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Different Types of Synchronization Between Different Fractional Order Chaotic Systems

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Abstract: In this paper complete synchronization, anti-synchronization and projective synchronization are achieved between two different fractional order chaotic systems, fractional order Lotka Volterra system and fractional order Lu system, via active control method. Numerical simulations have been done in Matlab by using Grunwald Letnikov method. Numerical results demonstrate the effectiveness and feasibility of the proposed control techniques.

Keywords: *synchronization; anti-synchronization; projective synchronization; fractional order chaotic systems; active control*

Mathematics Subject Classification (2010): 37B25, 37D45, 37N35, 70K99.

1 Introduction

A chaotic dynamical system is defined as the system which satisfies the properties of boundedness, infinite recurrence and sensitive dependence on initial conditions [2]. Chaos theory investigates the unstable behavior in deterministic nonlinear dynamical systems which cause 'chaos'. Sometimes chaotic behavior of a dynamical system is found useful like in secure communications [21, 37]. First time in 1963, Lorenz discovered a three dimensional chaotic system while studying weather model for atmospheric convection. After a decade, Rossler discovered a three dimensional chaotic system, which was constructed during the study of a chemical reaction. Synchronization is an important and famous phenomenon which can be understood within the unifying framework of the nonlinear sciences. Due to its potential applications in the field of nonlinear dynamics it has been hot topic of research. Since the pioneer work of Pecora and Carroll [26] it has been

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an active area for researchers in applied sciences. Synchronization in the language of the nonlinear dynamics is defined as an adjustment of rhythms of oscillating objects due to their weak interaction. So many types of synchronization have been achieved: generalized synchronization [3], phase synchronization and anti-phase synchronization [4, 9, 34], lag synchronization [5], Q-S synchronization [24], etc. So many techniques have been developed for synchronization like adaptive control [14], feedback control [22], fuzzy control [7], nonlinear control [25], active backstepping [23], adaptive sliding mode control [38] etc. Chaos synchronization has so many applications in different fields like power systems [18], physics [39], chemistry [19], medicine [20], diffusion [12] etc.

Recently, fractional differential equations have been used to study different dynamical systems and chaos have been analyzed in different fractional order systems. So many numerical methods have been developed for the solution of fractional differential equation [29–31]. Also, fractional calculus is a 300 years old subject which can be traced back to Leibniz, Riemann, Grunwald and Letnikov. So many systems have been found in real life which can be represented more accurately by fractional order systems. But for the last few decades fractional calculus with chaos has been an attractive field and so many works have been done on synchronization and control of chaos in fractional order systems like Lorenz system [10], Rossler system [17], Volta System [28], Chua system [11], Chen system [16] etc. Recently so many new chaotic and hyperchaotic systems [8,27,32,33,35] have also been developed and analyzed by the researchers.

Synchronization has so many applications in which secure communication is very important. Synchronization between integer order and fractional order system via tracking control [13] and sliding mode control [6], synchronization of fractional order systems with different dimensions [36] and hybrid projective synchronization [15] of fractional order chaotic systems between order (1,2) have also been obtained in recent years. The aim of this paper is to achieve different synchronizations between different fractional order systems which is important for secure communication. Amongst different types of chaos synchronization, projective synchronization has been found to be more secure because of its unpredictable scaling factor and this is why it has received so much attention in the last few years.

In this paper we achieve three types of synchronization between two chaotic systems, fractional order Lotka Volterra system (master system) and fractional order Lu system (slave system) via active control. The achieved synchronizations are complete synchronization, anti-synchronization and projective synchronization. For numerical simulations Matlab software has been used and to solve fractional differential equation Grunwald Letnikov method has been used.

2 Fractional Order Derivatives [29]

Fractional calculus generalizes differentiation and integration to non integer order fundamental operator ${}_aD_t^\alpha$ where a and t are the limits of the operator and α is the non integer order. This operator is defined as

$$ {}_aD_t^\alpha = \begin{cases} \frac{d^\alpha}{dt^\alpha}, & : \alpha > 0 \\ 0, & : \alpha = 0 \\ \int\limits_a^t (d\tau)^\alpha, & : \alpha < 0. \end{cases} $$

The most known three definitions for fractional integro differential operator are Riemann-Liouville definition, Grunwald-Letnikov definition and Caputo's definition.

The Riemann-Liouville definition is given as

$$ {}_aD_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t (t-\tau)^{n-\alpha-1} f(\tau) d(\tau), \quad n-1 < \alpha < n. $$

The Caputo's fractional derivative is defined as

$$ {}_0D_t^\alpha = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^n(\tau)}{(t-\tau)^{\alpha+1-n}} d\tau, \quad : n-1 < \alpha < n, $$

where $\Gamma(.)$ is the Gamma function, and the Grunwald Letnikov definition is

$$ {}_aD_t^\alpha f(t) = \lim_{h\to 0} \frac{1}{h^\alpha} \sum_{j=0}^{[\frac{(t-\alpha)}{h}]} (-1)^j \left(\begin{array}{c} \alpha \\ j \end{array} \right) f(t-jh), $$

where [.] denotes the greatest integer part.

3 Stability of Fractional Order System [13]

Theorem: *We consider the following linear fractional order system*

$$ D^\alpha x = Ax, \quad x(0) = x_0. \tag{1} $$

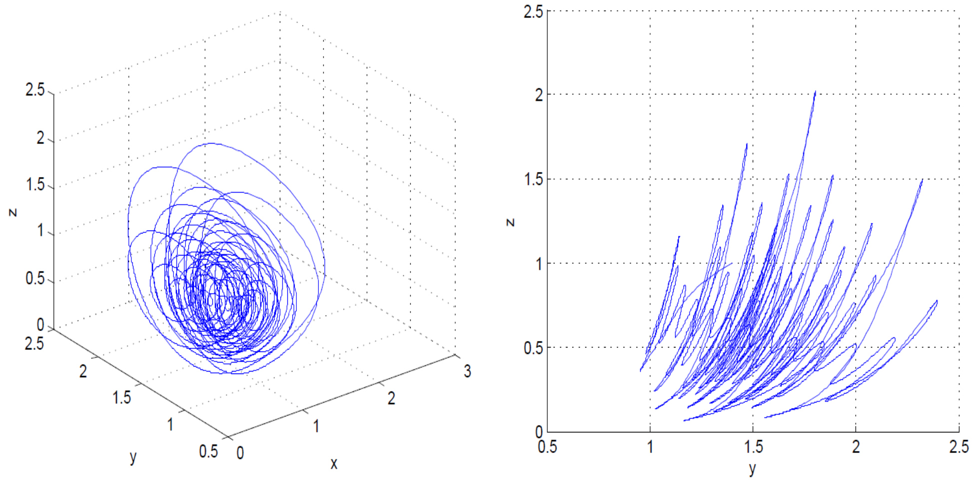
Here, $A \in R^{n\times n}$*, and* $\alpha = (\alpha_1, \alpha_2, ., \alpha_n), (0 < \alpha_i \leq 1)$*. System (1) is asymptotically stable if and only if* $|\ arg(\lambda_i)\ | > \alpha\pi/2$ *is satisfied for all eigenvalues* λ_i *of the matrix* A*. Furthermore, this system is stable if and only if* $|arg(\lambda_i)| \geq \alpha\pi/2$ *is satisfied for all eigenvalues* λ_i *of the matrix* A *and those critical eigenvalues that satisfy the condition* $|arg(\lambda_i)| = \alpha\pi/2$ *and have geometric multiplicity one. The geometric multiplicity of an eigenvalue is defined as the dimension of the associated eigenspace, i.e., the number of linearly independent eigenvectors with that eigenvalue.*

Master and Slave systems. The fractional order Lotka-Volterra system [1] is considered as master system

$$ \begin{aligned} \frac{d^{q_1}x_1}{dt^{q_1}} &= ax_1 - bx_1y_1 + ex_1^2 - sz_1x_1^2, \\ \frac{d^{q_2}y_1}{dt^{q_2}} &= dx_1y_1 - cy_1, \\ \frac{d^{q_3}z_1}{dt^{q_3}} &= sz_1x_1^2 - pz_1. \end{aligned} \tag{2} $$

This system exhibits chaotic behavior for parameter values $a = b = c = d = 1, e = 2, p = 3, s = 2.7$ and order $q_1 = q_2 = q_3 = 0.95$ for these values behavior of the system (2) is shown in Figure 1. Consider the following fractional order Lu chaotic system [29] as slave system

$$ \begin{aligned} \frac{d^{q_1}x_2}{dt^{q_1}} &= \alpha(y_2 - x_2), \\ \frac{d^{q_2}y_2}{dt^{q_2}} &= \gamma y_2 - x_2z_2, \\ \frac{d^{q_3}z_2}{dt^{q_3}} &= x_2y_2 - \beta z_2, \end{aligned} \tag{3} $$



(a) (b)

Figure 1: (a) Chaotic attractor of Lotka Volterra system in $x-y-z$ space for order $\alpha = 0.95$. (b) $y-z$ view of the Lotka-Volterra system for order $\alpha = 0.95$.

which exhibits chaotic behavior for parameters $\alpha = 36, \beta = 3, \gamma = 20$, and order $q_1 = q_2 = q_3 = 0.95$, as shown in Figure 2.

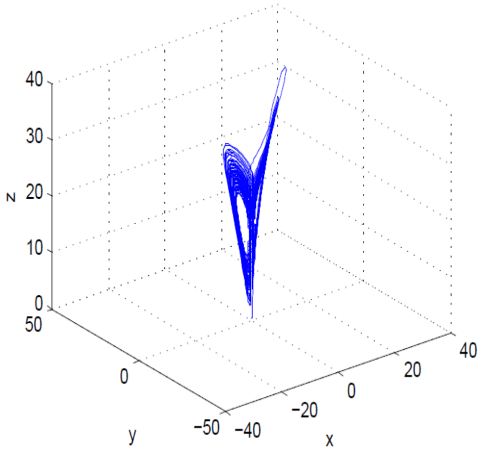
4 Synchronization Methodology and Numerical Simulations

To achieve different synchronizations between the considered two chaotic systems via active control method the error is defined as $e = y - \chi x$. For complete synchronization we take $\chi = 1$, for anti-synchronization $\chi = -1$, for projective synchronization arbitrary value of χ may be chosen. In this paper we took $\chi = 2$. Our aim is to design an effective controller $u(t)$ so that error e converges to zero. The master system is described by

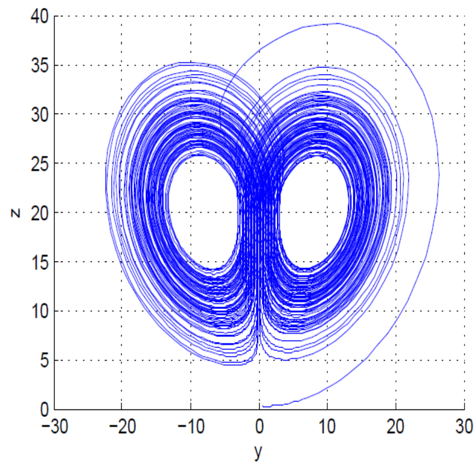
$$\begin{aligned}\frac{d^{q_1}x_1}{dt^{q_1}} &= ax_1 - bx_1y_1 + ex_1^2 - sz_1x_1^2,\\ \frac{d^{q_2}y_1}{dt^{q_2}} &= dx_1y_1 - cy_1,\\ \frac{d^{q_3}z_1}{dt^{q_3}} &= sz_1x_1^2 - pz_1.\end{aligned} \tag{4}$$

The slave system with controllers is described by

$$\begin{aligned}\frac{d^{q_1}x_2}{dt^{q_1}} &= \alpha(y_2 - x_2) + u_1(t),\\ \frac{d^{q_2}y_2}{dt^{q_2}} &= \gamma y_2 - x_2z_2 + u_2(t),\\ \frac{d^{q_3}z_2}{dt^{q_3}} &= x_2y_2 - \beta z_2 + u_3(t),\end{aligned} \tag{5}$$



(a)



(b)

Figure 2: (a) Chaotic attractor of Lu system in $x-y-z$ space for $\alpha = 0.95$. (b) $y-z$ view of the Lu system for order $\alpha = 0.95$.

where $u_1(t), u_2(t), u_3(t)$ are three control functions. For complete synchronization, the error dynamical system is

$$\begin{aligned}
\frac{d^{q_1}e_1}{dt^{q_1}} &= \alpha(e_2 - e_1) - (\alpha + a)x_1 + \alpha y_1 + bx_1y_1 - ex_1^2 + sz_1x_1^2 + u_1(t), \\
\frac{d^{q_2}e_2}{dt^{q_2}} &= \gamma e_2 - dx_1y_1 + (\gamma + c)y_1 - x_2z_2 + u_2(t), \\
\frac{d^{q_3}e_3}{dt^{q_3}} &= -\beta e_3 + x_2y_2 - sz_1x_1^2 + (p - \beta)z_1 + u_3(t).
\end{aligned} \tag{6}$$

Define the active control functions $u_1(t), u_2(t), u_3(t)$ as

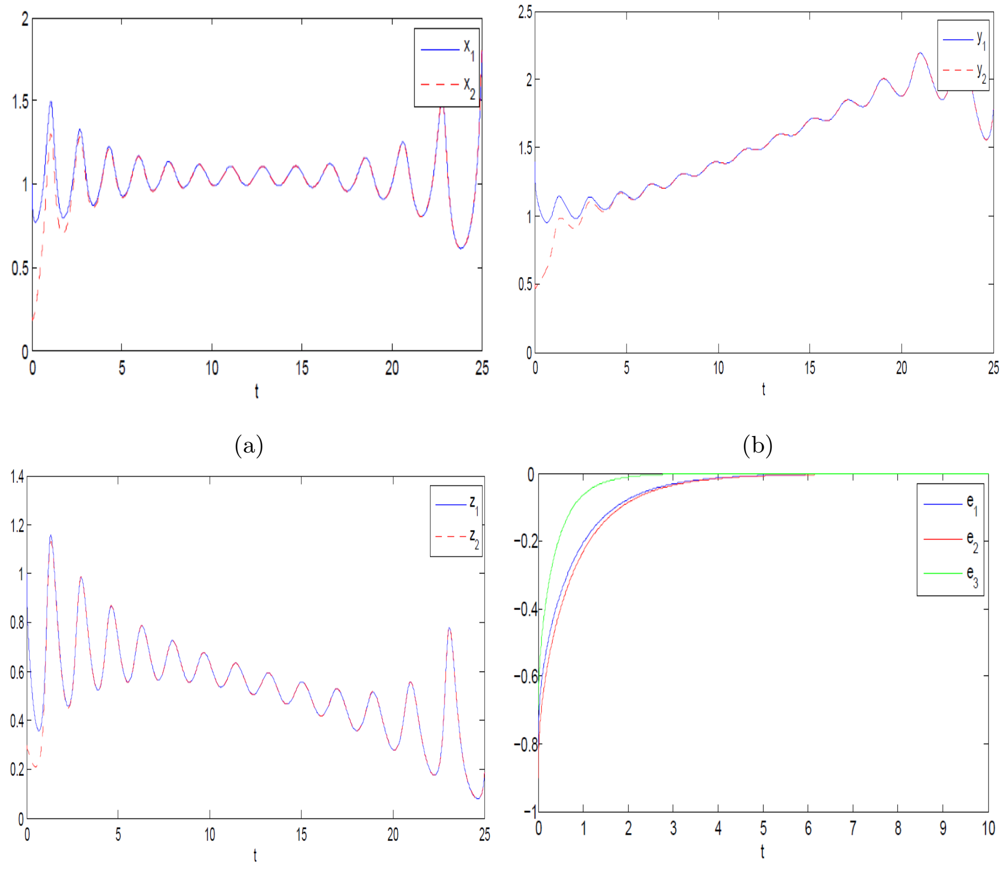
$$\begin{aligned}
u_1(t) &= (\alpha + a)x_1 - \alpha y_1 - bx_1y_1 + ex_1^2 - sz_1x_1^2 + v_1(t), \\
u_2(t) &= dx_1y_1 - (\gamma + c)y_1 + x_2z_2 + v_2(t), \\
u_3(t) &= -x_2y_2 + sz_1x_1^2 - (p - \beta)z_1 + v_3(t).
\end{aligned} \tag{7}$$

With these controllers the error dynamical system becomes

$$\begin{aligned}
\frac{d^{q_1}e_1}{dt^{q_1}} &= \alpha(e_2 - e_1) + v_1(t), \\
\frac{d^{q_2}e_2}{dt^{q_2}} &= \gamma e_2 + v_2(t), \\
\frac{d^{q_3}e_3}{dt^{q_3}} &= -\beta e_3 + v_3(t).
\end{aligned} \tag{8}$$

Define the suitable control inputs $v_1(t), v_2(t), v_3(t)$ to obtain the stable error dynamical system. Choosing $v_1(t), v_2(t), v_3(t)$ as

$$\begin{pmatrix} v_1(t) \\ v_2(t) \\ v_3(t) \end{pmatrix} = C \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}, \tag{9}$$



(a) (b)

(c) (d)

Figure 3: (a) Complete synchronization between state variables x_1, x_2. (b) Complete synchronization between state variables y_1, y_2. (c) Complete synchronization between state variables z_1, z_2. (d) Complete synchronization errors e_1, e_2, e_3 converging to zero.

where C is a 3×3 constant matrix, we choose C as

$$\begin{pmatrix} (\alpha-1) & -\alpha & 0 \\ 0 & -(\gamma+1) & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then the error dynamical system is

$$\begin{aligned} \frac{d^{q_1}e_1}{dt^{q_1}} &= -e_1, \\ \frac{d^{q_2}e_2}{dt^{q_2}} &= -e_2, \\ \frac{d^{q_3}e_3}{dt^{q_3}} &= -2e_3. \end{aligned} \tag{10}$$

Eigenvalues for this error dynamical system are $-1, -1, -2$ which satisfies stability condition. Figures 3(a),(b),(c) show complete synchronization between state variables of master system and slave system, and 3(d) shows errors of complete synchronization converging to zero at the initial conditions $(-0.8, -0.9, -0.7)$. For anti-synchronization, we choose $\chi = -1$, and the error dynamical system for the same master and slave systems becomes

$$\begin{aligned}
\frac{d^{q_1}e_1}{dt^{q_1}} &= \alpha(e_2 - e_1) + (\alpha + a)x_1 - \alpha y_1 - bx_1y_1 + ex_1^2 - sz_1x_1^2 + u_1(t), \\
\frac{d^{q_2}e_2}{dt^{q_2}} &= \gamma e_2 + dx_1y_1 - (\gamma + c)y_1 - x_2z_2 + u_2(t), \\
\frac{d^{q_3}e_3}{dt^{q_3}} &= -\beta e_3 + x_2y_2 + sz_1x_1^2 - (p - \beta)z_1 + u_3(t).
\end{aligned} \tag{11}$$

The active control functions $u_1(t), u_2(t), u_3(t)$ are defined as

$$\begin{aligned}
u_1(t) &= -(\alpha + a)x_1 + \alpha y_1 + bx_1y_1 - ex_1^2 + sz_1x_1^2 + v_1(t), \\
u_2(t) &= -dx_1y_1 + (\gamma + c)y_1 + x_2z_2 + v_2(t), \\
u_3(t) &= -x_2y_2 - sz_1x_1^2 + (p - \beta)z_1 + v_3(t),
\end{aligned} \tag{12}$$

the error dynamical system becomes

$$\begin{aligned}
\frac{d^{q_1}e_1}{dt^{q_1}} &= \alpha(e_2 - e_1) + v_1(t), \\
\frac{d^{q_2}e_2}{dt^{q_2}} &= \gamma e_2 + v_2(t), \\
\frac{d^{q_3}e_3}{dt^{q_3}} &= -\beta e_3 + v_3(t).
\end{aligned} \tag{13}$$

Now we have to choose suitable control inputs $v_1(t), v_2(t), v_3(t)$ so that the stable error dynamical system could be obtained. Choosing $v_1(t), v_2(t), v_3(t)$ as

$$\begin{pmatrix} v_1(t) \\ v_2(t) \\ v_3(t) \end{pmatrix} = C \begin{pmatrix} e_1(t) \\ e_2(t) \\ e_3(t) \end{pmatrix}, \tag{14}$$

where C is a constant matrix and C is

$$\begin{pmatrix} (\alpha - 1) & -\alpha & 0 \\ 0 & -(\gamma + 1) & 0 \\ 0 & 0 & (\beta - 1) \end{pmatrix}.$$

Then the error dynamical system becomes

$$\begin{aligned}
\frac{d^{q_1}e_1}{dt^{q_1}} &= -e_1, \\
\frac{d^{q_2}e_2}{dt^{q_2}} &= -e_2, \\
\frac{d^{q_3}e_3}{dt^{q_3}} &= -e_3.
\end{aligned} \tag{15}$$

Eigenvalues for this error dynamical system are $-1, -1, -1$ which satisfies stability condition and initial conditions are $(1.2, 1.9, 1.3)$.

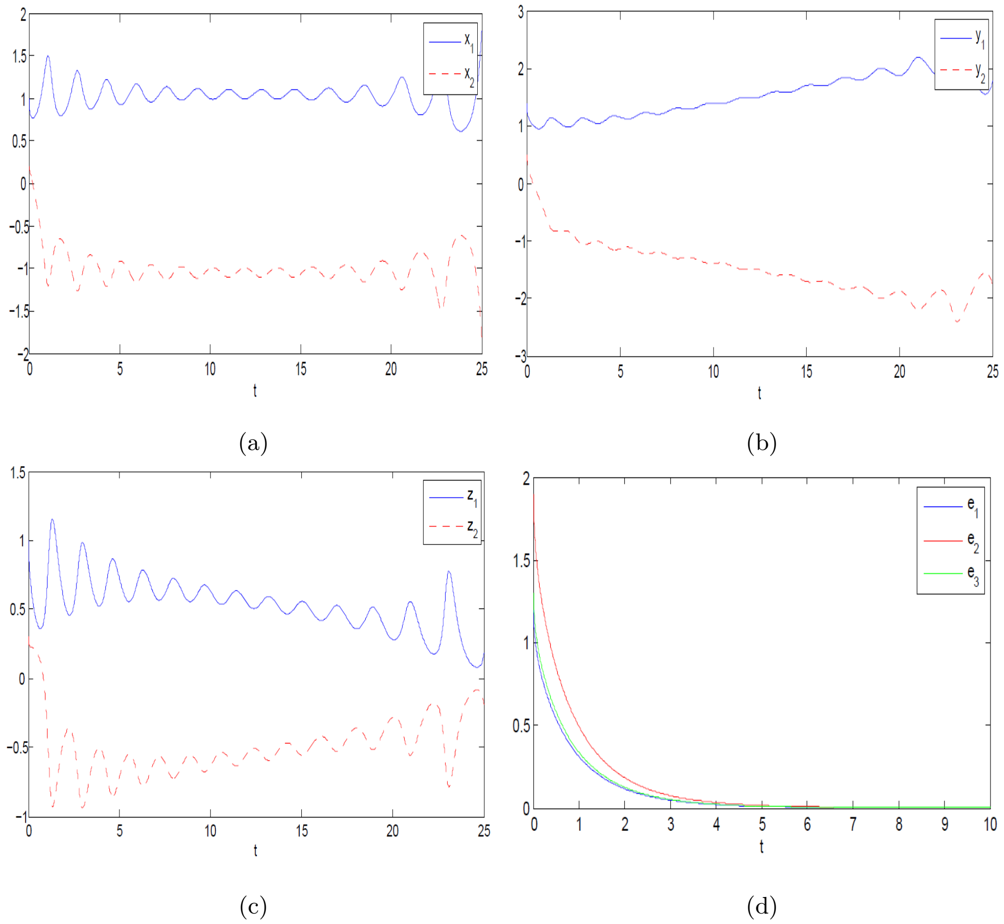


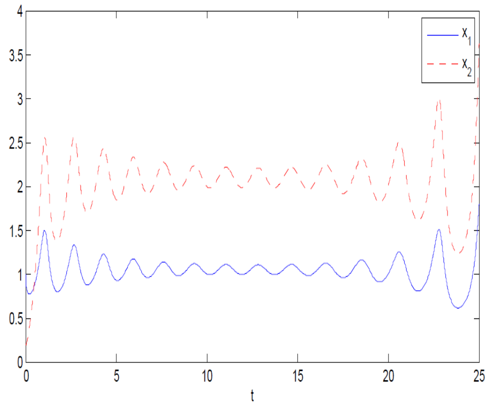
Figure 4: (a) Anti-synchronization between state variables x_1, x_2. (b) Anti-synchronization between state variables y_1, y_2. (c) Anti-synchronization between state variables z_1, z_2. (d) Anti-synchronization errors e_1, e_2, e_3 converging to zero.

Figure 4(a),(b) shows anti-synchronization between state variables x_1, x_2 and y_1, y_2, and Figure 4(c),(d) exhibits anti-synchronization between state variables z_1, z_2 and anti-synchronization errors tending to zero.

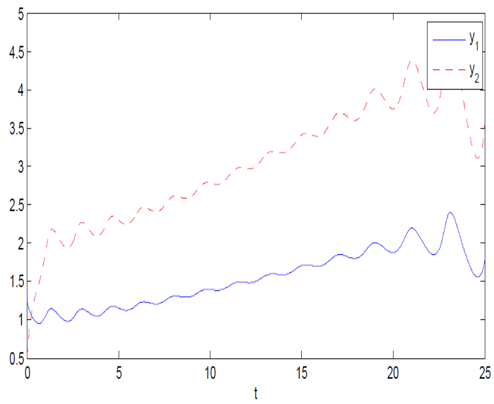
For projective synchronization the choice of the scalar χ is arbitrary, but we choose $\chi = 2$ and the error dynamical system with the same master and slave system becomes

$$\begin{aligned}
\frac{d^{q_1} e_1}{dt^{q_1}} &= \alpha(e_2 - e_1) - 2(\alpha + a)x_1 + 2\alpha y_1 + 2b x_1 y_1 - 2e x_1^2 + 2s z_1 x_1^2 + u_1(t), \\
\frac{d^{q_2} e_2}{dt^{q_2}} &= \gamma e_2 - 2d x_1 y_1 + 2(\gamma + c) y_1 - x_2 z_2 + u_2(t), \\
\frac{d^{q_3} e_3}{dt^{q_3}} &= -\beta e_3 + x_2 y_2 - 2s z_1 x_1^2 + u_3(t).
\end{aligned} \tag{16}$$

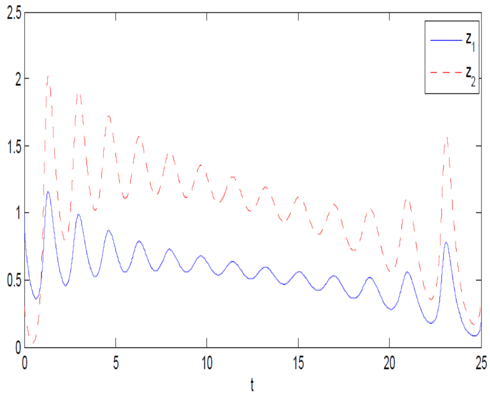
The active control functions $u_1(t), u_2(t), u_3(t)$ are defined as



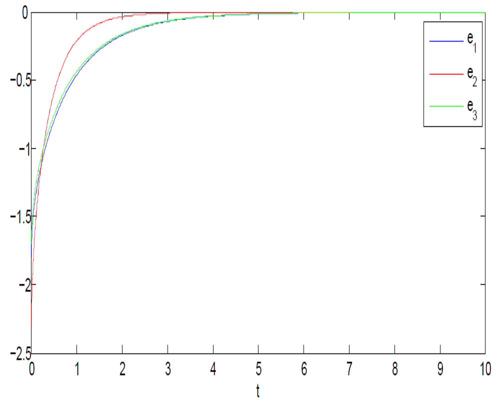
(a)



(b)



(c)



(d)

Figure 5: (a) Projective synchronization for state variables x_1, x_2. (b) Projective synchronization for state variables y_1, y_2. (c) Projective synchronization for state variables z_1, z_2. (d) Projective synchronization errors e_1, e_2, e_3 converging to zero.

$$\begin{aligned} u_1(t) &= 2(\alpha + a)x_1 - 2\alpha y_1 - 2bx_1y_1 + 2ex_1^2 - 2sz_1x_1^2 + v_1(t), \\ u_2(t) &= 2dx_1y_1 - 2(\gamma + c)y_1 + x_2z_2 + v_2(t), \\ u_3(t) &= -x_2y_2 + 2sz_1x_1^2 + v_3(t). \end{aligned} \tag{17}$$

Then the error dynamical system becomes

$$\begin{aligned} \frac{d^{q_1}e_1}{dt^{q_1}} &= \alpha(e_2 - e_1) + v_1(t), \\ \frac{d^{q_2}e_2}{dt^{q_2}} &= \gamma e_2 + v_2(t), \\ \frac{d^{q_3}e_3}{dt^{q_3}} &= -\beta e_3 + v_3(t). \end{aligned} \tag{18}$$

Now we have to choose suitable control inputs $v_1(t), v_2(t), v_3(t)$ so that the stable error dynamical system could be obtained. We choose $v_1(t), v_2(t), v_3(t)$ as follows

$$\begin{pmatrix} v_1(t) \\ v_2(t) \\ v_3(t) \end{pmatrix} = C \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}, \tag{19}$$

where C is a 3×3 constant matrix

$$\begin{pmatrix} (\alpha - 1) & -\alpha & 0 \\ 0 & -(\gamma + 2) & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Then the error dynamical system becomes

$$\begin{aligned} \frac{d^{q_1} e_1}{dt^{q_1}} &= -e_1, \\ \frac{d^{q_2} e_2}{dt^{q_2}} &= -2e_2, \\ \frac{d^{q_3} e_3}{dt^{q_3}} &= -e_3. \end{aligned} \tag{20}$$

Eigenvalues for this error dynamical system are $-1, -2, -1$ which satisfies stability condition and initial conditions for this error dynamical system are $(-1.8, -2.3, -1.7)$. Figure 5 (a),(b),(c) shows projective synchronization between state variables of master system and slave system and Figure 5(d) exhibits projective synchronization errors converging to zero .

5 Conclusion

Different types of synchronizations have been achieved between two different fractional order chaotic systems Lotka-Volterra system and Lu system by using active control method. The method is easy to apply. The results are validated by numerical simulations using Matlab. Numerical simulations are a witness to achievement of desired goal between these two systems. The analytical and computational results are in an excellent agreement.

References

[1] Agrawal, S.K., Srivastava, M. and Das. S. Synchronization of fractional order chaotic systems via active control. *Chaos Solitons Fractals* **45** (6) (2012) 737–752.

[2] Azar, A.T. and Vaidyanathan, S. *Chaos Modeling and Control Systems Design.* Springer, Newyork, 2015.

[3] Banerjee, T. Biswas, D. and Sarkar, B.C. Complete and generalized synchronizations of chaos and hyperchaos in a coupled first order time delayed system. *Nonlinear Dyn.* **71** (1) (2013) 279–290.

[4] Boutefnouchet, M., Taghavafard, H. and Erjaee, G.H. Global stability of phase synchronization in coupled chaotic systems. *Nonlinear Dynamics and Systems Theory* **15** (2) (2015) 141–147.

[5] Cai, G., Hu, P. and Li, Y. Modified function lag projective synchronization of a financial hyperchaotic system. *Nonlinear Dyn.* **69** (3) (2012) 1457–1464.

[6] Chen, D. Zhang, R. Sprott, J.C., Chen, H. Ma, X. Synchronization between integer-order chaotic systems and a class of fractional-order chaotic systems via sliding mode control. *Chaos* **22** (2) (2012) 023130.

[7] Chen, D., Zhao, W., Sprott, J.C. and Ma, X. Application of Takagi Sugeno fuzzy model to a class of chaotic synchronization and anti-synchronization. *Nonlinear Dyn.* **73** (3) (2013) 1495–1505.

[8] Chen, D.Y., Wu, C., Liu, C.F., Ma, X.Y., You, Y.J. and Zhang, R.F. Synchronization and circuit simulation of a new double wing chaos. *Nonlinear Dyn.* **67** (2) (2012) 1481–1504.

[9] Erjaee, G.H. and Taghvafard, H. Stability analysis of phase synchronization in coupled chaotic system presented by fractional differential equations. *Nonlinear Dynamics and Systems Theory* **11** (2) (2011) 147–154.

[10] Grigorenko, I. and Grigorenko, E. Chaotic dynamics of fractional Lorenz system. *Phys. Rev. Lett.* **91** (3) (2003) 034101.

[11] Harley, T.T., Lorenzo, C.F. and Qammer H.K. Chaos on a fractional Chua's system.*IEEE Trans. Circuits Syst. Fundamental Theory Appl.* **42** (8) (1995) 485–490.

[12] Jesus, I.S. and Machado, J. A. T. Fractional control of heat diffusion system. *Nonlinear Dyn.* **54** (3) (2008) 263–282.

[13] Khan, A. and Tripathi, P. Synchronization between a fractional order chaotic system and integer order chaotic system. *Nonlinear Dynamics and Systems Theory* **13** (4) (2013) 425–436.

[14] Khan, A. and Pal, R. Adaptive hybrid function projective synchronization of chaotic system Space-Tether system. *Nonlinear Dynamics and Systems Theory* **14** (1) (2014) 44–57.

[15] Khan, A. and Bhat, M. A. Hybrid projective synchronization of fractional order chaotic systems with fractional order in the Interval (1,2). *Nonlinear Dynamics and Systems Theory* **16** (4) (2016) 350–365.

[16] Li, C. and Peng G. Chaos in Chen's system with a fractional order. *Chaos Solitons Fractals* **22** (2) (2004) 443–450.

[17] Li, C. and Chen, G. Chaos and hyperchaos in the fractional order rossler equations. *Physica A* **341** (2004) 55–61.

[18] Lin, Q. and Wu, X. The sufficient criteria for global synchronization of chaotic power systems under linear state error feedback control. *Nonlinear Anal. Real World Appl.* **12** (3) (2011) 1500–1509.

[19] Liu, L.C., Tian, B., Xue, Y.C, Wang, M. and Liu, W.J. Analytic solution for a nonlinear chemistry system of ordinary differential equations. *Nonlinear Dyn.* **68** (1) (2012) 17–21.

[20] Losa, G. A., Merlini, D., Nonnenmacher, T.F. and Weibel, E.R. *Fractals in Biology and Medicine*. Birkhauser Basel, 1998.

[21] Mengue, A.D., Essimbi, B.Z. Secure Communication using chaotic synchronization in mutually coupled semiconductor lasers. *Nonlinear Dyn.* **70** (2) (2012) 1241–1253.

[22] Morgul, O. Further stability results for a generalization of delayed feedback control. *Nonlinear Dyn.* **70** (2) (2012) 1255–1262.

[23] Ojo, K.S., Njah, A.N., Ogunjo, S.T. and Olusola, O.I. Reduced order function projective combination synchronization of three Josephson junctions using backstepping technique. *Nonlinear Dynamics and Systems Theory* **1**4 (2) (2014) 119–133.

[24] Ouannas, A. A Simple Approach for Q-S Synchronization of chaotic dynamical systems in continous time. *Nonlinear Dyn. Syst. Theory* **17** (1) (2017) 86–94.

[25] Park, J. H. Chaos synchronization between two different chaotic dynamical systems. *Chaos Solitons Fractals* **27** (2) (2006) 549–554.

[26] Pecora, L.M, and Carroll, T.L. Synchronization in Chaotic Systems. *Phys. Rev. Lett.* **64** (8) (1990) 821–824.

[27] Pehlivan, I., Moroz, I. M., and Vaidyanathan, S. Analysis, Synchronization and circuit design of a novel butterfly attractor. *J. Sound Vibration* **333** (20) (2014) 5077–5096.

[28] Petras, I. Chaos in fractional order Volta's system: modeling and simulation. *Nonlinear Dyn.* **57** (1) (2009) 157–170.

[29] Petras, I. *Fractional order nonlinear systems, modeling analysis and simulation.* Springer-Verlag, Berlin-Heildberg, 2011.

[30] Podlubny, I. *Fractional Differential Equation.* Academic Press, New York, 1999.

[31] Sun, H.H., Abdelwahid, A.A, Ornal, B. Linear approximation of transfer function with a pole of fractional order. *IEEE Trans. Automat. Control* **29** (5)(1984) 441–444.

[32] Sundarapandian, V. and Pehlivan, I. Analysis, control, Synchronization and circuit design of a novel chaotic system. *Math. and Comp. Modelling* **55** (7-8) (2012) 1904–1915.

[33] Sundarapandian, V. Analysis and anti-synchronization of a novel chaotic system via active and adaptive controllers. *Journal of Engg Sci and Tech. Rev* **6** (4) (2013) 45–52.

[34] Taghvafard, H. and Erjaee G. H. Phase and antiphase synchronization of fractional order chaotic system via active control. *Commun. Nonlinear Sci. Numer. Simul* **16** (10) (2011) 4079–4088.

[35] Vaidyanathan, S., Volos, C. and Pham, V.T. Hyperchaos, adaptive control and synchronization of a novel 5-D hyperchaotic system with three positive Lyapunov exponents and its SPICE implementation. *Archives of Control Sciences* **24** (LX) (2014) 409–446.

[36] Wang, S., Yu, Y., and Diao, M. Hybrid projective synchronization of chaotic fractional order systems with different dimensions. *Physica A* **389** (21) (2010) 4981–4988.

[37] Wu, X., Wang, H. and Lu, H. Modified Generalized projective synchronization of a new hyperchaotic system and its application to secure communication. *Nonlinear Anal. Real World Appl.* **13** (3) (2012) 1441–1450.

[38] Zhang, R., Yang, S. Robust synchronization of two different fractional order chaotic systems with unknown parameters using adaptive sliding mode approach. *Nonlinear Dyn.* **71** (1) (2013) 269–278.

[39] Zou, W. and Zhen, M. Complete synchronization in coupled limit cycle systems. *Europhys Lett.* **81** (1) (2008) 10006.