Nonlinear Dynamics and Systems Theory, **17** (4) (2017) 409–420

Pseudo Almost Automorphic Mild Solutions to Some Fractional Differential Equations with Stepanov-like Pseudo Almost Automorphic Forcing Term

Vikram Singh [1*] and Dwijendra N. Pandey [1]

[1] *Department of Mathematics, Indian Institute of Technology Roorkee Roorkee-247667, India*

Received: January 12, 2017; Revised: October 23, 2017

Abstract: In this paper, we show the existence of (μ, ν)-pseudo almost automorphic mild solutions to some fractional differential equations in light of measure theory in a Banach space with new concept of Stepanov-like (μ, ν)-pseudo almost automorphy by virtue of Leray-Schauder alternate theorem.

Keywords: *almost automorphic and (μ, ν)-pseudo almost automorphic functions; Stepanov-like (μ, ν)-pseudo almost automorphic functions; mild solution; sectorial operator; solution operator; fractional differential equation.*

Mathematics Subject Classification (2010): 43A60, 26A33, 34C27, 28D05.

1 Introduction

In recent years, fractional differential equations with almost automorphic solutions have gained considerable interest. This is due to the fact that fractional differential equations are powerful tools to describe the hereditary properties and memory of various materials. Fractional differential equations have great applications in nonlinear oscillations of earthquakes, fractal theory, diffusion in porous media, viscoelastic panel in super sonic gas flow. For more details, we refer to the papers [2, 3, 8, 9, 18] and references therein.

The concept of almost automorphy was first introduced by Bochner [6]. Afterwards, being a most attractive topic in qualitative theory of differential equations, the theory of classical almost automorphy has been studied extensively by numerous authors and

* Corresponding author: mailto:vikramiitr1@gmail.com

generalized further in different ways using measure theory and weighted functions, see [4, 5, 14–16].

More recently, a new concept of the so-called (μ, ν)-pseudo almost automorphy was introduced by Diagana et. al. [10] and Abdelkarim et. al. [1], which is an interesting generalization of both μ-pseudo almost automorphy and weighted pseudo almost automorphy. Further, Chang et. al. [8] proposed the concept of Stepanov-like μ-pseudo almost automorphic mild solutions to semilinear functional differential equations. In this paper, stimulated by [1, 4, 8, 10], we will introduce the concept of Stepanov-like (μ, ν)-almost automorphic functions.

In this paper, we investigate the existence of (μ, ν)-pseudo almost automorphic mild solutions to the following fractional differential equation of order $1 < \eta < 2$,

$$D_t^{\eta} y(t) = Ay(t) + D_t^{\eta-1}\mathcal{F}\bigg(t, y(t), \int_{-\infty}^{t} \mathcal{K}(t-s)h(s, y(s))ds\bigg), \quad t \in \mathbb{R}, \tag{1}$$

where $A : D(A) \subseteq E \to E$ is a densely defined linear operator of sectorial type $\omega < 0$ on a complex Banach space E. The functions h, $\mathcal{F}$ are Stepanov-like (μ, ν)-pseudo almost automorphic. Here the derivative is taken in Riemann-Liouville sense and $\mathcal{K} \in L^1(\mathbb{R})$ with $|\mathcal{K}(t)| \leq C_{\mathcal{K}} e^{-bt}, b > 0$.

The rest of this paper is organized as follows: Section 2 provides some basic definitions, lemmas and theorems. In Section 3, we obtain main results by using Leray-Schauder alternate theorem fixed point theorem.

2 Preliminaries

Let $(E, \|\cdot\|)$ be a Banach space and $\mathbb{C}, \mathbb{R}$, and $\mathbb{N}$ stand for complex number, real number and natural numbers respectively. $C(\mathbb{R}, E)$ and $BC(\mathbb{R}, E)$ represent the sets of continuous functions and bounded continuous functions, respectively. For a linear operator A on E, let $\varrho(A), \rho(A), \mathcal{D}(A)$ and $\mathcal{R}(\mathcal{A})$ stand for the spectrum, the resolvent set, the domain and the range of A, respectively.

Now, we recall some definitions on fraction calculus (for more details, see [18]).

Definition 2.1 The fractional integral of a function $\phi : \mathbb{R}^+ \to E$ with the lower limit zero of order $\eta > 0$ is given by

$$I^{\eta}\phi(t) = \frac{1}{\Gamma(\eta)} \int_0^t (t-\tau)^{\eta-1}\phi(\tau)d\tau,$$

where $\Gamma(\cdot)$ denotes the Gamma function.

Definition 2.2 The Riemann-Liouville fractional derivative of a function $\phi : \mathbb{R}^+ \to E$ with the lower limit zero of order $\eta > 0$ is given by

$$D^{\eta}\phi(t) = \frac{1}{\Gamma(n-\eta)} \frac{d^n}{dt^n} \int_0^t (t-\tau)^{n-\eta-1}\phi(\tau)d\tau, \quad n-1 < \eta < n, n \in \mathbb{N}.$$

Definition 2.3 A densely defined closed linear operator A with domain $\mathcal{D}(A)$ in a Banach space E is said to be sectorial of type ω and angle θ if there exists

$$\theta \in (0, \frac{\pi}{2}), \quad \mathcal{M} > 0, \quad \omega \in \mathbb{R},$$

such that its resolvent exists outside the sector $\omega+\Sigma_\theta := \{\omega+\lambda : \lambda \in \mathbb{C}, |arg(-\lambda)| < \theta\}$, and

$$\|(\lambda - A)^{-1}\| \leq \frac{\mathcal{M}}{|\lambda-\omega|}, \quad \lambda \notin \omega + \Sigma_\theta.$$

It is easy to verify that an operator A is sectorial of type ω if and only if $\omega I - A$ is sectorial of type 0. For more details on sectorial operators see [13].

Definition 2.4 Let $1 < \eta < 2$ and A be a closed linear operator defined on the domain $\mathcal{D}(A)$ in a Banach space E. Then we say A is the generator of solution operator if there exists a $\omega \in \mathbb{R}$ and a strongly continuous function $\mathcal{S}_\eta : \mathbb{R}^+ \to \mathcal{L}(E)$ such that $\{\lambda^\eta : Re\lambda > \omega\} \subset \varrho(A)$ and

$$\lambda^{\eta-1}(\lambda^\eta - A)^{-1}y = \int_0^\infty e^{-\lambda t}\mathcal{S}_\eta(t)ydt, \quad Re\lambda > \omega,\ y \in E.$$

In this case, $\mathcal{S}_\eta(t)$ is called the solution operator generated by A and one can deduce that if A is sectorial of type ω with $0 < \theta < \pi(1-\frac{\eta}{2})$, then A generates the solution operator given by

$$\mathcal{S}_\eta(t)y = \frac{1}{2\pi i}\int_\Gamma e^{-\lambda t}\lambda^{\eta-1}(\lambda^\eta - A)^{-1}ydt, \tag{2}$$

where Γ is a suitable path lying outside the sector $\omega + \Sigma_\theta$ (see [9]).
Recently, Cuesta in [9] has shown that if A is a sectorial operator of type ω for some $\mathcal{M} > 0$ and $0 < \theta < \pi(1-\frac{\eta}{2})$, then there exists a constant $\mathcal{C} > 0$ depending solely on θ and η such that

$$\|\mathcal{S}_\eta(t)\|_{\mathcal{L}(E)} \leq \frac{\mathcal{C}\mathcal{M}}{1+|\omega|t^\eta}, \quad t \geq 0.$$

In boundary case, when $\eta = 1$, this is analogous to the statement that A is the generator of exponentially stable C_0-semigroup. Next, if $\eta > 1$, then solution family $\mathcal{S}_\eta(t)$ decays $t^{-\eta}$, in fact, $\mathcal{S}_\eta(t)$ is integrable on $(0,\infty)$ i.e.

$$\int_0^\infty \frac{1}{1+|\omega|s^\eta}ds = \frac{|\omega|^{-\frac{1}{\eta}}\pi}{\eta\sin(\frac{\pi}{\eta})}, \quad 1 < \eta < 2. \tag{3}$$

Definition 2.5 A continuous function $f : \mathbb{R} \to E$ is almost automorphic (in Bochner's sense) if for each sequence of real numbers $\{\tau'_n\}$, there exist a subsequence $\{\tau_n\}$ and a function $: \mathbb{R} \to E$ such that

$$g(t) = \lim_{n\to\infty} f(t+\tau_n), \quad \text{is well defined for each } t \in \mathbb{R}, \text{ and} \quad f(t) = \lim_{n\to\infty} g(t-\tau_n).$$

The set of all almost automorphic functions is denoted by $AA(E)$ and constitutes a Banach space endowed with the supnorm.

Definition 2.6 A function $f : \mathbb{R} \times E \to E$ is said to be almost automorphic if $f(\cdot, x) \in AA(\mathbb{R}, E)$ for all $x \in E$, and f is uniformly continuous in second variable on each compact set K of E. The set of all such functions is denoted by $AA(\mathbb{R} \times E, E)$.

Next we recall some definitons and basic results on Stepanov-like almost automorphic functions(for more details, see [8, 11, 20]).

Definition 2.7 The Bochner transform $f^b(t,s)$, $s \in [0,1], t \in \mathbb{R}$, of a function $f : \mathbb{R} \to E$ is defined by $f^b(t,s) = f(t+s)$.

Definition 2.8 The space of all Stepanov-like bounded functions denoted by $BS^p(\mathbb{R}, E)$ consists of all measurable functions $f : \mathbb{R} \to E$, with exponent $p \in [1, \infty)$ such that $f^b \in L^\infty(\mathbb{R}, L^p([0,1], E))$ and constitutes a Banach space with the norm

$$\|f\|_{S^p} = \|f^b\|_{L^\infty(\mathbb{R},L^p)} = \sup_{t\in\mathbb{R}} \left(\int_t^{t+1} \|f(\xi)\|^p d\xi \right)^{\frac{1}{p}}.$$

Definition 2.9 The space of Stepanov-like almost automorphic functions denoted by $S^pAA(\mathbb{R}, E)$ consists of all $f \in BS^p(\mathbb{R}, E)$ such that

$$f^b \in AA(\mathbb{R}, L^p([0,1], E)).$$

In other words, a function $f \in L^p_{loc}(\mathbb{R}, E)$ is a Stepanov-like almost automorphic function if its Bochner transform $f^b : \mathbb{R} \to L^p([0,1], E)$ is almost automorphic in the sense that every sequence $\{\tau'_n\}$ of real numbers contains a subsequence $\{\tau_n\}$ and a function $g \in L^p_{loc}([0,1], E)$ such that

$$\lim_{n\to\infty} \left[\int_t^{t+1} \|f(s+\tau_n) - g(s)\|^p ds \right]^{\frac{1}{p}} \to 0, \quad \text{and} \quad \lim_{n\to\infty} \left[\int_t^{t+1} \|g(s-\tau_n) - f(s)\|^p ds \right]^{\frac{1}{p}} \to 0,$$

for all $t \in \mathbb{R}$.

Definition 2.10 A function $f : \mathbb{R} \times E \to E$, with $f(\cdot, y) \in L^p(\mathbb{R}, E)$ for each $y \in K$ is said to be Stepanov-like almost automorphic function in $t \in \mathbb{R}$, uniformly for $y \in K$, if $t \to f(t,y)$ is Stepanov-like almost automorphic for each $y \in K$.

Remark 2.1 [7] It can be observed that if f is almost automorphic, then f is Stepanov-like almost automorphic, i.e. $AA(\mathbb{R}, E) \subset S^pAA(\mathbb{R}, E)$ [1]. Moreover, let $1 \le p \le q < \infty$, if $f \in S^qAA(\mathbb{R}, E)$ implies that $f \in S^pAA(\mathbb{R}, E)$.

Throughout this paper, we denote the Lebesgue σ-field of $\mathbb{R}$ by $\mathfrak{B}$, and the set of all positive measures μ on $\mathfrak{B}$ by $\mathfrak{M}$ satisfying $\mu(\mathbb{R}) = \infty$ and $\mu([a,b]) < \infty$, for all $a, b \in \mathbb{R}(a \le b)$.

Next, we define new ergodic space and the notion of Stepanov-like (μ,ν)-pseudo almost automorphic functions with positive measures $\mu, \nu \in \mathfrak{M}$.

Definition 2.11 [10] Let $\mu, \nu \in \mathfrak{M}$ and $p \in [1, \infty)$. A function $\psi \in BS^p(\mathbb{R}, E)$ is said to be (μ,ν)-ergodic if

$$\lim_{\gamma\to\infty} \frac{1}{\nu(\mathcal{Q}_\gamma)} \int_{\mathcal{Q}_\gamma} \left(\int_t^{t+1} \|\psi(s)\|^p ds \right)^{\frac{1}{p}} d\mu(t) = 0,$$

where $\mathcal{Q}_\gamma = [-\gamma, \gamma]$ and $\mu(\mathcal{Q}_\gamma) = \int_{\mathcal{Q}_\gamma} d\mu(t)$. We denote all such functions by $\mathcal{E}^p(\mathbb{R}, E, \mu, \nu)$.

Definition 2.12 Let $\mu, \nu \in \mathfrak{M}$. A function $f \in C(\mathbb{R}, E)$ is said to be (μ,ν)-pseudo almost automorphic function, if it can be decomposed as $f = \phi + \psi$, where $\phi \in AA(\mathbb{R}, E)$ and $\psi \in \mathcal{E}^1(\mathbb{R}, E, \mu, \nu)$. The collection of all such functions by $PAA(\mathbb{R}, E, \mu, \nu)$ is a Banach space equipped with sup norm.

Definition 2.13 Let $\mu, \nu \in \mathfrak{M}$. A function $f \in BS^p(\mathbb{R}, E)$ is said to be Stepanov-like (μ,ν)-pseudo almost automorphic function, if it can be decomposed as $f = \phi + \psi$, where $\phi \in S^pAA(\mathbb{R}, E)$ and $\psi \in \mathcal{E}^p(\mathbb{R}, E, \mu, \nu)$. We denote the collection of all such functions by $S^pPAA(\mathbb{R}, E, \mu, \nu)$.

Definition 2.14 [10] A continuous function $f : \mathbb{R} \times E \to E$ is said to be (μ,ν)-ergodic in $t \in \mathbb{R}$ uniformly with respect to $y \in E$, if the following conditions are true:

(i) $f(.,y) \in \mathcal{E}^p(\mathbb{R} \times E, E, \mu, \nu)$, for all $y \in E$,

(ii) The function $f(.,y)$ is uniformly continuous with the second variable in a compact set K in E.

We denote the collection of all such functions by $\mathcal{E}^pU(\mathbb{R} \times E, E, \mu, \nu)$.

Definition 2.15 The function $f \in BS^p(\mathbb{R} \times E, E)$ is said to be Stepanov-like (μ,ν)-pseudo almost automorphic, if it has decomposition of the form $f = \phi + \psi$, where $\phi \in S^pAAU(\mathbb{R} \times E, E)$ and $\psi \in \mathcal{E}^pU(\mathbb{R} \times E, E, \mu, \nu)$. We denote the set of all such functions by $S^pPAAU(\mathbb{R} \times E, E, \mu, \nu)$.

We assume the following:

(M_1) Let $\mu, \nu \in \mathfrak{M}$, then $\lim\limits_{\gamma\to\infty} \dfrac{\mu(\mathcal{Q}_\gamma)}{\nu(\mathcal{Q}_\gamma)} < \infty$.

(M_2) For all $s \in \mathbb{R}$ and $\nu \in \mathfrak{M}$, there exist a bounded interval I and $\alpha > 0$ such that $\mu(\{a+s, a \in D\}) \le \alpha\mu(D)$ if $D \in \mathfrak{B}$ satisfies $D \cap I = \emptyset$.

Theorem 2.1 *[10] Assume that $\mu, \nu \in \mathfrak{M}$ and $(M_1) - (M_2)$ hold. Then $S^pPAA(\mathbb{R}, E, \mu, \nu)$ is translation invariant and the set $(S^pPAA(\mathbb{R}, E, \mu, \nu), \|.\|_{S^p})$ is the Banach space.*

Theorem 2.2 *Let $\mu, \nu \in \mathfrak{M}$, $f = \phi + \psi \in S^pPAAU(\mathbb{R} \times E \times E, E, \mu, \nu)$ with $\phi \in S^pAAU(\mathbb{R}\times E\times E, E)$, $\psi \in \mathcal{E}^pU(\mathbb{R}\times E\times E, E, \mu, \nu)$. Suppose that the following conditions hold:*

(i) *ϕ is uniformly continuous on a bounded subset $\Omega \subset E \times E$ for all $t \in \mathbb{R}$.*

(ii) *f is uniformly continuous on a bounded subset $\Omega \subset E \times E$ for all $t \in \mathbb{R}$.*

(iii) *$\xi = \alpha + \beta, \chi = u + v \in S^pPAA(\mathbb{R}, E, \mu, \nu)$ with $\alpha, u \in S^pAA(\mathbb{R}, E)$ and $\beta, v \in \mathcal{E}^p(\mathbb{R}, E, \mu, \nu)$ and $\overline{\{\alpha(t) \in \mathbb{R}\}}, \overline{\{u(t) \in \mathbb{R}\}}$ are compact in E.*

Then $t \mapsto f(t, \xi(t), \chi(t)) \in S^pPAA(\mathbb{R}, E, \mu, \nu)$.

Proof. The proof is similar to the proof of Theorem 3.2 in [21] and hence the details are omitted here.

Lemma 2.1 *Let $y = y_1 + y_2, \in S^pPAA(\mathbb{R}, E, \mu, \nu)$ and $\mathcal{R}y = \overline{\{y_1(t) : t \in \mathbb{R}\}}$ be a compact set in E. Suppose that $h = \phi + \psi, \in S^pPAAU(\mathbb{R} \times E, E, \mu, \nu)$, with $\phi \in S^pAAU(\mathbb{R} \times E, E)$, $\psi \in \mathcal{E}^pU(\mathbb{R} \times E, E, \mu, \nu)$ satisfying*

$$\|h(t,y) - h(t,z)\| \le L_h\|y - z\| \quad \text{and} \quad \|\phi(t,y) - \phi(t,z)\| \le L_\phi\|y - z\|, \quad y, z \in E,\ t \in \mathbb{R},$$

where $L_\phi, L_h > 0$ are constants. Then

$$\Psi_h(t) := \int_{-\infty}^{t} \mathcal{K}(t-s)h(s, y(s))ds \in S^pPAA(\mathbb{R}, E, \mu, \nu). \tag{4}$$

Proof. The proof is similar to the proof of Lemma 3.2 in [19] and hence the details are omitted here.

Lemma 2.2 *Let* (M_1) *and* (M_2) *hold and let* $f \in S^p PAA(\mathbb{R}, E, \mu, \nu)$. *Then the function is defined by*

$$\Lambda_f(t) = \int_{-\infty}^{t} \mathcal{S}_\eta(t-s) f(s) ds \in PAA(\mathbb{R}, E, \mu, \nu).$$

Proof. Since $f \in S^p PAA(\mathbb{R}, E, \mu, \nu)$, there exist $\phi \in S^p AA(\mathbb{R}, E)$ and $\psi \in \mathcal{E}^p(\mathbb{R}, E, \mu, \nu)$, such that $f(t) = \phi(t) + \psi(t)$. Now consider

$$\Lambda_f(t) = \int_{-\infty}^{t} \mathcal{S}_\eta(t-s) f(s) ds = \Lambda_\phi(t) + \Lambda_\psi(t),$$

where

$$\Lambda_\phi(t) = \int_{-\infty}^{t} \mathcal{S}_\eta(t-s)\phi(s) ds \quad \text{and} \quad \Lambda_\psi(t) = \int_{-\infty}^{t} \mathcal{S}_\eta(t-s)\psi(s) ds.$$

First, we show $\Lambda_\phi \in AA(\mathbb{R}, E)$. Define a sequence of integral operators for $n = 1, 2, 3, \ldots,$

$$\Lambda_\phi^n(t) = \int_{t-n}^{t-n+1} \mathcal{S}_\eta(t-s)\phi(s) ds.$$

Using Holder's inequality, we have $\|\Lambda_\phi^n(t)\| < \infty$. Now by Weierstrass' theorem, the series $\Lambda_\phi(t) = \sum_{n=1}^{\infty} \Lambda_\phi^n = \int_{-\infty}^{t} \mathcal{S}_\eta(t-s)\phi(s) ds$ converges uniformly on $\mathbb{R}$. Moreover,

$$\|\Lambda_\phi(t)\| \le \sum_{n=1}^{\infty} \|\Lambda_\phi^n\| \le \|\phi_n\|_{S^p} \mathcal{C}\mathcal{M} \sum_{n=1}^{\infty} \left(\frac{1}{1 + |\omega|(n-1)^\eta} \right) < \infty \Rightarrow \Lambda_\phi \in C(\mathbb{R}, E).$$

Further, for $n = 1, 2, 3, \ldots$ we show that $\Lambda_\phi^n \in AA(\mathbb{R}, E)$. Since $\phi \in S^p AA(\mathbb{R}, E)$, this implies that every sequence $\{\tau_n'\}$ of real numbers contains a subsequence $\{\tau_n\}$ and a function $\widetilde{\phi} \in L^p_{loc}([0,1], E)$ such that

$$\left[\int_t^{t+1} \|\phi(s+\tau_n) - \widetilde{\phi}(s)\|^p ds \right]^{\frac{1}{p}} \to 0, \quad \text{and} \quad \left[\int_t^{t+1} \|\widetilde{\phi}(s-\tau_n) - \phi(s)\|^p ds \right]^{\frac{1}{p}} \to 0, \quad (5)$$

as $n \to 0$ and $t \in \mathbb{R}$. Consider

$$\begin{aligned} \|\Lambda_\phi^n(t+\tau_n) - \Lambda_{\widetilde{\phi}}^n(t)\| \le & \int_{n-1}^{n} \|\mathcal{S}_\eta(s)[\phi(t+\tau_n - s) - \widetilde{\phi}(t-s)]\| ds \\ \le & \left(\int_{n-1}^{n} \|\mathcal{S}_\eta(s)\|^q \right)^{\frac{1}{q}} \left(\int_{n-1}^{n} \|\phi(t+\tau_n - s) - \widetilde{\phi}(t-s)\|^p \right)^{\frac{1}{p}} \\ \le & \mathcal{C}\mathcal{M} \left(\frac{1}{1 + |\omega|(n-1)^\eta} \right) \left(\int_{n-1}^{n} \|\phi(t+\tau_n - s) - \widetilde{\phi}(t-s)\|^p \right)^{\frac{1}{p}}. \end{aligned}$$

It is obvious from (5), that the last inequality goes to 0 as $n \to \infty$ on $\mathbb{R}$. Similarly one can show that

$$\|\Lambda_{\widetilde{\phi}}(s-\tau_n) - \Lambda_\phi(s)\| \to 0, \tag{6}$$

as $n \to \infty$ on $\mathbb{R}$. Thus we conclude that $\Lambda_\phi \in S^p AA(\mathbb{R}, E)$.

Next, we show that $\Lambda_\psi \in \mathcal{E}(\mathbb{R}, E, \mu, \nu)$. To complete this task we consider the integral operator for $n = 1, 2, 3, \ldots$

$$\Lambda_\psi^n(t) = \int_{t-n}^{t-n+1} \mathcal{S}_\eta(t-s)\psi(s)ds = \int_{n-1}^{n} \mathcal{S}_\eta(s)\psi(t-s)ds.$$

Now, we get

$$\begin{aligned}\|\Lambda_\psi^n(t)\| \leq & \Big(\int_{n-1}^{n} \|\mathcal{S}_\eta(s)\|^q ds\Big)^{\frac{1}{q}} \Big(\int_{n-1}^{n} \|\psi(t-s)\|^p ds\Big)^{\frac{1}{p}} \\ \leq & \|\psi\|_{S^p} \mathcal{CM} \Bigg[\int_{n-1}^{n} \Big(\frac{1}{1+|\omega|(s)^\eta}\Big)^q ds\Bigg]^{\frac{1}{q}} \\ \leq & \|\psi\|_{S^p} \mathcal{CM} \Bigg[\frac{1}{1+|\omega|(n-1)^\eta}\Bigg] \\ < & \infty,\end{aligned}$$

where $q = p/(p-1)$. Further, for $\gamma > 0$,

$$\begin{aligned}&\lim_{\gamma\to\infty} \frac{1}{\nu(\mathcal{Q}_\gamma)} \int_{\mathcal{Q}_\gamma} \|\Lambda_\psi^n(t)\| d\mu(t) \\ &\qquad \leq \frac{\mathcal{CM}}{1+|\omega|(n-1)^\eta} \lim_{\gamma\to\infty} \frac{1}{\nu(\mathcal{Q}_\gamma)} \int_{\mathcal{Q}_\gamma} \Big(\int_{t-n}^{t-n+1} \|\psi(s)\|^p ds\Big)^{\frac{1}{p}} d\mu(t).\end{aligned}$$

Since $\psi \in \mathcal{E}^p(\mathbb{R}, E, \mu, \nu)$, the above estimation leads to $\Lambda_\psi^n \in \mathcal{E}^p(\mathbb{R}, E, \mu, \nu)$ for $n = 1, 2, 3, \ldots$. The above inequality also implies that the series $\mathcal{CM} \sum_{n=1}^{\infty} \Big[\frac{1}{1+|\omega|(n-1)^\eta}\Big]$ is convergent, then we deduce in view of Weierstrass test that the series $\sum_{n=1}^{\infty} \Lambda_\psi^n(t)$ converges uniformly on $\mathbb{R}$ and

$$\Lambda_\psi(t) = \sum_{n=1}^{\infty} \Lambda_\psi^n(t) = \int_{\infty}^{t} \mathcal{S}_\eta(t-s)\psi(s)ds.$$

Further, from $\Lambda_\psi^n \in \mathcal{E}^p(\mathbb{R}, E, \mu, \nu)$ and

$$\begin{aligned}\frac{1}{\nu(\mathcal{Q}_\gamma)} \int_{\mathcal{Q}_\gamma} \|\Lambda(t)\| d\mu(t) \leq & \frac{\mathcal{CM}}{1+|\omega|(n-1)^\eta} \frac{1}{\nu(\mathcal{Q}_\gamma)} \int_{\mathcal{Q}_\gamma} \Big\|\Lambda_\psi(s) - \sum_{n=1}^{N} \Lambda_\psi^n(s)\Big\| d\mu(s) \\ & + \sum_{n=1}^{N} \frac{\mathcal{CM}}{1+|\omega|(n-1)^\eta} \frac{1}{\nu(\mathcal{Q}_\gamma)} \int_{\mathcal{Q}_\gamma} \|\Lambda_\psi^n(s)\| d\mu(s),\end{aligned}$$

it follows that uniform limit $\Lambda(t) = \sum_{n=1}^{\infty} \Lambda_{\psi}^{n}(t) \in \mathcal{E}(\mathbb{R}, E, \mu, \nu)$.

Now, before moving further we briefly describe compactness criteria and the Leray-Schauder alternate theorem. Let $\mathcal{H} : \mathbb{R} \to \mathbb{R}$ be continuous such that $\mathcal{H}(t) \to \infty$ as $|t| \to \infty$ and $\mathcal{H}(t) \geq 1$ for all $t \in \mathbb{R}$. We define a Banach space

$$C_{\mathcal{H}}(\mathbb{R}, E) = \{v \in C(\mathbb{R}, E) : \lim_{|t|\to\infty} v(t)/\mathcal{H}(t) = 0\},$$

equipped with the norm $\|v\|_{\mathcal{H}} = \sup_{t\in\mathbb{R}}(\|v(t)\|/\mathcal{H}(t))$.

Lemma 2.3 *[17] A set $K \subseteq C_{\mathcal{H}}(\mathbb{R}, E)$ is relative compact in $C_{\mathcal{H}}(\mathbb{R}, E)$, if the following conditions hold:*

(a_1) *The set $K(t) = \{v(t) : v \in K, t \in \mathbb{R}\}$ is relative compact in E.*

(a_2) *The set K is equicontinuous.*

(a_3) *For each $\epsilon > 0$, there exists a constant $L > 0$ such that $\|v(t)\|_{\mathcal{H}} \leq \epsilon\mathcal{H}(t)$ for all $|t| > L$ and $u \in K$.*

Lemma 2.4 *([12]Leray-Schauder Alternate Theorem) Let $\mathcal{D}$ be a closed convex subset of a Banach space E such that $0 \in \mathcal{D}$. Let $f : \mathcal{D} \to \mathcal{D}$ be a completely continuous map. Then the set $\{y \in \mathcal{D} : y = \lambda f(y), 0 < \lambda < 1\}$ is unbounded or the map f has a fixed point in $\mathcal{D}$.*

3 Main Results

In this section, we investigate the existence of (μ, ν)-pseudo almost automorphic mild solutions to (1).

Definition 3.1 [2] A function $y \in C(\mathbb{R}, E)$ is said to be a mild solution of (1) if the function $s \mapsto \mathcal{S}_{\eta}(s)\mathcal{F}(s, y(s), \Psi y(s))$ is integrable on $(-\infty, s)$ for each $s \in \mathbb{R}$ and

$$y(t) = \int_{-\infty}^{t} \mathcal{S}_{\eta}(t-s)\mathcal{F}(s, y(s), \Psi_h y(s))ds,$$

where $\mathcal{S}_{\eta}(t)$ is a solution operator and Ψ_h is defined by $\Psi_h y(t) = \int_{-\infty}^{t} \mathcal{K}(t-s)h(s, y(s))ds$.

To establish the existence results, we consider the following assumptions:

(L_1) Suppose that $\mathcal{F} = \phi + \psi \in S^p PAAU(\mathbb{R} \times E \times E, E, \mu, \nu)$ with $\phi \in S^p AAU(\mathbb{R} \times E \times E, E)$, $\psi \in \mathcal{E}^p U(\mathbb{R} \times E \times E, E, \mu, \nu)$ is uniformly continuous on a bounded set $V \subset X \times X$ for all $t \in \mathbb{R}$ and $\{\mathcal{F}(t, y, z) : y, z \in V\}$ is bounded in $S^p PAAU(\mathbb{R} \times E \times E, E, \mu, \nu)$.

(L_2) There exist a nondecreasing continuous function $\mathcal{W} : [0, \infty) \to [0, \infty)$ such that

$$\|\mathcal{F}(t, y, z)\| \leq \mathcal{W}(\|y\| + \|z\|), \quad \text{for each } t \in \mathbb{R}, \quad y, z \in E.$$

Theorem 3.1 *Let A be a sectorial operator of type $\omega < 0$ and (M_1) and (M_2) hold. Assume that $\mathcal{F} : \mathbb{R} \times E \times E \to E$ is a function satisfying (L_1) and (L_2) and the following additional conditions hold:*

(L_3) *For $k, a \geq 0$,*

$$\lim_{|t|\to\infty} \int_{-\infty}^{t} \frac{\mathcal{W}((1+k)a\mathcal{H}(s))}{1+|\omega|(t-s)^{\eta}} ds = 0,$$

where $\mathcal{H}$ is defined in Lemma 2.3. We set

$$\beta(a) := \mathcal{C}\mathcal{M}\left\| \int_{-\infty}^{t} \frac{\mathcal{W}((1+k)a\mathcal{H}(s))}{1+|\omega|(t-s)^{\eta}} ds \right\|.$$

(L_4) *For every $y, z \in C_{\mathcal{H}}(\mathbb{R}, E)$ and each $\epsilon > 0$ there exists a $\delta > 0$ such that $\|y - z\| \leq \delta$ implies that*

$$\mathcal{C}\mathcal{M} \int_{-\infty}^{t} \frac{\|\mathcal{F}(s, y(s), \Psi_h y(s)) - \mathcal{F}(s, z(s), \Psi_h z(s))\|}{1+|\omega|(t-s)^{\eta}} ds \leq \epsilon.$$

(L_5) *$lim \inf_{s\to\infty} \dfrac{s}{\beta(s)} > 1.$*

(L_6) *The set $\{f(s, y(s), \Psi_h y(s)) : c \leq s \leq d, y \in C_{\mathcal{H}}, \|y\|_{\mathcal{H}} \leq \lambda\}$ is relatively compact in E for $c, d \in \mathbb{R}, c < d$ and $\lambda > 0$.*

Then equation (1) admits a (μ, ν)-pseudo almost automorphic mild solution.

Proof. Let us define an operator $\Lambda_{\mathcal{F}} : C_{\mathcal{H}}(\mathbb{R}, E) \to C_{\mathcal{H}}(\mathbb{R}, E)$ by

$$\Lambda_{\mathcal{F}} y(t) = \int_{-\infty}^{t} \mathcal{S}_{\eta}(t-s)\mathcal{F}(s, y(s), \Psi_h y(s)) ds.$$

Now, we need only to show that $\Lambda_{\mathcal{F}}$ has a fixed point in $PAA(\mathbb{R}, E, \mu, \nu)$. For the sake of convenience, we provide the proof in several steps.
Step 1 :$\Lambda_{\mathcal{F}}$ is well defined.
For $y \in C_{\mathcal{H}}(\mathbb{R}, E)$ with (L_1) we have

$$\begin{aligned}\|\Lambda_{\mathcal{F}} y(t)\| \leq& \mathcal{C}\mathcal{M} \int_{-\infty}^{t} \frac{\mathcal{W}(\|y(s)\| + \|\Psi_h y(s)\|)}{1+|\omega|(t-s)^{\eta}} ds \\ \leq& \mathcal{C}\mathcal{M} \int_{-\infty}^{t} \frac{\mathcal{W}[(1+\|\Psi_h\|)\|y\|_{\mathcal{H}}\mathcal{H}(s)]}{1+|\omega|(t-s)^{\eta}} ds.\end{aligned}$$

Hence by (L_3) $\Lambda_{\mathcal{F}}$ is well defined.
Step 2 : The operator $\Lambda_{\mathcal{F}}$ is continuous. In fact, let $y, z \in C_{\mathcal{H}}(\mathbb{R}, E)$. For any $\epsilon > 0$ we take $\delta > 0$ such that $\|y - z\| \leq \delta$, then

$$\|\Lambda_{\mathcal{F}} y(t) - \Lambda_{\mathcal{F}} z(t)\| \leq \mathcal{C}\mathcal{M} \int_{-\infty}^{t} \frac{\|\mathcal{F}(s, y(s), \Psi_h y(s)) - \mathcal{F}(s, z(s), \Psi_h z(s))\|}{1+|\omega|(t-s)^{\eta}} ds \leq \epsilon,$$

which shows the assertion.
Step 3 : Next, we show that $\Lambda_{\mathcal{F}}$ is completely continuous. Let $B_{\lambda}(E)$ denote a closed

ball in a space E with radius λ and center at 0. Let us denote $U = \Lambda_{\mathcal{F}}(B_\lambda(C_{\mathcal{H}}(E)))$ and $w = \Lambda_{\mathcal{F}}(v)$ for $v \in B_\lambda(C_{\mathcal{H}}(E))$. Now, we show that U is a relative compact subset of E. The condition (L_3) implies that $\frac{\mathcal{W}((1+\|\Psi_h\|)\lambda\mathcal{H}(t-s))}{1+|\omega|(s)^\eta}$ is integrable on $[0,\infty)$. Hence, for $\epsilon > 0$, we can chose $\alpha \geq 0$ such that $\mathcal{CM}\int_0^\infty \frac{\mathcal{W}((1+\|\Psi_h\|)\lambda\mathcal{H}(t-s))}{1+|\omega|(s)^\eta}ds \leq \epsilon$.

Since

$$\begin{aligned}w(t) = &\int_0^\alpha \mathcal{S}_\eta(s)\mathcal{F}(t-s, y(t-s), \Psi_h y(t-s))ds\\ &+ \int_\alpha^\infty \mathcal{S}_\eta(s)\mathcal{F}(t-s, y(t-s), \Psi_h y(t-s))ds,\end{aligned}$$

and

$$\left\| \int_0^\alpha \mathcal{S}_\eta(s)\mathcal{F}(t-s, y(t-s), \Psi_h y(t-s))ds \right\| \leq \mathcal{CM}\int_\alpha^\infty \frac{\mathcal{W}((1+\|\Psi_h\|)a\mathcal{H}(t-s))}{1+|\omega|(s)^\eta}ds \leq \epsilon,$$

we deduce that $w(t) \in \alpha\overline{C_0(M)} + B_\epsilon(E)$, where $C_0(M)$ denotes the convex hull of M and

$$M = \{\mathcal{S}_\eta(s)f(\xi, (\xi)y, \Psi_h(\xi)y) : 0 \leq s \leq \alpha, t-\alpha \leq \xi \leq t, \|y\|_{\mathcal{H}} \leq \lambda\}.$$

By the strong continuity of $\mathcal{S}_\eta$ and (L_6) we deduce that M is relatively compact set and $U \in \alpha\overline{C_0(M)}$ $+B_\epsilon(E)$ which establishes the assertion.

Further, we show that U is equicontinuous. In fact, we can decompose

$$\begin{aligned}w(t+h) - w(t) = &\int_0^h \mathcal{S}_\eta(s)\mathcal{F}(t+h-s, y(t+h-s), \Psi_h y(t+h-s))ds\\ &+ \int_0^\alpha [\mathcal{S}_\eta(h+s) - \mathcal{S}_\eta(s)]\mathcal{F}(t-s, y(t-s), \Psi_h y(t-s))ds\\ &+ \int_\alpha^\infty [\mathcal{S}_\eta(h+s) - \mathcal{S}_\eta(s)]\mathcal{F}(t-s, y(t-s), \Psi_h y(t-s))ds.\end{aligned}$$

For each $\epsilon > 0$, we can take $\alpha > 0$ and δ_1 such that

$$\begin{aligned}\Big\| \int_0^h &\mathcal{S}_\eta(s)\mathcal{F}(t+h-s, y(t+h-s), \Psi_h y(t+h-s))ds\\ &+ \int_\alpha^\infty [\mathcal{S}_\eta(h+s) - \mathcal{S}_\eta(s)]\mathcal{F}(t-s, y(t-s), \Psi_h y(t-s))\Big\|\\ &\leq \mathcal{CM}\Big[\int_0^s \frac{\mathcal{W}((1+\|\Psi_h\|)\lambda\mathcal{H}(t+h-s))}{1+|\omega|(s)^\eta}ds\\ &\quad + \int_\alpha^\infty \frac{\mathcal{W}((1+\|\Psi_h\|)\lambda\mathcal{H}(t-s))}{1+|\omega|(s)^\eta}ds\Big] \leq \frac{\epsilon}{2},\end{aligned}$$

for $h \leq \delta_1$. Moreover, since $\mathcal{S}_\eta$ is strongly continuous and $\{\mathcal{F}(t-s, y(t-s), \Psi_h y(t-s)) : 0 \leq s \leq \alpha, y \in (B_\lambda(C_{\mathcal{H}}(E)))\}$is relative compact, we can take $\delta_2 > 0$ such that

$$\|[\mathcal{S}_\eta(h+s) - \mathcal{S}_\eta(s)]\mathcal{F}(t-s, y(t-s), \Psi_h y(t-s))\| \leq \frac{\epsilon}{2\alpha},$$

for $h \leq \delta_2$. We have from the above estimation that $\|w(t+h) - w(t)\| \leq \epsilon$ for small ϵ and is independent of $y \in B_\lambda(C_{\mathcal{H}}(E))$.

Finally, from (L_3) we deduce

$$\frac{\|w(t)\|}{\mathcal{H}(t)} \leq \frac{\mathcal{C}\mathcal{M}}{\mathcal{H}(t)} \int_0^\infty \frac{\mathcal{W}((1+\|\Psi_h\|)\lambda\mathcal{H}(s))}{1+|\omega|(t-s)^\eta} ds \to 0, \quad \text{as} \quad |t| \to \infty,$$

uniformly and is independent of $y \in B_\lambda(C_\mathcal{H}(E))$. Thus, by Lemma 2.3, U is a relatively compact set in $C_\mathcal{H}(E)$.
Step 4 : Let for some $0 < \tau < 1$, $y^\tau(\cdot)$ be a solution of the equation $y = \tau\Lambda_\mathcal{F}(y^\tau)$.

Then, we have the estimate

$$\begin{aligned} \|y^\tau(t)\| \leq & \tau \int_{-\infty}^t \|\mathcal{S}_\eta(t-s)\mathcal{F}(s, y^\tau(s), \Psi_h y^\tau(s))\| ds \\ \leq & \mathcal{C}\mathcal{M} \int_{-\infty}^t \frac{\mathcal{W}[(1+\|\Psi_h\|)\|y^\tau\|_\mathcal{H}\mathcal{H}(s)]}{1+|\omega|(t-s)^\eta} ds \\ \leq & \beta(\|y^\tau\|_\mathcal{H})\mathcal{H}(t). \end{aligned}$$

It leads to

$$\frac{\|y^\tau(t)\|}{\beta(\|y^\tau\|_\mathcal{H})} < 1.$$

We deduce from the above relation and (L_5) that the set $\{y^\tau : y^\tau = \tau\Lambda_\mathcal{F}(y^\tau), 0 < \tau < 1\}$ is a bounded set.
Step 5 : We deduce form Remark 2.1, (L_1) and Theorem 2.2 that the function $t \mapsto \mathcal{F}(t, y(t), \Psi_h y(t)) \in S^p PAA(\mathbb{R}, E, \mu, \nu)$, whenever $y \in PAA(\mathbb{R}, E, \mu, \nu) \subset S^p PAA(\mathbb{R}, E, \mu, \nu)$. Further, by Lemma 2.2, we get $\Lambda_\mathcal{F}(PAA(\mathbb{R}, E, \mu, \nu)) \subset PAA(\mathbb{R}, E, \mu, \nu)$ and notice that $PAA(\mathbb{R}, E, \mu, \nu)$ is a closed subspace of $C_\mathcal{H}(\mathbb{R}, E)$. Now, using the Steps $1-4$, we obtain that the map $\Lambda_\mathcal{F}$ is completely continuous. Applying Lemma 2.4, we infer that mapping $\Lambda_\mathcal{F}$ has a fixed point in $PAA(\mathbb{R}, E, \mu, \nu)$.

Acknowledgment

The authors would like to express their sincere gratitude to the reviewers for their helpful and valuable comments in improving the presentation and quality of the paper. This research was completed with the support of "Ministry of Human Resource and Development, India under grant number: MHR-02-23-200-44".

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