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Random Impulsive Partial Hyperbolic Fractional Differential Equations

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Abstract: This paper deals with the existence of random solutions of Darboux problem of impulsive fractional differential equations. The main results are based on the measure of noncompactness and a fixed point theorem for random operators.

Keywords: *Darboux problem, differential equation; Caputo fractional derivative; random solution; impulses; measure of noncompactness.*

Mathematics Subject Classification (2010): 26A33.

1 Introduction

Fractional calculus is generalization of ordinary differentiation and integration to arbitrary non-integer order. The subject is as old as the differential calculus, starting from some speculations of G.W. Lebeniz (1967) and L. Euler (1730) and since then, it has continued to be developed up to nowadays. Integral equations are one of the most useful mathematical tools in both pure and applied analysis. This is particulary true for problems in mechanical vibrations and the related fields of engineering and mathematical physics. We can find numerous applications of differential and integral equations of fractional order in finance, hydrology, biophysics, thermodynamics, control theory, statistical mechanics, astrophysics, cosmology and bioengineering [10,14,19,20,23]. There has been a significant development in ordinary and partial fractional differential equations in recent years; see the monographs of Abbas *et al.* [5,6], Baleanu *et al.* [10], Kilbas *et al.* [16], Zhou [25], the papers of Abbas *et al.* [1–3,7], Sowmya and Vatsala [21], Stutson and Vatsala [22], Vityuk and Golushkov [24], and the references therein.

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There has been a significant development in impulse theory in recent years, especially in the area of impulsive differential equations with fixed moments. Recently some results on the Darboux problem for fractional order impulsive hyperbolic differential equations and inclusions have been obtained by Abbas *et al.* [3, 5].

The initial value problems of ordinary random differential equations have been studied in the literature on bounded as well as unbounded internals of the real line for different aspects of the solution. See, for example, Burton and Furumochi [11] and the references therein.

In this paper, we discuss the existence of random solutions for the following impulsive partial fractional random differential equations

$$\begin{cases} {}^cD^r_{x_k}u(x,y,w) = f(x,y,u(x,y,w),w); \text{ if } (x,y) \in J_k,\ k = 0,\dots,m, w \in \Omega, \\ u(x_k^+,y,w) = u(x_k^-,y,w) + I_k(u(x_k^-,y,w)); \text{ if } y \in [0,b],\ k = 1,\dots,m, w \in \Omega, \\ u(x,0,w) = \varphi(x,w);\ x \in [0,a], w \in \Omega, \\ u(0,y,w) = \psi(y,w);\ y \in [0,b], w \in \Omega, \\ \varphi(0,w) = \psi(0,w), \end{cases} \tag{1}$$

where $J_0 = [0,x_1] \times [0,b]$, $J_k := (x_k,x_{k+1}] \times [0,b]$; $k = 1,\dots,m$, $a,b > 0$, $\theta_k = (x_k,0)$; $k = 0,\dots,m$, ${}^cD^r_{x_k}$ is the fractional Caputo derivative of order $r = (r_1,r_2) \in (0,1] \times (0,1]$, $0 = x_0 < x_1 < \cdots < x_m < x_{m+1} = a$, $(\Omega,\mathcal{A})$ is a measurable space, $f : J \times E \times \Omega \to E$; $I_k : E \to E$; $k = 1,\dots,m$ are given continuous functions, $\varphi : [0,a] \times \Omega \to E$ and $\psi : [0,b] \times \Omega \to E$ are given absolutely continuous functions. Here $u(x_k^+,y,w)$ and $u(x_k^-,y,w)$ denote the right and left limits of $u(x,y,w)$ at $x = x_k$, respectively.

This paper initiates the study of random solutions for impulsive partial hyperbolic fractional differential equations.

2 Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper. Let E be a Banach space and let $J := [0,a] \times [0,b]$; $a,b > 0$. Denote by $L^1(J)$ the space of Bochner-integrable functions $u : J \to E$ with the norm

$$\|u\|_{L^1} = \int_0^a \int_0^b \|u(x,y)\|_E dydx,$$

where $\|\cdot\|_E$ denotes a suitable complete norm on E.

As usual, by $AC(J)$ we denote the space of absolutely continuous functions from J into E, and $\mathcal{C} := C(J)$ is the Banach space of continuous functions from J into E with the norm $\|\cdot\|_\infty$ defined by

$$\|u\|_\infty = \sup_{(x,y)\in J} \|u(x,y)\|_E.$$

Consider the space

$$\begin{aligned} PC = PC(J \times \Omega) = \big\{ & u : J \times \Omega \to E : u(\cdot,\cdot,w) \text{ is continuous on } J_k;\ k = 0,1,\dots,m, \text{ and} \\ & \text{there exist } u(x_k^-,y,w) \text{ and } u(x_k^+,y,w);\ k = 1,\dots,m, \\ & \text{with } u(x_k^-,y,w) = u(x_k,y,w) \text{ for each } y \in [0,b], w \in \Omega \big\}. \end{aligned}$$

This set is a Banach space with the norm

$$\|u\|_{PC} = \sup_{(x,y)\in J} \|u(x,y,w)\|_E.$$

Let β_E be the σ-algebra of Borel subsets of E. A mapping $v : \Omega \to E$ is said to be measurable if for any $B \in \beta_E$, one has

$$v^{-1}(B) = \{w \in \Omega : v(w) \in B\} \subset \mathcal{A}.$$

To define integrals of sample paths of random process, it is necessary to define a jointly measurable map.

Definition 2.1 A mapping $T : \Omega \times E \to E$ is called jointly measurable if for any $B \in \beta_E$, one has

$$T^{-1}(B) = \{(w,v) \in \Omega \times E : T(w,v) \in B\} \subset \mathcal{A} \times \beta_E,$$

where $\mathcal{A} \times \beta_E$ is the direct product of the σ-algebras $\mathcal{A}$ and β_E that are defined in Ω and E respectively.

Lemma 2.1 *Let $T : \Omega \times E \to E$ be a mapping such that $T(\cdot, v)$ is measurable for all $v \in E$, and $T(w, \cdot)$ is continuous for all $w \in \Omega$. Then the map $(w,v) \mapsto T(w,v)$ is jointly measurable.*

Definition 2.2 *A function $f : J \times E \times \Omega \to E$ is called random Carathéodory if the following conditions are satisfied:*

(i) The map $(x,y,w) \to f(x,y,u,w)$ is jointly measurable for all $u \in E$, and

(ii) The map $u \to f(x,y,u,w)$ is continuous for almost all $(x,y) \in J$ and $w \in \Omega$.

Let $T : \Omega \times E \to E$ be a mapping. Then T is called a random operator if $T(w,u)$ is measurable in w for all $u \in E$ and it is expressed as $T(w)u = T(w,u)$. In this case we also say that $T(w)$ is a random operator on E. A random operator $T(w)$ on E is called continuous (compact, totally bounded and completely continuous) if $T(w,u)$ is continuous (compact, totally bounded and completely continuous, respectively) in u for all $w \in \Omega$. The details of completely continuous random operators in Banach spaces and their properties appear in Itoh [15].

Definition 2.3 [13] Let $\mathcal{P}(Y)$ be the family of all nonempty subsets of Y and C be a mapping from Ω into $\mathcal{P}(Y)$. A mapping $T : \{(w,y) : w \in \Omega,\ y \in C(w)\} \to Y$ is called random operator with stochastic domain C if C is measurable (i.e., for all closed $A \subset Y$, $\{w \in \Omega, C(w) \cap A \neq \emptyset\}$ is measurable) and for all open $D \subset Y$ and all $y \in Y$, $\{w \in \Omega : y \in C(w), T(w,y) \in D\}$ is measurable. T will be called continuous if every $T(w)$ is continuous. For a random operator T, a mapping $y : \Omega \to Y$ is called random (stochastic) fixed point of T if for $P-$almost all $w \in \Omega$, $y(w) \in C(w)$ and $T(w)y(w) = y(w)$ and for all open $D \subset Y$, $\{w \in \Omega : y(w) \in D\}$ is measurable.

Let $\mathcal{M}_X$ denote the class of all bounded subsets of a metric space X.

Definition 2.4 Let X be a complete metric space. A map $\alpha : \mathcal{M}_X \to [0,\infty)$ is called a measure of noncompactness on X if it satisfies the following properties for all $B, B_1, B_2 \in \mathcal{M}_X$:

(MNC.1) $\alpha(B) = 0$ if and only if B is precompact (regularity),

(MNC.2) $\alpha(B) = \alpha(\overline{B})$ (invariance under closure),

(MNC.3) $\alpha(B_1 \cup B_2) = \max\{\alpha(B_1), \alpha(B_2)\}$ (semi-additivity).

For more details on measure of noncompactness and its properties, see [8, 9].

Let $\theta = (0,0)$, $r_1, \ r_2 > 0$ and $r = (r_1, r_2)$. For $f \in L^1(J)$, the expression

$$(I_\theta^r f)(x,y) = \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} f(s,t)dsdt$$

is called the left-sided mixed Riemann-Liouville integral of order r, where $\Gamma(\cdot)$ is the (Euler's) gamma function defined by $\Gamma(\xi) = \int_0^\infty t^{\xi-1}e^{-t}dt, \ \xi > 0$.

In particular,

$$(I_\theta^\theta u)(x,y) = u(x,y), \ (I_\theta^\sigma u)(x,y) = \int_0^x \int_0^y u(s,t)dtds; \text{ for almost all } (x,y) \in J,$$

where $\sigma = (1,1)$. For instance, $I_\theta^r u$ exists for all $r_1, r_2 \in (0,\infty)$, when $u \in L^1(J)$. Note also that when $u \in C(J)$, then $(I_\theta^r u) \in C(J)$. Moreover

$$(I_\theta^r u)(x,0) = (I_\theta^r u)(0,y) = 0; \ x \in [0,a], \ y \in [0,b].$$

Example 2.1 Let $\lambda, \omega \in (-1,\infty)$ and $r = (r_1, r_2) \in (0,\infty) \times (0,\infty)$, then

$$I_\theta^r x^\lambda y^\omega = \frac{\Gamma(1+\lambda)\Gamma(1+\omega)}{\Gamma(1+\lambda+r_1)\Gamma(1+\omega+r_2)} x^{\lambda+r_1} y^{\omega+r_2}, \text{ for almost all } (x,y) \in J.$$

By $1-r$ we mean $(1-r_1, 1-r_2) \in [0,1) \times [0,1)$. Denote by $D_{xy}^2 := \frac{\partial^2}{\partial x \partial y}$ the mixed second order partial derivative.

Definition 2.5 [24] Let $r \in (0,1] \times (0,1]$ and $u \in L^1(J)$. The Caputo fractional-order derivative of order r of u is defined by the expression

$${}^cD_\theta^r u(x,y) = (I_\theta^{1-r} D_{xy}^2 u)(x,y).$$

The case $\sigma = (1,1)$ is included and we have

$$({}^cD_\theta^\sigma u)(x,y) = (D_{xy}^2 u)(x,y); \text{ for almost all } (x,y) \in J.$$

Example 2.2 Let $\lambda, \omega \in (-1,\infty)$ and $r = (r_1, r_2) \in (0,1] \times (0,1]$, then

$${}^cD_\theta^r x^\lambda y^\omega = \frac{\Gamma(1+\lambda)\Gamma(1+\omega)}{\Gamma(1+\lambda-r_1)\Gamma(1+\omega-r_2)} x^{\lambda-r_1} y^{\omega-r_2}; \text{ for almost all } (x,y) \in J.$$

Let $a_1 \in [0,a],\ z^+ = (a_1, 0) \in J,\ J_z = (a_1, a] \times [0, b],\ r_1, r_2 > 0$ and $r = (r_1, r_2)$. For $u \in L^1(J_z)$, the expression

$$(I^r_{z^+} u)(x,y) = \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{a_1^+}^{x} \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} u(s,t)\,dtds$$

is called the left-sided mixed Riemann-Liouville integral of order r of u.

Definition 2.6 [24]. For $u \in L^1(J_z)$ where $D^2_{xy}u$ is Lebesque integrable on $[x_k, x_{k+1}] \times [0,b],\ k = 0, \dots, m$, the Caputo fractional order derivative of order r of u is defined by the expression

$$({}^cD^r_{z^+} f)(x,y) = (I^{1-r}_{z^+} D^2_{xy} f)(x,y).$$

Lemma 2.2 *[12] If Y is a bounded subset of Banach space X, then for each $\epsilon > 0$, there is a sequence $\{y_k\}_{k=1}^{\infty} \subset Y$ such that*

$$\alpha(Y) \le 2\alpha(\{y_k\}_{k=1}^{\infty}) + \epsilon.$$

Lemma 2.3 *[18] If $\{u_k\}_{k=1}^{\infty} \subset L^1(J)$ is uniformly integrable, then $\alpha(\{u_k\}_{k=1}^{\infty})$ is measurable and for each $(x,y) \in J$,*

$$\alpha\left(\left\{\int_0^x \int_0^y u_k(s,t)\,dtds\right\}_{k=1}^{\infty}\right) \le 2\int_0^x \int_0^y \alpha(\{u_k(s,t)\}_{k=1}^{\infty})\,dtds.$$

Lemma 2.4 *[17] Let F be a closed and convex subset of a real Banach space, let $G : F \to F$ be a continuous operator and $G(F)$ be bounded. If there exists a constant $k \in [0,1)$ such that for each bounded subset $B \subset F$,*

$$\alpha(G(B)) \le k\alpha(B),$$

then G has a fixed point in F.

3 Existence Results

We need the following auxiliary lemma.

Lemma 3.1 *[4] Let $0 < r_1, r_2 \le 1,\ \mu(x,y) = \varphi(x) + \psi(y) - \varphi(0)$ and let $f : J \times E \to E$ be continuous. A function $u \in PC(J)$ is a solution of the fractional integral equation*

$$u(x,y) = \begin{cases} \mu(x,y) + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} f(s,t,u(s,t))\,dtds; \\ \quad \text{if } (x,y) \in [0, x_1] \times [0,b], \\ \mu(x,y) + \sum_{i=1}^{k} (I_i(u(x_i^-, y)) - I_i(u(x_i^-, 0))) \\ + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \sum_{i=1}^{k} \int_{x_{i-1}}^{x_i} \int_0^y (x_i - s)^{r_1-1}(y-t)^{r_2-1} f(s,t,u(s,t))\,dtds \\ + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_k}^{x} \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} f(s,t,u(s,t))\,dtds; \\ \quad \text{if } (x,y) \in (x_k, x_{k+1}] \times [0,b],\ k = 1, \dots, m, \end{cases}$$

if and only if u is a solution of the problem

$$\begin{cases} {}^cD^r_{x_k} u(x,y) = f(x,y,u(x,y)); \quad \text{if } (x,y) \in J_k,\ k = 0, \dots, m, \\ u(x_k^+, y) = u(x_k^-, y) + I_k(u(x_k^-, y)); \quad \text{if } y \in [0,b],\ k = 1, \dots, m, \\ u(x,0) = \varphi(x);\ x \in [0,a], \\ u(0,y) = \psi(y);\ y \in [0,b], \\ \varphi(0) = \psi(0). \end{cases}$$

As a consequence, we have the following lemma.

Lemma 3.2 *Let $0 < r_1, r_2 \leq 1$, $\mu(x,y,w) = \varphi(x,w) + \psi(y,w) - \varphi(0,w)$. A function $u \in PC$ is a solution of the random fractional integral equation*

$$u(x,y,w) = \begin{cases} \mu(x,y,w) + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} f(s,t,u(s,t,w),w)dtds; \\ \text{if } (x,y) \in [0,x_1] \times [0,b],\ w \in \Omega, \\ \mu(x,y,w) + \sum_{i=1}^k (I_i(u(x_i^-,y,w)) - I_i(u(x_i^-,0,w))) \\ + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \sum_{i=1}^k \int_{x_{i-1}}^{x_i} \int_0^y (x_i-s)^{r_1-1}(y-t)^{r_2-1} f(s,t,u(s,t,w),w)dtds \\ + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_k}^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} f(s,t,u(s,t,w),w)dtds; \\ \text{if } (x,y) \in (x_k, x_{k+1}] \times [0,b],\ k = 1,\ldots,m,\ w \in \Omega, \end{cases} \tag{2}$$

if and only if u is a solution of the random problem (1).

The following hypotheses will be used in the sequel.

Hypothesis 3.1 *The functions $w \mapsto \varphi(x,0,w)$ and $w \mapsto \psi(0,y,w)$ are measurable and bounded for almost each $x \in [0,a]$ and $y \in [0,b]$ respectively.*

Hypothesis 3.2 *The function f is random Carathéeodory on $J \times E \times \Omega$.*

Hypothesis 3.3 *There exist functions $p_1, p_2, p_3 : J \times \Omega \to [0,\infty)$ with $p_i(\cdot,w) \in L^\infty(J,[0,\infty))$; $i = 1,2,3$ such that for each $w \in \Omega$,*

$$\|f(x,y,u,w)\|_E \leq p_1(x,y,w) + p_2(x,y,w)\|u\|_E,$$

and

$$\|I_k(u)\|_E \leq p_3(x,y,w)\|u\|_E,$$

for all $u \in E$ and almost each $(x,y) \in J$.

Hypothesis 3.4 *For any bounded $B \subset E$,*

$$\alpha(f(x,y,B,w)) \leq p_2(x,y,w)\alpha(B), \quad \textit{for almost each } (x,y) \in J,$$

and

$$\alpha(I_k(B)) \leq p_3(x,y,w)\alpha(B), \ \textit{for almost each } (x,y) \in J.$$

Set

$$\mu^*(w) = \sup_{(x,y)\in J} \|\mu(x,y,w)\|_E,\ p_i^*(w) = \sup ess_{(x,y)\in J} p_i(x,y,w);\ i = 1,2,3.$$

Remark 3.1 Hypotheses 3.3 and 3.4 are equivalent [8].

Theorem 3.1 *Assume that hypotheses 3.1-3.3 hold. If*

$$\ell := 2mp_3^*(w) + \frac{4(m+1)p_2^*(w)a^{r_1}b^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)} < 1,$$

then the problem (1) has a random solution defined on J.

Proof. By Lemma 3.2, the problem (1) is equivalent to the integral equation

$$u(x,y,w) = \begin{cases} \mu(x,y,w) + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} f(s,t,u(s,t,w),w)dtds; \\ \text{ if } (x,y) \in [0,x_1] \times [0,b], \ w \in \Omega, \\ \mu(x,y,w) + \sum_{i=1}^k (I_i(u(x_i^-,y,w)) - I_i(u(x_i^-,0,w))) \\ + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \sum_{i=1}^k \int_{x_{i-1}}^{x_i} \int_0^y (x_i-s)^{r_1-1}(y-t)^{r_2-1} f(s,t,u(s,t,w),w)dtds \\ + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_k}^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} f(s,t,u(s,t,w),w)dtds; \\ \text{ if } (x,y) \in (x_k,x_{k+1}] \times [0,b], \ k=1,\dots,m, \ w \in \Omega, \end{cases}$$

for each $w \in \Omega$ and almost each $(x,y) \in J$.

Define the operator $N : PC \to PC$ by

$$\begin{aligned}(Nu)(x,y) &= \mu(x,y,w) + \sum_{i=1}^k (I_i(u(x_i^-,y,w)) - I_i(u(x_i^-,0,w))) \\ &+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \sum_{i=1}^k \int_{x_{i-1}}^{x_i} \int_0^y (x_i-s)^{r_1-1}(y-t)^{r_2-1} f(s,t,u(s,t,w),w)dtds \\ &+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_k}^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} f(s,t,u(s,t,w),w)dtds.\end{aligned}$$

Since the functions φ, ψ and I_k and f are absolutely continuous, the function μ and the indefinite integral are absolutely continuous for all $w \in \Omega$ and almost all $(x,y) \in J$. Again, as the maps μ and I_k are continuous for all $w \in \Omega$ and the indefinite integral is continuous on J, then $N(w)$ defines a mapping $N : PC \to PC$. Hence u is a solution for the problem (1) if and only if $u = Nu$. We shall show that the operator N satisfies all conditions of Lemma 2.4. The proof will be given in several steps.

Step 1: N is a random operator with stochastic domain on PC.
Since $f(x,y,u,w)$ is random Carathéodory, the map $w \to f(x,y,u,w)$ is measurable in view of Definition 2.1. Similarly, the product $(x-s)^{r_1-1}(y-t)^{r_2-1}f(s,t,u(s,t,w),w)$ of a continuous and a measurable function is again measurable. Further, the integral is a limit of a finite sum of measurable functions and I_k is measurable. Therefore, the map

$$\begin{aligned}w \mapsto & \mu(x,y,w) + \sum_{i=1}^k (I_i(u(x_i^-,y,w)) - I_i(u(x_i^-,0,w))) \\ &+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \sum_{i=1}^k \int_{x_{i-1}}^{x_i} \int_0^y (x_i-s)^{r_1-1}(y-t)^{r_2-1} f(s,t,u(s,t,w),w)dtds \\ &+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_k}^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} f(s,t,u(s,t,w),w)dtds\end{aligned}$$

is measurable. As a result, N is a random operator from PC into PC.

Let $W : \Omega \to \mathcal{P}(PC)$ be defined by

$$W(w) = \{u \in PC : \|u\|_{PC} \le R(w)\}$$

with $R(\cdot)$ being chosen appropriately. For instance, we assume that

$$R(w) \geq \frac{\mu^* + \frac{(m+1)p_1^*(w)a^{r_1}b^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)}}{1 - 2mp_3^*(w) - (m+1)p_2^*(w)\frac{a^{r_1}b^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)}}.$$

The set $W(w)$ is bounded, closed, convex and solid for all $w \in \Omega$. Then W is measurable (Lemma 17 ([13]). Let $w \in \Omega$ be fixed, then from Hypothesis 3.4 for any $u \in w(w)$, we get

$$\begin{aligned}
&\|(Nu)(x,y)\|_E \\
&\leq \|\mu(x,y,w)\|_E + \sum_{i=1}^{k} \|I_i(u(x_i^-, y, w))\| + \|I_i(u(x_i^-, 0, w))\| \\
&\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \sum_{i=1}^{k} \int_{x_{i-1}}^{x_i} \int_0^y (x_i - s)^{r_1 - 1}(y-t)^{r_2-1} \|f(s,t,u(s,t,w),w)\|_E dtds \\
&\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_k}^{x} \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} \|f(s,t,u(s,t,w),w)\|_E dtds, \\
&\leq \|\mu(x,y,w)\|_E + \sum_{i=1}^{k} (p_3(x,y,w)\|u\| + (p_3(x_i,0,w))\|u\|) \\
&\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \sum_{i=1}^{k} \left(\int_{x_{i-1}}^{x_i} \int_0^y (x_i - s)^{r_1-1}(y-t)^{r_2-1} p_1(s,t,w) dtds \right. \\
&\quad \left. + \int_{x_{i-1}}^{x_i} \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} p_2(s,t,w) \|u(s,t,w)\|_E dtds \right) \\
&\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_k}^{x} \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} p_1(s,t,w) dtds \\
&\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_k}^{x} \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} p_2(s,t,w) \|u(s,t,w)\|_E dtds \\
&\leq \mu^*(w) + 2mp_3^*(w)R(w) \\
&\quad + \sum_{i=1}^{k} \left(\frac{p_1^*(w)}{\Gamma(r_1)\Gamma(r_2)} \int_{x_{i-1}}^{x_i} \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} dtds \right. \\
&\quad \left. + \frac{p_2^*(w)R(w)}{\Gamma(r_1)\Gamma(r_2)} \int_{x_{i-1}}^{x_i} \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} dtds \right) \\
&\quad + \frac{p_1^*(w)}{\Gamma(r_1)\Gamma(r_2)} \int_{x_k}^{x} \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} dtds \\
&\quad + \frac{p_2^*(w)R(w)}{\Gamma(r_1)\Gamma(r_2)} \int_{x_k}^{x} \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} dtds \\
&\leq \mu^*(w) + 2mp_3^*(w)R(w) + \frac{(p_1^*(w) + p_2^*(w)R(w))(m+1)a^{r_1}b^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)} \\
&\leq R(w).
\end{aligned}$$

Therefore, N is a random operator with stochastic domain W and $N : W(w) \to W(w)$. Furthermore, N maps bounded sets into bounded sets in PC.

Step 2: N is continuous.
Let $\{u_n\}$ be a sequence such that $u_n \to u$ in PC. Then, for each $(x, y) \in J$ and $w \in \Omega$, we have

$$\begin{aligned}
&\|(Nu_n)(x,y) - (N(w)u)(x,y)\|_E \\
&\leq \sum_{i=1}^{k}(\|I_i(u_n(x_i^-, y, w)) - I_i(u(x_i^-, y, w))\| + \|I_i(u_n(x_i^-, 0, w)) - I_i(u(x_i^-, 0, w))\|) \\
&+\frac{1}{\Gamma(r_1)\Gamma(r_2)} \sum_{i=1}^{k} \int_{x_{i-1}}^{x_i} \int_0^y (x_i - s)^{r_1 - 1}(y - t)^{r_2 - 1}\|f(s, t, u_n(s, t, w), w) \\
&-f(s, t, u(s, t, w), w)\|_E dtds \\
&+\frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_k}^{x} \int_0^y (x - s)^{r_1 - 1}(y - t)^{r_2 - 1}\|f(s, t, u_n(s, t, w), w) \\
&-f(s, t, u(s, t, w), w)\|_E dtds.
\end{aligned}$$

Using the Lebesgue dominated convergence theorem, we get

$$\|Nu_n - Nu\|_\infty \to 0 \text{ as } n \to \infty.$$

As a consequence of Steps 1 and 2, we can conclude that $N : W(w) \to W(w)$ is a continuous random operator with stochastic domain W, and $N(W(w))$ is bounded.
Step 3: For each bounded subset B of $W(w)$ we have

$$\alpha(NB) \leq \ell\alpha(B).$$

Let $w \in \Omega$ be fixed. From Lemmas 2.2 and 2.3, for any $B \subset W$ and any $\epsilon > 0$, there exists a sequence $\{u_n\}_{n=0}^{\infty} \subset B$, such that for all $(x, y) \in J$, we have

$$\begin{aligned}
&\alpha((NB)(x,y)) \\
&= \alpha\Bigg\{\mu(x, y, w) + \sum_{i=1}^{k}(I_i(u(x_i^-, y, w)) - I_i(u(x_i^-, 0, w))) \\
&+\sum_{i=1}^{k} \int_{x_{i-1}}^{x_i} \int_0^y \frac{(x_i - s)^{r_1 - 1}(y - t)^{r_2 - 1}}{\Gamma(r_1)\Gamma(r_2)} f(s, t, u(s, t, w), w)dtds \\
&+\int_{x_k}^{x} \int_0^y \frac{(x - s)^{r_1 - 1}(y - t)^{r_2 - 1}}{\Gamma(r_1)\Gamma(r_2)} f(s, t, u(s, t, w), w)dtds;\ u \in B\Bigg\} \\
&\leq \alpha\Bigg\{\sum_{i=1}^{k}(I_i(u_n(x_i^-, y, w)) - I_i(u_n(x_i^-, 0, w))) \\
&+\sum_{i=1}^{k} \int_{x_{i-1}}^{x_i} \int_0^y \frac{(x_i - s)^{r_1 - 1}(y - t)^{r_2 - 1}}{\Gamma(r_1)\Gamma(r_2)} f(s, t, u_n(s, t, w), w)dtds
\end{aligned}$$

$$
\begin{aligned}
&+\int_{x_k}^{x}\int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} f(s,t,u_n(s,t,w),w)dtds\Bigg\}_{n=1}^{\infty}+\epsilon\\
&\le \alpha\left\{\sum_{i=1}^{k}(I_i(u_n(x_i^-,y,w))-I_i(u_n(x_i^-,0,w)))\right\}_{n=1}^{\infty}\\
&+2\sum_{i=1}^{k}\int_{x_{i-1}}^{x_i}\int_0^y \frac{(x_i-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)}\alpha\{f(s,t,u_n(s,t,w),w)\}_{n=1}^{\infty}dtds\\
&+2\int_{x_k}^{x}\int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)}\alpha\{f(s,t,u_n(s,t,w),w)\}_{n=1}^{\infty}dtds+\epsilon\\
&\le 2mp_3(x,y,w)\alpha\left(\{u_n(s,t,w)\}_{n=1}^{\infty}\right)\\
&+4\sum_{i=1}^{k}\int_{x_{i-1}}^{x_i}\int_0^y \frac{(x_i-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)}p_2(s,t,w)\alpha\left(\{u_n(s,t,w)\}_{n=1}^{\infty}\right)dtds\\
&+4\int_{x_k}^{x}\int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)}p_2(s,t,w)\alpha\left(\{u_n(s,t,w)\}_{n=1}^{\infty}\right)dtds+\epsilon\\
&\le 2mp_3(x,y,w)\alpha\left(\{u_n\}_{n=1}^{\infty}\right)\\
&+\left(4\sum_{i=1}^{k}\int_{x_{i-1}}^{x_i}\int_0^y \frac{(x_i-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)}p_2(s,t,w)\right)\alpha\left(\{u_n\}_{n=1}^{\infty}\right)dtds\\
&+\left(4\int_{x_k}^{x}\int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)}p_2(s,t,w)dsdt\right)\alpha\left(\{u_n\}_{n=1}^{\infty}\right)+\epsilon\\
&\le 2mp_3(x,y,w)\alpha(B)\\
&+\left(4\sum_{i=1}^{k}\int_{x_{i-1}}^{x_i}\int_0^y \frac{(x_i-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)}p_2(s,t,w)dtds\right)\alpha(B)\\
&+\left(4\int_{x_k}^{x}\int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)}p_2(s,t,w)dtds\right)\alpha(B)+\epsilon\\
&\le\left(2mp_3^*(w)+\frac{4(m+1)p_2^*(w)a^{r_1}b^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)}\right)\alpha(B)+\epsilon\\
&=\ell\alpha(B)+\epsilon.
\end{aligned}
$$

Since $\epsilon>0$ is arbitrary, we have

$$\alpha(N(B))\le\ell\alpha(B).$$

It follows from Lemma 2.4 that for each $w\in\Omega$, N has at least one fixed point in W. Since $\bigcap_{w\in\Omega} intW(w)\neq\emptyset$, there exists a measurable selector of $intW$, thus N has a stochastic fixed point, i.e., the problem (1) has at least one random solution.

4 An Example

Let $E=\mathbb{R}$, $\Omega=(-\infty,0)$ be equipped with the usual σ-algebra consisting of Lebesgue measurable subsets of $(-\infty,0)$. Given a measurable function $u:\Omega\to AC([0,1]\times[0,1])$, consider the following impulsive partial fractional random differential equations of the form

$$\begin{cases} {}^cD^r_{x_k}u(x,y,w) = \frac{w^2e^{-x-y-3}}{1+w^2+5|u(x,y,w)|}; \ \text{ if } (x,y)\in J_k,\ k=0,\dots,m, \\ u(x_k^+,y,w) = u(x_k^-,y,w) + \frac{w^2}{(1+w^2+10|u(x,y,w)|)e^{x+y+10}}; \ \text{ if } y\in[0,1],\ k=1,\dots,m, \end{cases} \tag{3}$$

where $w\in\Omega,\ J=[0,1]\times[0,1],\ (r_1,r_2)\in(0,1]\times(0,1]$ with the initial conditions

$$\begin{cases} u(x,0,w) = x\sin w;\ x\in[0,1], \\ u(0,y,w) = y^2\cos w;\ y\in[0,1]. \end{cases} \quad w\in\Omega, \tag{4}$$

Set

$$f(x,y,u(x,y,w),w) = \frac{w^2}{(1+w^2+5|u(x,y,w)|)e^{x+y+10}},\ (x,y)\in[0,1]\times[0,1],\ w\in\Omega,$$

and

$$I_k(u(x_k^-,y,w)) = \frac{w^2}{(1+w^2+10|u(x,y,w)|)e^{x+y+10}},\ y\in[0,1],\ k=1,\dots,m,\ w\in\Omega.$$

The functions $w\mapsto\varphi(x,0,w)=x\sin w$ and $w\mapsto\psi(0,y,w)=y^2\cos w$ are measurable and bounded with

$$|\varphi(x,0,w)|\le 1,\ |\psi(0,y,w)|\le 1,$$

hence, Hypothesis 3.1 is satisfied.

Clearly, the map $(x,y,w)\mapsto f(x,y,u,w)$ is jointly continuous for all $u\in\mathbb{R}$ and hence jointly measurable for all $u\in\mathbb{R}$. Also the map $u\mapsto f(x,y,u,w)$ is continuous for all $(x,y)\in J$ and $w\in\Omega$. So the function f is Carathéodory on $[0,1]\times[0,1]\times\mathbb{R}\times\Omega$.
For each $u\in\mathbb{R},\ (x,y)\in[0,1]\times[0,1]$ and $w\in\Omega$, we have

$$|f(x,y,u,w)|\le 1+\frac{5}{e^{10}}|u|,$$

and

$$|I_k(u)|\le\frac{10}{e^{10}}|u|.$$

Hence Hypothesis 3.4 is satisfied with

$$p_1(x,y,w)=p_1^*(w)=1,\ p_2(x,y,w)=p_2^*(w)=\frac{5}{e^{10}},\ p_3(x,y,w)=p_3^*(w)=\frac{10}{e^{10}}.$$

We shall show that condition $\ell<1$ holds with $a=b=1$. Indeed, if we assume, for instance, that the number of impulses $m=3$, then we have

$$\begin{aligned} \ell &= 2mp_3^*(w)+\frac{4(m+1)p_2^*(w)a^{r_1}b^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)} \\ &= \frac{60}{e^{10}}+\frac{80}{e^{10}\Gamma(1+r_1)\Gamma(1+r_2)} \\ &< 1, \end{aligned}$$

which is satisfied for each $(r_1,r_2)\in(0,1]\times(0,1]$. Consequently, Theorem 3.1 implies that the problem (3)-(4) has a random solution defined on $[0,1]\times[0,1]$.

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