}\sum_{l=1}^{r_i}|d_{i_l}|q_{i_l} - \frac{1}{2}|c_{ii_k}|M_{1i_k} \\ & -\frac{1}{2}|\overline{c}_{ii_k}|\overline{M}_{2i_k} - |\overline{c}_{ii_k}|\overline{M}_{1i_k}\Big](y_{i_k} - y_{i_k}^*)^2. \end{aligned}$$

Then by assumptions, V' is negative definite, and hence, the conclusion follows employing standard arguments as in earlier case (e.g., [3, 19, 26]).

We shall now provide examples to illustrate these results and establish the criteria provided in these two results are independent.

Example 3.3. Consider

$$\begin{aligned} \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = & -\begin{pmatrix} 1.49\ x_1 \\ 3.79\ x_2 \end{pmatrix} + \begin{pmatrix} 0.32 & 0.43 \\ 0.18 & 0.24 \end{pmatrix}\begin{pmatrix} f_1(x_1) \\ f_2(x_2) \end{pmatrix} \\ & + \begin{pmatrix} 0.25 & 0.53 \\ 0.85 & 0.95 \end{pmatrix}\begin{pmatrix} g_{1_1}(x_1, y_{1_1}) & g_{2_1}(x_2, y_{2_1}) \\ g_{1_2}(x_1, y_{1_2}) & g_{2_2}(x_2, y_{2_2}) \end{pmatrix} + \begin{pmatrix} I_1 \\ I_2 \end{pmatrix}, \end{aligned}$$

$$\begin{aligned} \begin{pmatrix} y_{1_1}' \\ y_{1_2}' \end{pmatrix}, = & -\begin{pmatrix} 1.25\ y_{1_1} \\ 1.02\ y_{1_2} \end{pmatrix} + \begin{pmatrix} 0.5 & 0.25 \\ 0.3 & 0.1 \end{pmatrix}\begin{pmatrix} h_{1_1}(y_{1_1}) \\ h_{1_2}(y_{1_2}) \end{pmatrix} + \begin{pmatrix} J_{1_1} \\ J_{1_2} \end{pmatrix} \\ & -\begin{pmatrix} 0.15\ g_{1_1}(x_1, y_{1_1}) \\ 0.05\ g_{1_2}(x_1, y_{1_2}) \end{pmatrix}, \end{aligned}$$

$$\begin{aligned} \begin{pmatrix} y_{2_1}' \\ y_{2_2}' \end{pmatrix} = & -\begin{pmatrix} 2.01\ y_{2_1} \\ 1.72\ y_{2_2} \end{pmatrix} + \begin{pmatrix} 0.25 & 0.12 \\ 0.15 & 0.05 \end{pmatrix}\begin{pmatrix} h_{2_1}(y_{2_1}) \\ h_{2_2}(y_{2_2}) \end{pmatrix} + \begin{pmatrix} J_{2_1} \\ J_{2_2} \end{pmatrix} \\ & -\begin{pmatrix} 0.75\ g_{2_1}(x_2, y_{2_1}) \\ 0.53\ g_{2_2}(x_2, y_{2_2}) \end{pmatrix}. \end{aligned}$$

Let $f_i(x_i) = tanh(x_i), h_{i_k} = tanh(y_{i_k})$ and $g_{i_k}(x_i, y_{i_k}) = x_i + y_{i_k}$. Then $p_j = q_{i_k} = M_{1i_k} = M_{2i_k} = 1$. Choose $I_i = 10,\ J_{i_k} = 10,\ i = 1, 2,\ k = 1, 2.$

For the above system, the equilibrium pattern is given by (11.68, 4.33, 6.67, 9.40, 2.69, 3.89). It may be seen that all the conditions of Theorem 3.1 are satisfied, and hence,the equilibrium pattern of the system is globally asymptotically stable by virtue of Theorem 3.1. Also some of the parametric conditions of Theorem 3.2 are violated, it can not be applied here.

Example 3.4. Consider

$$\begin{aligned} \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = & -\begin{pmatrix} 2.45\ x_1 \\ 3.85\ x_2 \end{pmatrix} + \begin{pmatrix} 0.4 & 0.6 \\ 0.7 & 1.3 \end{pmatrix}\begin{pmatrix} f_1(x_1) \\ f_2(x_2) \end{pmatrix} \\ & + \begin{pmatrix} 0.6 & 0.3 \\ 0.8 & 0.5 \end{pmatrix}\begin{pmatrix} g_{1_1}(x_1, y_{1_1}) & g_{2_1}(x_2, y_{2_1}) \\ g_{1_2}(x_1, y_{1_2}) & g_{2_2}(x_2, y_{2_2}) \end{pmatrix} + \begin{pmatrix} I_1 \\ I_2 \end{pmatrix}, \end{aligned}$$

$$\begin{aligned}\begin{pmatrix} y'_{1_1} \\ y'_{1_2} \end{pmatrix} &= -\begin{pmatrix} 1.4\ y_{1_1} \\ 1.7\ y_{1_2} \end{pmatrix} + \begin{pmatrix} 0.2 & 0.4 \\ 0.4 & 0.6 \end{pmatrix} \begin{pmatrix} h_{1_1}(y_{1_1}) \\ h_{1_2}(y_{1_2}) \end{pmatrix} + \begin{pmatrix} J_{1_1} \\ J_{1_2} \end{pmatrix} \\ &\quad - \begin{pmatrix} 0.3\ g_{1_1}(x_1, y_{1_1}) \\ 0.6\ g_{1_2}(x_1, y_{1_2}) \end{pmatrix},\end{aligned}$$

$$\begin{aligned}\begin{pmatrix} y'_{2_1} \\ y'_{2_2} \end{pmatrix} &= -\begin{pmatrix} 1.65\ y_{2_1} \\ 2.5\ y_{2_2} \end{pmatrix} + \begin{pmatrix} 0.2 & 0.4 \\ 0.4 & 0.6 \end{pmatrix} \begin{pmatrix} h_{2_1}(y_{2_1}) \\ h_{2_2}(y_{2_2}) \end{pmatrix} + \begin{pmatrix} J_{2_1} \\ J_{2_2} \end{pmatrix} \\ &\quad - \begin{pmatrix} 0.4\ g_{2_1}(x_2, y_{2_1}) \\ 0.8\ g_{2_2}(x_2, y_{2_2}) \end{pmatrix}.\end{aligned}$$

Choosing $f_i(x_i) = tanh(x_i), h_{i_k} = tanh(y_{i_k})$ and $g_{i_k}(x_i, y_{i_k}) = x_i + y_{i_k}$, we have $p_j = q_{i_k} = M_{1i_k} = M_{2i_k} = 1$. Let $I_i = 10,\ J_{i_k} = 10,\ i = 1, 2,\ k = 1, 2$.

The equilibrium pattern of the above system is given by (6.21, 4.38, 5.51, 4.06, 4.74, 2.18). Clearly, all the conditions of Theorem 3.2 are satisfied here while some of the parametric conditions in Theorem 3.1 are violated. Thus, the unique equilibrium pattern of system is stable by virtue of Theorem 3.2.

It may be concluded from Examples 3.3 and 3.4 that Theorems 3.1 and 3.2 are independent of each other. The examples are simulated using ODE23 of MATLAB and Figures 1 and 2 picturize our theoretical conclusions. We now consider the case where all

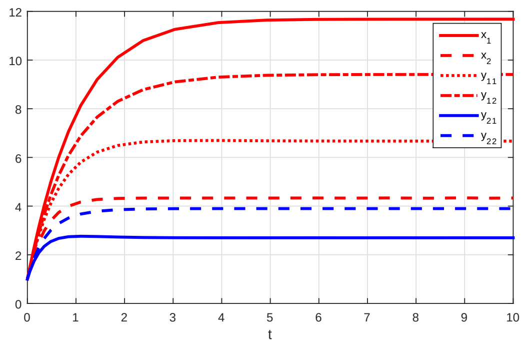

Figure 1: Behaviour of solutions in Example 3.3.

contribution of y_{i_k} has been completely received and utilized by x_i as it is. That means, we assume that $\overline{c}_{ii_k}\overline{g}_{i_k}(x_i, y_{i_k}) \equiv c_{ii_k} g_{i_k}(x_i, y_{i_k})$ for all x_i and y_{i_k}. Our next result studies the global stability of equilibrium in this case.

Theorem 3.5. *Assume that the parameters of the system satisfy the following conditions:*

$$a_i - \sum_{j=1}^{n} |b_{ji}| p_i > 0, \quad c_{i_k} - \sum_{l=1}^{r_i} |d_{i_l}| q_{i_l} > 0,$$

for all i and i_k and the response functions satisfy (3)-(6). Then the equilibrium pattern of (2) is globally asymptotically stable.

Proof. We employ the same functional as in Theorem 3.1,

$$V(t) = \sum_{i=1}^{n} \Big[|x_i - x_i^*| + \sum_{k=1}^{r_i} |y_{i_k} - y_{i_k}^*| \Big]. \qquad (12)$$

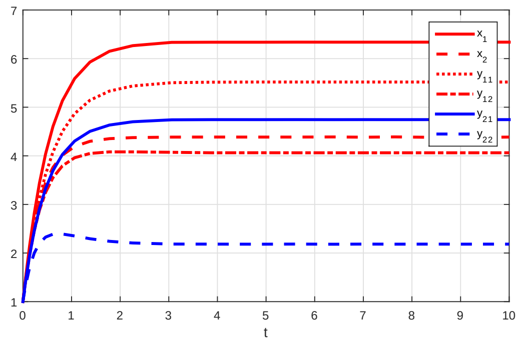

Figure 2: Solutions converging to equilibrium values in Example 3.4.

Then we have

$$\begin{aligned} D^+V(t) \quad &\leq \quad \sum_{i=1}^{n}\Big[-a_i|x_i-x_i^*|+\sum_{j=1}^{n}|b_{ij}||f_j(x_j)-f_j(x_j^*)| \\ &+ \quad \sum_{k=1}^{r_i}|c_{ii_k}||g_{i_k}(x_i,y_{i_k})-g_{i_k}(x_i^*,y_{i_k}^*)|+\sum_{k=1}^{r_i}\Big[-c_{i_k}|y_{i_k}-y_{i_k}^*| \\ &+ \quad \sum_{l=1}^{r_i}|d_{i_l}||h_{i_l}(y_{i_l})-h_{i_l}(y_{i_l}^*)|-\sum_{k=1}^{r_i}|\overline{c}_{ii_k}||\overline{g}_{i_k}(x_i,y_{i_k})-\overline{g}_{i_k}(x_i^*,y_{i_k}^*)|\Big]\Big] \\ &\leq \quad \sum_{i=1}^{n}\Big[-a_i|x_i-x_i^*|+\sum_{j=1}^{n}|b_{ij}|p_j|x_j-x_j^*| \\ &+ \quad \sum_{k=1}^{r_i}\Big[-c_{i_k}|y_{i_k}-y_{i_k}^*|+\sum_{l=1}^{r_i}|d_{i_l}|q_{i_l}|y_{i_l}-y_{i_l}^*|\Big]\Big]. \end{aligned}$$

Therefore,

$$D^+V(t) \leq -\sum_{i=1}^{n}\Big[[a_i-\sum_{j=1}^{n}|b_{ji}|p_i]\;|x_i-x_i^*|+\sum_{k=1}^{r_i}[c_{i_k}-\sum_{l=1}^{r_i}|d_{i_l}|q_{i_l}]\;|y_{i_k}-y_{i_k}^*|\Big]$$

Negative definiteness of D^+V follows from assumptions on parameters. The rest of the argument is similar to that of Theorem 3.1, and thus, omitted.

Remark 3.6. Two types of approaches are possible here. For system (1), where the dynamics of subnetwork neurons y_{i_k} (i.e., second equation of (1)) do not include terms of main components x_i, the subnetworks are allowed to converge first, x_i waits to receive this contribution and then starts working on its own for a convergence - as worked out in Theorem 4.1 of [26]. Secondly, the case where x_i works together with y_{i_k} and interacts continuously with them for a simultaneous convergence was discussed in Corollary 2.3 of [19]. First situation may be called as a 'serial processing' – elongates the convergence process but the strain on the parameters is considerably less when compared to that in second situation which may be termed as a 'parallel processing'.

It may be noticed from Theorem 3.5 here that the strain on parameters is very less as compared to that of Theorem 4.1 of [26] at the same time allows interactions of y_{i_k}'s with x_i as Corollary 2.3 of [19]. Thus, influence of deactivation term $\overline{g}_{i_k}(x_i, y_{i_k})$ in second equation is clear. This also indicates that when the subcomponents contribute exactly what their main components require and the main components receive what they need with a proper interaction with their subcomponents then the system parameters are strained less and thus, paving way for a better performance of the system.

4 Discussion

In the present paper, we studied the influence of deactivation dynamics introduced into the supportive subnetwork of a cooperative and supportive network system. We established sufficient conditions for global asymptotic stability of the equilibrium pattern. Examples are provided to establish that the criteria presented are independent of each other. It was assumed in [26] that the subnetworks of the main group always support it. If the subnetwork is an ancillary unit established independently of the main system (but always supports it) and survives on its own (has independent, own dynamics – second equation of system(1)), then main system has no burden. In case if the subnetwork is an ancillary unit that survives only because of main network or is an integral part of the main system which needs to be defunct as soon as the task of main network is finished either to reduce or to avoid unnecessary use of y_{i_k}'s, then system (2) comes into play and the study in this paper becomes very relevant and useful. A look at Theorems 3.1 and 3.2 shows that the parameters have to be strained much when the contribution from subnetwork is not utilized as it is or is not known to be the same as that required by main network. On the other hand, the strain on parameters is much less for systems which utilize contributions of its subnetworks completely or equivalently, the subnetworks are contributing exactly what their main group is expecting from them. This is what Theorem 3.5 says. Thus, systems with perfect coordination and cooperation among groups perform well with less strain on constituent components and resources.

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