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Existence of Positive Periodic Solutions for a Second-Order Nonlinear Neutral Differential Equation by the Krasnoselskii's Fixed Point Theorem

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Abstract: This work is devoted to the study of the existence of positive periodic solutions of the second order nonlinear neutral differential equation

$$\frac{d^2}{dt^2}x(t) + p(t)\frac{d}{dt}x(t) + q(t)x(t) = \frac{d^2}{dt^2}Q(t, x(t-\tau(t))) + f(t, h(x(t)), g(x(t-\tau(t)))).$$

The method used here is one of the most efficient techniques for studying this type of equations since it combines some useful properties of Green's function together with Krasnoselskii's fixed point theorem.

Keywords: *positive periodic solutions; nonlinear neutral differential equations; fixed point theorem.*

Mathematics Subject Classification (2010): 34K13, 34A34, 34K30, 34L30.

1 Introduction

In this work we are essentially interested in the study of the existence of positive periodic solutions for certain classes of second order nonlinear neutral differential equations which are ubiquitous in different scientific disciplines and arise specially in beam theory, viscoelastic and inelastic flows and electric circuits.

There is a sizeable literature related to this topic, for instance in the middle of the previous century, the existence of solutions of differential equations was extensively studied by many investigators, see, for example, the papers and books [1]- [9], [11], [12]. During

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the last two decades, there has been an increasing activity in the study of periodic problems of second-order nonlinear neutral differential equations (see [1]- [3], [9], [11], [12] and references therein).

Some mathematicians used transformation in order to reduce the equation into more simple equation or system of equations or used synthetic division, others gave the solution in a series form which converges to the exact solution and some of them dealt with second-order nonlinear neutral differential equations by using some numerical techniques such as Ritz method, finite difference method, finite element method, cubic spline method and multiderivative method. In this paper, these usual methods may seem inefficient to establish the existence of positive periodic solutions of the second-order nonlinear neutral differential equations

$$\frac{d^2}{dt^2}x(t)+p(t)\frac{d}{dt}x(t)+q(t)x(t)=\frac{d^2}{dt^2}Q(t,x(t-\tau(t)))+f(t,h\left(x(t)\right),g\left(x(t-\tau(t))\right)), \quad (1)$$

where p, q are positive continuous real-valued functions. The functions $Q:\mathbb{R}\times\mathbb{R}\longrightarrow\mathbb{R}$, $h,g:\mathbb{R}\longrightarrow\mathbb{R}$ and $f:\mathbb{R}\times\mathbb{R}\times\mathbb{R}\longrightarrow\mathbb{R}$ are continuous with respect to their arguments. Our ideas are inspired by the ones given in the recent papers [1, 3, 9, 11, 12], we will convert the nonlinear neutral differential equation into an integral equation before using the Krasnoselskii's fixed point theorem.

This paper is organized as follows. In the next section, we start by providing some background definitions, lemmas and some preliminary results, then we give Green's function of a second order differential equation and some of their useful properties. We introduce Green's functions of a second order differential equation and we show that the solution of a given equation can be explicitly expressed in terms of Green's function of the corresponding homogeneous equation. Next, we present the inversion of (1) and we assert without proof the well-known Krasnoselskii's fixed point theorem which will be useful in what follows.

Finally, in the last section, we study the neutral functional differential equation (1) and present an existence result for positive periodic solutions for this equation by combining some properties of Green's function together with Krasnoselskii fixed point theorem.

2 Preliminaries

For $T>0$, let P_T be the set of all continuous scalar functions x, periodic in t of period T. Then $(P_T,\|.\|)$ is a Banach space with the supremum norm

$$\|x\|=\sup_{t\in\mathbb{R}}\left|x\left(t\right)\right|=\sup_{t\in[0,T]}\left|x\left(t\right)\right|.$$

Since we are searching for the existence of periodic solutions for equation (1), it is natural to assume that

$$p\left(t+T\right)=p\left(t\right),\ q\left(t+T\right)=q\left(t\right),\ \tau\left(t+T\right)=\tau\left(t\right), \quad (2)$$

with τ being scalar function, continuous, and $\tau\left(t\right)\geq\tau^{*}>0$. Also, we assume

$$\int_0^T p\left(s\right)ds>0,\ \int_0^T q\left(s\right)ds>0. \quad (3)$$

We also assume that the functions $Q\left(t,x\right)$ and $f\left(t,x,y\right)$ are periodic in t with period T, that is,

$$Q\left(t+T,x\right)=Q\left(t,x\right),\ f\left(t+T,x,y\right)=f\left(t,x,y\right). \quad (4)$$

Lemma 2.1 ([9]) *Suppose that (2) and (3) hold and*

$$\frac{R_1\left[\exp\left(\int_0^T p(u)\,du\right)-1\right]}{Q_1 T}\geq 1, \tag{5}$$

where

$$R_1=\max_{t\in[0,T]}\left|\int_t^{t+T}\frac{\exp\left(\int_t^s p(u)\,du\right)}{\exp\left(\int_0^T p(u)\,du\right)-1}q(s)\,ds\right|,\ Q_1=\left(1+\exp\left(\int_0^T p(u)\,du\right)\right)^2 R_1^2.$$

Then there are continuous T-periodic functions a and b such that $b(t)>0$, $\int_0^T a(u)\,du>0$ and

$$a(t)+b(t)=p(t),\ \frac{d}{dt}b(t)+a(t)\,b(t)=q(t),\ \text{for } t\in\mathbb{R}.$$

Lemma 2.2 ([11]) *Suppose the conditions of Lemma 2.1 hold and $\phi\in P_T$. Then the equation*

$$\frac{d^2}{dt^2}x(t)+p(t)\frac{d}{dt}x(t)+q(t)\,x(t)=\phi(t)$$

has a T-periodic solution. Moreover, the periodic solution can be expressed by

$$x(t)=\int_t^{t+T}G(t,s)\,\phi(s)\,ds,$$

where

$$G(t,s)=\frac{\int_t^s\exp\left[\int_t^u b(v)\,dv+\int_u^s a(v)\,dv\right]du+\int_s^{t+T}\exp\left[\int_t^u b(v)\,dv+\int_u^{s+T}a(v)\,dv\right]du}{\left[\exp\left(\int_0^T a(u)\,du\right)-1\right]\left[\exp\left(\int_0^T b(u)\,du\right)-1\right]}.$$

Corollary 2.1 ([11]) *Green's function G satisfies the following properties*

$$G(t,t+T)=G(t,t),\ G(t+T,s+T)=G(t,s),$$

$$\frac{\partial}{\partial s}G(t,s)=a(s)\,G(t,s)-\frac{\exp\left(\int_t^s b(v)\,dv\right)}{\exp\left(\int_0^T b(v)\,dv\right)-1},$$

$$\frac{\partial}{\partial t}G(t,s)=-b(t)\,G(t,s)+\frac{\exp\left(\int_t^s a(v)\,dv\right)}{\exp\left(\int_0^T a(v)\,dv\right)-1},$$

$$\frac{\partial^2}{\partial s^2}G(t,s)=\left(a'(s)+a^2(s)\right)G(t,s)-p(t)\,\frac{\exp\left(\int_t^s b(v)\,dv\right))}{\exp\left(\int_0^T b(v)\,dv\right))-1}.$$

The following lemma is fundamental to our results.

Lemma 2.3 *Suppose (2)-(4) and (5) hold. If $x\in P_T$, then x is a solution of equation (1) if and only if*

$$x(t)=Q(t,x(t-\tau(t)))-\int_t^{t+T}p(s)\,E(t,s)\,Q(s,x(s-\tau(s)))\,ds$$
$$+\int_t^{t+T}G(t,s)\left[f(s,h(x(s)),g(x(s-\tau(s))))+\left(a'(s)+a^2(s)\right)Q(s,x(s-\tau(s)))\right]ds, \tag{6}$$

where

$$E(t,s) = \frac{\exp\left(\int_t^s b(v)\,dv\right)}{\exp\left(\int_0^T b(v)\,dv\right) - 1}. \tag{7}$$

Proof. Let $x \in P_T$ be a solution of (1). From Lemma 2.2, we have

$$x(t) = \int_t^{t+T} G(t,s)\left[\frac{\partial^2}{\partial s^2} Q(s, x(s-\tau(s))) + f(s, h(x(s)), g(x(s-\tau(s))))\right] ds. \tag{8}$$

Using the integration by parts, we have

$$\begin{aligned}
&\int_t^{t+T} G(t,s)\,\frac{\partial^2}{\partial s^2} Q(s, x(s-\tau(s)))\,ds \\
&= \left[G(t,s)\frac{\partial}{\partial s}Q(s,x(s-\tau(s)))\right]_t^{t+T} \\
&= -\int_t^{t+T}\left(\frac{\partial}{\partial s}G(t,s)\right)\left(\frac{\partial}{\partial s}Q(s,x(s-\tau(s)))\right) ds.
\end{aligned}$$

But

$$\left[G(t,s)\frac{\partial}{\partial s}Q(s,x(s-\tau(s)))\right]_t^{t+T} = 0.$$

So

$$\int_t^{t+T} G(t,s)\frac{\partial^2}{\partial s^2}Q(s,x(s-\tau(s)))ds = -\int_t^{t+T}\left(\frac{\partial}{\partial s}G(t,s)\right)\left(\frac{\partial}{\partial s}Q(s,x(s-\tau(s)))\right) ds.$$

A second integration by parts gives

$$\begin{aligned}
&\int_t^{t+T} G(t,s)\frac{\partial^2}{\partial s^2}Q(s,x(s-\tau(s)))ds \\
&= \left[-Q(s,x(s-\tau(s)))\left(\frac{\partial}{\partial s}G(t,s)\right)\right]_t^{t+T} \\
&+ \int_t^{t+T} Q(s,x(s-\tau(s)))\frac{\partial^2}{\partial s^2}G(t,s)ds.
\end{aligned}$$

Since

$$\left[-Q(s,x(s-\tau(s)))\left(\frac{\partial}{\partial s}G(t,s)\right)\right]_t^{t+T}$$
$$=\left[-Q(s,x(s-\tau(s)))\left(a(s)G(t,s)-\frac{\exp\left(\int_t^s b\,(v)\,dv\right)}{\exp\left(\int_0^T b\,(v)\,dv\right)-1}\right)\right]_t^{t+T}$$
$$=-Q(t+T,x(t+T-\tau(t+T)))\left(a(t+T)G(t,t+T)-\frac{\exp\left(\int_t^{t+T} b\,(v)\,dv\right)}{\exp\left(\int_0^T b\,(v)\,dv\right)-1}\right)$$
$$+\,Q(t,x(t-\tau(t)))\left(a(t)G(t,t)-\frac{\exp\left(\int_t^t b\,(v)\,dv\right)}{\exp\left(\int_0^T b\,(v)\,dv\right)-1}\right)$$
$$=-Q(t,x(t-\tau(t)))\left(a(t)G(t,t)-\frac{\exp\left(\int_0^T b\,(v)\,dv\right)}{\exp\left(\int_0^T b\,(v)\,dv\right)-1}\right)$$
$$+\,Q(t,x(t-\tau(t)))\left(a(t)G(t,t)-\frac{1}{\exp\left(\int_0^T b\,(v)\,dv\right)-1}\right)$$
$$=Q(t,x(t-\tau(t))),$$

and

$$\int_t^{t+T}\left(\frac{\partial^2}{\partial s^2}G(t,s)\right)Q(s,x(s-\tau(s)))ds$$
$$=\int_t^{t+T}\left\{\left(a'(s)+a^2(s)\right)Q(s,x(s-\tau(s)))G(t,s)-p\,(s)\,E\,(t,s)\,Q(s,x(s-\tau(s)))\right\}ds,$$

we obtain

$$\int_t^{t+T}G(t,s)\frac{\partial^2}{\partial s^2}Q(s,x(s-\tau(s)))ds$$
$$=Q(t,x(t-\tau(t)))$$
$$+\int_t^{t+T}\left\{\left(a'(s)+a^2(s)\right)Q(s,x(s-\tau(s)))G(t,s)-p\,(s)\,E\,(t,s)\,Q(s,x(s-\tau(s)))\right\}ds, \tag{9}$$

where E is given by (7). Then substituting (9) in (8) completes the proof.

Lemma 2.4 ([11]) *Let* $A=\int_0^T p\,(u)\,du$, $B=T^2\exp\left(\frac{1}{T}\int_0^T \ln\left(q\,(u)\right)du\right)$. *If*

$$A^2\geq 4B, \tag{10}$$

then we have

$$\min\left\{\int_0^T a(u)\,du, \int_0^T b(u)\,du\right\} \geq \frac{1}{2}\left(A - \sqrt{A^2 - 4B}\right) := l,$$

$$\max\left\{\int_0^T a(u)\,du, \int_0^T b(u)\,du\right\} \leq \frac{1}{2}\left(A + \sqrt{A^2 - 4B}\right) := m.$$

Corollary 2.2 ([11]) *Functions G and E satisfy*

$$\frac{T}{(e^m - 1)^2} \leq G(t,s) \leq \frac{T\exp\left(\int_0^T p(u)\,du\right)}{(e^l - 1)^2}, \; E(t,s) \leq \frac{e^m}{e^l - 1}.$$

Lastly in this section, we state Krasnoselskii's fixed point theorem which enables us to prove the existence of positive periodic solutions to (1). For its proof we refer the reader to [10].

Theorem 2.1 (Krasnoselskii) *Let $\mathbb{D}$ be a closed convex nonempty subset of a Banach space $(\mathbb{B}, \|.\|)$. Suppose that $\mathcal{A}$ and $\mathcal{B}$ map $\mathbb{D}$ into $\mathbb{B}$ such that*
(i) $x, y \in \mathbb{D}$, implies $\mathcal{A}x + \mathcal{B}y \in \mathbb{D}$,
(ii) $\mathcal{A}$ is compact and continuous,
(iii) $\mathcal{B}$ is a contraction mapping.
Then there exists $z \in \mathbb{D}$ with $z = \mathcal{A}z + \mathcal{B}z$.

3 Existence of Positive Periodic Solutions

To apply Theorem 2.1, we need to define a Banach space $\mathbb{B}$, a closed convex subset $\mathbb{D}$ of $\mathbb{B}$ and construct two mappings, one is a contraction and the other is compact. So, we let $(\mathbb{B}, \|.\|) = (P_T, \|.\|)$ and $\mathbb{D} = \{\varphi \in \mathbb{B} : K \leq \varphi \leq L\}$, where K is non-negative constant and L is positive constant. We express equation (6) as

$$\varphi(t) = (\mathcal{B}\varphi)(t) + (\mathcal{A}\varphi)(t) := (H\varphi)(t),$$

where $\mathcal{A}, \mathcal{B} : \mathbb{D} \to \mathbb{B}$ are defined by

$$(\mathcal{A}\varphi)(t) = \int_t^{t+T} G(t,s)\left[f(s, h(\varphi(s)), g(\varphi(s - \tau(s)))) + \left(a'(s) + a^2(s)\right)Q(s, \varphi(s - \tau(s)))\right] ds, \tag{11}$$

and

$$(\mathcal{B}\varphi)(t) = Q(t, \varphi(t - \tau(t))) - \int_t^{t+T} p(s)\,E(t,s)\,Q(s, \varphi(s - \tau(s)))\,ds. \tag{12}$$

To simplify notations, we introduce the following constants

$$\alpha = \frac{T\exp\left(\int_0^T p(u)\,du\right)}{(e^l - 1)^2}, \; \beta = \frac{e^m}{e^l - 1}, \; \gamma = \frac{T}{(e^m - 1)^2},$$

$$\theta = \max_{t\in[0,T]}\{b(t)\}, \; \mu = \min_{t\in[0,T]}\{p(t)\}, \; \lambda = \max_{t\in[0,T]}\{p(t)\}. \tag{13}$$

In this section we obtain the existence of a positive periodic solution of (1) by considering the two cases: $Q(t,x) \geq 0$ and $Q(t,x) \leq 0$ for all $t \in \mathbb{R}$, $x \in \mathbb{D}$. We assume that function $Q(t,x)$ is locally Lipschitz continuous in x. That is, there exists a positive constant k such that

$$|Q(t,x) - Q(t,y)| \leq k \|x-y\|, \text{ for all } t \in [0,T], \ x \in \mathbb{D}. \tag{14}$$

In the case $Q(t,x) \geq 0$, we assume that there exist a non-negative constant k_1 and positive constants k_2, σ such that

$$E(t,s) > \sigma, \text{ for all } (t,s) \in [0,T] \times [0,T], \tag{15}$$

$$k_1 x \leq Q(t,x) \leq k_2 x, \text{ for all } t \in [0,T], \ x \in \mathbb{D}, \tag{16}$$

$$k_2 < 1, \tag{17}$$

and for all $s \in [0,T]$, $x, y \in \mathbb{D}$

$$\frac{(1-k_1) K + \lambda \beta k_2 T L}{\gamma T} \leq f(s, h(x), g(y)) + \left(a'(s) + a^2(s)\right) Q(s,y) \leq \frac{(1-k_2) L + \mu \sigma k_1 T K}{\alpha T}. \tag{18}$$

Lemma 3.1 *Suppose that the conditions (2)-(5), (10) and (15)-(18) hold. Then* $\mathcal{A} : \mathbb{D} \to \mathbb{B}$ *is compact.*

Proof. Let $\mathcal{A}$ be defined by (11). Obviously, $\mathcal{A}\varphi$ is continuous and it is easy to show that $(\mathcal{A}\varphi)(t+T) = (\mathcal{A}\varphi)(t)$. For $t \in [0,T]$ and for $\varphi \in \mathbb{D}$, we have

$$\begin{aligned} |(\mathcal{A}\varphi)(t)| &\leq \left| \int_t^{t+T} G(t,s) \left[f(s, h(\varphi(s)), g(\varphi(s-\tau(s)))) - a(s) Q(s, \varphi(s-\tau(s))) \right] ds \right| \\ &\leq \alpha T \frac{(1-k_2) L + \mu \sigma k_1 T K}{\alpha T} = (1-k_2) L + \mu \sigma k_1 T K. \end{aligned}$$

Thus from the estimation of $|(\mathcal{A}\varphi)(t)|$ we have

$$\|\mathcal{A}\varphi\| \leq (1-k_2) L + \mu \sigma k_1 T K.$$

This shows that $\mathcal{A}(\mathbb{D})$ is uniformly bounded.

Let us that $\mathcal{A}(\mathbb{D})$ is equicontinuous. Let $\varphi_n \in \mathbb{D}$, where n is a positive integer. Next we calculate $\frac{d}{dt}(\mathcal{A}\varphi_n)(t)$ and show that it is uniformly bounded. By making use of (2), (3) and (4) we obtain by taking the derivative in (11) that

$$\begin{aligned} \frac{d}{dt}(\mathcal{A}\varphi_n)(t) = \int_t^{t+T} &\left[-b(t) G(t,s) + \frac{\exp\left(\int_t^s a(v)\,dv\right)}{\exp\left(\int_0^T a(v)\,dv\right) - 1} \right] \\ &\times \left[f(s, h(\varphi_n(s)), g(\varphi_n(s-\tau(s)))) - a(s) Q(s, \varphi_n(s-\tau(s))) \right] ds. \end{aligned}$$

Consequently, by invoking (13) and (18), we obtain

$$\left| \frac{d}{dt}(\mathcal{A}\varphi_n)(t) \right| \leq T(\theta\alpha + \beta) \frac{(1-k_2) L + \mu \sigma k_1 T K}{\alpha T} \leq D,$$

for some positive constant D. Hence the sequence $(\mathcal{A}\varphi_n)$ is equicontinuous. The Ascoli-Arzela theorem implies that a subsequence $(\mathcal{A}\varphi_{n_k})$ of $(\mathcal{A}\varphi_n)$ converges uniformly to a continuous T-periodic function. Thus $\mathcal{A}$ is continuous and $\mathcal{A}(\mathbb{D})$ is contained in a compact subset of $\mathbb{B}$.

Lemma 3.2 *Suppose that (14) holds. If* $\mathcal{B}$ *is given by (12) with*

$$k\left(1+\lambda\beta T\right)<1, \tag{19}$$

then $\mathcal{B}:\mathbb{D}\rightarrow\mathbb{B}$ *is a contraction.*

Proof. Let $\mathcal{B}$ be defined by (12). Obviously, $\mathcal{B}\varphi$ is continuous and it is easy to show that $\left(\mathcal{B}\varphi\right)\left(t+T\right)=\left(\mathcal{B}\varphi\right)\left(t\right)$. So, for any $\varphi,\psi\in\mathbb{D}$, we have

$$\begin{aligned}
&\left|\left(\mathcal{B}\varphi\right)\left(t\right)-\left(\mathcal{B}\psi\right)\left(t\right)\right| \\
&\leq\left|Q\left(t,\varphi\left(t-\tau\left(t\right)\right)\right)-Q\left(t,\psi\left(t-\tau\left(t\right)\right)\right)\right| \\
&+\int_t^{t+T}p\left(s\right)E\left(t,s\right)\left|Q\left(s,\varphi\left(s-\tau\left(s\right)\right)\right)-Q\left(s,\psi\left(s-\tau\left(s\right)\right)\right)\right|ds \\
&\leq k\left(1+\lambda\beta T\right)\left\|\varphi-\psi\right\|.
\end{aligned}$$

Then $\left\|\mathcal{B}\varphi-\mathcal{B}\psi\right\|\leq k\left(1+\lambda\beta T\right)\left\|\varphi-\psi\right\|$. Thus $\mathcal{B}:\mathbb{D}\rightarrow\mathbb{B}$ is a contraction by (19).

Theorem 3.1 *Suppose (2)-(5), (10) and (14)-(19) hold. Then equation (1) has a positive* T*-periodic solution* x *in the subset* $\mathbb{D}$.

Proof. By Lemma 3.1, the operator $\mathcal{A}:\mathbb{D}\rightarrow\mathbb{B}$ is compact and continuous. Also, from Lemma 3.2, the operator $\mathcal{B}:\mathbb{D}\rightarrow\mathbb{B}$ is a contraction. Moreover, if $\varphi,\psi\in\mathbb{D}$, we see that

$$\begin{aligned}
&\left(\mathcal{B}\psi\right)\left(t\right)+\left(\mathcal{A}\varphi\right)\left(t\right) \\
&=Q\left(t,\psi\left(t-\tau\left(t\right)\right)\right)-\int_t^{t+T}p\left(s\right)E\left(t,s\right)Q\left(s,\psi\left(s-\tau\left(s\right)\right)\right)ds \\
&+\int_t^{t+T}G\left(t,s\right)\left[f\left(s,h\left(\varphi\left(s\right)\right),g\left(\varphi\left(s-\tau\left(s\right)\right)\right)\right)+\left(a'(s)+a^2(s)\right)Q\left(s,\varphi\left(s-\tau\left(s\right)\right)\right)\right]ds \\
&\leq k_2L-\mu\sigma\int_t^{t+T}Q\left(s,\psi\left(s-\tau\left(s\right)\right)\right)ds \\
&+\alpha\int_t^{t+T}\left[f\left(s,h\left(\varphi\left(s\right)\right),g\left(\varphi\left(s-\tau\left(s\right)\right)\right)\right)+\left(a'(s)+a^2(s)\right)Q\left(s,\varphi\left(s-\tau\left(s\right)\right)\right)\right]ds \\
&\leq k_2L-\mu\sigma k_1TK+\alpha T\frac{\left(1-k_2\right)L+\mu\sigma k_1TK}{\alpha T}=L,
\end{aligned}$$

and

$$\begin{aligned}
&\left(\mathcal{B}\psi\right)\left(t\right)+\left(\mathcal{A}\varphi\right)\left(t\right) \\
&=Q\left(t,\psi\left(t-\tau\left(t\right)\right)\right)-\int_t^{t+T}p\left(s\right)E\left(t,s\right)Q\left(s,\psi\left(s-\tau\left(s\right)\right)\right)ds \\
&+\int_t^{t+T}G\left(t,s\right)\left[f\left(s,h\left(\varphi\left(s\right)\right),g\left(\varphi\left(s-\tau\left(s\right)\right)\right)\right)-a\left(s\right)Q\left(s,\varphi\left(s-\tau\left(s\right)\right)\right)\right]ds \\
&\geq k_1K-\lambda\beta\int_t^{t+T}Q\left(s,\psi\left(s-\tau\left(s\right)\right)\right)ds \\
&+\gamma\int_t^{t+T}\left[f\left(s,h\left(\varphi\left(s\right)\right),g\left(\varphi\left(s-\tau\left(s\right)\right)\right)\right)-a\left(s\right)Q\left(s,\varphi\left(s-\tau\left(s\right)\right)\right)\right]ds \\
&\geq k_1K-\lambda\beta k_2TL+\gamma T\frac{\left(1-k_1\right)K+\lambda\beta k_2TL}{\gamma T}=K.
\end{aligned}$$

Clearly, all the hypotheses of the Krasnoselskii theorem are satisfied. Thus there exists a fixed point $x \in \mathbb{D}$ such that $x = \mathcal{A}x + \mathcal{B}x$. By Lemma 2.3 this fixed point is a solution of (1) and the proof is complete.

In the case $Q(t,x) \leq 0$, we substitute conditions (16)-(18) with the following conditions respectively. We assume that there exist a negative constant k_3 and a non-positive constant k_4 such that

$$k_3 x \leq Q(t,x) \leq k_4 x, \text{ for all } t \in [0,T], \ x \in \mathbb{D}, \tag{20}$$

$$-k_3 \lambda \beta T < 1, \tag{21}$$

and for all $s \in [0,T], \ x, y \in \mathbb{D}$

$$\frac{K(1+k_4\mu\sigma T)-k_3L}{\gamma T} \leq f(s,h(x),g(y))+\left(a'(s)+a^2(s)\right)Q(s,y) \leq \frac{L(1+k_3\lambda\beta T)-k_4K}{\alpha T}. \tag{22}$$

Theorem 3.2 *Suppose (2)-(5), (10), (14), (15) and (19)-(22) hold. Then equation (1) has a positive T-periodic solution x in the subset* $\mathbb{D}$.

The proof follows along the lines of Theorem 3.2, and hence we omit it.

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