

## NONLINEAR DYNAMICS AND SYSTEMS THEORY

An International Journal of Research and Surveys

Volume 17 Number 4 2017

## CONTENTS

Random Impulsive Partial Hyperbolic Fractional Differential Equations.....	327
<i>Mouffak Benchohra and Amel Heris</i>	
On the Hyers-Ulam Stability of Laguerre and Bessel Equations by Laplace Transform Method.....	340
<i>E. Bicer and C. Tunc</i>	
Output Tracking of Some Class Non-minimum Phase Nonlinear Uncertain Systems .....	347
<i>Firman, Janson Naiborhu, Roberd Saragih and Sapto Wahyu Indratno</i>	
Passivity Based Control of Continuous Bioreactors.....	357
<i>M. Makkar and J.-Y. Dieulot</i>	
Mathematical Models of Nonlinear Oscillations of Mechanical Systems with Several Degrees of Freedom.....	369
<i>G.I. Melnikov, N.A. Dudarenko, K.S. Malykh, L.N. Ivanova and V.G. Melnikov</i>	
Existence Results for Mild Solution for a Class of Impulsive Fractional Stochastic Problems with Nonlocal Conditions .....	376
<i>Mohd Nadeem and Jaydev Dabas</i>	
Refinements of Some Pachpatte and Bihari Inequalities on Time Scales .....	388
<i>B. Ben Nasser, K. Boukerrioua, M. Defoort, M. Djemai and M.A. Hammami</i>	
Boundedness of the New Modified Hyperchaotic Pan System .....	402
<i>Samia Rezzag</i>	
Pseudo Almost Automorphic Mild Solutions to Some Fractional Differential Equations with Stepanov-like Pseudo Almost Automorphic Forcing Term .....	409
<i>Vikram Singh and Dwijendra N. Pandey</i>	
Nonlinear Parabolic Equations with Singular Coefficient and Diffuse Data .....	421
<i>K. Zaki and H. Redwane</i>	
Contents of Volume 17, 2017 .....	433

NONLINEAR DYNAMICS &amp; SYSTEMS THEORY

Volume 17, No. 4, 2017

# Nonlinear Dynamics and Systems Theory

An International Journal of Research and Surveys

EDITOR-IN-CHIEF A.A.MARTYNYUK

*S.P.Timoshenko Institute of Mechanics  
National Academy of Sciences of Ukraine, Kiev, Ukraine*

## REGIONAL EDITORS

P.BORNE, Lille, France  
M.FABRIZIO, Bologna, Italy  
*Europe*

M.BOHNER, Rolla, USA  
*North America*

T.A.BURTON, Port Angeles, USA  
C.CRUIZ-HERNANDEZ, Ensenada, Mexico  
*USA, Central and South America*

AI-GUO WU, Harbin, China  
*China and South East Asia*

K.L.TEO, Perth, Australia  
*Australia and New Zealand*

# Nonlinear Dynamics and Systems Theory

An International Journal of Research and Surveys

## EDITOR-IN-CHIEF A.A.MARTYNYUK

The S.P.Timoshenko Institute of Mechanics, National Academy of Sciences of Ukraine,  
Nesterov Str. 3, 03680 MSP, Kiev-57, UKRAINE / e-mail: journalndst@gmail.com

## MANAGING EDITOR I.P.STAVROULAKIS

Department of Mathematics, University of Ioannina  
451 10 Ioannina, HELLAS (GREECE) / e-mail: ipstav@cc.uoi.gr

## REGIONAL EDITORS

AI-GUO WU (China), e-mail: agwu@163.com  
P.BORNE (France), e-mail: Pierre.Borne@ec-lille.fr  
M.BOHNER (USA), e-mail: bohner@mst.edu  
T.A.BURTON (USA), e-mail: taburton@olypen.com  
C. CRUZ-HERNANDEZ (Mexico), e-mail: ccruz@cicese.mx  
M.FABRIZIO (Italy), e-mail: mauro.fabrizio@unibo.it  
K.L.TEO (Australia), e-mail: K.L.Teo@curtin.edu.au

## EDITORIAL BOARD

Aleksandrov, A.Yu. (Russia)	Kokologiannaki, C. (Greece)
Artstein, Z. (Israel)	Lamarque C.-H. (France)
Awrejcewicz J. (Poland)	Lazar, M. (The Netherlands)
Benrejeb M. (Tunisia)	Leonov, G.A. (Russia)
Bevilaqua, R. (USA)	Limarchenko, O.S. (Ukraine)
Braiek, N.B. (Tunisia)	Lopez-Gutierrez, R.M. (Mexico)
Chen Ye-Hwa (USA)	Nguang Sing Kiong (New Zealand)
Corduneanu, C. (USA)	Okninski, A. (Poland)
D'Anna, A. (Italy)	Peng Shi (Australia)
De Angelis, M. (Italy)	Peterson, A. (USA)
Dshalalow, J.H. (USA)	Radziszewski, B. (Poland)
Eke, F.O. (USA)	Shi Yan (Japan)
Enciso G. (USA)	Siljak, D.D. (USA)
Georgiou, G. (Cyprus)	Sivasundaram S. (USA)
Guang-Ren Duan (China)	Sree Hari Rao, V. (India)
Honglei Xu (Australia)	Stavrakakis, N.M. (Greece)
Izobov, N.A. (Belarussia)	Vassilyev, S.N. (Russia)
Karimi, H.R. (Italy)	Vatsala, A. (USA)
Khusainov, D.Ya. (Ukraine)	Wang Hao (Canada)
Kloeden, P. (Germany)	Zuyev A.L. (Germany)

## ADVISORY EDITOR

A.G.MAZKO, Kiev, Ukraine  
e-mail: mazko@imath.kiev.ua

## ADVISORY COMPUTER SCIENCE EDITORS

A.N.CHERNIENKO and L.N.CHERNETSKAYA, Kiev, Ukraine

## ADVISORY LINGUISTIC EDITOR

S.N.RASSHYVALOVA, Kiev, Ukraine

© 2017, InforMath Publishing Group, ISSN 1562-8353 print, ISSN 1813-7385 online, Printed in Ukraine  
No part of this Journal may be reproduced or transmitted in any form or by any means without  
permission from InforMath Publishing Group.

## INSTRUCTIONS FOR CONTRIBUTORS

**(1) General.** Nonlinear Dynamics and Systems Theory (ND&ST) is an international journal devoted to publishing peer-refereed, high quality, original papers, brief notes and review articles focusing on nonlinear dynamics and systems theory and their practical applications in engineering, physical and life sciences. Submission of a manuscript is a representation that the submission has been approved by all of the authors and by the institution where the work was carried out. It also represents that the manuscript has not been previously published, has not been copyrighted, is not being submitted for publication elsewhere, and that the authors have agreed that the copyright in the article shall be assigned exclusively to InforMath Publishing Group by signing a transfer of copyright form. Before submission, the authors should visit the website:

<http://www.e-ndst.kiev.ua>

for information on the preparation of accepted manuscripts. Please download the archive Sample\_NDST.zip containing example of article file (you can edit only the file Samplefilename.tex).

**(2) Manuscript and Correspondence.** Manuscripts should be in English and must meet common standards of usage and grammar. To submit a paper, send by e-mail a file in PDF format directly to

Professor A.A. Martynyuk, Institute of Mechanics,  
Nesterov str.3, 03057, MSP 680, Kiev-57, Ukraine  
e-mail: journalndst@gmail.com

or to one of the Regional Editors or to a member of the Editorial Board. Final version of the manuscript must typeset using LaTeX program which is prepared in accordance with the style file of the Journal. Manuscript texts should contain the title of the article, name(s) of the author(s) and complete affiliations. Each article requires an abstract not exceeding 150 words. Formulas and citations should not be included in the abstract. AMS subject classifications and key words must be included in all accepted papers. Each article requires a running head (abbreviated form of the title) of no more than 30 characters. The sizes for regular papers, survey articles, brief notes, letters to editors and book reviews are: (i) 10-14 pages for regular papers, (ii) up to 24 pages for survey articles, and (iii) 2-3 pages for brief notes, letters to the editor and book reviews.

**(3) Tables, Graphs and Illustrations.** Each figure must be of a quality suitable for direct reproduction and must include a caption. Drawings should include all relevant details and should be drawn professionally in black ink on plain white drawing paper. In addition to a hard copy of the artwork, it is necessary to attach the electronic file of the artwork (preferably in PCX format).

**(4) References.** References should be listed alphabetically and numbered, typed and punctuated according to the following examples. Each entry must be cited in the text in form of author(s) together with the number of the referred article or in the form of the number of the referred article alone.

Journal: [1] Poincare, H. Title of the article. *Title of the Journal* **Vol. 1** (No.1), Year, Pages. [Language]

Book: [2] Liapunov, A.M. *Title of the book*. Name of the Publishers, Town, Year.

Proceeding: [3] Bellman, R. Title of the article. In: *Title of the book*. (Eds.). Name of the Publishers, Town, Year, Pages. [Language]

**(5) Proofs and Sample Copy.** Proofs sent to authors should be returned to the Editorial Office with corrections within three days after receipt. The corresponding author will receive a sample copy of the issue of the Journal for which his/her paper is published.

**(6) Editorial Policy.** Every submission will undergo a stringent peer review process. An editor will be assigned to handle the review process of the paper. He/she will secure at least two reviewers' reports. The decision on acceptance, rejection or acceptance subject to revision will be made based on these reviewers' reports and the editor's own reading of the paper.

# NONLINEAR DYNAMICS AND SYSTEMS THEORY

An International Journal of Research and Surveys

Published by InforMath Publishing Group since 2001

Volume 17

Number 4

2017

# CONTENTS

Random Impulsive Partial Hyperbolic Fractional Differential Equations	327
<i>Mouffak Benchohra and Amel Heris</i>	
On the Hyers-Ulam Stability of Laguerre and Bessel Equations by Laplace Transform Method	340
<i>E. Bicer and C. Tunc</i>	
Output Tracking of Some Class Non-minimum Phase Nonlinear Uncertain Systems	347
<i>Firman, Janson Naiborhu, Roberd Saragih and Sapto Wahyu Indratno</i>	
Passivity Based Control of Continuous Bioreactors	357
<i>M. Makkar and J.-Y. Dieulot</i>	
Mathematical Models of Nonlinear Oscillations of Mechanical Systems with Several Degrees of Freedom	369
<i>G.I. Melnikov, N.A. Dudarenko, K.S. Malykh, L.N. Ivanova and V.G. Melnikov</i>	
Existence Results for Mild Solution for a Class of Impulsive Fractional Stochastic Problems with Nonlocal Conditions	376
<i>Mohd Nadeem and Jaydev Dabas</i>	
Refinements of Some Pachpatte and Bihari Inequalities on Time Scales	388
<i>B. Ben Nasser, K. Boukerrioua, M. Defoort, M. Djemai and M.A. Hammami</i>	
Boundedness of the New Modified Hyperchaotic Pan System	402
<i>Samia Rezzag</i>	
Pseudo Almost Automorphic Mild Solutions to Some Fractional Differential Equations with Stepanov-like Pseudo Almost Automorphic Forcing Term	409
<i>Vikram Singh and Dwijendra N. Pandey</i>	
Nonlinear Parabolic Equations with Singular Coefficient and Diffuse Data	421
<i>K. Zaki and H. Redwane</i>	
Contents of Volume 17, 2017	433

*Founded by A.A. Martynyuk in 2001.*

Registered in Ukraine Number: KB 5267 / 04.07.2001.

# NONLINEAR DYNAMICS AND SYSTEMS THEORY

An International Journal of Research and Surveys

*Impact Factor from SCOPUS for 2017: SNIP – 0.707, SJR – 0.316*

**Nonlinear Dynamics and Systems Theory** (ISSN 1562–8353 (Print), ISSN 1813–7385 (Online)) is an international journal published under the auspices of the S.P. Timoshenko Institute of Mechanics of National Academy of Sciences of Ukraine and Curtin University of Technology (Perth, Australia). It aims to publish high quality original scientific papers and surveys in areas of nonlinear dynamics and systems theory and their real world applications.

## AIMS AND SCOPE

**Nonlinear Dynamics and Systems Theory** is a multidisciplinary journal. It publishes papers focusing on proofs of important theorems as well as papers presenting new ideas and new theory, conjectures, numerical algorithms and physical experiments in areas related to nonlinear dynamics and systems theory. Papers that deal with theoretical aspects of nonlinear dynamics and/or systems theory should contain significant mathematical results with an indication of their possible applications. Papers that emphasize applications should contain new mathematical models of real world phenomena and/or description of engineering problems. They should include rigorous analysis of data used and results obtained. Papers that integrate and interrelate ideas and methods of nonlinear dynamics and systems theory will be particularly welcomed. This journal and the individual contributions published therein are protected under the copyright by International InforMath Publishing Group.

## PUBLICATION AND SUBSCRIPTION INFORMATION

**Nonlinear Dynamics and Systems Theory** will have 4 issues in 2018, printed in hard copy (ISSN 1562–8353) and available online (ISSN 1813–7385), by InforMath Publishing Group, Nesterov str., 3, Institute of Mechanics, Kiev, MSP 680, Ukraine, 03057. Subscription prices are available upon request from the Publisher, EBSCO Information Services (<mailto:journals@ebSCO.com>), or website of the Journal: <http://e-ndst.kiev.ua>. Subscriptions are accepted on a calendar year basis. Issues are sent by airmail to all countries of the world. Claims for missing issues should be made within six months of the date of dispatch.

## ABSTRACTING AND INDEXING SERVICES

Papers published in this journal are indexed or abstracted in: Mathematical Reviews / MathSciNet, Zentralblatt MATH / Mathematics Abstracts, PASCAL database (INIST–CNRS) and SCOPUS.



# Random Impulsive Partial Hyperbolic Fractional Differential Equations

Mouffak Benchohra\* and Amel Heris

*Laboratory of Mathematics, Djillali Liabes University of Sidi Bel-Abbès,  
P.O. Box 89, Sidi Bel-Abbès 22000, Algeria*

Received: March 31, 2016; Revised: October 29, 2017

**Abstract:** This paper deals with the existence of random solutions of Darboux problem of impulsive fractional differential equations. The main results are based on the measure of noncompactness and a fixed point theorem for random operators.

**Keywords:** *Darboux problem, differential equation; Caputo fractional derivative; random solution; impulses; measure of noncompactness.*

**Mathematics Subject Classification (2010):** 26A33.

## 1 Introduction

Fractional calculus is generalization of ordinary differentiation and integration to arbitrary non-integer order. The subject is as old as the differential calculus, starting from some speculations of G.W. Lebeniz (1667) and L. Euler (1730) and since then, it has continued to be developed up to nowadays. Integral equations are one of the most useful mathematical tools in both pure and applied analysis. This is particularly true for problems in mechanical vibrations and the related fields of engineering and mathematical physics. We can find numerous applications of differential and integral equations of fractional order in finance, hydrology, biophysics, thermodynamics, control theory, statistical mechanics, astrophysics, cosmology and bioengineering [10, 14, 19, 20, 23]. There has been a significant development in ordinary and partial fractional differential equations in recent years; see the monographs of Abbas *et al.* [5, 6], Baleanu *et al.* [10], Kilbas *et al.* [16], Zhou [25], the papers of Abbas *et al.* [1–3, 7], Sowmya and Vatsala [21], Stutson and Vatsala [22], Vityuk and Golushkov [24], and the references therein.

---

\* Corresponding author: <mailto:benchohra@univ-sba.dz>

There has been a significant development in impulse theory in recent years, especially in the area of impulsive differential equations with fixed moments. Recently some results on the Darboux problem for fractional order impulsive hyperbolic differential equations and inclusions have been obtained by Abbas *et al.* [3, 5].

The initial value problems of ordinary random differential equations have been studied in the literature on bounded as well as unbounded intervals of the real line for different aspects of the solution. See, for example, Burton and Furumochi [11] and the references therein.

In this paper, we discuss the existence of random solutions for the following impulsive partial fractional random differential equations

$$\begin{cases} {}^c D_{x_k}^r u(x, y, w) = f(x, y, u(x, y, w), w); & \text{if } (x, y) \in J_k, \ k = 0, \dots, m, w \in \Omega, \\ u(x_k^+, y, w) = u(x_k^-, y, w) + I_k(u(x_k^-, y, w)); & \text{if } y \in [0, b], \ k = 1, \dots, m, w \in \Omega, \\ u(x, 0, w) = \varphi(x, w); & x \in [0, a], w \in \Omega, \\ u(0, y, w) = \psi(y, w); & y \in [0, b], w \in \Omega, \\ \varphi(0, w) = \psi(0, w), \end{cases} \quad (1)$$

where  $J_0 = [0, x_1] \times [0, b]$ ,  $J_k := (x_k, x_{k+1}] \times [0, b]$ ;  $k = 1, \dots, m$ ,  $a, b > 0$ ,  $\theta_k = (x_k, 0)$ ;  $k = 0, \dots, m$ ,  ${}^c D_{x_k}^r$  is the fractional Caputo derivative of order  $r = (r_1, r_2) \in (0, 1] \times (0, 1]$ ,  $0 = x_0 < x_1 < \dots < x_m < x_{m+1} = a$ ,  $(\Omega, \mathcal{A})$  is a measurable space,  $f : J \times E \times \Omega \rightarrow E$ ;  $I_k : E \rightarrow E$ ;  $k = 1, \dots, m$  are given continuous functions,  $\varphi : [0, a] \times \Omega \rightarrow E$  and  $\psi : [0, b] \times \Omega \rightarrow E$  are given absolutely continuous functions. Here  $u(x_k^+, y, w)$  and  $u(x_k^-, y, w)$  denote the right and left limits of  $u(x, y, w)$  at  $x = x_k$ , respectively.

This paper initiates the study of random solutions for impulsive partial hyperbolic fractional differential equations.

## 2 Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper. Let  $E$  be a Banach space and let  $J := [0, a] \times [0, b]$ ;  $a, b > 0$ . Denote by  $L^1(J)$  the space of Bochner-integrable functions  $u : J \rightarrow E$  with the norm

$$\|u\|_{L^1} = \int_0^a \int_0^b \|u(x, y)\|_E dy dx,$$

where  $\|\cdot\|_E$  denotes a suitable complete norm on  $E$ .

As usual, by  $AC(J)$  we denote the space of absolutely continuous functions from  $J$  into  $E$ , and  $\mathcal{C} := C(J)$  is the Banach space of continuous functions from  $J$  into  $E$  with the norm  $\|\cdot\|_\infty$  defined by

$$\|u\|_\infty = \sup_{(x, y) \in J} \|u(x, y)\|_E.$$

Consider the space

$$\begin{aligned} PC = PC(J \times \Omega) = \{ & u : J \times \Omega \rightarrow E : u(\cdot, \cdot, w) \text{ is continuous on } J_k; \ k = 0, 1, \dots, m, \text{ and} \\ & \text{there exist } u(x_k^-, y, w) \text{ and } u(x_k^+, y, w); \ k = 1, \dots, m, \\ & \text{with } u(x_k^-, y, w) = u(x_k, y, w) \text{ for each } y \in [0, b], w \in \Omega \}. \end{aligned}$$

This set is a Banach space with the norm

$$\|u\|_{PC} = \sup_{(x,y) \in J} \|u(x, y, w)\|_E.$$

Let  $\beta_E$  be the  $\sigma$ -algebra of Borel subsets of  $E$ . A mapping  $v : \Omega \rightarrow E$  is said to be measurable if for any  $B \in \beta_E$ , one has

$$v^{-1}(B) = \{w \in \Omega : v(w) \in B\} \subset \mathcal{A}.$$

To define integrals of sample paths of random process, it is necessary to define a jointly measurable map.

**Definition 2.1** A mapping  $T : \Omega \times E \rightarrow E$  is called jointly measurable if for any  $B \in \beta_E$ , one has

$$T^{-1}(B) = \{(w, v) \in \Omega \times E : T(w, v) \in B\} \subset \mathcal{A} \times \beta_E,$$

where  $\mathcal{A} \times \beta_E$  is the direct product of the  $\sigma$ -algebras  $\mathcal{A}$  and  $\beta_E$  that are defined in  $\Omega$  and  $E$  respectively.

**Lemma 2.1** Let  $T : \Omega \times E \rightarrow E$  be a mapping such that  $T(\cdot, v)$  is measurable for all  $v \in E$ , and  $T(w, \cdot)$  is continuous for all  $w \in \Omega$ . Then the map  $(w, v) \mapsto T(w, v)$  is jointly measurable.

**Definition 2.2** A function  $f : J \times E \times \Omega \rightarrow E$  is called random Carathéodory if the following conditions are satisfied:

- (i) The map  $(x, y, w) \rightarrow f(x, y, u, w)$  is jointly measurable for all  $u \in E$ , and
- (ii) The map  $u \rightarrow f(x, y, u, w)$  is continuous for almost all  $(x, y) \in J$  and  $w \in \Omega$ .

Let  $T : \Omega \times E \rightarrow E$  be a mapping. Then  $T$  is called a random operator if  $T(w, u)$  is measurable in  $w$  for all  $u \in E$  and it is expressed as  $T(w)u = T(w, u)$ . In this case we also say that  $T(w)$  is a random operator on  $E$ . A random operator  $T(w)$  on  $E$  is called continuous (compact, totally bounded and completely continuous) if  $T(w, u)$  is continuous (compact, totally bounded and completely continuous, respectively) in  $u$  for all  $w \in \Omega$ . The details of completely continuous random operators in Banach spaces and their properties appear in Itoh [15].

**Definition 2.3** [13] Let  $\mathcal{P}(Y)$  be the family of all nonempty subsets of  $Y$  and  $C$  be a mapping from  $\Omega$  into  $\mathcal{P}(Y)$ . A mapping  $T : \{(w, y) : w \in \Omega, y \in C(w)\} \rightarrow Y$  is called random operator with stochastic domain  $C$  if  $C$  is measurable (i.e., for all closed  $A \subset Y$ ,  $\{w \in \Omega, C(w) \cap A \neq \emptyset\}$  is measurable) and for all open  $D \subset Y$  and all  $y \in Y$ ,  $\{w \in \Omega : y \in C(w), T(w, y) \in D\}$  is measurable.  $T$  will be called continuous if every  $T(w)$  is continuous. For a random operator  $T$ , a mapping  $y : \Omega \rightarrow Y$  is called random (stochastic) fixed point of  $T$  if for  $P$ -almost all  $w \in \Omega$ ,  $y(w) \in C(w)$  and  $T(w)y(w) = y(w)$  and for all open  $D \subset Y$ ,  $\{w \in \Omega : y(w) \in D\}$  is measurable.

Let  $\mathcal{M}_X$  denote the class of all bounded subsets of a metric space  $X$ .

**Definition 2.4** Let  $X$  be a complete metric space. A map  $\alpha : \mathcal{M}_X \rightarrow [0, \infty)$  is called a measure of noncompactness on  $X$  if it satisfies the following properties for all  $B, B_1, B_2 \in \mathcal{M}_X$  :

- (MNC.1)  $\alpha(B) = 0$  if and only if  $B$  is precompact (regularity),
- (MNC.2)  $\alpha(B) = \alpha(\overline{B})$  (invariance under closure),
- (MNC.3)  $\alpha(B_1 \cup B_2) = \max\{\alpha(B_1), \alpha(B_2)\}$  (semi-additivity).

For more details on measure of noncompactness and its properties, see [8, 9].

Let  $\theta = (0, 0)$ ,  $r_1, r_2 > 0$  and  $r = (r_1, r_2)$ . For  $f \in L^1(J)$ , the expression

$$(I_\theta^r f)(x, y) = \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1} (y-t)^{r_2-1} f(s, t) ds dt$$

is called the left-sided mixed Riemann-Liouville integral of order  $r$ , where  $\Gamma(\cdot)$  is the (Euler's) gamma function defined by  $\Gamma(\xi) = \int_0^\infty t^{\xi-1} e^{-t} dt$ ,  $\xi > 0$ .

In particular,

$$(I_\theta^\theta u)(x, y) = u(x, y), \quad (I_\theta^\sigma u)(x, y) = \int_0^x \int_0^y u(s, t) dt ds; \text{ for almost all } (x, y) \in J,$$

where  $\sigma = (1, 1)$ . For instance,  $I_\theta^r u$  exists for all  $r_1, r_2 \in (0, \infty)$ , when  $u \in L^1(J)$ . Note also that when  $u \in C(J)$ , then  $(I_\theta^r u) \in C(J)$ . Moreover

$$(I_\theta^r u)(x, 0) = (I_\theta^r u)(0, y) = 0; \quad x \in [0, a], \quad y \in [0, b].$$

**Example 2.1** Let  $\lambda, \omega \in (-1, \infty)$  and  $r = (r_1, r_2) \in (0, \infty) \times (0, \infty)$ , then

$$I_\theta^r x^\lambda y^\omega = \frac{\Gamma(1+\lambda)\Gamma(1+\omega)}{\Gamma(1+\lambda+r_1)\Gamma(1+\omega+r_2)} x^{\lambda+r_1} y^{\omega+r_2}, \text{ for almost all } (x, y) \in J.$$

By  $1-r$  we mean  $(1-r_1, 1-r_2) \in [0, 1] \times [0, 1]$ . Denote by  $D_{xy}^2 := \frac{\partial^2}{\partial x \partial y}$  the mixed second order partial derivative.

**Definition 2.5** [24] Let  $r \in (0, 1] \times (0, 1]$  and  $u \in L^1(J)$ . The Caputo fractional-order derivative of order  $r$  of  $u$  is defined by the expression

$${}^c D_\theta^r u(x, y) = (I_\theta^{1-r} D_{xy}^2 u)(x, y).$$

The case  $\sigma = (1, 1)$  is included and we have

$$({}^c D_\theta^\sigma u)(x, y) = (D_{xy}^2 u)(x, y); \text{ for almost all } (x, y) \in J.$$

**Example 2.2** Let  $\lambda, \omega \in (-1, \infty)$  and  $r = (r_1, r_2) \in (0, 1] \times (0, 1]$ , then

$${}^c D_\theta^r x^\lambda y^\omega = \frac{\Gamma(1+\lambda)\Gamma(1+\omega)}{\Gamma(1+\lambda-r_1)\Gamma(1+\omega-r_2)} x^{\lambda-r_1} y^{\omega-r_2}; \text{ for almost all } (x, y) \in J.$$



Let  $a_1 \in [0, a]$ ,  $z^+ = (a_1, 0) \in J$ ,  $J_z = (a_1, a] \times [0, b]$ ,  $r_1, r_2 > 0$  and  $r = (r_1, r_2)$ . For  $u \in L^1(J_z)$ , the expression

$$(I_{z^+}^r u)(x, y) = \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{a_1^+}^x \int_0^y (x-s)^{r_1-1} (y-t)^{r_2-1} u(s, t) dt ds$$

is called the left-sided mixed Riemann-Liouville integral of order  $r$  of  $u$ .

**Definition 2.6** [24]. For  $u \in L^1(J_z)$  where  $D_{xy}^2 u$  is Lebesgue integrable on  $[x_k, x_{k+1}] \times [0, b]$ ,  $k = 0, \dots, m$ , the Caputo fractional order derivative of order  $r$  of  $u$  is defined by the expression

$$({}^c D_{z^+}^r f)(x, y) = (I_{z^+}^{1-r} D_{xy}^2 f)(x, y).$$

**Lemma 2.2** [12] If  $Y$  is a bounded subset of Banach space  $X$ , then for each  $\epsilon > 0$ , there is a sequence  $\{y_k\}_{k=1}^\infty \subset Y$  such that

$$\alpha(Y) \leq 2\alpha(\{y_k\}_{k=1}^\infty) + \epsilon.$$

**Lemma 2.3** [18] If  $\{u_k\}_{k=1}^\infty \subset L^1(J)$  is uniformly integrable, then  $\alpha(\{u_k\}_{k=1}^\infty)$  is measurable and for each  $(x, y) \in J$ ,

$$\alpha\left(\left\{\int_0^x \int_0^y u_k(s, t) dt ds\right\}_{k=1}^\infty\right) \leq 2 \int_0^x \int_0^y \alpha(\{u_k(s, t)\}_{k=1}^\infty) dt ds.$$

**Lemma 2.4** [17] Let  $F$  be a closed and convex subset of a real Banach space, let  $G : F \rightarrow F$  be a continuous operator and  $G(F)$  be bounded. If there exists a constant  $k \in [0, 1)$  such that for each bounded subset  $B \subset F$ ,

$$\alpha(G(B)) \leq k\alpha(B),$$

then  $G$  has a fixed point in  $F$ .

### 3 Existence Results

We need the following auxiliary lemma.

**Lemma 3.1** [4] Let  $0 < r_1, r_2 \leq 1$ ,  $\mu(x, y) = \varphi(x) + \psi(y) - \varphi(0)$  and let  $f : J \times E \rightarrow E$  be continuous. A function  $u \in PC(J)$  is a solution of the fractional integral equation

$$u(x, y) = \begin{cases} \mu(x, y) + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1} (y-t)^{r_2-1} f(s, t, u(s, t)) dt ds; \\ \text{if } (x, y) \in [0, x_1] \times [0, b], \\ \mu(x, y) + \sum_{i=1}^k (I_i(u(x_i^-, y)) - I_i(u(x_i^-, 0))) \\ + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \sum_{i=1}^k \int_{x_{i-1}}^{x_i} \int_0^y (x_i-s)^{r_1-1} (y-t)^{r_2-1} f(s, t, u(s, t)) dt ds \\ + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_k}^x \int_0^y (x-s)^{r_1-1} (y-t)^{r_2-1} f(s, t, u(s, t)) dt ds; \\ \text{if } (x, y) \in (x_k, x_{k+1}] \times [0, b], \quad k = 1, \dots, m, \end{cases}$$

if and only if  $u$  is a solution of the problem

$$\begin{cases} {}^c D_{x_k}^r u(x, y) = f(x, y, u(x, y)); & \text{if } (x, y) \in J_k, \quad k = 0, \dots, m, \\ u(x_k^+, y) = u(x_k^-, y) + I_k(u(x_k^-, y)); & \text{if } y \in [0, b], \quad k = 1, \dots, m, \\ u(x, 0) = \varphi(x); & x \in [0, a], \\ u(0, y) = \psi(y); & y \in [0, b], \\ \varphi(0) = \psi(0). \end{cases}$$

As a consequence, we have the following lemma.

**Lemma 3.2** *Let  $0 < r_1, r_2 \leq 1$ ,  $\mu(x, y, w) = \varphi(x, w) + \psi(y, w) - \varphi(0, w)$ . A function  $u \in PC$  is a solution of the random fractional integral equation*

$$u(x, y, w) = \begin{cases} \mu(x, y, w) + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1} (y-t)^{r_2-1} f(s, t, u(s, t, w), w) dt ds; \\ \text{if } (x, y) \in [0, x_1] \times [0, b], \quad w \in \Omega, \\ \mu(x, y, w) + \sum_{i=1}^k (I_i(u(x_i^-, y, w)) - I_i(u(x_i^-, 0, w))) \\ + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \sum_{i=1}^k \int_{x_{i-1}}^{x_i} \int_0^y (x_i-s)^{r_1-1} (y-t)^{r_2-1} f(s, t, u(s, t, w), w) dt ds \\ + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_k}^x \int_0^y (x-s)^{r_1-1} (y-t)^{r_2-1} f(s, t, u(s, t, w), w) dt ds; \\ \text{if } (x, y) \in (x_k, x_{k+1}] \times [0, b], \quad k = 1, \dots, m, \quad w \in \Omega, \end{cases} \quad (2)$$

if and only if  $u$  is a solution of the random problem (1).

The following hypotheses will be used in the sequel.

**Hypothesis 3.1** *The functions  $w \mapsto \varphi(x, 0, w)$  and  $w \mapsto \psi(0, y, w)$  are measurable and bounded for almost each  $x \in [0, a]$  and  $y \in [0, b]$  respectively.*

**Hypothesis 3.2** *The function  $f$  is random Carathéodory on  $J \times E \times \Omega$ .*

**Hypothesis 3.3** *There exist functions  $p_1, p_2, p_3 : J \times \Omega \rightarrow [0, \infty)$  with  $p_i(\cdot, w) \in L^\infty(J, [0, \infty))$ ;  $i = 1, 2, 3$  such that for each  $w \in \Omega$ ,*

$$\|f(x, y, u, w)\|_E \leq p_1(x, y, w) + p_2(x, y, w)\|u\|_E,$$

and

$$\|I_k(u)\|_E \leq p_3(x, y, w)\|u\|_E,$$

for all  $u \in E$  and almost each  $(x, y) \in J$ .

**Hypothesis 3.4** *For any bounded  $B \subset E$ ,*

$$\alpha(f(x, y, B, w)) \leq p_2(x, y, w)\alpha(B), \quad \text{for almost each } (x, y) \in J,$$

and

$$\alpha(I_k(B)) \leq p_3(x, y, w)\alpha(B), \quad \text{for almost each } (x, y) \in J.$$

Set

$$\mu^*(w) = \sup_{(x, y) \in J} \|\mu(x, y, w)\|_E, \quad p_i^*(w) = \sup_{(x, y) \in J} p_i(x, y, w); \quad i = 1, 2, 3.$$

**Remark 3.1** Hypotheses 3.3 and 3.4 are equivalent [8].

**Theorem 3.1** *Assume that hypotheses 3.1-3.3 hold. If*

$$\ell := 2mp_3^*(w) + \frac{4(m+1)p_2^*(w)a^{r_1}b^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)} < 1,$$

then the problem (1) has a random solution defined on  $J$ .

**Proof.** By Lemma 3.2, the problem (1) is equivalent to the integral equation

$$u(x, y, w) = \begin{cases} \mu(x, y, w) + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1} (y-t)^{r_2-1} f(s, t, u(s, t, w), w) dt ds; \\ \text{if } (x, y) \in [0, x_1] \times [0, b], \quad w \in \Omega, \\ \mu(x, y, w) + \sum_{i=1}^k (I_i(u(x_i^-, y, w)) - I_i(u(x_i^-, 0, w))) \\ + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \sum_{i=1}^k \int_{x_{i-1}}^{x_i} \int_0^y (x_i-s)^{r_1-1} (y-t)^{r_2-1} f(s, t, u(s, t, w), w) dt ds \\ + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_k}^x \int_0^y (x-s)^{r_1-1} (y-t)^{r_2-1} f(s, t, u(s, t, w), w) dt ds; \\ \text{if } (x, y) \in (x_k, x_{k+1}] \times [0, b], \quad k = 1, \dots, m, \quad w \in \Omega, \end{cases}$$

for each  $w \in \Omega$  and almost each  $(x, y) \in J$ .

Define the operator  $N : PC \rightarrow PC$  by

$$\begin{aligned} (Nu)(x, y) &= \mu(x, y, w) + \sum_{i=1}^k (I_i(u(x_i^-, y, w)) - I_i(u(x_i^-, 0, w))) \\ &+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \sum_{i=1}^k \int_{x_{i-1}}^{x_i} \int_0^y (x_i-s)^{r_1-1} (y-t)^{r_2-1} f(s, t, u(s, t, w), w) dt ds \\ &+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_k}^x \int_0^y (x-s)^{r_1-1} (y-t)^{r_2-1} f(s, t, u(s, t, w), w) dt ds. \end{aligned}$$

Since the functions  $\varphi, \psi$  and  $I_k$  and  $f$  are absolutely continuous, the function  $\mu$  and the indefinite integral are absolutely continuous for all  $w \in \Omega$  and almost all  $(x, y) \in J$ . Again, as the maps  $\mu$  and  $I_k$  are continuous for all  $w \in \Omega$  and the indefinite integral is continuous on  $J$ , then  $N(w)$  defines a mapping  $N : PC \rightarrow PC$ . Hence  $u$  is a solution for the problem (1) if and only if  $u = Nu$ . We shall show that the operator  $N$  satisfies all conditions of Lemma 2.4. The proof will be given in several steps.

**Step 1:**  $N$  is a random operator with stochastic domain on  $PC$ .

Since  $f(x, y, u, w)$  is random Carathéodory, the map  $w \rightarrow f(x, y, u, w)$  is measurable in view of Definition 2.1. Similarly, the product  $(x-s)^{r_1-1} (y-t)^{r_2-1} f(s, t, u(s, t, w), w)$  of a continuous and a measurable function is again measurable. Further, the integral is a limit of a finite sum of measurable functions and  $I_k$  is measurable. Therefore, the map

$$\begin{aligned} w \mapsto & \mu(x, y, w) + \sum_{i=1}^k (I_i(u(x_i^-, y, w)) - I_i(u(x_i^-, 0, w))) \\ & + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \sum_{i=1}^k \int_{x_{i-1}}^{x_i} \int_0^y (x_i-s)^{r_1-1} (y-t)^{r_2-1} f(s, t, u(s, t, w), w) dt ds \\ & + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_k}^x \int_0^y (x-s)^{r_1-1} (y-t)^{r_2-1} f(s, t, u(s, t, w), w) dt ds \end{aligned}$$

is measurable. As a result,  $N$  is a random operator from  $PC$  into  $PC$ .

Let  $W : \Omega \rightarrow \mathcal{P}(PC)$  be defined by

$$W(w) = \{u \in PC : \|u\|_{PC} \leq R(w)\}$$

with  $R(\cdot)$  being chosen appropriately. For instance, we assume that

$$R(w) \geq \frac{\mu^* + \frac{(m+1)p_1^*(w)a^{r_1}b^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)}}{1 - 2mp_3^*(w) - (m+1)p_2^*(w)\frac{a^{r_1}b^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)}}.$$

The set  $W(w)$  is bounded, closed, convex and solid for all  $w \in \Omega$ . Then  $W$  is measurable (Lemma 17 ([13])). Let  $w \in \Omega$  be fixed, then from Hypothesis 3.4 for any  $u \in w(w)$ , we get

$$\begin{aligned} & \| (Nu)(x, y) \|_E \\ & \leq \| \mu(x, y, w) \|_E + \sum_{i=1}^k \| I_i(u(x_i^-, y, w)) \| + \| I_i(u(x_i^-, 0, w)) \| \\ & \quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \sum_{i=1}^k \int_{x_{i-1}}^{x_i} \int_0^y (x_i - s)^{r_1-1} (y - t)^{r_2-1} \| f(s, t, u(s, t, w), w) \|_E dt ds \\ & \quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_k}^x \int_0^y (x - s)^{r_1-1} (y - t)^{r_2-1} \| f(s, t, u(s, t, w), w) \|_E dt ds, \\ & \leq \| \mu(x, y, w) \|_E + \sum_{i=1}^k (p_3(x, y, w) \| u \| + (p_3(x_i, 0, w)) \| u \|) \\ & \quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \sum_{i=1}^k \left( \int_{x_{i-1}}^{x_i} \int_0^y (x_i - s)^{r_1-1} (y - t)^{r_2-1} p_1(s, t, w) dt ds \right. \\ & \quad \left. + \int_{x_{i-1}}^{x_i} \int_0^y (x - s)^{r_1-1} (y - t)^{r_2-1} p_2(s, t, w) \| u(s, t, w) \|_E dt ds \right) \\ & \quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_k}^x \int_0^y (x - s)^{r_1-1} (y - t)^{r_2-1} p_1(s, t, w) dt ds \\ & \quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_k}^x \int_0^y (x - s)^{r_1-1} (y - t)^{r_2-1} p_2(s, t, w) \| u(s, t, w) \|_E dt ds \\ & \leq \mu^*(w) + 2mp_3^*(w)R(w) \\ & \quad + \sum_{i=1}^k \left( \frac{p_1^*(w)}{\Gamma(r_1)\Gamma(r_2)} \int_{x_{i-1}}^{x_i} \int_0^y (x - s)^{r_1-1} (y - t)^{r_2-1} dt ds \right. \\ & \quad \left. + \frac{p_2^*(w)R(w)}{\Gamma(r_1)\Gamma(r_2)} \int_{x_{i-1}}^{x_i} \int_0^y (x - s)^{r_1-1} (y - t)^{r_2-1} dt ds \right) \\ & \quad + \frac{p_1^*(w)}{\Gamma(r_1)\Gamma(r_2)} \int_{x_k}^x \int_0^y (x - s)^{r_1-1} (y - t)^{r_2-1} dt ds \\ & \quad + \frac{p_2^*(w)R(w)}{\Gamma(r_1)\Gamma(r_2)} \int_{x_k}^x \int_0^y (x - s)^{r_1-1} (y - t)^{r_2-1} dt ds \\ & \leq \mu^*(w) + 2mp_3^*(w)R(w) + \frac{(p_1^*(w) + p_2^*(w)R(w))(m+1)a^{r_1}b^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)} \\ & \leq R(w). \end{aligned}$$

Therefore,  $N$  is a random operator with stochastic domain  $W$  and  $N : W(w) \rightarrow W(w)$ . Furthermore,  $N$  maps bounded sets into bounded sets in  $PC$ .

**Step 2:**  $N$  is continuous.

Let  $\{u_n\}$  be a sequence such that  $u_n \rightarrow u$  in  $PC$ . Then, for each  $(x, y) \in J$  and  $w \in \Omega$ , we have

$$\begin{aligned} & \| (Nu_n)(x, y) - (N(w)u)(x, y) \|_E \\ & \leq \sum_{i=1}^k (\|I_i(u_n(x_i^-, y, w)) - I_i(u(x_i^-, y, w))\| + \|I_i(u_n(x_i^-, 0, w)) - I_i(u(x_i^-, 0, w))\|) \\ & + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \sum_{i=1}^k \int_{x_{i-1}}^{x_i} \int_0^y (x_i - s)^{r_1-1} (y - t)^{r_2-1} \|f(s, t, u_n(s, t, w), w) \\ & - f(s, t, u(s, t, w), w)\|_E dt ds \\ & + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_k}^x \int_0^y (x - s)^{r_1-1} (y - t)^{r_2-1} \|f(s, t, u_n(s, t, w), w) \\ & - f(s, t, u(s, t, w), w)\|_E dt ds. \end{aligned}$$

Using the Lebesgue dominated convergence theorem, we get

$$\|Nu_n - Nu\|_\infty \rightarrow 0 \text{ as } n \rightarrow \infty.$$

As a consequence of Steps 1 and 2, we can conclude that  $N : W(w) \rightarrow W(w)$  is a continuous random operator with stochastic domain  $W$ , and  $N(W(w))$  is bounded.

**Step 3:** For each bounded subset  $B$  of  $W(w)$  we have

$$\alpha(NB) \leq \ell\alpha(B).$$

Let  $w \in \Omega$  be fixed. From Lemmas 2.2 and 2.3, for any  $B \subset W$  and any  $\epsilon > 0$ , there exists a sequence  $\{u_n\}_{n=0}^\infty \subset B$ , such that for all  $(x, y) \in J$ , we have

$$\begin{aligned} & \alpha((NB)(x, y)) \\ & = \alpha \left\{ \mu(x, y, w) + \sum_{i=1}^k (I_i(u(x_i^-, y, w)) - I_i(u(x_i^-, 0, w))) \right. \\ & + \sum_{i=1}^k \int_{x_{i-1}}^{x_i} \int_0^y \frac{(x_i - s)^{r_1-1} (y - t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} f(s, t, u(s, t, w), w) dt ds \\ & + \left. \int_{x_k}^x \int_0^y \frac{(x - s)^{r_1-1} (y - t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} f(s, t, u(s, t, w), w) dt ds; u \in B \right\} \\ & \leq \alpha \left\{ \sum_{i=1}^k (I_i(u_n(x_i^-, y, w)) - I_i(u_n(x_i^-, 0, w))) \right. \\ & + \sum_{i=1}^k \int_{x_{i-1}}^{x_i} \int_0^y \frac{(x_i - s)^{r_1-1} (y - t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} f(s, t, u_n(s, t, w), w) dt ds \end{aligned}$$

$$\begin{aligned}
& + \int_{x_k}^x \int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} f(s, t, u_n(s, t, w), w) dt ds \Big\}_{n=1}^\infty + \epsilon \\
& \leq \alpha \left\{ \sum_{i=1}^k (I_i(u_n(x_i^-, y, w)) - I_i(u_n(x_i^-, 0, w))) \right\}_{n=1}^\infty \\
& + 2 \sum_{i=1}^k \int_{x_{i-1}}^{x_i} \int_0^y \frac{(x_i-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} \alpha \{f(s, t, u_n(s, t, w), w)\}_{n=1}^\infty dt ds \\
& + 2 \int_{x_k}^x \int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} \alpha \{f(s, t, u_n(s, t, w), w)\}_{n=1}^\infty dt ds + \epsilon \\
& \leq 2mp_3(x, y, w) \alpha(\{u_n(s, t, w)\}_{n=1}^\infty) \\
& + 4 \sum_{i=1}^k \int_{x_{i-1}}^{x_i} \int_0^y \frac{(x_i-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} p_2(s, t, w) \alpha(\{u_n(s, t, w)\}_{n=1}^\infty) dt ds \\
& + 4 \int_{x_k}^x \int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} p_2(s, t, w) \alpha(\{u_n(s, t, w)\}_{n=1}^\infty) dt ds + \epsilon \\
& \leq 2mp_3(x, y, w) \alpha(\{u_n\}_{n=1}^\infty) \\
& + \left( 4 \sum_{i=1}^k \int_{x_{i-1}}^{x_i} \int_0^y \frac{(x_i-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} p_2(s, t, w) \right) \alpha(\{u_n\}_{n=1}^\infty) dt ds \\
& + \left( 4 \int_{x_k}^x \int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} p_2(s, t, w) ds dt \right) \alpha(\{u_n\}_{n=1}^\infty) + \epsilon \\
& \leq 2mp_3(x, y, w) \alpha(B) \\
& + \left( 4 \sum_{i=1}^k \int_{x_{i-1}}^{x_i} \int_0^y \frac{(x_i-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} p_2(s, t, w) dt ds \right) \alpha(B) \\
& + \left( 4 \int_{x_k}^x \int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} p_2(s, t, w) dt ds \right) \alpha(B) + \epsilon \\
& \leq \left( 2mp_3^*(w) + \frac{4(m+1)p_2^*(w)a^{r_1}b^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)} \right) \alpha(B) + \epsilon \\
& = \ell \alpha(B) + \epsilon.
\end{aligned}$$

Since  $\epsilon > 0$  is arbitrary, we have

$$\alpha(N(B)) \leq \ell \alpha(B).$$

It follows from Lemma 2.4 that for each  $w \in \Omega$ ,  $N$  has at least one fixed point in  $W$ . Since  $\bigcap_{w \in \Omega} \text{int}W(w) \neq \emptyset$ , there exists a measurable selector of  $\text{int}W$ , thus  $N$  has a stochastic fixed point, i.e., the problem (1) has at least one random solution.

#### 4 An Example

Let  $E = \mathbb{R}$ ,  $\Omega = (-\infty, 0)$  be equipped with the usual  $\sigma$ -algebra consisting of Lebesgue measurable subsets of  $(-\infty, 0)$ . Given a measurable function  $u : \Omega \rightarrow AC([0, 1] \times [0, 1])$ , consider the following impulsive partial fractional random differential equations of the

form

$$\begin{cases} {}^c D_{x_k}^r u(x, y, w) = \frac{w^2 e^{-x-y-3}}{1+w^2+5|u(x, y, w)|}; & \text{if } (x, y) \in J_k, \quad k = 0, \dots, m, \\ u(x_k^+, y, w) = u(x_k^-, y, w) + \frac{w^2}{(1+w^2+10|u(x, y, w)|)e^{x+y+10}}; & \text{if } y \in [0, 1], \quad k = 1, \dots, m, \end{cases} \quad (3)$$

where  $w \in \Omega$ ,  $J = [0, 1] \times [0, 1]$ ,  $(r_1, r_2) \in (0, 1] \times (0, 1]$  with the initial conditions

$$\begin{cases} u(x, 0, w) = x \sin w; & x \in [0, 1], \\ u(0, y, w) = y^2 \cos w; & y \in [0, 1]. \end{cases} \quad w \in \Omega, \quad (4)$$

Set

$$f(x, y, u(x, y, w), w) = \frac{w^2}{(1 + w^2 + 5|u(x, y, w)|)e^{x+y+10}}, \quad (x, y) \in [0, 1] \times [0, 1], \quad w \in \Omega,$$

and

$$I_k(u(x_k^-, y, w)) = \frac{w^2}{(1 + w^2 + 10|u(x, y, w)|)e^{x+y+10}}, \quad y \in [0, 1], \quad k = 1, \dots, m, \quad w \in \Omega.$$

The functions  $w \mapsto \varphi(x, 0, w) = x \sin w$  and  $w \mapsto \psi(0, y, w) = y^2 \cos w$  are measurable and bounded with

$$|\varphi(x, 0, w)| \leq 1, \quad |\psi(0, y, w)| \leq 1,$$

hence, Hypothesis 3.1 is satisfied.

Clearly, the map  $(x, y, w) \mapsto f(x, y, u, w)$  is jointly continuous for all  $u \in \mathbb{R}$  and hence jointly measurable for all  $u \in \mathbb{R}$ . Also the map  $u \mapsto f(x, y, u, w)$  is continuous for all  $(x, y) \in J$  and  $w \in \Omega$ . So the function  $f$  is Carathéodory on  $[0, 1] \times [0, 1] \times \mathbb{R} \times \Omega$ .

For each  $u \in \mathbb{R}$ ,  $(x, y) \in [0, 1] \times [0, 1]$  and  $w \in \Omega$ , we have

$$|f(x, y, u, w)| \leq 1 + \frac{5}{e^{10}}|u|,$$

and

$$|I_k(u)| \leq \frac{10}{e^{10}}|u|.$$

Hence Hypothesis 3.4 is satisfied with

$$p_1(x, y, w) = p_1^*(w) = 1, \quad p_2(x, y, w) = p_2^*(w) = \frac{5}{e^{10}}, \quad p_3(x, y, w) = p_3^*(w) = \frac{10}{e^{10}}.$$

We shall show that condition  $\ell < 1$  holds with  $a = b = 1$ . Indeed, if we assume, for instance, that the number of impulses  $m = 3$ , then we have

$$\begin{aligned} \ell &= 2mp_3^*(w) + \frac{4(m+1)p_2^*(w)a^{r_1}b^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)} \\ &= \frac{60}{e^{10}} + \frac{80}{e^{10}\Gamma(1+r_1)\Gamma(1+r_2)} \\ &< 1, \end{aligned}$$

which is satisfied for each  $(r_1, r_2) \in (0, 1] \times (0, 1]$ . Consequently, Theorem 3.1 implies that the problem (3)-(4) has a random solution defined on  $[0, 1] \times [0, 1]$ .

## References

- [1] Abbas, S. and Benchohra, M. Darboux problem for perturbed partial differential equations of fractional order with finite delay. *Nonlinear Anal. Hybrid Syst.* **3** (2009) 597–604.
- [2] Abbas, S. and Benchohra, M. Fractional order partial hyperbolic differential equations involving Caputo's derivative, *Stud. Univ. Babeş-Bolyai Math.* **57** (4) (2012) 469–479.
- [3] Abbas, S. and Benchohra, M. Upper and lower solutions method for Darboux problem for fractional order implicit impulsive partial hyperbolic differential equations. *Acta Univ. Palacki. Olomuc.* **51** (2) (2012) 5–18.
- [4] Abbas, S. Benchohra, M. and Gorniewicz, M. Existence theory for impulsive partial hyperbolic functional differential equations involving the Caputo fractional derivative. *Sci. Math. Jpn.* online e-2010, 271–282.
- [5] Abbas, S. Benchohra, M. and N'Guérékata, G.M. *Topics in Fractional Differential Equations*, Springer, New York, 2012.
- [6] Abbas, S. Benchohra, M. and N'Guérékata, G.M. *Advanced Fractional Differential and Integral Equations*, Nova Science Publishers, New York, 2015.
- [7] Abbas, S. Benchohra, M. and Vityuk, A.N. On fractional order derivatives and Darboux problem for implicit differential equations. *Frac. Calc. Appl. Anal.* **15** (2) (2012) 168–182.
- [8] Appell, J. Implicit functions, nonlinear integral equations, and the measure of noncompactness of the superposition operator. *J. Math. Anal. Appl.* **83** (1981) 251–263.
- [9] Ayerbe Toledano, J. M. Dominguez Benavides, T. and Lopez Acedo, G. *Measures of noncompactness in metric fixed point theory, Operator Theory, Advances and Applications*. Birkhäuser, Basel, 1997.
- [10] Baleanu, D. Diethelm, K. Scalas, E. and Trujillo, J.J. *Fractional Calculus Models and Numerical Methods*. World Scientific Publishing, New York, 2012.
- [11] Burton, T.A. and Furumochi, T. A note on stability by Schauder's theorem. *Funkcial. Ekvac.* **44** (2001), 73–82.
- [12] Bothe, D. Multivalued perturbation of  $m$ -accretive differential inclusions, *Isr. J. Math.* **108** (1998) 109–138.
- [13] Engl, H.W. A general stochastic fixed-point theorem for continuous random operators on stochastic domains. *J. Math. Anal. Appl.* **66** (1978) 220–231.
- [14] Hilfer, R. *Applications of Fractional Calculus in Physics*. World Scientific, Singapore, 2000.
- [15] Itoh, S. Random fixed point theorems with applications to random differential equations in Banach spaces. *J. Math. Anal. Appl.* **67** (1979) 261–273.
- [16] Kilbas, A.A. Srivastava, H.M. and Trujillo, J.J. *Theory and Applications of Fractional Differential Equations*. North-Holland Mathematics Studies, 204. Elsevier Science B.V., Amsterdam, 2006.
- [17] Liu, L. Guo, F. Wu, C. and Wu, Y. Existence theorems of global solutions for nonlinear Volterra type integral equations in Banach spaces. *J. Math. Anal. Appl.* **309** (2005) 638–649.
- [18] Mönch, H. Boundary value problems for nonlinear ordinary differential equations of second order in Banach spaces. *Nonlinear Anal. Theory Methods Appl.* **4** (1980) 985–999.
- [19] Ortigueira, M.D. *Fractional Calculus for Scientists and Engineers*. Lecture Notes in Electrical Engineering, **84**. Springer, Dordrecht, 2011.
- [20] Podlubny, I. *Fractional Differential Equations*. Academic Press, San Diego, 1999.



- [21] Sowmya, M. and Vatsala, A.S. Generalized iterative methods for Caputo fractional differential equations via coupled lower and upper solutions with superlinear convergence. *Nonlinear Dyn. Syst. Theory* **15** (2015) 198–208.
- [22] Stutson, D.S. and Vatsala, A.S. Riemann Liouville and Caputo fractional differential and integral inequalities. *Dynam. Systems Appl.* **23** (2014), 723–733.
- [23] Tarasov, V.E. *Fractional Dynamics. Applications of Fractional Calculus to Dynamics of Particles, Fields and Media*. Springer, Heidelberg, 2010.
- [24] Vityuk, A.N. and Golushkov, A.V. Existence of solutions of systems of partial differential equations of fractional order. *Nonlinear Oscil.* **7** (3) (2004) 318–325.
- [25] Zhou, Y. *Basic Theory of Fractional Differential Equations*. World Scientific Publishing, Singapore, 2014.



## On the Hyers-Ulam Stability of Laguerre and Bessel Equations by Laplace Transform Method

E. Biçer<sup>1\*</sup> and C. Tunç<sup>2</sup>

<sup>1</sup> *Department of Mathematics, Faculty of Arts and Sciences, Bingöl University, 12000, Bingöl, Turkey*

<sup>2</sup> *Department of Mathematics, Faculty of Sciences, Yuzuncu Yıl University, 65080, Van, Turkey*

Received: April 23, 2017; Revised: October 24, 2017

**Abstract:** The purpose of this paper is to obtain new sufficient conditions guaranteeing the Hyers-Ulam stability of Laguerre differential equation

$$xy'' + (1 - x)y' + ny = 0$$

and Bessel differential equation of order zero

$$xy'' + y' + xy = 0.$$

Our findings make a contribution to the topic and complete those in the relevant literature.

**Keywords:** *Hyers-Ulam stability; Laguerre equation; Bessel equation; Laplace transform.*

**Mathematics Subject Classification (2010):** 34A12, 34A30, 39B82, 44A10.

### 1 Introduction

Differential equations of second order can serve as excellent tools for description of mathematical modelling of systems and processes in the fields of engineering, physics, chemistry, economics, aerodynamics, and polymerrheology, etc. Therefore, the qualitative behaviors of solutions of differential equations of second order, stability, boundedness, oscillation, etc., play an important role in many real world phenomena related to the sciences and engineering technique fields. However, we would not like to give the details

---

\* Corresponding author: <mailto:cemtunc@yahoo.com>

of the applications related to differential equations of second order here. This information indicates the importance of investigating the qualitative properties, Hyers-Ulam stability, Lyapunov stability, etc., of solutions of differential equations of second order.

The stability of functional equations was originally raised by Ulam in 1940 in a talk given at Wisconsin University (Ulam [17]). He discussed a number of unsolved important problems in that presentation. Later, Hyers [5] answered to the questions of Ulam [17]. Hence, the concepts related to the Hyers-Ulam stability arose in the literature. Later, the result of Hyers [5] has been generalized by Rassias [15]. In 1998, Alsina and Ger [3] studied the Hyers-Ulam stability of the fundamental linear differential equation. They proved that the linear differential equation has the Hyers-Ulam stability. After that, many researchers have studied the Hyers-Ulam stability of the various linear and partially differential equations. For more details on the Hyers-Ulam stability of various linear ordinary and partially differential equations, one can see Abdollahpour et al. [1], Alqifiary [2], Alsina and Ger [3], Biçer and Tunç [4], Hyers [5], Jung [6-11], Liu and Zhao [12], Lungu and Popa [13-14], Rassias [15], Tunç and Biçer [16], Ulam [17] and the references therein.

In these sources, the Hyers-Ulam stability of solutions to various linear ordinary, functional and partially differential equations was discussed by direct method, iteration method, fixed point method with a Lipschitz condition, integrating factor method, open mapping theorem, the Gronwall inequality, power series method, the Laplace transform method and etc.

The following works are notable. Jung [11] investigated general solution of the inhomogeneous Bessel differential equation of the form

$$x^2 y''(x) + xy'(x) + (x^2 - \gamma^2)y(x) = \sum_{m=0}^{\infty} a_m x^m,$$

where the parameter  $\gamma$  is non-integral number.

Jung [10] solved the inhomogeneous differential equation of the form

$$xy'' + (1-x)y' + ny = \sum_{m=0}^{\infty} a_m x^m$$

by the power series method, where  $n$  is positive integer, and applied this result to obtain a partial solution to the Ulam stability of the differential equation

$$xy'' + (1-x)y' + ny = 0.$$

Abdollahpour et al. [1] discussed the Hyers-Ulam stability of the differential equation

$$xy'' + (1+v-x)y' + \lambda y = \sum_{m=0}^{\infty} a_m x^m$$

by means of the power series method. They studied the Hyers-Ulam stability of the associated homogeneous Laguerre differential equation in a subclass of analytic functions.

Alqifiary and Jung [2] investigated Hyers-Ulam stability of the differential equation

$$y^{(n)}(t) + \sum_{k=0}^{n-1} \alpha_k y^{(k)}(t) = f(t)$$

by applying the Laplace transform method, where  $\alpha_k$  is a scalar.

In this paper, we investigate the Hyers-Ulam stability of Laguerre differential equation of the form

$$xy'' + (1-x)y' + ny = 0, \quad (1)$$

where  $n$  is positive integer, and Bessel differential equation of order zero

$$xy'' + y' + xy = 0. \quad (2)$$

Motivated by the mentioned sources, the aim of this paper is to prove the Hyers-Ulam stability of Laguerre and Bessel equations given by (1) and (2) by the Laplace transform method. It is worth mentioning that, to the best of our knowledge, the Laplace transform method is a very effective method to discuss the Hyers-Ulam stability of these equations, equation (1) and equation (2). In addition, to the best of our information till now, the Hyers-Ulam stability of equation (1) and equation (2) was not discussed in the literature by the Laplace transform method. This paper is the first attempt in the literature on the topic for the mentioned equations. Our results will also be differ from those obtained in the literature (see, [1-19] and the references therein). By this way, we mean that this paper has made a contribution to the subject in the literature, and the paper may be useful for researchers working on the qualitative behaviors of solutions like the Hyers-Ulam stability to various differential and partially differential equations. In view of all the mentioned information, the novelty and originality of the current paper can be checked.

## 2 Hyers-Ulam Stability of Laguerre Equation

Let  $I = (0, \infty)$ . Our first main result is the following theorem.

**Theorem 1.** If the function  $y$  satisfies the differential inequality

$$|xy'' + (1-x)y' + ny| \leq \varepsilon \quad (3)$$

for all  $x \in I$  and for some  $\varepsilon > 0$ , then there exists a solution  $y_0 : I \rightarrow \mathbb{R}$  of equation (1) such that

$$|y(x) - y_0(x)| \leq \frac{1}{n}\varepsilon.$$

**Proof.** It is clear from (3) that

$$-\varepsilon \leq xy'' + (1-x)y' + ny \leq \varepsilon.$$

If we apply the Laplace transform to the last inequality, then we have

$$L(-\varepsilon) \leq L[xy'' + (1-x)y' + ny] \leq L(\varepsilon).$$

Hence, since a Laplace transform is linear, it is clear that

$$L(-\varepsilon) \leq L(xy'') + L((1-x)y') + L(ny) \leq L(\varepsilon).$$

In view of the basic information related to the properties of a Laplace transform, it can be written that

$$-\frac{\varepsilon}{s} \leq -\frac{d}{ds}[s^2Y(s) - sY(0) - Y'(0)] + sY(s) - Y(0) + \frac{d}{ds}[sY(s) - Y(0)] + nY(s) \leq \frac{\varepsilon}{s}$$

and

$$-\frac{\varepsilon}{s} \leq -2sY(s) - s^2 \frac{dY}{ds} + sY(s) + Y(s) + s \frac{dY}{ds} + nY(s) \leq \frac{\varepsilon}{s}$$

so that

$$-\frac{\varepsilon}{s} \leq (s - s^2) \frac{dY}{ds} + (n + 1 - s)Y(s) \leq \frac{\varepsilon}{s}.$$

Assume that  $(s^2 - s) > 0$ . Dividing the above inequality by  $(s^2 - s)$  and then multiplying the last inequality by the term  $\frac{s^{n+1}}{(s-1)^n}$ , we obtain

$$-\frac{\varepsilon s^{n-1}}{(s-1)^{n+1}} \leq \frac{dY}{ds} \frac{s^{n+1}}{(s-1)^n} + \frac{(s-n-1)}{(s^2-s)} \frac{s^{n+1}}{(s-1)^n} Y(s) \leq \frac{\varepsilon s^{n-1}}{(s-1)^{n+1}}.$$

From this, we have

$$-\frac{\varepsilon s^{n-1}}{(s-1)^{n+1}} \leq \frac{d}{ds} \left[ \frac{s^{n+1}}{(s-1)^n} Y(s) \right] \leq \frac{\varepsilon s^{n-1}}{(s-1)^{n+1}}.$$

For any  $s_1 > s$ , integrating the above inequality from  $s$  to  $s_1$ , we get

$$-\frac{\varepsilon}{n} \left[ \left( \frac{s}{s-1} \right)^n - \left( \frac{s_1}{s_1-1} \right)^n \right] \leq \frac{s_1^{n+1}}{(s_1-1)^n} Y(s_1) - \frac{s^{n+1}}{(s-1)^n} Y(s) \leq \frac{\varepsilon}{n} \left[ \left( \frac{s}{s-1} \right)^n - \left( \frac{s_1}{s_1-1} \right)^n \right]$$

so that

$$\begin{aligned} -\frac{\varepsilon}{n} \left[ \left( \frac{s}{s-1} \right)^n - 2 \left( \frac{s_1}{s_1-1} \right)^n \right] &\leq \frac{s_1^{n+1}}{(s_1-1)^n} Y(s_1) - \frac{s^{n+1}}{(s-1)^n} Y(s) + \frac{\varepsilon}{n} \left( \frac{s_1}{s_1-1} \right)^n \\ &\leq \frac{\varepsilon}{n} \left( \frac{s}{s-1} \right)^n, \end{aligned}$$

$$-\frac{\varepsilon}{n} \left( \frac{s}{s-1} \right)^n \leq \frac{s_1^{n+1}}{(s_1-1)^n} Y(s_1) - \frac{s^{n+1}}{(s-1)^n} Y(s) + \frac{\varepsilon}{n} \left( \frac{s_1}{s_1-1} \right)^n \leq \frac{\varepsilon}{n} \left( \frac{s}{s-1} \right)^n.$$

Multiplying the last inequality by the term  $\frac{(s-1)^n}{s^{n+1}}$ , we obtain

$$-\frac{\varepsilon}{ns} \leq \frac{s_1^{n+1}}{(s_1-1)^n} Y(s_1) \frac{(s-1)^n}{s^{n+1}} + \frac{\varepsilon}{n} \left( \frac{s_1}{s_1-1} \right)^n \frac{(s-1)^n}{s^{n+1}} - Y(s) \leq \frac{\varepsilon}{ns}.$$

Applying the inverse Laplace transform, we have

$$\begin{aligned} L^{-1} \left( -\frac{\varepsilon}{ns} \right) &\leq L^{-1} \left[ \frac{s_1^{n+1}}{(s_1-1)^n} Y(s_1) \frac{(s-1)^n}{s^{n+1}} \right] + L^{-1} \left[ \frac{\varepsilon}{n} \left( \frac{s_1}{s_1-1} \right)^n \frac{(s-1)^n}{s^{n+1}} \right] - L^{-1} [Y(s)] \\ &\leq L^{-1} \left( \frac{\varepsilon}{ns} \right) \end{aligned}$$

and

$$-\frac{\varepsilon}{n} \leq \left[ \frac{\varepsilon}{n} + s_1 Y(s_1) \right] \left( \frac{s_1}{s_1-1} \right)^n L^{-1} \left[ \frac{(s-1)^n}{s^{n+1}} \right] - y(x) \leq \frac{\varepsilon}{n}.$$

Since

$$L^{-1} \left[ \frac{(s-1)^n}{s^{n+1}} \right] = 1 - nx + \binom{n}{2} \frac{x^2}{2!} - \binom{n}{3} \frac{x^3}{3!} + \dots + (-1)^{n+1} \frac{x^n}{n!},$$

it follows that

$$\begin{aligned} -\frac{\varepsilon}{n} &\leq \left[\frac{\varepsilon}{n} + s_1 Y(s_1)\right] \left(\frac{s_1}{s_1 - 1}\right)^n \left[1 - nx + \binom{n}{2} \frac{x^2}{2!} - \binom{n}{3} \frac{x^3}{3!} + \dots + (-1)^{n+1} \frac{x^n}{n!}\right] - y(x) \\ &\leq \frac{\varepsilon}{n}. \end{aligned}$$

Then, we can write

$$|y(x) - y_0(x)| \leq \frac{\varepsilon}{n},$$

where

$$y_0(x) = (s_1 Y(s_1) - \frac{\varepsilon}{n}) \left(\frac{s_1}{s_1 - 1}\right)^n \left(1 - nx + \binom{n}{2} \frac{x^2}{2!} - \binom{n}{3} \frac{x^3}{3!} + \dots + (-1)^{n+1} \frac{x^n}{n!}\right).$$

This completes the proof of Hyers-Ulam stability of solutions of equation (1).

Our second and last main result is the following theorem.

**Theorem 2.** Let  $\varepsilon \in \mathfrak{R}, \varepsilon > 0$ . If the function  $y$  satisfies the differential inequality

$$|xy'' + y' + xy| \leq \varepsilon \quad (4)$$

for all  $x \in I$ , then there exists a solution  $y_0 : I \rightarrow \mathfrak{R}$  of equation (2) such that

$$|y(x) - y_0(x)| \leq 2\varepsilon.$$

**Proof.** It is clear from (4) that

$$-\varepsilon \leq xy'' + y' + xy \leq \varepsilon.$$

When we apply the Laplace transform to the last inequality, we get

$$L(-\varepsilon) \leq L(xy'') + L(y') + L(xy) \leq L(\varepsilon).$$

Then, it follows that

$$-\frac{\varepsilon}{s} \leq -\frac{d}{ds} [s^2 Y(s) - sY(0) - Y'(0)] + sY(s) - Y(0) - \frac{d}{ds} Y(s) \leq \frac{\varepsilon}{s}.$$

Hence

$$-\frac{\varepsilon}{s} \leq -s^2 Y'(s) - 2sY(s) + Y(0) + sY(s) - Y(0) - Y'(s) \leq \frac{\varepsilon}{s}$$

so that

$$-\frac{\varepsilon}{s} \leq -(s^2 + 1)Y'(s) - sY(s) \leq \frac{\varepsilon}{s}.$$

Multiplying the last inequality with the term  $-\frac{1}{\sqrt{s^2+1}}$ , we arrive at

$$-\frac{\varepsilon}{s\sqrt{s^2+1}} \leq \sqrt{s^2+1}Y'(s) + \frac{s}{\sqrt{s^2+1}}Y(s) \leq \frac{\varepsilon}{s\sqrt{s^2+1}}$$

so that

$$-\frac{\varepsilon}{s\sqrt{s^2+1}} \leq \frac{d}{ds} (\sqrt{s^2+1}Y(s)) \leq \frac{\varepsilon}{s\sqrt{s^2+1}}.$$

For any  $s_1 > s$ , integrating the above inequality from  $s$  to  $s_1$ , we get

$$\begin{aligned} -\varepsilon \left[ \ln \left( \frac{\sqrt{s^2+1}+1}{s} \right) - \ln \left( \frac{\sqrt{s_1^2+1}+1}{s_1} \right) \right] &\leq \sqrt{s_1^2+1}Y(s_1) - \sqrt{s^2+1}Y(s) \\ &\leq \varepsilon \left[ \ln \left( \frac{\sqrt{s^2+1}+1}{s} \right) - \ln \left( \frac{\sqrt{s_1^2+1}+1}{s_1} \right) \right]. \end{aligned}$$

In view of the last inequality, we can write

$$-\varepsilon \ln \left( \frac{\sqrt{s^2+1}+1}{s} \right) \leq \sqrt{s_1^2+1}Y(s_1) - \sqrt{s^2+1}Y(s) \leq \varepsilon \ln \left( \frac{\sqrt{s^2+1}+1}{s} \right).$$

Multiplying the last inequality with term  $\frac{1}{\sqrt{s^2+1}}$ , we obtain

$$-\frac{\varepsilon}{\sqrt{s^2+1}} \ln \left( \frac{\sqrt{s^2+1}+1}{s} \right) \leq \frac{\sqrt{s_1^2+1}}{\sqrt{s^2+1}}Y(s_1) - Y(s) \leq \frac{\varepsilon}{\sqrt{s^2+1}} \ln \left( \frac{\sqrt{s^2+1}+1}{s} \right).$$

Since  $s > 0$ , we can write

$$-\frac{\varepsilon}{\sqrt{s^2+1}} \frac{\sqrt{s^2+1}+1}{s} \leq \frac{\sqrt{s_1^2+1}}{\sqrt{s^2+1}}Y(s_1) - Y(s) \leq \frac{\varepsilon}{\sqrt{s^2+1}} \frac{\sqrt{s^2+1}+1}{s}$$

so that

$$-\frac{2\varepsilon}{s} \leq \frac{\sqrt{s_1^2+1}}{\sqrt{s^2+1}}Y(s_1) - Y(s) \leq \frac{2\varepsilon}{s}.$$

If we apply the inverse Laplace transform, then we obtain

$$L^{-1} \left( -\frac{2\varepsilon}{s} \right) \leq L^{-1} \left( \frac{\sqrt{s_1^2+1}}{\sqrt{s^2+1}}Y(s_1) \right) - L^{-1}(Y(s)) \leq L^{-1} \left( \frac{2\varepsilon}{s} \right)$$

so that

$$-2\varepsilon \leq \sqrt{s_1^2+1}Y(s_1)J_0(x) - y(x) \leq 2\varepsilon,$$

where

$$J_0(x) = 1 - \frac{1}{1!} \binom{x}{2}^2 + \frac{1}{(2!)^2} \binom{x}{2}^4 - \frac{1}{(3!)^2} \binom{x}{2}^6 + \dots$$

From this, we can obtain

$$|y(x) - y_0(x)| \leq 2\varepsilon,$$

where

$$y_0(x) = -\sqrt{s_1^2+1}Y(s_1)J_0(x).$$

This completes the proof of Hyers-Ulam stability of solutions of equation (2).

### 3 Conclusion

A kind of linear differential equations of second order, namely Laguerre and Bessel equations, is considered. Sufficient conditions are established guaranteeing the Hyers-Ulam stability of solutions of these equations. To prove the main results here, we benefit from the Laplace transform method. The results obtained essentially complement the results in the literature.

## References

- [1] Abdollahpour, M. R. and Aghayari, R. and Rassias, M. Th. Hyers-Ulam stability of associated Laguerre differential equations in a subclass of analytic functions. *J. Math. Anal. Appl.* **1** (2016) 605–612.
- [2] Alqifiary, Q.H. and Jung, S.M. Laplace transform and generalized Hyers-Ulam stability of linear differential equations. *Electron. J. Differential Equations* **80** (2014) 11 pp.
- [3] Alsina, C. and Ger, R. On some inequalities and stability results related to the exponential function. *J. Inequal. Appl.* **2** (1998) 373–380.
- [4] Bicer, E. and Tunç, C. On the Hyers-Ulam stability of certain partial differential equations of second order. *Nonlinear Dyn. Syst. Theory* **17** (2) (2017) 150–157.
- [5] Hyers, D.H. On the Stability of the Linear Functional Equation. *Proc. Nat. Acad. Sci.* **27** (1941) 222–224.
- [6] Jung, S.M. Hyers Ulam stability of linear differential equations of first order (III). *J. Math. Anal. Appl.* **311** (2005) 139–146.
- [7] Jung, S.M. Hyers Ulam stability of linear differential equations of first order (II). *Appl. Math. Lett.* **19** (2006) 854–858.
- [8] Jung, S.M. Hyers Ulam stability of first order linear partial differential equations with constant coefficients. *Math. Inequal. Appl.* **10** (2007) 261–266.
- [9] Jung, S.M. Hyers Ulam stability of linear partial differential equations of first order. *Appl. Math. Lett.* **22** (2009) 70–74.
- [10] Jung, S.M. Approximation of analytic functions by Legendre functions. *Nonlinear Anal.* **71** (2009) 103–108.
- [11] Jung, S.M. Approximation of analytic functions by Laguerre functions. *Appl. Math. Comput.* **218** (2011) 832–835.
- [12] Liu, H. and Zhao, X. Hyers-Ulam-Rassias stability of second order partial differential equations. *Ann. Differential Equations* **29** (4) (2013) 430–437.
- [13] Lungu, N. and Popa, D. Hyers-Ulam stability of a first order partial differential equation. *J. Math. Anal. Appl.* **385** (2012) 86–91.
- [14] Lungu, N. and Popa, D. Hyers-Ulam stability of some partial differential equations. *Carpathian J. Math.* **30** (3) (2014) 327–334.
- [15] Rassias, T.M. On the Stability of the Linear Mapping in Banach Spaces. *Proc. Amer. Math. Soc.* **72** (2) (1978) 297–300.
- [16] Tunc, C. and Bicer, E. Hyers-Ulam-Rassias stability for a first order functional differential equation. *J. Math. Fundam. Sci.* **47** (2) (2015) 143–153.
- [17] Ulam, S.M. *Problems in Modern Mathematics*. Science Editions John Wiley & Sons, Inc. New York. 1964.
- [18] Vlasov, V. Asymptotic behavior and stability of the solutions of functional differential equations in Hilbert space. *Nonlinear Dyn. Syst. Theory* **2** (2) (2002) 215–232.
- [19] Vasundhara D.J. Stability in terms of two measures for matrix differential equations and graph differential equations. *Nonlinear Dyn. Syst. Theory* **16** (2) (2016) 179–191.





# Output Tracking of Some Class Non-minimum Phase Nonlinear Uncertain Systems

Firman<sup>1\*</sup>, Janson Naiborhu<sup>2</sup> Roberd Saragih<sup>2</sup>  
and Sapto Wahyu Indratno<sup>3</sup>

<sup>1</sup> *Department of Mathematics, Faculty of Mathematics and Natural Science,  
Universitas Hasanuddin, Indonesia*

<sup>2</sup> *Industrial and Financial Research Group, Faculty of Mathematics and Natural Science,  
Institut Teknologi Bandung, Indonesia*

<sup>3</sup> *Statistics Research Group, Faculty of Mathematics and Natural Science, Institut Teknologi  
Bandung, Indonesia*

Received: September 28, 2016;    Revised: October 30, 2017

**Abstract:** In this paper, we present the output tracking for a class of non-minimum phase nonlinear uncertain systems. To achieve the output tracking, we will apply the modified steepest descent control. To apply the modified steepest descent control, the output of the system will be redefined so that the system will become minimum phase with respect to a new output.

**Keywords:** *relative degree of system; minimum phase system; non-minimum phase system; modified steepest descent control.*

**Mathematics Subject Classification (2010):** 93C10, 93D21.

## 1 Introduction

In the output tracking theory, the input-output linearization is one of the most available methods [1]. Output tracking problems for nonlinear non-minimum phase systems are a rather difficult issue in control theory. Most of researchers restrict their research to some special nonlinear classes only. The stable inversion proposed in [2], [3] is an iterative solution to the tracking problem with the unstable zero dynamics. This method requires the system to have well defined relative degree and hyperbolic dynamics, i.e. no eigenvalues on the imaginary axis. In [4], control design procedure for the output tracking was proposed. The design procedure consists of two steps. At the first step, the

---

\* Corresponding author: [mailto:firmam.math11@gmail.com](mailto:mailto:firman.math11@gmail.com)

standard input-output linearization is applied. At the second step, we group an output with the internal dynamics as one subsystem, which is usually nonlinear, and the rest of the output as the other subsystem that is linear, the nonlinear subsystem is linearized about its equilibrium. In [5], the asymptotic output tracking which is a class of causal non-minimum phase uncertain nonlinear systems is achieved by using higher order sliding modes (HOSM) without reduction of the input-output dynamics order. Results on stabilization of non-minimum phase system in the output feedback form have been presented in [6], [7], [8]. The main idea in [6], [7], [8] is output reconstruction such that the system becomes minimum phase with respect to a new output. Results on output tracking of some class non-minimum phase nonlinear system have been presented in [9], [10]. In [9], the design of the input control is based on the exact linearization.

In this paper, we will modify the steepest descent control for output tracking of a class of non-minimum phase nonlinear uncertain systems, with relative degree being  $n - 1$ ,  $n$  is the dimension of the system. The modification is the addition of an artificial input of the steepest descent control. The design of descent control can not be initiated from the output causing the system to be non-minimum phase. In this paper, to solve the problem, we transform the system into a normal form which is minimum phase with respect to a virtual output, which is a linear combination of state variables.

## 2 Problem Statement

Consider nonlinear uncertain system

$$\dot{x} = \mathbf{A}x + \phi(y) + \theta\psi(y) + bu, \quad x(t) \in \mathbf{R}^n, \quad u(t) \in \mathbf{R}, \quad (1)$$

$$y = x_1, \quad (2)$$

in which  $\phi(x)$  is smooth vector field in  $\mathbf{R}^n$ , with  $\phi(0) = 0$ ,  $\phi(y) = [\phi_1(y), \phi_2(y), \dots, \phi_n(y)]^T$ ,  $\psi(0) = 0$ ,  $\theta\psi(y) = [\theta_1(t)\psi_1(y), \theta_2(t)\psi_2(y), \dots, \theta_n(t)\psi_n(y)]$ ,  $b = [0, \dots, 0, b_{n-1}, b_n]^T$ ,

$$b_{n-1} \neq 0, \quad b_{n-1} = -b_n \quad \text{and} \quad \mathbf{A} = \begin{pmatrix} 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \\ 0 & 0 & \dots & 0 \end{pmatrix}.$$

The relative degree of the system (1)-(2) is  $n - 1$ .

The system (1)-(2) can be transformed to

$$\dot{z}_1 = z_2 + \theta_1(t)\psi(x_1), \quad (3)$$

$$\dot{z}_k = z_{k+1} + \varphi_{k-1}(t, x_1, \dots, x_{k-1}), \quad k = 2, \dots, n-2, \quad (4)$$

$$\dot{z}_{n-1} = a(z, \eta) + b(z, \eta)u + \varphi(t, x_1, \dots, x_{n-2}), \quad (5)$$

$$\begin{aligned} \dot{\eta} &= \dot{x}_1 + \dot{x}_2 + \dots + \dot{x}_n \\ &= \eta - z_1 + \phi_1(y) + \dots + \phi_n(y) + \theta_1(t)\psi_1(y) + \dots + \theta_n(t)\psi_n(y), \\ y &= z_1, \end{aligned}$$

with the internal dynamics

$$\dot{\eta} = \eta - z_1 + \phi_1(y) + \dots + \phi_n(y) + \theta_1(t)\psi_1(y) + \dots + \theta_n(t)\psi_n(y). \quad (6)$$

Then the zero dynamics of the system (1)-(2) is

$$\dot{\eta} = \eta.$$

Thus the system (1)-(2) is non-minimum phase.

Our objective is to make the output system (2) track the desired output. To make the system (1)-(2) track the desired output, we will use the dynamic feedback control. The design of the dynamic control is based on the modification of the steepest descent control. By "Trajectory Following Method" [11], the steepest descent control is determined from the differential equation  $\dot{u} = -\frac{\partial F}{\partial u}$ , where  $F$  is a descent function which has a variable as the solution of internal dynamics system. So, the modification of the steepest descent control can not be initiated from the output causing the system to be non-minimum phase. Therefore, the output of the system will be redefined so that the system will become minimum phase with respect to a new output.

### 3 Main Results

We consider system (1). Consider now a new output  $\mu = t_1 x$ , with  $t_1 = (\alpha \ 1 \ 1 \ \dots \ 1)$ . The relative degree of system (1) with respect to  $\mu$  is  $n - 1$ . The system (1) with respect to  $\mu$ , can be transformed to

$$\dot{z}_1 = z_2 + c\theta(t)\psi(x_1), \quad (7)$$

$$\dot{z}_k = z_{k+1} + \omega_{i-1}(t, x_1, \dots, x_{i-1}), \quad k = 2, \dots, n-2, \quad (8)$$

$$\dot{z}_{n-1} = a(z, \eta) + b(z, \eta)u + \omega(t, x_1, \dots, x_{n-2}), \quad (9)$$

$$\begin{aligned} \dot{\eta} &= \dot{x}_1 + \dot{x}_2 + \dots + \dot{x}_n \\ &= \eta - x_1 + \phi_1(x_1) + \dots + \phi_n(x_1) + \theta_1(t)\psi_1(x_1) + \dots + \theta_n(t)\psi_n(x_1), \\ y &= \mu = z_1. \end{aligned}$$

Furthermore

$$\eta\dot{\eta} = \eta(\eta - x_1 + \phi_1(x_1) + \dots + \phi_n(x_1) + \theta_1(t)\psi_1(x_1) + \dots + \theta_n(t)\psi_n(x_1)). \quad (10)$$

**Assumption 3.1**  $\psi_i(x_1) \leq |x_1|$ ,  $\forall x_1$ ,  $i = 1, 2, \dots, n$ .

Case 1 : if  $\phi_1(x_1) + \phi_2(x_1) + \dots + \phi_n(x_1) = 0$ .

Then

$$\begin{aligned} \eta\dot{\eta} &= \eta^2 - \eta x_1 + \eta\theta_1(t)\psi_1(x_1) + \dots + \eta\theta_n(t)\psi_n(x_1) \\ &\leq \eta^2 - \eta x_1 + |\eta||x_1| (|\theta_1(t)| + |\theta_2(t)| + \dots + |\theta_n(t)|) \\ &= \eta^2 - \eta \left( \frac{z_1 - \eta}{\alpha - 1} \right) + |\eta| \left| \frac{z_1 - \eta}{\alpha - 1} \right| (|\theta_1(t)| + |\theta_2(t)| + \dots + |\theta_n(t)|). \end{aligned}$$

Then if  $z_1 = 0$  and  $0 < \alpha < 1$ , we have

$$\eta\dot{\eta} \leq \eta^2 \left( \frac{-\alpha + (|\theta_1(t)| + |\theta_2(t)| + \dots + |\theta_n(t)|)}{|\alpha - 1|} \right). \quad (11)$$

If  $|\theta_1(t)| + |\theta_2(t)| + \dots + |\theta_n(t)| < \alpha$ , then  $\eta\dot{\eta} < 0$ . Therefore, the zero dynamics (1) with respect to output  $\mu$  is asymptotically stable. Thus the system (1) with respect to output  $\mu$  is minimum phase.

Case 2 : if  $\phi_1(x_1) + \phi_2(x_1) + \dots + \phi_n(x_1) = h(x_1) \neq 0$ .

We have

$$\begin{aligned}\eta\dot{\eta} &= \eta(\eta - x_1 + h(x_1) + \theta_1(t)\psi(x_1) + \dots + \theta_n(t)\psi_n(x_1)) \\ &\leq \eta^2 - \eta x_1 + \eta h(x_1) + |\eta||x_1|(|\theta_1(t)| + |\theta_2(t)| + \dots + |\theta_n(t)|) \\ &= \eta^2 - \eta \left( \frac{z_1 - \eta}{\alpha - 1} \right) + \eta h \left( \frac{z_1 - \eta}{\alpha - 1} \right) + |\eta| \left| \frac{z_1 - \eta}{\alpha - 1} \right| (|\theta_1(t)| + \dots + |\theta_n(t)|).\end{aligned}$$

If  $z_1 = 0$ ,  $\forall t$  and  $0 < \alpha < 1$ , then

$$\eta\dot{\eta} \leq \eta^2 \left( \frac{-\alpha + (|\theta_1(t)| + |\theta_2(t)| + \dots + |\theta_n(t)|)}{|\alpha - 1|} \right) + \eta h \left( \frac{-\eta}{\alpha - 1} \right). \quad (12)$$

**Assumption 3.2** We consider system (1). Choose  $\phi_1(x_1)$ ,  $\phi_2(x_1)$ , ...,  $\phi_n(x_1)$  so that

$$\eta h \left( \frac{-\eta}{\alpha - 1} \right) < 0.$$

If  $|\theta_1(t)| + |\theta_2(t)| + \dots + |\theta_n(t)| \leq \alpha$  and by Assumption 3.2, we have  $\eta\dot{\eta} < 0$ . Therefore the system (1) with respect to output  $\mu$  is minimum phase.

**Lemma 3.1** Consider system (1). Then there exists a linear combination of the state variables  $\mu = \alpha x_1 + x_2 + x_3 + \dots + x_n$  such that the relative degree of the system (1) with respect to output  $\mu$  is  $n - 1$ . Furthermore due to Assumption 3.1 we obtain

- (i) If  $\phi(x_1) + \dots + \phi_n(x_1) = 0$ , the system (1) with respect to output  $\mu$  is minimum phase, with  $|\theta_1(t)| + |\theta_2(t)| + \dots + |\theta_n(t)| < \alpha$ ,  $0 < \alpha < 1$ .
- (ii) If  $\phi(x_1) + \dots + \phi_n(x_1) \neq 0$  and by Assumption 3.2, the system (1) with respect to output  $\mu$  is minimum phase, with  $0 < \alpha < 1$  and  $|\theta_1(t)| + \dots + |\theta_n(t)| \leq \alpha$ .

Let  $\mu_d$  be the desired output of the new output.

**Assumption 3.3** Let  $x_i = x_{id}$ ,  $i = 1, 2, \dots, n - 2$ .

Based on Assumption 3.3, we have  $x_{2d}$ ,  $x_{3d}$ , ...,  $x_{(n-1)d}$ , respectively. Then  $\dot{x}_n = f(x_1, x_{n-1}, x_n)$  can be solved by substituting  $x_{n-1} = x_{(n-1)d}$ . Thus  $x_n = x_{nd}$ . Furthermore the definition error  $e = \mu - \mu_d$ , with  $\mu_d = \alpha x_{1d} + x_{2d} + \dots + x_{nd}$ .

We design a control law  $u$  in terms of the properties of the solution of higher order ordinary differential equation. Consider a differential equation

$$a_r e^{(r)}(t) + a_{r-1} e^{(r-1)}(t) + \dots + a_1 \dot{e}(t) + a_0 e(t) = 0, \quad (13)$$

where  $r$  is the relative degree of the system. If a polynomial

$$p(s) = a_r s^r + a_{r-1} s^{r-1} + \dots + a_1 s + a_0 \quad (14)$$

is Hurwitz, then the solution of differential equation (13) tends to zero if  $t \rightarrow \infty$ . In this case, for the purpose of designing the control law, an explicit relationship between input and output is required. To this end, we define a descent function as follows :

$$\begin{aligned}F(\mu, \mu_d, \dot{\mu}, \dot{\mu}_d, \dots, \mu^{(n-1)}(t), \mu_d^{(n-1)}(t)) &= \left( \sum_{j=0}^{n-1} a_j (\mu - \mu_d)^{(j)} \right)^2 \\ &= \left( \sum_{j=0}^{n-1} a_j (e)^{(j)} \right)^2.\end{aligned} \quad (15)$$

By "Trajectory Following Method" [11], the control  $u$  is determined from the differential equation

$$\dot{u} = -\frac{\partial F}{\partial u} = -2a_{n-1} \left( \sum_{j=0}^{n-1} a_j(e)^{(j)} \right) \frac{\partial(e)^{(n-1)}}{\partial u}. \quad (16)$$

The control law in equation (16) is called the steepest descent control.

Calculate the time derivative of the descent function (15) along the trajectory of the extended system

$$\dot{x} = \mathbf{A}x + \phi(y) + \theta\psi(y) + bu, \quad (17)$$

$$\dot{u} = -2a_{n-1} \left( \sum_{j=0}^{n-1} a_j(e)^{(j)} \right) \frac{\partial(e)^{(n-1)}}{\partial u}. \quad (18)$$

Then we have

$$\begin{aligned} \dot{F}(e, \dot{e}, \dots, e^{(n-1)}) &= 2 \left( \sum_{j=0}^{n-1} a_j(e)^{(j)} \right) \left( \sum_{j=0}^{n-2} a_j(e)^{(j+1)} \right) \\ &+ 2a_{n-1} \left( \sum_{j=0}^{n-1} a_j(e)^{(j)} \right) \left( \frac{\partial a(e + \beta_d, \eta)}{\partial t} + \frac{\partial b(e + \beta_d, \eta)}{\partial t} u \right) \\ &- 2a_{n-1} \left( \sum_{j=0}^{n-1} a_j(e)^{(j)} \right) y_d^{(n)} - \left( \frac{\partial F}{\partial u} \right)^2. \end{aligned} \quad (19)$$

From equation (19), the value of the time derivative of the descent function (15) along the trajectory of (17) can not be guaranteed to be less than zero  $t \geq 0$ .

Now we modify the steepest descent control (16) by adding an artificial input  $v$ . Then the extended system (17) becomes

$$\begin{aligned} \dot{x} &= \mathbf{A}x + \phi(y) + \theta\psi(y) + bu, \\ \dot{u} &= -\frac{\partial F}{\partial u} + v. \end{aligned} \quad (20)$$

In the same way, let us calculate the time derivative of the descent function (15) along the trajectory of the extended system (20) yielding

$$\begin{aligned} \dot{F}(e, \dot{e}, \dots, e^{(n-1)}) &= 2 \left( \sum_{j=0}^{n-1} a_j(e)^{(j)} \right) \left( \sum_{j=0}^{n-2} a_j(e)^{(j+1)} \right) \\ &+ 2a_{n-1} \left( \sum_{j=0}^{n-1} a_j(e)^{(j)} \right) \left( \frac{\partial a(e + \beta_d, \eta)}{\partial t} + \frac{\partial b(e + \beta_d, \eta)}{\partial t} u \right) \\ &- 2a_{n-1} \left( \sum_{j=0}^{n-1} a_j(e)^{(j)} \right) y_d^{(n)} - \left( \frac{\partial F}{\partial u} \right)^2 + \frac{\partial F}{\partial u} v. \end{aligned} \quad (21)$$

Consider equation (21). We will choose the artificial input  $v$  so that  $\dot{F}(e, \dot{e}, \dots, e^{(r)})$  be less then zero. We take

$$v = \frac{1}{\frac{\partial F}{\partial u}} \left( -k(e, \dot{e}, \dots, e^{(n-1)}) \right), \quad (22)$$

where

$$\begin{aligned} k(e, \dot{e}, \dots, e^{(n-1)}) &= 2 \left( \sum_{j=0}^{n-1} a_j(e)^{(j)} \right) \left( \sum_{j=0}^{n-2} a_j(e)^{(j+1)} \right) \\ &+ 2a_{n-1} \left( \sum_{j=0}^{n-1} a_j(e)^{(j)} \right) \left( \frac{\partial a(e + \beta_d, \eta)}{\partial t} + \frac{\partial b(e + \beta_d, \eta)}{\partial t} u \right) \\ &- 2a_{n-1} \left( \sum_{j=0}^{n-1} a_j(e)^{(j)} \right) y_d^{(n)}. \end{aligned} \quad (23)$$

Then

$$\dot{F}(e, \dot{e}, \dots, e^{(n-1)}) = - \left( \frac{\partial F}{\partial u} \right)^2. \quad (24)$$

We have  $\dot{F}(e, \dot{e}, \dots, e^{(n-1)}) < 0$ , if  $\sum_{j=0}^{n-1} a_j(e_1)^{(j)} \neq 0$ . Let  $\sum_{j=0}^{n-1} a_j(e_1)^{(j)} = 0$ . From equation (24)  $\dot{F}(e, \dot{e}, \dots, e^{(n-1)}) = 0$ . Thus, the descent function (15) becomes minimum. The minimum value of descent function (15) is zero. Therefore  $F(e, \dot{e}, \dots, e^{(n-1)}) = 0$ , then  $\sum_{j=0}^{n-1} a_j(e)^{(j)} = 0$ . Thus, we choose  $a_j, j = 0, \dots, n-1$  so that the polynomial  $p(s) = a_0 + a_1 s + \dots + a_{n-1} s^{n-2} + s^{n-1}$  is Hurwitz, then the error  $e(t) \rightarrow 0$ , if  $t \rightarrow \infty$ . Thus  $\mu$  tends to  $\mu_d$  if time  $t \rightarrow \infty$ . Hence the output of the original system  $y = x_1$  tracks to the desired output  $y_d(t)$ .

### Example 3.1

$$\begin{aligned} \dot{x}_1 &= x_2 + x_1^2, \\ \dot{x}_2 &= x_3 - u + x_1^2 + k_1 \cos(t) \frac{x_1}{1 + x_1^2}, \end{aligned} \quad (25)$$

$$\begin{aligned} \dot{x}_3 &= u - 2x_1^2 + k_2 \sin(t) \sin(x_1), \\ y &= x_1. \end{aligned} \quad (26)$$

The zero dynamic system (25)-(26) is  $\eta = \dot{\eta}$ . Thus the system (25)-(26) is non-minimum phase. Now redefine the output :  $z_1 = \alpha x_1 + x_2 + x_3$ , with  $0 < \alpha < 1$ . Furthermore

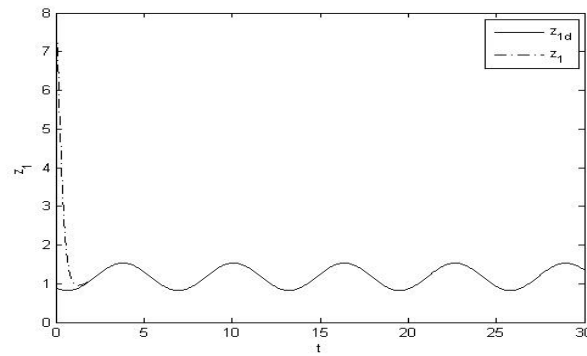
$$\dot{z}_1 = \alpha x_2 + (\alpha - 1)x_1^2 + x_3 + k_1 \cos(t) \frac{x_1}{1 + x_1^2} + k_2 \sin(t) \sin(x_1),$$

$$\ddot{z}_1 = \alpha \dot{x}_2 + 2(\alpha - 1)x_1 \dot{x}_1 + \dot{x}_3 + \frac{d}{dt} \left( k_1 \cos(t) \frac{x_1}{1 + x_1^2} \right) + \frac{d}{dt} (k_2 \sin(t) \sin(x_1)).$$

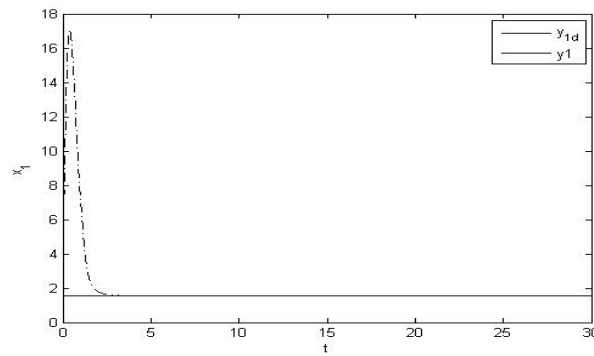
Thus the relative degree of the systems (25) with respect to the output  $z_1$  is 2. If  $z_1 = 0$ , we have

$$\begin{aligned} \eta \dot{\eta} &\leq \frac{-\alpha}{|\alpha - 1|} \eta^2 + \frac{\eta^2}{|\alpha - 1|} (|k_1 \cos(t)| + |k_2 \sin(t)|) \\ &= \frac{\eta^2}{|\alpha - 1|} (-\alpha + (|k_1 \cos(t)| + |k_2 \sin(t)|)). \end{aligned} \quad (27)$$

If  $|k_1 \cos(t)| + |k_2 \sin(t)| < \alpha$ , then  $\eta \dot{\eta} < 0$ . Thus system (25) with respect to the output  $z_1$  is minimum phase. Let  $y_d(t) = \pi/2$ . Next, we choose  $z_{1d}$  so that if  $z_1$  tracks  $z_{1d}(t)$ , then  $y(t)$  tracks the desired output  $y_d(t)$ . By replacing  $x_1$  with  $x_{1d} = y_d = \pi/2$ , we get  $x_{2d} = -(\pi/2)^2$ . By replacing  $x_2$  with  $x_{2d}$ , we have the differential equation  $\dot{x}_3 - x_3 = -(\pi/2)^2 + k_2 \sin(t) + k_1 \left( \frac{\pi/2}{1+(\pi/2)^2} \right) \cos(t)$ . Thus  $x_{3d} = (\pi/2)^2 +$



**Figure 1:** Output tracking  $z_1$  to  $z_{1d}$ .



**Figure 2:** Output tracking  $y$  to  $y_d = \pi/2$ .

$0.5 \sin(t) \left( k_1 \left( \frac{\pi/2}{1+(\pi/2)^2} \right) - k_2 \right) + 0.5 \cos(t) \left( -k_1 \left( \frac{\pi/2}{1+(\pi/2)^2} \right) - k_2 \right)$ . Now,  $z_{1d} = \alpha x_{1d} + x_{2d} + x_{3d} = \alpha(\pi/2) + x_{3d}$ . The modified steepest descent control with respect to the output  $z_1$  is

$$\dot{u} = -\frac{\partial F}{\partial u} = -2a_2(a_0(z_1 - z_{1d}) + a_1(\dot{z}_1 - \dot{z}_{1d}) + a_2(\ddot{z}_1 - \ddot{z}_{1d}))(1 - \alpha) + v, \quad (28)$$

where  $v$  is the same as in equation (22). Simulation results are shown in Figure 1 and in Figure 2 for the constants  $a_0 = 35$ ,  $a_1 = 12$ ,  $a_2 = 1$ ,  $\alpha = 0.75$ ,  $k_1 = 0.1$ ,  $k_2 = 0.5$ . The initial values  $x_1(0) = 5$ ,  $x_2(0) = 4$ ,  $x_3(0) = 0$ ,  $x_4(0) = 0$ . In Figure 1, the output which

has been selected so that the system becomes minimum phase tracks the desired output  $z_{1d}$ . In Figure 2, the output of the original system tracks the desired output  $y_d = \pi/2$ .

**Example 3.2**

$$\begin{aligned} \dot{x}_1 &= x_2 - x_1^3, \\ \dot{x}_2 &= x_3 - u + 2x_1^3, \end{aligned} \quad (29)$$

$$\begin{aligned} \dot{x}_3 &= \theta \sin(x_1) + u - 2x_1^3, \\ y &= x_1. \end{aligned} \quad (30)$$

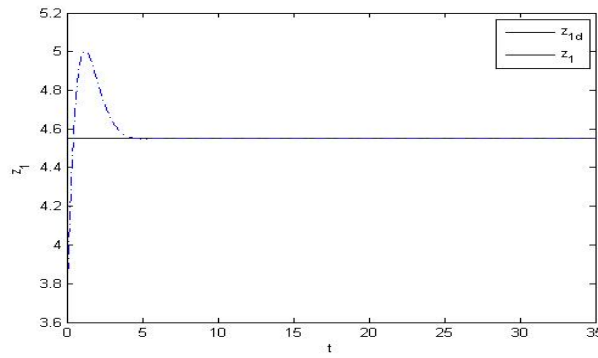
The zero dynamic system (25)-(26) is  $\eta = \dot{\eta}$ . Thus the system (25)-(26) is non-minimum phase. Now redefine the output :  $z_1 = \alpha x_1 + x_2 + x_3$ , with  $0 < \alpha < 1$ . The zero dynamic system (25)-(26) with respect to the output  $z_1$  is

$$\dot{\eta} = \eta - \left( \frac{-\eta}{\alpha - 1} \right) - \left( \frac{-\eta}{\alpha - 1} \right)^3 + \theta \sin \left( \frac{-\eta}{\alpha - 1} \right).$$

We have

$$\begin{aligned} \eta \dot{\eta} &= \eta^2 + \frac{\eta^2}{\alpha - 1} + \frac{\eta^4}{(\alpha - 1)^3} + \eta \theta \sin \left( \frac{-\eta}{\alpha - 1} \right) \\ &\leq \eta^2 + \frac{\eta^2}{\alpha - 1} + \frac{\eta^4}{(\alpha - 1)^3} + |\eta| |\theta| \left| \frac{-\eta}{\alpha - 1} \right| \\ &= \frac{\eta^2(|\theta| - \alpha)}{|\alpha - 1|} + \frac{\eta^4}{\alpha - 1}. \end{aligned} \quad (31)$$

If  $|\theta| \leq \alpha$ , then  $\eta \dot{\eta} < 0$ . Thus the system (29) with respect to the output  $z_1$  is minimum phase. Let  $y_d(t) = \pi/2$ . By replacing  $x_1$  with  $x_{1d} = y_d = \pi/2$ , we get  $x_{2d} = (\pi/2)^3$ . By replacing  $x_2$  with  $x_{2d}$ , we have the differential equation  $\dot{x}_3 - x_3 = \theta$ . Thus  $x_{3d} = -\theta$ . Now,  $z_{1d} = \alpha x_{1d} + x_{2d} + x_{3d} = \alpha(\pi/2) + (\pi/2)^3 - \theta$ . The modified steepest descent

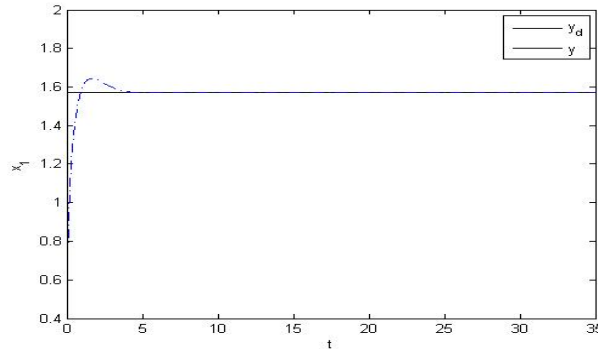


**Figure 3:** Output tracking  $z_1$  to  $z_{1d}$ .

control with respect to the output  $z_1$  is

$$\dot{u} = -\frac{\partial F}{\partial u} = -2a_2(a_0(z_1 - z_{1d}) + a_1(\dot{z}_1 - \dot{z}_{1d}) + a_2(\ddot{z}_1 - \ddot{z}_{1d}))(1 - \alpha) + v, \quad (32)$$





**Figure 4:** Output tracking  $y$  to  $y_d = \pi/2$ .

where  $v$  is the same as in equation (22). Simulation results are shown in Figure 3 and in Figure 4 for the constants  $a_0 = 12$ ,  $a_1 = 14$ ,  $a_2 = 6$ ,  $\alpha = 0.75$ . The initial values  $x_1(0) = 0, 5$ ,  $x_2(0) = 1$ ,  $x_3(0) = 0$ ,  $u(0) = 0$ ,  $\theta(t) = 0.6$ . In Figure 3, the output which has been selected so that the system become minimum phase tracks the desired output  $z_{1d}$ . In Figure 4, the output of the original system tracks the desired output  $y_d = \pi/2$ .

#### 4 Conclusion

In this paper, we have designed the dynamic feedback control for output tracking of some class non-minimum phase nonlinear uncertain system (1)-(2). The design of the dynamic control is based on the modification of the steepest descent control. To apply the modified steepest descent control the system (1) is required to be minimum phase with respect to a new output, where the new output is the linear combination of the state variables. Furthermore, the new desired output will be set based on the desired output of the original system. By applying the modified steepest descent control, the system output tracks the desired output.

#### Acknowledgment

This research is supported by the Indonesian Directorate General of Higher Education (DIKTI).

#### References

- [1] Isidori, A. *Nonlinear Control Systems: An Introduction*. Springer, Berlin, Heidelberg, 1989.
- [2] Chen, D. and Paden, B. Stable inversion of nonlinear non-minimum phase systems. *Int. J. Control* **64** (1) (1996) 45–54.
- [3] Chen, D. Iterative solution to stable inversion of nonlinear non-minimum Phase systems. In: *Proc. American Control Conference*. June, 1993, 2960–2964.
- [4] Dong Li. Output Tracking of Nonlinear Nonminimum Phase Systems: an Engineering Solution. In: *Proceedings of the 44th IEEE Conference on Decision and control, and the European Control Conference*. Sevilla, Spain, Dec 12-15, 1995, 3462–3467.

- [5] Baev, S., Shtessel, Y. and Shkolnikov, I. HOSM driven output tracking in the nonminimum-phase causal nonlinear Systems. In: *Proceeding of the 46th IEEE Conference on Decision and Control*. New Orleans, USA, Dec 12-14, 2007, 3715–3720.
- [6] Riccardo Marino. and Patrizio Tomei. A class of Globally Output Feedback Stabilizable Nonlinear Nonminimum Phase Systems. *IEEE Transactions on Automatic Control* **50** (12) (2005) 2097–2101.
- [7] Z.Y. Zili Li. and Zengqiang Chen. The stability analysis and control of nonminimum phase nonlinear systems. *International Journal of Nonlinear Science* **3** (2) (2007) 103–110.
- [8] Wang, F.C.N. and Xu, W. Adaptive global output feedback stabilisation of some non-minimum phase nonlinear uncertain systems. *IET Control Theory Appl.* **2** (2) (2008) 117–125.
- [9] Naiborhu, J., Firman and Mu'tamar, K. Particle Swarm Optimization in the Exact Linearization Technic for Output Tracking of Non-Minimum Phase Nonlinear Systems. *Applied Mathematical Science* **7** (1) (2013) 5427–5442.
- [10] Firman, Naiborhu, J. and Roberd Saragih. Modification of a Steepest Descent Control for Output Tracking of Some Class Non-minimum Phase Nonlinear Systems. *Applied Mathematics and Computation* **269** (2015) 497–506.
- [11] Vincent, T.L. and Grantham, W.J. Trajectory Following Methods in Control System Design. *J. of Global Optimization* **23** (2002) 267–282.



# Passivity Based Control of Continuous Bioreactors

M. Makkar<sup>1</sup> and J.-Y. Dieulot<sup>2\*</sup>

<sup>1</sup> LMNIIT, Jaipur, India

<sup>2</sup> CRISTAL CNRS 9189, Polytech-Lille Cité Scientifique 59650 Villeneuve d'Ascq, France

Received: December 9, 2016; Revised: November 1, 2017

**Abstract:** In this paper, a passivity based model of a general set of bio-reactions in open reactors with new energy functions is derived. A change of coordinates is done, based on the stoichiometric invariance principle, which simplifies the number of equations to be taken care of and shows directly the passivity of the system. The passivity based control will be obtained in terms of systematic controller design techniques. The energy functions can be said to be in close proximity with the Gibbs free energy function used in port-Hamiltonian model of enzymatic reactions and are far from the traditional non-physical quadratic functions.

**Keywords:** *Port-Hamiltonian systems; passivity; nonlinear control; bioreactors.*

**Mathematics Subject Classification (2010):** 92C45, 70S05, 93C40.

## 1 Introduction

Passivity is a fundamental property of physical systems which are able to transform and dissipate energy. For such systems, passivity balances the energy of a system quantifying the external input and generated output. Hence, passivity is also related to the stability of the system by the fact that the system is said to be passive if the input energy is always more than or equal to the stored energy (closed systems) or output energy (open systems). Port-Hamiltonian (PH) modelling has been one of the most physical passivity based modelling technique which has inherent structural properties clearly defining the interconnection and dissipation of energy. Bond graph (BG) modelling technique can be considered as the graphical representation of the PH models. However, it is possible to propose only quasi-port-Hamiltonian representations for chemical and enzymatic systems using different energy functions and subsequent controllers (entropy, enthalpy, Gibbs free energy, etc., see e.g. [1], [2]) or pseudo bond graph models, e.g. [3].

---

\* Corresponding author: <mailto:jean-yves.dieulot@polytech-lille.fr>

When it comes to bioreactions, a true energetic representation becomes impossible, as these involve a high number of microbial reactions, which are generally lumped into a mathematical reaction term without any thermodynamical meaning. On a macroscopic level, different kinetics are being proposed based on empirical data fitting, e.g. Monod kinetics, which reflect energy dissipation phenomena and can contribute to passivity based structure. [4] have tried different coordinate transformations allowing a generic but artificial obtention of a passive system where the examples use again quadratic energy functions. [5] explored different possibilities of unphysical Hamiltonian functions such as constant, logarithmic and quadratic functions. Nevertheless, adequate coordinate transformation is needed for better understanding of the mechanisms. The authors in this paper contributed through a new specific passivity based model taking advantage of the structure, based on decoupling of dynamics and the use of invariants extended to continuous reactors in [6]. It is shown that the passivity-based model involves non quadratic storage functions. A general formulation leads to easy application for a large number of systems. The case of multiple equilibria and bifurcation analysis can be seen e.g. in [7].

Passivity based control (PBC), as discussed above, exploits system's physical properties while exploring the possibilities of managing its energy and takes into account physical terms while choosing the control action. PBC of continuous chemical reactors generally relies on non-physical energy functions (e.g. quadratic functions) [1]. Subsequently, in [4] the authors proposed a systematic design of a real PH structure with an efficient control design. However, the energy function is given as a pure meaningless quadratic form, and the PH model is given by an artificial decomposition of the nonlinear model without any real world insight. In [8] it was shown that internal entropy production can be used as a storage function and also, a quasi port-controlled Hamiltonian representation of chemical reactors was formulated. Hence, an original and physical-based control design presented in this paper exploits the new passive model and is applied to aniline degradation by *Pseudomonas putida* cells.

## 2 The General Dynamical Model of a Single Stream Bioreactor

Suppose there are  $j$  independent reactions involving  $n$  components, taking place inside a perfectly mixed continuous reactor at constant volume and temperature. The bioreactor has only one single stream for all the concentrations coming in or going out (e.g. wastewater treatment). The inlet dilution rate is equal to outlet dilution rate to maintain constant volume. Dilution rate  $D$  is the control parameter. The state space of the concentrations is:

$$[\mathbf{z}] = [\xi_1, \xi_2 \cdots \xi_n]^T.$$

$[\mathbf{z}]$  comprises a set of  $\begin{bmatrix} S & X & P \end{bmatrix}^T$ .  $S$  represent substrates,  $X$  are biomasses,  $P$  are products of reaction. The general dynamical model (GDM) of bioreactions is as follows:

$$\left[ \frac{d\mathbf{z}}{dt} \right] = [\mathbf{c}] [\mathbf{r}(\mathbf{z})] + [\mathbf{F}] - [D\mathbf{z}], \quad (1)$$

where  $\mathbf{z}$  represents the concentration of components,  $\mathbf{F}$  represents the inlet flow rate of component  $\mathbf{z}$ ,  $\mathbf{c}$  represents the yield coefficients and  $\mathbf{r}(\mathbf{z})$  is the rate of reaction.

**Remark 2.1** The GDM in this paper can be said to be a specific case of the GDM shown in (1) in which there is only single inlet stream  $D\mathbf{z}$  with only one feed instead of multiple inlet flow rates ( $\mathbf{F}$ ).

A generalised first order time derivative of concentration model of a set of bioreactions in an open reactor with single dilution rate at constant volume and temperature can be written as:

$$\left[ \frac{dz}{dt} \right] = [c][r(z)] + [Dz_{in} - Dz_{out}], \quad (2)$$

where  $z$  are the  $n$  components,  $c$  is the matrix of constant yield coefficients associated with the reaction.  $r$  are the rates of reaction.  $z_{in}$  and  $z_{out}$  are the inlet and outlet concentrations of  $n$  components.  $z_{in}$  are mostly substrates altogether coming in one stream with dilution rate  $D$ . For the concentrations not fed from outside, such as products and biomasses,  $z_{in}$  will be zero. Similarly,  $z_{out}$  is the concentration coming out of the reactor which will be the same as the concentration inside the reactor i.e.  $z$ . The model (2) is valid for all types of microbial kinetics. The inputs  $u$  will be:  $u \in [D, Dz_{in}]$ .

## 2.1 A useful coordinate transformation

This coordinate transformation is chosen to simplify the model by finding invariants, making it easier to passivate. The important point here is that the new set of coordinates will be independent of kinetics which are restricted to appear in the kinetics, extending the work of [6] to the general dynamical model of bioreactors [9].

Suppose, state vector  $z$  can be divided into two vectors of dimensions  $j$  and  $k = n - j$ ,  $[z] = [\xi \ \phi]^T$ ,  $[c] = [c_j \ c_k]$  so that:

$$\left[ \dot{\xi} \right] = [c_j][r(z)] + [D\xi_{in} - D\xi], \quad (3)$$

$$\left[ \dot{\phi} \right] = [c_k][r(z)] + [D\phi_{in} - D\phi]. \quad (4)$$

The coordinate transformation will lead to a new vector of  $k = n - j$  elements and will be represented by state  $W$ , where  $[A]$  is a constant matrix:

$$[W]_{n-j \times 1} = [A]_{n-j \times n} \left( [\xi_{in} - \xi]_{j \times 1} \right) + [\phi_{in} - \phi]_{n-j \times 1}. \quad (5)$$

**Proposition 2.1** *For the relation of  $W$  proposed in (5),  $j$  independent reactions ( $c_j$  is full rank), if matrix  $[A]$  and functions of  $\xi_{in}$  and  $\phi_{in}$  are chosen in a way that  $[A][c_j] + [c_k] = 0$  and  $[A]\dot{\xi}_{in} + \dot{\phi}_{in} = 0$ , the state space model takes the form:*

$$\begin{bmatrix} \dot{\xi} \\ \dot{W} \end{bmatrix} = \begin{bmatrix} [c_j]_{j \times j} & [0]_{j \times n-j} \\ [0]_{n-j \times j} & [-DI]_{n-j \times n-j} \end{bmatrix}_{n \times n} \begin{bmatrix} r(\xi, W) \\ W \end{bmatrix}_{n \times 1} + \begin{bmatrix} D\xi_{in} - D\xi \\ 0 \end{bmatrix}. \quad (6)$$

**Proof:** On differentiating (5) with respect to time we get:

$$\left[ \dot{W} \right] = [A] \left( \dot{\xi}_{in} - \dot{\xi} \right) + \left( \dot{\xi}_{in} - \dot{\phi} \right). \quad (7)$$

Further substitution for  $\left[ \dot{\xi} \right]$ ,  $\left[ \dot{\phi} \right]$  from (3) and (4) respectively will lead to:

$$\begin{aligned} \left[ \dot{W} \right] &= [A]_{n-j \times j} \left( -[c_j]_{j \times j} [r(\xi, W)]_{j \times 1} + [D\xi_{in} - D\xi]_{j \times 1} \right) \\ &\quad - [c_k]_{n-j \times j} [r(\xi, W)]_{j \times 1} + [D\phi_{in} - D\phi]. \end{aligned} \quad (8)$$

Substituting  $[A][c_j] = -[c_k]$  and  $[A]\dot{\xi}_{in} = -\dot{\phi}_{in}$  in (8) will give:

$$\dot{W} = -DW. \quad (9)$$

With state space as  $[\xi \ W]^T$ , the bioreactor model becomes same as shown in (6).

Note that this solution necessarily needs  $c_j$  to be a full rank square matrix by careful choice of the components of  $\xi$ . It is always possible to find such a matrix  $A$  by the stoichiometric invariance principle if the  $j$  reactions are truly independent. Further, other assumptions on inlet concentrations  $\phi_{in}, \xi_{in}$  are weak, since they are always verified when these are constant, which will be assumed in the sequel.

**Corollary 2.1** *If  $\forall D : D > 0$ ,  $W$  is a reaction invariant, i.e.  $W$  will exponentially converge to zero.*

**Proof:** Consider a continuously differentiable non-negative storage function  $H = \frac{1}{2}W^2$  and  $H : W \rightarrow R$  with  $H(0) = 0$ . Differentiating  $H$  w.r.t. time and substituting (9) will give  $\dot{H} = -DW^2 = -2DH$ . Hence for  $D > 0$ ,  $H$  and  $W \rightarrow 0$  as  $t \rightarrow \infty$ .

**Remark 2.2** For the general model (1), the representation after coordinate transformation, originated from the stoichiometric invariance principle, was independent of kinetics and was referred to as a "nice" representation in [9]. However, the model in (9), which considers the case of a single stream input flow, also allows to find a reaction invariant. Hence, the model splits the dynamics into a stable bilinear subsystem ( $W$ ) and a control affine subsystem ( $\xi$ ) which are weakly coupled. The convergence of  $W$  to zero extends the so-called "useful" change of coordinates in [9].

### 3 Passivity Based Model

A passive system is a system which cannot store more energy than is supplied by some source. The difference between the stored energy and supplied energy is the dissipated energy:

**Definition 3.1** [4] Consider the system:

$$\dot{x} = f(x) + g(x)u, \quad y = h(x), \quad (10)$$

where  $u, y$  are the input and output of the system respectively,  $f(x)$ ,  $g(x)$  and  $h(x)$  are matrices and vector fields that define the interconnection between physical-meaning elements (state, inputs, and outputs). With a storage function  $V(x): V(x^*) = 0$ , where  $x^*$  is the steady state value of  $x$  and  $V(x) > 0$  at  $x \neq x^*$ , this system is passive if:

$$\frac{dV}{dt} \leq u^T y. \quad (11)$$

The passive system satisfying the condition presented in Definition 3.1 is written as:

$$\dot{x} = Q(x, u) \frac{\partial V}{\partial x} + \gamma(x)v, \quad y = \gamma^T(x) \frac{\partial V}{\partial x}. \quad (12)$$

Here  $v$  is the modified input,  $Q$  and  $\gamma$  are the modified interconnection matrices.

**Lemma 3.1** [4], Consider the system shown in equation (12), which with a storage function  $V(\mathbf{x})$ :  $V(\mathbf{x}^*) = 0$ , where  $\mathbf{x}^*$  is the steady state value of  $\mathbf{x}$  and  $V(\mathbf{x}) > 0$  at  $\mathbf{x} \neq \mathbf{x}^*$ , will be passive if  $Q \prec 0$ .

In biochemistry, most of the microbial reactions are coupled but can be turned into decoupled reactions either as a linear combination of functions of single state variable or such a transformation can be achieved through decoupling process. The proposed passivization methodology is suitable for such reactions in terms of physical and structural understanding. Decoupling also leads to further simplification of the model by getting rid of many complex terms using minor assumptions without considerable change in the actual kinetics. The following section will explain the general process of decoupling of coupled bioreactions and derive their passivity based model.

### 3.1 Decoupling of coupled bioreactions

A decoupled reaction has its rate terms depending only on single state or many states if they can be separated (decoupled) algebraically so that they become a linear combination of functions of single state only. It is supposed that there exist  $j$  independent reactions with a full rank stoichiometric  $\mathbf{c}_j$  which allow for the nice representation described above. It would also be possible to achieve a partial stabilization of the system using passivity properties [10], [11].

The bioreactor systems chosen here are single stream bioreactors having inlet concentration of each component to be constant. Dilution rate  $D$  is the only control input in such systems. We assume that we can split the rate term of  $\boldsymbol{\xi}$ , i.e.  $\mathbf{c}_j \mathbf{r}(\boldsymbol{\xi}, \mathbf{W})$  into two parts,  $u$  and  $c$  standing for uncoupled and coupled  $\mathbf{c}_u \mathbf{r}_u(\boldsymbol{\xi}_u, \mathbf{W})$  and  $\mathbf{c}_c \mathbf{r}_c(\boldsymbol{\xi}, \mathbf{W})$ , where  $\mathbf{c}_u \mathbf{r}_u(\boldsymbol{\xi}_u, \mathbf{W})$  is the sum of decoupled rate terms  $\bar{\mathbf{c}}_u \mathbf{p}_u(\boldsymbol{\xi}_u)$  and function  $\mathbf{f}_u(\boldsymbol{\xi}_u, \mathbf{W})$ , with  $\frac{\partial(\mathbf{p}_u)^i}{\partial(\mathbf{x})^j} = \frac{\partial(\mathbf{f}_u)^i}{\partial(\mathbf{x})^j} = 0$  if  $j \neq i$ ,  $(.)^i$  standing for the  $i^{th}$  component of a vector.  $\mathbf{c}_c \mathbf{r}_c(\boldsymbol{\xi}, \mathbf{W})$  is the sum of a decoupled modified rate term  $\bar{\mathbf{c}}_c \mathbf{r}_c(\boldsymbol{\xi})$ , where  $\frac{\partial(\mathbf{p}_c)^i}{\partial(\mathbf{x})^j} = 0$ , if  $j \neq i$  and a remaining coupled term depends on the whole  $\boldsymbol{\xi}$ ,  $\mathbf{f}_c(\boldsymbol{\xi}, \mathbf{W})$ . Concisely, one can write:

$$\begin{bmatrix} \dot{\boldsymbol{\xi}}_u \\ \dot{\boldsymbol{\xi}}_c \\ \dot{\mathbf{W}} \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{c}}_j & 0 \\ 0 & -D\mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{p}_u(\boldsymbol{\xi}_u) \\ \mathbf{p}_c(\boldsymbol{\xi}_c) \\ \mathbf{W} \end{bmatrix} + \begin{bmatrix} \mathbf{f}_u(\boldsymbol{\xi}_u, \mathbf{W}) \\ \mathbf{f}_c(\boldsymbol{\xi}, \mathbf{W}) \\ 0 \end{bmatrix} + \begin{bmatrix} D(\boldsymbol{\xi}_{uin} - \boldsymbol{\xi}_u) \\ D(\boldsymbol{\xi}_{cin} - \boldsymbol{\xi}_c) \\ 0 \end{bmatrix}. \quad (13)$$

At this stage, equation (13) shows the decoupling process, as the  $d$  first equations are only coupled by the vanishing reaction invariant  $\mathbf{W}$ . In practical applications, the corresponding variables are substrates concentrations, for which the kinetics is only coupled with one or several biomass concentrations. Now, the input concentrations can be controlled to obtain a more interesting configuration for the coupled dynamics  $\boldsymbol{\xi}_c$ .

**Lemma 3.2** Let us consider the equilibrium point  $\boldsymbol{\xi}^*$  of the system (13). If  $(D - D^*)(\boldsymbol{\xi}_{cin} - \boldsymbol{\xi}_c) + D^*(\boldsymbol{\xi}_{cin}^* - \boldsymbol{\xi}_c) + \mathbf{f}_c(\boldsymbol{\xi}, \mathbf{W}) - \mathbf{f}_c(\boldsymbol{\xi}^*, \mathbf{W}) = 0$ ,

then system (13) can be written as:

$$\begin{bmatrix} \dot{\xi}_u \\ \dot{\xi}_c \\ \dot{W} \end{bmatrix} = \begin{bmatrix} \bar{c}_j & 0 \\ -\bar{\theta} & -D^*I \end{bmatrix} \begin{bmatrix} p_u(\xi_u) - p_u(\xi_u^*) \\ p_c(\xi_c) - p_c(\xi_c^*) \\ W \end{bmatrix} + \begin{bmatrix} (D - D^*)(\xi_{uin} - \xi_u) + D^*(\xi_u^* - \xi_u) \\ 0 \\ 0 \end{bmatrix}. \quad (14)$$

**Proof:** At equilibrium,  $\xi^* = [\xi_u^* \ \xi_c^* \ 0]$ . As  $\dot{\xi} = 0$ , this in turn implies  $\bar{c}_u p_u(\xi_u^*) + f_u(\xi_u^*) = -D^*(\xi_{uin} - \xi_u^*)$  and  $\bar{c}_c p_c(\xi_c^*) + f_c(\xi_c^*) = -D^*(\xi_{cin}^* - \xi_c^*)$ .

Adding and subtracting  $p_u(\xi_u^*)$ ,  $p_c(\xi_c^*)$  in the corresponding equation (13) and replacing the compensation yield the final result. The above set of equations will be decoupled if one can cancel the  $f_c$  term, using control terms. These control terms can be either the dilution rate  $D$ , or the inlet concentrations  $\xi_{cin}^*$  (provided that the equation  $[A]\dot{\xi}_{in} + \dot{\phi}_{in} = 0$  is verified). The states are only coupled by the stoichiometric matrix  $\bar{c}_j$  and  $W$  ( $W \rightarrow 0$ ). The next section will show the passivization procedure using a physical energy (storage) function.

### 3.2 Passivity based model of a general decoupled bioreactor

**Proposition 3.1** Suppose the system:

$$\dot{\xi}_u = \bar{c}_j p_u(\xi_u) - \bar{c}_j p_u(\xi_u^*) + f_u(\xi_u, W) - f_u(\xi_u^*) + \underbrace{(\xi_{uin} - \xi_u)}_g \underbrace{(D - D^*)}_u + D^*(\xi_u^* - \xi_u) \quad (15)$$

is passive with storage function  $V(\xi_u, t)$ , input  $u$  and output  $y : y = g^T \frac{\partial V}{\partial \xi_u}$ , and  $f_u(\xi_u, W) - f_u(\xi_u^*)$  is a vanishing perturbation:  $\lim_{t \rightarrow \infty} f_u(\xi_u, W) - f_u(\xi_u^*) = 0$ . Assume that there exists a neighbourhood  $Z$  of  $\xi_u^*$  such that the reduced system:

$$\dot{\xi}_u = \bar{c}_j p_u(\xi_u) - \bar{c}_j p_u(\xi_u^*) + (\xi_{uin} - \xi_u)(D - D^*) \quad (16)$$

has  $\xi_u^*$  as an exponentially stable equilibrium point and for  $\bar{\xi}_u = \xi_u - \xi_u^*$ , the storage function  $V(\xi_u, t)$  satisfies the following conditions:

$$\exists k_3, k_4 > 0, k_3 \|\bar{\xi}_u\| \leq \frac{\partial V}{\partial \xi_u} \leq k_4 \|\bar{\xi}_u\|,$$

$$\exists \gamma : \gamma + D^* > 0 \ \|f'(\xi_u, W) - f_u(\xi_u^*)\| \leq (\gamma + D^*) \|\bar{\xi}_u\|.$$

Then the full system (15) is also locally exponentially stable at  $\xi^*$  if:

$(-\lambda_{\min} k_3 - k_3 + k_4(\gamma + D^*)\lambda_{\max}) < 0$ , where  $\lambda_{\min}$ ,  $\lambda_{\max}$  are the minimum and maximum eigenvalues of  $-\bar{c}_j$ .

**Proof:** One knows from the exponential stability conditions that  $\exists k_1, k_2 > 0$ ,  $k_1 \|\bar{\xi}_u\| \leq V \leq k_2 \|\bar{\xi}_u\|$ . Since  $\frac{dV}{dt} = \frac{\partial V}{\partial \xi_u} \frac{\partial \xi_u}{\partial t}$ , it follows from the assumption:

$$\frac{\partial V}{\partial \xi_u}^T \bar{c}_j (p_u(\xi_u) - p_u(\xi_u^*)) \leq (-\lambda_{\min} - 1) k_3 \|\bar{\xi}_u\|^2,$$

$$\frac{\partial V}{\partial \xi_u}^T (f_u(\xi_u, W) - f_u(\xi_u^*, 0)) \leq k_4(\gamma + D^*)\lambda_{\max} \|\bar{\xi}_u\|^2.$$

Now,  $\frac{dV}{dt} \leq (-\lambda_{\min} k_3 - k_3 + k_4(\gamma + D^*)\lambda_{\max}) \|\bar{\xi}_u\|^2 + u^T y \leq u^T y$  if



$(-\lambda_{\min} k_3 - k_3 + k_4(\gamma + D^*)\lambda_{\max}) < 0$ , Hence, the reduced system (16) is exponentially stable and according to *Theorem 3.12* in [12], the full system will be exponentially stable.

Now, *Proposition 3.1* tells that the full system will be exponentially stable if the reduced unperturbed system is exponentially stable. From *Proposition 3.1* one can take  $\mathbf{f}_u(\xi_u, \mathbf{W}) - \mathbf{f}_u(\xi_u^*, 0) + D^*(\xi_u^* - \xi_u) = 0$  and the system (13) is written as (17).

$$\begin{bmatrix} \dot{\xi}_u \\ \dot{\xi}_c \\ \dot{\mathbf{W}} \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{c}}_j & 0 \\ 0 & -D\mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{p}_u(\xi_u) - \mathbf{p}_u(\xi_u^*) \\ \mathbf{p}_c(\xi_c) - \mathbf{p}_c(\xi_c^*) \\ \mathbf{W} \end{bmatrix} + \begin{bmatrix} (D - D^*)(\xi_{uin} - \xi_u) \\ 0 \\ 0 \end{bmatrix}. \quad (17)$$

This presentation is straightforward and physically linked to passivity.

**Proposition 3.2** Consider the system (17) with  $\bar{\mathbf{c}}_j \prec 0$ . Assume that there exists a neighbourhood  $Z$  of  $\xi = \xi^*$  such that:

1.  $\sum \left( \int_0^{(\xi_u)^i} (\mathbf{p}_u)^i((\xi_u)^i) - \int_0^{(\xi_u)^i} (\mathbf{p}_u)^i((\xi_u^*)^i) \right) > 0$
2.  $\sum \left( \int_0^{(\xi_c)^i} (\mathbf{p}_c)^i((\xi_c)^i) - \int_0^{(\xi_c)^i} (\mathbf{p}_c)^i((\xi_c^*)^i) \right) > 0,$

then the storage function  $V' = \sum_{i=1}^n V'_i = \sum_{i=1}^{n_u} \int ((\mathbf{p}_u)^i(\xi_u)^i - (\mathbf{p}_u)^i(\xi_u^*)^i) \partial(\xi_u)^i + \sum_{i=1}^{n_c} \int ((\mathbf{p}_c)^i(\xi_c)^i - (\mathbf{p}_c)^i(\xi_c^*)^i) \partial(\xi_c)^i + \sum_{i=1}^{n-j} \frac{1}{2} \mathbf{W}_i^2$  will make the reduced system (17) asymptotically stable at  $\xi = \xi^*$ .

**Proof:** One has  $V'$  being always positive around  $\xi^*$ . On partially differentiating  $V'$  w.r.t. states  $\xi_u, \xi_c$  and  $\mathbf{W}$ :

$$\frac{\partial V'}{\partial \xi_u} = \mathbf{p}_u(\xi_u) - \mathbf{p}_u(\xi_u^*) : \frac{\partial V'}{\partial \xi_c} = \mathbf{p}_c(\xi_c) - \mathbf{p}_c(\xi_c^*) : \frac{\partial V'}{\partial \mathbf{W}} = \mathbf{W} \quad (18)$$

the system in (17) can be written in the form:

$$\underbrace{\begin{bmatrix} \dot{\xi}_u \\ \dot{\xi}_c \\ \dot{\mathbf{W}} \end{bmatrix}}_{\dot{\xi}} = \underbrace{\begin{bmatrix} \bar{\mathbf{c}}_j & 0 \\ 0 & -D\mathbf{I} \end{bmatrix}}_{\mathbf{Q}'} \underbrace{\begin{bmatrix} \frac{\partial V'}{\partial \xi_u} \\ \frac{\partial V'}{\partial \xi_c} \\ \frac{\partial V'}{\partial \mathbf{W}} \end{bmatrix}}_{\frac{\partial V'}{\partial \xi}} + \underbrace{\begin{bmatrix} (\xi_{uin} - \xi_u) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\mathbf{g}} \underbrace{\begin{bmatrix} (D - D^*) \\ 0 \\ 0 \end{bmatrix}}_{\mathbf{u}'}. \quad (19)$$

The output of the system will be  $\mathbf{y}' = \mathbf{g}^T \frac{\partial V'}{\partial \xi}$ . The time derivative of  $V'$  is:

$$\dot{V}' = \frac{\partial V'}{\partial \xi} \dot{\xi} = \frac{\partial V'}{\partial \xi} \mathbf{Q}' \frac{\partial V'}{\partial \xi} + \frac{\partial V'}{\partial \xi} \mathbf{g} \mathbf{u}' = \frac{\partial V'}{\partial \xi} \mathbf{Q} \frac{\partial V'}{\partial \xi} + \mathbf{y}'^T \mathbf{u}'. \quad (20)$$

Since  $\bar{\mathbf{c}}_j \prec 0$  and  $D > 0$  is making matrix  $\mathbf{Q}'$  negative definite, (19) is passive.  $V'_i$  is minimum i.e. 0 at  $(\xi^*)^i$ , the system (19) has a passive equilibrium point  $\xi = \xi^*$ .

#### 4 Passivity Based Control

Passivity based control is a generic design method which is extensively used in electro-mechanical systems.

**Proposition 4.1** [4] *Consider the passive system of the form:*

$$\dot{\mathbf{x}} = \mathbf{Q}(\mathbf{x}, \mathbf{u}) \frac{\partial V}{\partial \mathbf{x}} + \gamma(\mathbf{x}) \mathbf{v}; \mathbf{y} = \gamma^T(\mathbf{x}) \frac{\partial V}{\partial \mathbf{x}}, \quad (21)$$

where  $V(\mathbf{x})$  is the specified closed-loop storage function  $V(\mathbf{x})$ :  $V(\mathbf{x}^d) = 0$ ,  $\mathbf{x}^d \neq 0$  is the desired steady state value of  $\mathbf{x}$  and  $V(\mathbf{x}) > 0$ ,  $\mathbf{Q} < 0$ . Suppose that the model is zero state detectable, then the feedback  $\mathbf{v} = -\mathbf{C}(\mathbf{x}, t)\mathbf{y}$  with  $\mathbf{C}(\mathbf{x}, t) \geq e\mathbf{I} > 0$  and constant  $e$  renders  $\mathbf{x} = \mathbf{x}^d$  globally asymptotically stable.

##### 4.1 Passivity based control of a general decoupled bioreactor

The following proposition will give general formulations of passivity based control of a decoupled bioreactor system.

**Proposition 4.2** *Consider the desired storage function  $\bar{V}$ , with conditions following:*

$$\begin{aligned} \bar{V} &= \sum_{i=1}^n \bar{V}_i = \sum_{i=1}^{n_u} \int \left( p_u(\xi_u) - p_u(\xi_u^d) \right)^i (\partial \xi_u)^i \\ &+ \sum_{i=1}^{n_c} \int \left( p_c(\xi_c) - p_c(\xi_c^d) \right)^i (\partial \xi_c)^i + \sum_{i=1}^{n-j} \frac{1}{2} (W^2)^i, \end{aligned} \quad (22)$$

1.  $\sum_{i=1}^{n_u} \left( \int_0^{(\xi_u)^i} (p_u)^i ((\xi_u)^i) - \int_0^{(\xi_u^d)^i} (p_u)^i ((\xi_u^d)^i) \right) > 0$ ,
2.  $\sum_{i=1}^{n_c} \left( \int_0^{(\xi_c)^i} (p_c)^i ((\xi_c)^i) - \int_0^{(\xi_c^d)^i} (p_c)^i ((\xi_c^d)^i) \right) > 0$ ,
3.  $\bar{V}(\xi^d) = 0$

Hence, the system (19) is passive and the feedback  $\bar{\mathbf{u}} = -\mathbf{C}(\mathbf{x}, t)\bar{\mathbf{y}}$  with  $\mathbf{C}(\mathbf{x}, t) \geq e\mathbf{I} > 0$  renders (19) globally asymptotically stable at  $\xi = \xi^d$ .

**Proof:** After replacing the equilibrium point  $\xi^*$  with desired equilibrium point  $\xi^d$ , the system (19) can take the form:

$$\underbrace{\begin{bmatrix} \dot{\xi}_u \\ \dot{\xi}_c \\ \dot{W} \end{bmatrix}}_{\dot{\xi}} = \underbrace{\begin{bmatrix} \bar{\mathbf{C}}\mathbf{j} & 0 \\ 0 & -D\bar{\mathbf{I}} \end{bmatrix}}_{\mathbf{Q}'} \underbrace{\begin{bmatrix} \frac{\partial \bar{V}}{\partial \xi_u} \\ \frac{\partial \bar{V}}{\partial \xi_c} \\ \frac{\partial \bar{V}}{\partial W} \end{bmatrix}}_{\frac{\partial \bar{V}}{\partial \xi}} + \underbrace{\begin{bmatrix} (\xi_{uin} - \xi_u) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\mathbf{g}} \underbrace{\begin{bmatrix} (D - D^d) \\ 0 \\ 0 \end{bmatrix}}_{\bar{\mathbf{u}}}. \quad (23)$$

According to Proposition 3.2, this system is passive. The input of the system is  $\bar{\mathbf{u}}$  and the output  $\bar{\mathbf{y}}$  is:  $\bar{\mathbf{y}} = [\mathbf{g}]^T \frac{\partial \bar{V}}{\partial \xi}$ . By Proposition 4.1, the feedback  $\bar{\mathbf{u}} = -\mathbf{C}(\mathbf{x}, t)\bar{\mathbf{y}}$  with  $\mathbf{C}(\mathbf{x}, t) \geq e\mathbf{I} > 0$  will render (23) globally asymptotically stable at  $\xi = \xi^d$ .

## 5 Application to a Single Reaction with Monod Kinetics: Aniline Degradation by *Pseudomonas Putida* in CSTR

Aniline is among the toxic constituents of many industrial effluents (e.g. wastewaters in chemical and dyeing industries). Biological processing for aniline degradation is a cheap and green alternative to chemical removal processes such as solvent extraction, chemical oxidation, etc. In [13] the author has studied the model of aniline degradation by *Pseudomonas putida* ATCC 21812 cells in batch reactors following a Monod model. *Pseudomonas putida* growth  $X$  and simultaneous aniline degradation  $S$  in a CSTR equations are:

$$\dot{X} = \mu X - DX, \quad \dot{S} = -\frac{\mu X}{Y} + D(S_{in} - S), \quad (24)$$

where  $D$  is the dilution rate,  $Y$  is the cell/substrate yield coefficient and  $\mu$  is the specific growth rate. For Monod kinetics:

$$\mu = \frac{\mu_m S}{K_s + S}, \quad (25)$$

here  $\mu_m$  is the maximum specific growth rate and  $K_s$  is the half velocity constant. The state space will be  $[z] = [S \ X]^T$  and the model can be represented as:

$$\underbrace{\begin{bmatrix} \dot{S} \\ \dot{X} \end{bmatrix}}_{\dot{w}} = \underbrace{\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}}_c \underbrace{\begin{bmatrix} \frac{\mu X}{Y} \\ \mu X \end{bmatrix}}_r + \underbrace{\begin{bmatrix} DS_{in} - DS \\ -DX \end{bmatrix}}_{D(z_{in} - z_{out})}. \quad (26)$$

### 5.1 Coordinate transformation and a passivity based model

Divide the state space into two parts  $\xi_a$  and  $\xi_b$  such that:

$$\begin{bmatrix} \dot{\xi} \end{bmatrix} = \begin{bmatrix} \dot{S} \end{bmatrix} = \underbrace{\begin{bmatrix} -1 \end{bmatrix}}_{c_j} \underbrace{\begin{bmatrix} \frac{\mu X}{Y} \end{bmatrix}}_r + \underbrace{\begin{bmatrix} DS_{in} - DS \end{bmatrix}}_{D(\xi_{in} - \xi)}, \quad \begin{bmatrix} \dot{\phi} \end{bmatrix} = \begin{bmatrix} \dot{X} \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \end{bmatrix}}_{c_k} \underbrace{\begin{bmatrix} \mu X \end{bmatrix}}_r + \underbrace{\begin{bmatrix} -DX \end{bmatrix}}_{D(\psi_{in} - \psi)}. \quad (27)$$

The new coordinate  $W$  can be written as:

$$W = A(S_{in} - S) + Y(X_{in} - X), \quad (28)$$

where  $A = 1$ ,  $X_{in} = 0$  and  $S_{in}$  is a constant. Hence, differentiating (28) w.r.t. time and substituting (27) will give  $\dot{W} = -DW$ . With the new state space  $[S \ W]^T$  and the substitution  $X = S_{in} - S - W$  the bioreactor model becomes:

$$\begin{bmatrix} \dot{S} \\ \dot{W} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -D \end{bmatrix} \begin{bmatrix} \mu \frac{(S_{in} - S)}{Y} \\ \frac{W}{W} \end{bmatrix} + \begin{bmatrix} \mu \frac{W}{Y} \\ 0 \end{bmatrix} + \begin{bmatrix} D(S_{in} - S) \\ 0 \end{bmatrix}. \quad (29)$$

Taking the steady state points of  $(S, W)$  as  $(S^*, 0)$  and then adding and subtracting equilibrium rate term  $\mu(S^*) \frac{(S_{in} - S^*)}{Y}$  in (29), (29) can be written as:

$$\begin{bmatrix} \dot{S} \\ \dot{W} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -D \end{bmatrix} \begin{bmatrix} \mu \frac{(S_{in} - S)}{Y} - \mu(S^*) \frac{(S_{in} - S^*)}{Y} \\ \frac{W}{W} \end{bmatrix} + \begin{bmatrix} \mu \frac{W}{Y} \\ 0 \end{bmatrix} + \begin{bmatrix} (D - D^*)(S_{in} - S) + D^*(S^* - S) \\ 0 \end{bmatrix}. \quad (30)$$

From *Proposition 3.1*,  $\mu \frac{W}{Y} + D^*(S^* - S) = 0$ . Using the storage function:

$$V' = \int \mu(S) \frac{(S_{in} - S)}{Y} \partial S - \int \mu^*(S^*) \frac{(S_{in} - S^*)}{Y} \partial S + \frac{1}{2} W^2, \quad (31)$$

where  $\mu^*, S^*$  are the steady state values of  $\mu, S$ , and doing some algebraic modifications, the bioreactor model can be rewritten as:

$$\underbrace{\begin{bmatrix} \dot{S} \\ \dot{W} \end{bmatrix}}_{\xi} = \underbrace{\begin{bmatrix} -1 & 0 \\ 0 & -D \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} \frac{\partial V'}{\partial S} \\ \frac{\partial V'}{\partial W} \end{bmatrix}}_{\frac{\partial V'}{\partial \xi}} + \underbrace{\begin{bmatrix} S_{in} - S & 0 \\ 0 & 1 \end{bmatrix}}_{\gamma} \underbrace{\begin{bmatrix} (D - D^*) \\ 0 \end{bmatrix}}_{u'}. \quad (32)$$

The matrix  $Q$  will always be negative definite and it can be seen through careful observation that  $V' \geq 0$  and 0 at  $S = S^*$ , making the system (32) passive.

## 5.2 Passivity based control design

Replacing the steady state  $S^*$  with desired steady state  $S^d$  and the new storage function  $\bar{V}$ :

$$\bar{V} = \int \mu(S) \frac{(S_{in} - S)}{Y} \partial S - \int \mu^d(S^d) \frac{(S_{in} - S^d)}{Y} \partial S + \frac{1}{2} W^2, \quad (33)$$

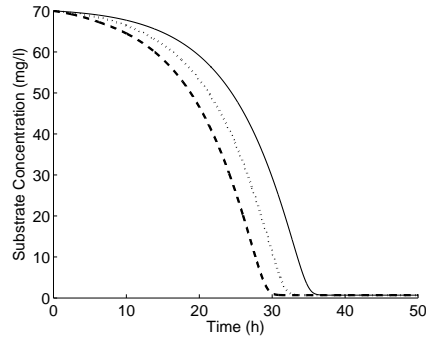
where  $\mu^d$  is the desired steady state values of  $\mu$ , and doing some algebraic modifications, the bioreactor model can be rewritten as:

$$\underbrace{\begin{bmatrix} \dot{S} \\ \dot{W} \end{bmatrix}}_{\xi} = \underbrace{\begin{bmatrix} -1 & 0 \\ 0 & -D \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} \frac{\partial \bar{V}}{\partial S} \\ \frac{\partial \bar{V}}{\partial W} \end{bmatrix}}_{\frac{\partial \bar{V}}{\partial \xi}} + \underbrace{\begin{bmatrix} S_{in} - S & 0 \\ 0 & 1 \end{bmatrix}}_{\gamma} \underbrace{\begin{bmatrix} (D - D^d) \\ 0 \end{bmatrix}}_{\bar{u}}; \bar{y} = \gamma^T \frac{\partial \bar{V}}{\partial \xi}. \quad (34)$$

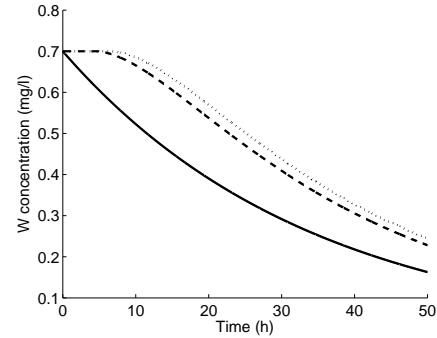
Matrix  $Q \prec 0$  and if  $\bar{V} \geq 0$ , the system (34) is passive.  $\bar{V} = 0$  at  $S = S^d$  and  $W = 0$ . Since the system (34) is zero state detectable if the desired concentration of substrate  $S^d = 0$ , the feedback  $\bar{u} = -C\bar{y}$  ensures asymptotical stability at  $S = S^d$ .

## 5.3 Simulations

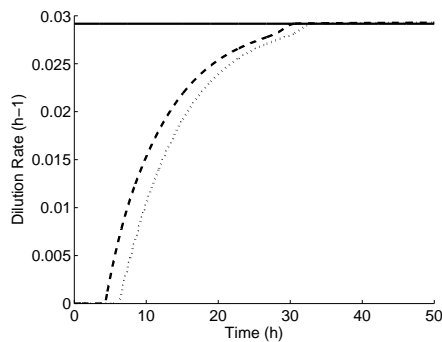
An industrial incident, where 9 tons of aniline at 70 mg/l leaked from a chemical plant into a river is considered, and 1 mg/l or less must be reached. Monod parameters are  $K_s = 3.1$  mg/l,  $\mu_m = .12h^{-1}$ ,  $Y = 0.74$ . The dilution rate  $D$  is the control input and substrate concentration is the only measurement. The simulation results compare three control strategies i.e. chemostat control with steady state dilution rate, passivity based control and passivity based adaptive control (not discussed here but similar to the control designed in [14]). The new coordinate  $W$  converges to zero as shown in Figure 2, ensuring proper control. The cell concentration will obviously increase at a similar rate as substrate concentration will decrease as can be seen in Figure 4.



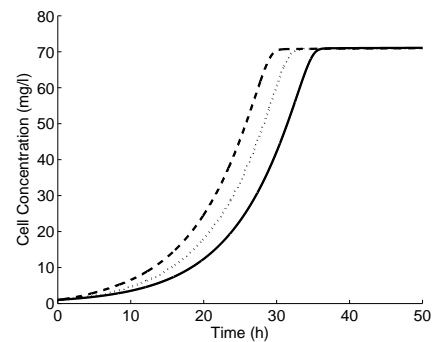
**Figure 1:** Substrate Concentration; Bold: Chemostat; Dotted: Passivity Based; Dashed: Adaptive.



**Figure 2:** W Concentration; Bold: Chemostat; Dotted: Passivity Based; Dashed: Adaptive.



**Figure 3:** Dilution Rate; Bold: Steady state, Dotted: Passivity Based; Dashed: Adaptive.



**Figure 4:** Cell Concentration; Bold: Chemostat; Dotted: Passivity Based; Dashed: Adaptive.

## 6 Conclusion

This paper is a successful attempt to maintain the structure and physical meaning of the passivity based model of microbial reactions with Monod kinetics in continuous reactors by using meaningful storage functions and obvious coordinate transformation on the grounds of passivity. The general model implies that this technique can be directly applied to a huge set of reactions. This paper is providing a physical view for all issues related to robust control of a bioreaction. Simulations obtained justify and validate the model. In the future, this technique can be extended to other kinetics involved and to different types of reactors such as plug flow, etc. The physical meaning given to the design of observers (as in e.g. [15]) and parameter estimation could be an interesting job to work on.

## Acknowledgment

The authors would like to thank Dr. Iordan Nikov and Dr. Ludovic Montastruc who have provided the experimental data for validation of results.

## References

- [1] Hoang, H., Couenne, F., Jallut, C. and Le Gorrec, Y. The Port Hamiltonian approach to modeling and control of continuous stirred tank reactors. *Journal of Process Control* **21** (2011) 1449–1458.
- [2] Makkar, M. and Dieulot, J.-Y. Passivity based control of a chemical process in isothermal reactors: Application to enzymatic hydrolysis of cellulose. In: *Proc. IEEE Conference on Control Applications* Antives, France, 2014, 753–758.
- [3] Selişteanu, D., Roman, M. and Şendrescu, D. Pseudo Bond Graph modelling and on-line estimation of unknown kinetics for a wastewater biodegradation process. *Simulation Modelling Practice and Theory* **18** (2010) 1297–1313.
- [4] Fossas, E., Ros, R.M. and Sira-Ramírez, H. Passivity-based control of a bioreactor system. *Journal of Mathematical Chemistry* **36** (2004) 347–360.
- [5] Dieulot, J.-Y. and Makkar, M. A pseudo-Port-Hamiltonian representation and control of a continuous bioreactor. In: *Proc. 1st Conference on Modelling, Identification and Control of Nonlinear Systems*, Saint Petersburg, Russia, 2015, 1300–1306.
- [6] Fjeld, M., Asbjørnsen, A.O. and Åström, K.J. Reaction invariants and their importance in the analysis of eigenvectors, state observability and controllability of the continuous stirred tank reactor. *Chemical Engineering Science* **29** (1974) 1917–1926.
- [7] Villa, J., Olivar, G. and Angulo, F. Transcritical-like Bifurcation in a Model of a Bioreactor. *Nonlinear Dynamics and Systems Theory* **15** (2015) 90–99.
- [8] García-Sandoval, J., Hudon, N., Dochain, D. and González-Álvarez, V. Stability analysis and passivity properties of a class of thermodynamic processes: An internal entropy production approach. *Chemical Engineering Science* **139** (2016) 261–272.
- [9] Bastin, G. and Dochain, D. On-line estimation of microbial specific growth rates. *Automatica* **22** (1986) 705–709.
- [10] Binazadeh, T. and Yazdanpanah, M.J. Application of Passivity Based Control for Partial Stabilization. *Nonlinear Dynamics and Systems Theory* **11** (2011) 373–383.
- [11] Shafiei, M.H. and Binazadeh, T. Partial Control Design for Nonlinear Control Systems. *Nonlinear Dynamics and Systems Theory* **12** (2012) 269–279.
- [12] Khalil, H.K. and Grizzle, J. *Nonlinear Systems*. 3rd Edition, Prentice Hall: New Jersey, 1996.
- [13] Montastruc, L. and Nikov, I. Modeling of aromatic compound degradation by pseudomonas putida atcc 21812. *Chemical Industry and Chemical Engineering Quarterly* **12** (2006) 220–224.
- [14] Dirks, D.A. and Scherpen, J.M. Structure preserving adaptive control of Port-Hamiltonian systems. *IEEE Transactions on Automatic Control* **57** (2012) 2880–2885.
- [15] Iben Warrad, B., Bouafoura, M.K. and Benhadj Braiek, N. Observer Based Output Tracking Control for Bounded Linear Time Variant Systems. *Nonlinear Dynamics and Systems Theory* **15** (2015) 428–441.



# Mathematical Models of Nonlinear Oscillations of Mechanical Systems with Several Degrees of Freedom

G.I. Melnikov<sup>1</sup>, N.A. Dudarenko<sup>1</sup>, K.S. Malykh<sup>1</sup>, L.N. Ivanova<sup>1</sup> and V.G. Melnikov<sup>1,2\*</sup>

<sup>1</sup> ITMO University, St. Petersburg, Russia

<sup>2</sup> National Mineral Resources University, St. Petersburg, Russia

Received: May 14, 2017; Revised: October 27, 2017

**Abstract:** A nonlinear dynamic system with several degrees of freedom, which is represented by a system of differential equations with polynomial structure, is considered. The system contains non-linear polynomials. It is assumed that the spectrum of the eigenvalues of the linear part matrix starts with a pair of complex conjugate eigenvalues having negative real parts with minimum modulus. A polynomial transformation of the equations is performed in order to simplify the mathematical model by reducing the number of non-linear terms in the differential equations. Nonlinear oscillations of an object with constant parameters are investigated. Estimations of motion are obtained by the method of differential inequalities for positive definite Lyapunov function at different ratios between the constant parameters of the system. An example is presented.

**Keywords:** *autonomous dynamical system; degrees of freedom; phase state variables; nonlinear oscillations; polynomial transformation of variables; Lyapunov function; differential inequality.*

**Mathematics Subject Classification (2010):** 74H45, 70K75, 70K05, 34C10, 34C15, 45G10, 41A10, 37B25, 34K13.

## 1 Introduction

The paper deals with nonlinear analysis in classical and modern mechanics [1–5].

We use a Poincaré-Dulac approach [6–9] and consider a nonoscillatory nonlinear stationary mechanical system with one degree of freedom. The system has autonomous nonlinear polynomial characteristics associated with its phase variables. This fact leads to the linear form, alternative to the extended model method shown in [10].

---

\* Corresponding author: <mailto:v.g.melnikov@yandex.ru>

## 2 Transformation of Polynomial Equations

We consider a nonlinear autonomous system of equations of perturbed motion in the case when all roots of the characteristic equation of the corresponding linear system are different [11–13]. Let us transform it into a canonical form

$$\dot{y}_s = \lambda_s y_s + \sum_{k=2}^m \sum_{\nu_1 + \dots + \nu_n = k} p_s^{(\nu_1, \dots, \nu_n)} y_1^{\nu_1} \dots y_n^{\nu_n} \quad (s = \overline{1, n}), \quad (1)$$

where  $y_s$  are real and complex variables;  $\lambda_s$  are roots of the characteristic equation of the linear part of the system;  $p_s^{(\nu_1, \dots, \nu_n)}$  are small coefficients;  $m$  are odd numbers.

Suppose  $\lambda_1$  and  $\lambda_2$  are complex-conjugate pure imaginary roots or roots with real parts much less than those of the other roots and the imaginary parts of these roots

$$\lambda_{1,2} = \alpha \pm \beta i, \quad \text{where} \quad \alpha \gtrless 0, \quad |\alpha| < \beta.$$

The real parts of the other roots are essentially negative  $|\operatorname{Re} \lambda_s| < 0$ ,  $s = \overline{3, n}$ . It should be noted that in such case the variables  $y_1, y_2$  will be complex-conjugated. Such systems are often used to describe nonlinear oscillations in engineering and physics. Suppose the roots  $\lambda_1 \dots \lambda_n$  are such that within the limits of some number of digit order numbers  $k = \overline{3, m}$  they do not vanish at any values of indices  $\nu_1, \dots, \nu_n$  complying with the condition (2), except for the values (3) where  $\operatorname{Re} \lambda_1, \operatorname{Re} \lambda_2$  are small.

$$\lambda_s^{(\nu_1, \dots, \nu_n)} = \nu_1 \lambda_1 + \nu_2 \lambda_2 + \dots + (\nu_s - 1) \lambda_s + \nu_{s+1} \lambda_{s+1} + \dots + \nu_n \lambda_n, \quad (2)$$

$$(s = \overline{1, n}; \quad \nu_1 + \dots + \nu_n = k, \quad \nu_i \geq 0).$$

$$\begin{aligned} s = 1, \quad \nu_1 &= (k+1)/2, \quad \nu_2 = (k-1)/2, \quad k - \text{odd}, \\ s = 2, \quad \nu_1 &= (k-1)/2, \quad \nu_2 = (k+1)/2, \quad \nu_3 = \dots = \nu_n = 0, \quad k = \overline{3, m}, \\ s = \overline{3, n}; \quad \nu_1 &= \nu_2 = (k-1)/2, \quad \nu_s = 1, \\ \nu_3 &= \dots = \nu_{s-1} = \nu_{s+1} = \nu_{s+2} = \dots = \nu_n = 0. \end{aligned} \quad (3)$$

With these hypotheses, we approximately integrate system (1) in the neighborhood  $\sum_{s=1}^n |y_s|^2 \leq \varepsilon^2$ . Let us make polynomial transformation of variables.

$$z_s = y_s + \sum_{k=3}^m \sum_{\nu_1 + \dots + \nu_n = k} A_s^{(\nu_1, \dots, \nu_n)} y_1^{\nu_1} \dots y_n^{\nu_n} \quad (s = \overline{1, n}). \quad (4)$$

The transformation coefficients are constant and are defined from the condition that the system (1) in new variables has the following form

$$\dot{z}_s = \left( \lambda_s + \sum_{k=3}^m a_s^{(k)} r^{k-1} \right) z_s + Z_s^{(m+1)}, \quad r = |z_1| = \sqrt{z_1 z_2}. \quad (5)$$

The right-hand parts of this system contain linear as well as nonlinear terms corresponding to special values of indices with undefined coefficients and remainder terms of  $(m+1)$  smallness order. We make coefficients  $A_s^{(\nu_1, \dots, \nu_n)}$  corresponding to special values (3) equal to zero (instead of them the coefficients  $a_s^{(k)}$  are introduced). In order to calculate all undefined coefficients, we apply

$$r^2 = \sum B^{(\nu_1, \dots, \nu_n)} y_1^{\nu_1} \dots y_n^{\nu_n}, \quad B^{(\nu_1, \dots, \nu_n)} = \sum_{\nu'_r} A_1^{(\nu'_1, \dots, \nu'_n)} A_2^{(\nu_1 - \nu'_1, \dots, \nu_n - \nu'_n)}. \quad (6)$$



Having put every sum in the form  $\sum Q^{(\nu_1, \dots, \nu_n)} y_1^{\nu_1} \dots y_n^{\nu_n}$  and equating the coefficients of similar powers  $C_1^{\nu_1} \dots C_n^{\nu_n}$ , we obtain the following equations:

$$\begin{aligned} \lambda_s^{(\nu_1, \dots, \nu_n)} A_s^{(\nu_1, \dots, \nu_n)} + p_s^{(\nu_1, \dots, \nu_n)} = & \sum_{k'=\nu}^k a_s^{(k')} \sum_{\left(\nu_r^{(i)}, \nu_r^{(i')}\right)} \prod_{i=1}^{\frac{k'-1}{2}} B^{(\nu_1^{(i)}, \dots, \nu_n^{(i)})} A_s^{(\nu_1^{(i')}, \dots, \nu_n^{(i')})} - \\ & - \sum_{i=1}^n \sum_{\nu_j'} \nu_i' p_i^{(\nu_1 - \nu_1', \dots, \nu_i - \nu_i' + 1, \nu_{i+1} - \nu_{i+1}', \dots, \nu_n - \nu_n')} A_s^{(\nu_1', \dots, \nu_n')}, \quad s = \overline{1, n}, \quad k' \text{ is odd.} \end{aligned} \quad (7)$$

We designate the sum of upper indices in undefined coefficients  $A_s^{(\nu_1, \dots, \nu_n)}$  as a coefficient decade. In the right-hand part of the equations the sums depend on the coefficients with decade smaller than  $k$ , as every factor “takes” its decade from the total “stock” of  $k$ . The high-order digit  $\{k'\}$  of the coefficients  $a_s^{(k')}$  is reached when the number of factors under the product sign is the largest, which is possible if every factor has the lowest order. By adding correlations we define that  $k' = k$ , and by analysing every correlation we can make sure that indexes  $\nu$  have special values (3). So the high-order digit of the coefficients equals  $k$ ; besides, it may be obtained only with special values of indexes. It is obvious that the coefficient  $a_s^{(k)}$  equals one.

System (7) represents a chain of linear algebraic equations which is solved starting from the lower order  $k = 2$  and from the lower number  $s = 1$  to further ones. Indeed, all equations corresponding to non-special values of indices are satisfied when choosing undefined coefficients from the first term of the formula (7), and all “special” equations where the factor  $A_s^{(\nu_1, \dots, \nu_n)}$  equals zero or is very small are satisfied when choosing  $a_s^{(k)}$ .

The remainder functions  $Z_s^{(m+1)}$  should be equated to nonlinear terms of not lower than  $(m + 1)$  order that are contained in the equations obtained by means of formulas (4) and (1) in (5). These functions may be transformed to  $z_s$  variables by correlations (4) previously solved with respect to  $y_s$ .

### 3 Transformed System Analysis

Suppose that by means of (4), (7) the system (1) is transformed to (5). The latter system may be integrated if the remainder terms of  $(m + 1)$  order are ignored. From the first two equations we obtain the equation for variables module:

$$\dot{r} = \alpha r + \sum_{k=3}^m \alpha^{(k)} r^k \quad \text{where} \quad \alpha = \operatorname{Re} \lambda_1 < 0, \quad \alpha^{(k)} = \operatorname{Re} a_1^{(k)}. \quad (8)$$

The special points of the equation (8) are defined in  $[t_0, t]$  by equating the right-hand part to zero, and general solution is defined by means of variables separation.

$$\int_{r_0}^r \left( \alpha r + \sum \alpha^{(k)} r^k \right)^{-1} dr = t - t_0, \quad (k \text{ are odd numbers}). \quad (9)$$

The second way of equality (8) integration is given in [14]. Suppose that using one of the methods, we found the solution  $r = r(t, r_0, t_0)$ . Then the solutions of the first and second equations of system (5) are as follows:

$$z_{1,2} = r e^{\pm i\theta} \quad \text{at} \quad \theta = \beta(t - t_0) + \sum_{k=3}^m \beta^{(k)} \int_{t_0}^t r^{k-1} dt + \theta_0, \quad \beta^{(k)} = \operatorname{Im} a_1^{(k)}. \quad (10)$$

The solution of the other equations is obtained according to the following formulas:

$$z_s = z_{s0} \exp \left( \lambda_s(t - t_0) + \sum_k a_s^{(k)} \int_{t_0}^t r^{k-1} dt \right), \quad s = \overline{3, n}. \quad (11)$$

In order to find an approximate solution in variables  $y_s$  we must solve the transformation (4) with respect to  $y_s$ :

$$y_s \approx z_s + \sum B_s^{(\nu_1, \dots, \nu_n)} z_1^{\nu_1} \dots z_n^{\nu_n}, \quad s = \overline{1, n}. \quad (12)$$

The coefficients  $B_s^{(\nu_1, \dots, \nu_n)}$  may be expressed in terms of  $A_s^{(\nu_1, \dots, \nu_n)}$  by using (12) in (4) and equating the coefficients with similar terms. Namely, the coefficients of lower orders in (12) differ from the coefficients in (4) only by sign.

With regard to the obtained approximated solution, there is an idea that: if it is allowed to ignore or add the terms of  $(m+1)$  order and more in equations (1), then it is always possible to select terms such that the obtained system will be integrated quite accurately. These terms are selected from the condition of  $Z_s^{(m+1)}$  being equal to zero.

**Remark 3.1** An additional condition for the characteristic coefficients (2) may be annulled, if we introduce additional terms corresponding to the special values of indices into the transformed system (5), as Dulak did in a non-special case. In this case, instead of system (5), we have:

$$\dot{z}_s = \left( \lambda_s + \sum_{k=1}^m a_s^{(k)} r^{k-1} \right) z_s + \sum_{\tilde{\nu}_j}^m a_s^{(\tilde{\nu}_1, \dots, \tilde{\nu}_{s-1})} z_1^{\tilde{\nu}_1} \dots z_{s-1}^{\tilde{\nu}_{s-1}} + Z_s^{(m+1)}. \quad (13)$$

The equations that must be in accord with the undefined coefficients are calculated from formulas (4) and (1) in (13) and by equating the coefficients of the corresponding terms; they differ from equations (7) by the additional terms. The system (13) also represents a chain of consequent approximately integrated equations.

We note that I.G. Malkin [15] analyzed the transformation of two equations system to the form similar to the first two equations of system (5) by means of substitution reverse to substitution (4).

From the first two equations we can obtain an equation similar to (8) but with an additional term of  $(m+1)$  order

$$\dot{r} = rf(r^2) + R^{(m+1)}, \quad f(r^2) = \alpha + \sum_{k=3}^m \alpha^{(k)} r^{k-1}, \quad (14)$$

where  $k, m$  are odd. Let us take Lyapunov's function and its derivative

$$V = r^2 + \sum_{s=3}^n z_s \bar{z}_s, \quad \dot{V} = 2r\dot{r} + \sum_{s=3}^m (\dot{z}_s \bar{z}_s + \dot{\bar{z}}_s z_s).$$

Taking into consideration the equations (14) and (5), we obtain the inequality:

$$\dot{V} < 2[f(V) + KV^{\frac{m}{2}}]V, \quad 0 < V \leq \varepsilon^2. \quad (15)$$

Function  $V$  decreases in this ring in accordance with the law

$$\int_{V_0}^V \frac{dV}{[f(V + KV^{\frac{m}{2}})]V} > 2(t - t_0), \quad 0 < V_0 < \varepsilon^2. \quad (16)$$

**Example 3.1** Let us integrate approximately the Van-der-Pol equation

$$\dot{x} = \varepsilon x - y - \varepsilon x y^2, \quad \dot{y} = x. \quad (17)$$

Here,  $\lambda_{1,2} \approx \alpha \pm i$ , where  $\alpha = \varepsilon/2 > 0$  are the complex-conjugated roots of the characteristic equation of the corresponding linear system with small real part. Let us put system (17) into the canonical form:

$$\dot{y}_1 = \lambda_1 y_1 + p_1^{(3,0)} y_1^3 + p_1^{(2,1)} y_1^2 y_2 + p_1^{(1,2)} y_1 y_2^2 + p_1^{(0,3)} y_2^3, \quad y_2 = \overline{y_1}. \quad (18)$$

Accurate to the  $\varepsilon^2$  order, we have:

$$p_1^{(3,0)} = p_1^{(0,3)} = -p_1^{(2,1)} = -p_1^{(1,2)} = \frac{\varepsilon}{2}; \quad y_1 = x + \left(-\frac{\varepsilon}{2} + i\right) y.$$

Let us make transformation of variables

$$z_1 = y_1 + A_1^{(3,0)} y_1^3 + A_1^{(1,2)} y_1 y_2^2 + A_1^{(0,3)} y_2^3 \quad (19)$$

where the coefficients are defined from formulas (7), (2)

$$(\nu_1 \lambda_1 + \nu_2 \lambda_2 - \lambda_1) A_1^{(\nu_1, \nu_2)} + p_1^{(\nu_1, \nu_2)} = 0, \quad p_1^{(2,1)} = a_1^{(3)} \quad (\nu_1 = 3, 1, 0; \quad \nu_2 = 3 - \nu_1).$$

Therefrom, accurate to the 2nd order, we find:

$$A_1^{(3,0)} = A_1^{(1,2)} = -A_1^{(0,3)} = -\frac{\varepsilon i}{32}, \quad a_1^{(3)} = -\frac{\varepsilon}{8}, \quad (20)$$

$z_1$  being a variable module, according to (8), satisfies the equation:

$$\frac{dr}{dt} \approx \frac{\varepsilon}{2} r - \frac{\varepsilon}{8} r^3,$$

which coincides with the equation for amplitude obtained by the method of Krylov and Bogolyubov [16]. General solution is as follows:

$$r = 2 \left(1 + c e^{-\varepsilon t}\right)^{-\frac{1}{2}}, \quad \text{where} \quad c = \frac{4}{r_0^2} - 1. \quad (21)$$

From formula (10), we obtain:

$$z_1 = r e^{i\theta}, \quad z_2 = \overline{z_1} = r e^{-i\theta}, \quad \text{where} \quad \theta = t + \theta_0, \quad (22)$$

where the initial value  $\theta_0$  can be defined on the basis of (18), (20), (21).

Formulas (21), (22) show that in complex plane  $z_1 = \xi + i\eta$  the paths of representation point and spiral coil from the inside and outside on the circumference  $r = 2$ . As this takes place, the angular speed of vector radius  $r$  is  $\theta = 1$ . Based on (19) and (21) we have:

$$y_1 \approx z_1 - A_1^{(3,0)} z_1^3 - A_1^{(1,2)} z_1 z_2^2 - A_1^{(0,3)} z_2^3 = r e^{i\theta} - \frac{\varepsilon i}{32} r^3 (2e^{3i\theta} + 2e^{-i\theta} - e^{-3i\theta}).$$

The original variable is determined as:

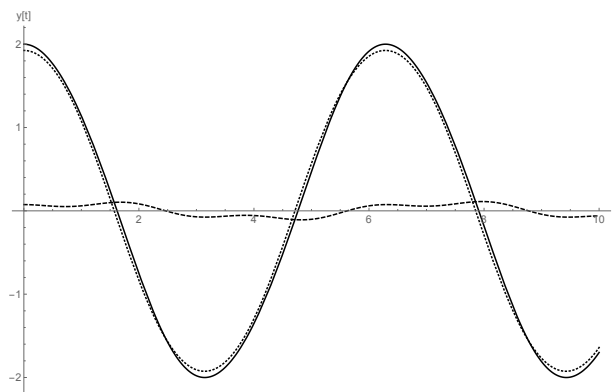
$$y = \frac{y_1 - y_2}{2i} = \operatorname{Im}(y_1) = r \sin \theta - \frac{\varepsilon}{32} r^3 (\cos 2\theta + 2\cos \theta), \quad \theta = t + \theta_0.$$

The results are shown in Figure 1.

As  $t \rightarrow \infty$ , all solutions, except for zero, asymptotically tend to the periodic one

$$y = 2 \sin \theta - \frac{\varepsilon}{2} \cos \theta - \frac{\varepsilon}{4} \cos 3\theta, \quad \theta = t + \theta_0.$$

This solution accurate to  $\varepsilon^2$  terms coincides with the solution defined by the method of Krylov and Bogolyubov [6, 14, 17].



**Figure 1:** (—) denotes an exact solution  $y(t)$  of (3.1), ( $\cdots$ ) stands for an approximate solution, ( $- -$ ) means an approximation error.

#### 4 Conclusion

Using the non-linear transformation of the polynomial model with adopted precision we investigate a nonlinear vibrational autonomous system with finite degrees of freedom at different ratios between the constants.

This transformation simplifies the form of differential equations, ultimately reduces the number of non-linear terms in the model and forms a small number of high-quality constant coefficients of monomials. The method is modified in order to exclude small divisors. Nonlinear oscillations are investigated by means of analytical integration of the transformed recurrence equations, as well as by integrating the differential inequalities for the Lyapunov function. This method can be applied to a wide range of problems.

#### Acknowledgment

The authors gratefully acknowledge the reviewers' comments and the contribution of the RFFI research projects no. 17-01-00672 and 16-08-00997, the state financial support of the leading universities, project no. 074-U01, the Ministry of Education and Science, projects no. 14.Z50.31.0031, 14.578.21.0203.

#### References

- [1] Kovalev, A.M., Martynyuk, A.A., Boichuk, O.A., Mazko, A.G., Petryshyn, R.I., Slyusarchuk, V.Y., Zuyev, A.L. and Slyn'ko, V.I. Novel qualitative methods of nonlin-

- ear mechanics and their application to the analysis of multifrequency oscillations, stability, and control problems. *Nonlinear Dynamics and Systems Theory* **9** (2) (2009) 117–145.
- [2] Dshalalow, J.H., Izobov, N.A. and Vassilyev, S.N. Academician A.A. Martynyuk to the 70th birthday anniversary. *Nonlinear Dynamics and Systems Theory* **11** (1) (2011) 1–5.
- [3] Martynyuk, A.A., Dshalalow, J.H. and Zhukovskii, V.I. Professor V.M. Starzhinskii. *Nonlinear Dynamics and Systems Theory* **8** (1) (2008) 1–6.
- [4] Vassilyev, S.N., Martynyuk, A.A. and Siljak, D.D. Academician V.M. Matrosov in memoriam. *Nonlinear Dynamics and Systems Theory* **13** (4) (2013) 325–331.
- [5] Martynyuk, A.A., Mishchenko, E.F., Samoilenko, A.M. and Sukhanov, A.D. Personage in science:academician N.N. Bogoliubov (to the 100th birthday anniversary). *Nonlinear Dynamics and Systems Theory* **9** (2) (2009) 109–115.
- [6] Kyzio, J. and Okniski, A. The Duffing-van der Pol equation: Metamorphoses of resonance curves. *Nonlinear Dynamics and Systems Theory* **15** (1) (2015) 25–31.
- [7] Leonov, G. Strange attractors and classical stability theory. *Nonlinear Dynamics and Systems Theory* **8** (1) (2008) 49–96.
- [8] Melnikov, V. Chebyshev economization in Poincare-Dulac transformations of nonlinear systems. *Nonlinear Analysis, Theory, Methods and Applications* **63** (5-7) (2005) e1351–e1355.
- [9] Melnikov, G., Ivanov, S. and Melnikov, V. The modified Poincare-Dulac method in analysis of autooscillations of nonlinear mechanical systems. *J. Phys. Conf. Ser.* **570** (2014) 022002.
- [10] Melnikov, V. Chebyshev economization in transformations of nonlinear systems with polynomial structure. *Int. Conf. on Systems – Proc.* **1** (2010) 301–303.
- [11] Aleksandrov, A.Y., Martynyuk, A.A. and Zhabko, A.P. The problem of stability by nonlinear approximation. *Nonlinear Dynamics and Systems Theory* **15** (3) (2015) 221–230.
- [12] Melnikov, V.G., Melnikov, G.I., Malykh, K.S. and Dudarenko, N.A. Poincare-Dulac method with Chebyshev economization in autonomous mechanical systems simulation problem. *2015 International Conference on Mechanics – Seventh Polyakhov's Reading* (2015) 7106757.
- [13] Ivanov, S. Mathematical modeling of nonlinear dynamic system of the truck crane. *Contemporary Engineering Sciences* **9** (10) (2016) 487–495.
- [14] Mitropol'skii, Y.A. and Martynyuk, A.A. Several results and real problems in the theory of periodic motions and nonlinear mechanics (review). *Soviet Applied Mechanics* **24** (3) (1988) 205–216.
- [15] Mitropol'skii, Y.A. and Martynyuk, A.A. Certain directions of investigations into stability of periodic motions and the theory of nonlinear vibrations. *Soviet Applied Mechanics* **14** (3) (1978) 223–232.
- [16] Martynyuk, A.A. and Nikitina, N.V. Complex oscillations revisited. *International Applied Mechanics* **41** (2) (2005) 179–186.
- [17] Martynyuk, A.A. On an averaging theorem of the first N. N. Bogolyubov type. *Polish Academy of Sciences, Inst. of Fundamental Technical Research, Nonlinear Vibration Probl.* (21) (1983) 7–10.



# Existence Results for Mild Solution for a Class of Impulsive Fractional Stochastic Problems with Nonlocal Conditions

Mohd Nadeem<sup>1\*</sup> and Jaydev Dabas<sup>1</sup>

<sup>1</sup> *Department of Applied Science and Engineering, Indian Institute of Technology Roorkee, Saharanpur Campus, Saharanpur-247001, INDIA.*

Received: October 12, 2016; Revised: October 28, 2017

**Abstract:** This paper is concerned with the existence of mild solutions for nonlocal impulsive fractional order functional stochastic differential equations with delay. The existence and uniqueness results are shown by using the fixed point technique in a real Hilbert space. Finally, we illustrate the uniqueness result by an example involving partial derivatives.

**Keywords:** *fractional differential equation, existence and uniqueness, impulsive effects, stochastic differential equation.*

**Mathematics Subject Classification (2010):** 26A33, 34A12, 34A37, 34K50.

## 1 Introduction

The modeling with stochastic differential equations has attracted many authors due to its various applications in physics, biology, mathematical finance, etc (see [29, 31, 33] and references therein). The issues related to the existence and uniqueness for such model are widely studied by many authors and one can see the contribution in [5, 7, 18, 19, 34, 35, 37] and references therein. Recently, Das et al. [15] studied a fractional stochastic model with deviating argument and successfully applied the Faedo-Galerkin approximation method to prove the existence results. Benaabane et al. [7] examined the Sobolev-type fractional stochastic model and established the existence and uniqueness of mild solutions via Picard's iteration technique.

Recently, the modeling with fractional differential equations has gained considerable importance due to its numerous applications in various fields of science and engineering,

---

\* Corresponding author: <mailto:mohdnadeem.jmi@gmail.com>

such as physics, chemistry, mechanics, system identification, etc (see the monographs [26, 30, 32]). A significant and systematic development on the existence and uniqueness of solutions for nonlinear type model of fractional differential equations can be seen in [3, 4, 10, 24, 25] and references therein. The model with impulsive nature is found in many real world problems which describe the phenomena of evolution of processes that are subject to sudden changes in their states, for details and update work, we cite the papers [12–14, 17, 20, 23, 28, 39, 40].

In some phenomena, the rate of change of the system and current status often depends not only on the current state but also on the history of the system. Such type of problem models are in the form of functional differential equations and arise in many important fields such as cell biology, electrodynamics, position control, etc. For more details, we refer the reader to the monographs [22, 27] and the papers [11, 35–37].

The nonlocal type initial condition, which is generalization of classical initial condition, was firstly initiated by Byszewski [8]. Further, Byszewski and Lakshmikantham in [9], remarked that the nonlocal condition can be more useful than the standard initial condition to describe some physical phenomena. In [41] Jhou et al. considered more general nonlocal condition and established the existence and uniqueness of mild solutions by using Krasnoselskii's fixed point theorem and Banach contraction principle.

As far as solution technique is concerned, Fečkan et al. [16] established a concept of solutions for the class of impulsive fractional differential equations which is claimed to be more suitable than the concept given by Agarwal et al. [2]. Recently, many authors followed this concept and improved the existing results (see [12, 14, 36]). In this work, we define the mild solution of the system (1)–(3) using the concept introduced in papers [16, 38]. The mild solution is associated with the solution operator reformed by Mittag-Leffler function on a Hilbert space.

Motivated by the above mentioned works as well as the papers [11, 16, 35, 36, 39, 41], we consider the following impulsive fractional functional stochastic differential equation with nonlocal condition:

$${}^C D_t^\alpha u(t) = Au(t) + t^n f(t, u_t) + t^n g(t, u_t) \frac{dw(t)}{dt}, t \in J, t \neq t_k, \quad (1)$$

$$u(t) + (h(u_{t_1}, u_{t_2}, \dots, u_{t_p}))(t) = \phi(t), \quad t \in [-d, 0], \quad (2)$$

$$\Delta u(t_k) = I_k(u(t_k^-)), \quad k = 1, 2, \dots, m, \quad (3)$$

where  $J = [0, T]$ ,  $n \in \mathbb{Z}^+$ , and  ${}^C D_t^\alpha$  denotes Caputo's fractional derivative of order  $\alpha \in (0, 1)$ .  $A : D(A) \subset \mathbb{H} \rightarrow \mathbb{H}$  is a closed linear sectorial operator defined on a Hilbert space  $(\mathbb{H}, \|\cdot\|)$  and  $u(\cdot)$  takes the values in the real separable Hilbert space  $\mathbb{H}$ ;  $f : J \times PC_{\mathcal{L}}^0 \rightarrow \mathbb{H}$ ,  $g : J \times PC_{\mathcal{L}}^0 \rightarrow \mathcal{L}(\mathbb{K}, \mathbb{H})$ ,  $h : PC_{\mathcal{L}}^{0p} \rightarrow \mathbb{H}$  and  $I_k : \mathbb{H} \rightarrow \mathbb{H}$  are appropriate functions;  $\phi(t)$  is  $\mathbb{F}_0$ -measurable  $\mathbb{H}$ -valued random variable independent of  $w$ . The functions  $u_\theta$  are defined as  $u_\theta(t) = u(\theta + t)$  for  $\theta \in [-d, 0]$ .

In the problem under consideration, the equation (1) is very important due to its appearance in the mathematical modeling of viscoelasticity. This fact prompts us to study the existence and uniqueness of solutions of system (1)–(3). To the best of our knowledge, the study of sufficient conditions for the existence of the problem (1)–(3) in Hilbert space is an untreated topic yet.

This work has been divided in four sections, the second section provides some basic definitions and preliminary results. The third section is equipped with main results for the problem (1)–(3) and in the last section an example is presented to verify the established results.

## 2 Preliminaries

Let  $\mathbb{H}, \mathbb{K}$  be two real separable Hilbert spaces and  $\mathcal{L}(\mathbb{K}, \mathbb{H})$  be the space of bounded linear operators from  $\mathbb{K}$  into  $\mathbb{H}$ . For convenience, we will use the same notation  $\|\cdot\|$  to denote the norms in  $\mathbb{H}, \mathbb{K}$  and  $\mathcal{L}(\mathbb{K}, \mathbb{H})$ , and use  $(\cdot, \cdot)$  to denote the inner product of  $\mathbb{H}$  and  $\mathbb{K}$  without any confusion. Let  $(\Omega, \mathbb{F}, \{\mathbb{F}_t\}_{t \geq 0}, \mathbb{P})$  be a complete filtered probability space satisfying the condition that  $\mathbb{F}_0$  contains all  $\mathbb{P}$ -null sets of  $\mathbb{F}$ . An  $\mathbb{H}$ -valued random variable is an  $\mathbb{F}$ -measurable function  $u(t) : \Omega \rightarrow \mathbb{H}$  and a collection of random variables  $S = \{u(t, \omega) : \Omega \rightarrow \mathbb{H} \setminus t \in J\}$  is called the stochastic process. Usually, we write  $u(t)$  instead of  $u(t, \omega)$  and  $u(t) : J \rightarrow \mathbb{H}$  in the space of  $S$ . Let  $\mathbb{W} = (\mathbb{W}_t)_{t \geq 0}$  be a  $\mathbb{Q}$ -Wiener process defined on  $(\Omega, \mathbb{F}, \{\mathbb{F}_t\}_{t \geq 0}, \mathbb{P})$  with the covariance operator  $\mathbb{Q}$  such that  $Tr \mathbb{Q} < \infty$ . We assume that there exist a complete orthonormal system  $\{e_k\}_{k \geq 1}$  in  $\mathbb{K}$ , a bounded sequence of nonnegative real numbers  $\lambda_k$  such that  $\mathbb{Q}e_k = \lambda_k e_k$ ,  $k = 1, 2, \dots$ , and a sequence of independent Brownian motions  $\{\beta_k\}_{k \geq 1}$  such that

$$(w(t), e)_{\mathbb{K}} = \sum_{k=1}^{\infty} \sqrt{\lambda_k} (e_k, e)_{\mathbb{K}} \beta_k(t), e \in \mathbb{K}, t \geq 0.$$

Let  $\mathcal{L}_0^2 = \mathcal{L}^2(\mathbb{Q}^{\frac{1}{2}} \mathbb{K}, \mathbb{H})$  be the space of all Hilbert-Schmidt operators from  $\mathbb{Q}^{\frac{1}{2}} \mathbb{K}$  to  $\mathbb{H}$  with the inner product  $\langle \varphi, \psi \rangle_{\mathcal{L}_0^2} = Tr[\varphi \mathbb{Q} \psi^*]$ .

The collection of all strongly measurable, square integrable,  $\mathbb{H}$ -valued random variables, denoted by  $\mathcal{L}^2(\Omega, \mathbb{F}, \{\mathbb{F}_t\}_{t \geq 0}, \mathbb{P}; \mathbb{H}) = \mathcal{L}^2(\Omega; \mathbb{H})$ , is a Banach space equipped with the norm  $\|u(\cdot)\|_{\mathcal{L}^2}^2 = E\|u(\cdot, w)\|_{\mathbb{H}}^2$ , where  $E$  denotes expectation defined by  $E(h) = \int_{\Omega} h(w) d\mathbb{P}$ . An important subspace is given by  $\mathcal{L}_0^2(\Omega; \mathbb{H}) = \{f \in \mathcal{L}^2(\Omega; \mathbb{H}) : f \text{ is } \mathbb{F}_0\text{-measurable}\}$ .

We consider the space

$$PC_{\mathcal{L}}^0 = PC([-d, 0], \mathcal{L}^2(\Omega; \mathbb{H}))$$

as a Banach space of all continuous functions  $u : [-d, 0] \rightarrow \mathcal{L}^2(\Omega; \mathbb{H})$ , endowed with the norm

$$\|u\|_{PC_{\mathcal{L}}^0}^2 = \sup_{t \in J} \{E\|u(t)\|_{\mathbb{H}}^2, u \in PC_{\mathcal{L}}^0\}.$$

To study the impulsive conditions, we consider

$$PC_{\mathcal{L}} = PC([-d, T], \mathcal{L}^2(\Omega; \mathbb{H}))$$

as a Banach space of all such continuous functions  $u : [-d, T] \rightarrow \mathcal{L}^2(\Omega; \mathbb{H})$ , which are continuous on  $[0, T]$  except for a finite number of points  $t_i \in (0, T)$ ,  $i = 1, 2, \dots, m$ , at which  $u(t_i^+)$  and  $u(t_i^-) = u(t_i)$  exist, endowed with the norm

$$\|u\|_{PC_{\mathcal{L}}}^2 = \sup_{t \in J} \{E\|u(t)\|_{\mathbb{H}}^2, u \in PC_{\mathcal{L}}\}.$$

**Remark 2.1** ([21]) If  $\alpha \in (0, 1)$  and  $A \in \mathbb{A}^{\alpha}(\theta_0, \omega_0)$ , then for any  $u \in \mathbb{H}$  and  $t > 0$  we have  $\|T_{\alpha}(t)\| \leq Me^{\omega t}$  and  $\|S_{\alpha}(t)\| \leq Ce^{\omega t}(1 + t^{\alpha-1})$ ,  $\omega > \omega_0$ . Thus we have

$$\|T_{\alpha}(t)\| \leq \widetilde{M}_T \text{ and } \|S_{\alpha}(t)\| \leq t^{\alpha-1} \widetilde{M}_S,$$

where  $\widetilde{M}_T = \sup_{0 \leq t \leq T} \|T_{\alpha}(t)\|$  and  $\widetilde{M}_S = \sup_{0 \leq t \leq T} Ce^{\omega t}(1 + t^{1-\alpha})$ .



Now, we state the definition of mild solution of the system (1)-(3) based on the concept introduced in [38].

**Definition 2.1** A measurable  $\mathbb{F}_t$ -adapted stochastic process  $u : [-d, T] \rightarrow \mathbb{H}$  such that  $u \in PC_{\mathcal{L}}$  is called a mild solution of the system (1)-(3) if  $u(0) = \phi(0) - (h(u_{t_1}, u_{t_2}, \dots, u_{t_p}))(0)$  on  $[-d, 0]$ ,  $\Delta u|_{t=t_k} = I_k(u(t_k^-))$ ,  $k = 1, 2, \dots, m$ , the restriction of  $u(\cdot)$  to the interval  $[0, T] \setminus \{t_1, \dots, t_m\}$ , is continuous and  $u(t)$  satisfies the following integral equation

$$u(t) = \begin{cases} S_{\alpha}(t)[\phi(0) - (h(u_{t_1}, u_{t_2}, \dots, u_{t_p}))(0)] \\ \quad + \int_0^t T_{\alpha}(t-s)s^n f(s, u_s)ds \\ \quad + \int_0^t T_{\alpha}(t-s)s^n g(s, u_s)dw(s), & t \in (0, t_1], \\ S_{\alpha}(t)[\phi(0) - (h(u_{t_1}, u_{t_2}, \dots, u_{t_p}))(0)] \\ \quad + S_{\alpha}(t-t_1)I_1(u(t_1^-)) + \int_0^t T_{\alpha}(t-s)s^n f(s, u_s)ds \\ \quad + \int_0^t T_{\alpha}(t-s)s^n g(s, u_s)dw(s), & t \in (t_1, t_2], \\ \vdots \\ S_{\alpha}(t)[\phi(0) - (h(u_{t_1}, u_{t_2}, \dots, u_{t_p}))(0)] \\ \quad + \sum_{i=1}^m S_{\alpha}(t-t_i)I_i(u(t_i^-)) + \int_0^t T_{\alpha}(t-s)s^n f(s, u_s)ds \\ \quad + \int_0^t T_{\alpha}(t-s)s^n g(s, u_s)dw(s), & t \in (t_m, T]. \end{cases}$$

To avoid the repetitions of the basic definitions, we cite them from appropriate papers and books: for Reimann-Liouville integral operator, Mittag-Leffler function and Caputo's derivative see [32], for  $\alpha$ -resolvent family see [4], for sectorial operator see [21] and for solution operator see [1].

### 3 Existence and Uniqueness of Solutions

For the forthcoming analysis, we introduce the following assumption.

(H1) Functions  $f; g; h$  and  $I_k$  are continuous and there exist positive constants  $L_f; L_g; L_h$  and  $L_I$  such that

$$\begin{aligned} E\|f(t, \phi) - f(t, \varphi)\|_{\mathbb{H}}^2 &\leq L_f \|\phi - \varphi\|_{PC_{\mathcal{L}}^0}^2, \\ E\|g(t, \phi) - g(t, \varphi)\|_{\mathbb{H}}^2 &\leq L_g \|\phi - \varphi\|_{PC_{\mathcal{L}}^0}^2, \\ E\|(h(u_{t_1}, u_{t_2}, \dots, u_{t_p}))(t) - (h(v_{t_1}, v_{t_2}, \dots, v_{t_p}))(t)\|_{\mathbb{H}}^2 &\leq L_h E\|u - v\|_{\mathbb{H}}^2, \\ E\|I_k(u) - I_k(v)\|_{\mathbb{H}}^2 &\leq L_I E\|u - v\|_{\mathbb{H}}^2, \end{aligned}$$

for all  $u, v \in \mathbb{H}$  and  $\phi, \varphi \in PC_{\mathcal{L}}^0$ .

Our first result is based on the Banach contraction principle.

**Theorem 3.1** Let the assumption (H1) hold with the positive constant

$$\Theta = \begin{cases} [4\widetilde{M}_S^2 L_h + 4m\widetilde{M}_S^2 L_I + 4\widetilde{M}_T^2 \frac{T^{2\alpha+n}}{\alpha} \frac{\Gamma(\alpha)\Gamma(n+1)}{\Gamma(\alpha+n+1)} L_f \\ \quad + 4\widetilde{M}_T^2 T^{2\alpha-1+n} \frac{\Gamma(2\alpha-1)\Gamma(n+1)}{\Gamma(2\alpha+n)} L_g] < 1, \end{cases}$$

then the system (1)-(3) has a unique mild solution.

**Proof.** Define the operator  $P : PC_{\mathcal{L}} \rightarrow PC_{\mathcal{L}}$  so that

$$(Pu)(t) = \begin{cases} S_{\alpha}(t)[\phi(0) - (h(u_{t_1}, u_{t_2}, \dots, u_{t_p}))(0)] \\ + \int_0^t T_{\alpha}(t-s)s^n f(s, u_s)ds \\ + \int_0^t T_{\alpha}(t-s)s^n g(s, u_s)dw(s), & t \in (0, t_1], \\ S_{\alpha}(t)[\phi(0) - (h(u_{t_1}, u_{t_2}, \dots, u_{t_p}))(0)] \\ + S_{\alpha}(t-t_1)I_1(u(t_1^-)) + \int_0^t T_{\alpha}(t-s)s^n f(s, u_s)ds \\ + \int_0^t T_{\alpha}(t-s)s^n g(s, u_s)dw(s), & t \in (t_1, t_2], \\ \vdots \\ S_{\alpha}(t)[\phi(0) - (h(u_{t_1}, u_{t_2}, \dots, u_{t_p}))(0)] \\ + \sum_{i=1}^m S_{\alpha}(t-t_i)I_i(u(t_i^-)) + \int_0^t T_{\alpha}(t-s)s^n f(s, u_s)ds \\ + \int_0^t T_{\alpha}(t-s)s^n g(s, u_s)dw(s), & t \in (t_m, T]. \end{cases}$$

Now, we show that  $P$  is a contraction map. To this end we take two points  $u, u^* \in PC_{\mathcal{L}}$ , then for all  $t \in (0, t_1]$ , we have

$$\begin{aligned} E\|(Pu)(t) - (Pu^*)(t)\|_{\mathbb{H}}^2 &\leq 3E\|S_{\alpha}(t)[(h(u_{t_1}, u_{t_2}, \dots, u_{t_p}))(0) \\ &\quad - (h(u_{t_1}^*, u_{t_2}^*, \dots, u_{t_p}^*))(0)]\|_{\mathbb{H}}^2 \\ &\quad + 3E\|\int_0^t T_{\alpha}(t-s)s^n[f(s, u_s) - f(s, u_s^*)]ds\|_{\mathbb{H}}^2 \\ &\quad + 3E\|\int_0^t T_{\alpha}(t-s)s^n[g(s, u_s) - g(s, u_s^*)]dw(s)\|_{\mathbb{H}}^2, \\ &\leq [3\widetilde{M}_S^2 L_h + 3\widetilde{M}_T^2 \frac{T^{2\alpha+n}}{\alpha} \frac{\Gamma(\alpha)\Gamma(n+1)}{\Gamma(\alpha+n+1)} L_f \\ &\quad + 3\widetilde{M}_T^2 T^{2\alpha-1+n} \frac{\Gamma(2\alpha-1)\Gamma(n+1)}{\Gamma(2\alpha+n)} L_g] \|u - u^*\|_{PC_{\mathcal{L}}}. \end{aligned}$$

For  $t \in (t_1, t_2]$ , we get the estimate

$$\begin{aligned} E\|(Pu)(t) - (Pu^*)(t)\|_{\mathbb{H}}^2 &\leq 4E\|S_{\alpha}(t)[(h(u_{t_1}, u_{t_2}, \dots, u_{t_p}))(0) \\ &\quad - (h(u_{t_1}^*, u_{t_2}^*, \dots, u_{t_p}^*))(0)]\|_{\mathbb{H}}^2 \\ &\quad + 4E\|S_{\alpha}(t-t_1)[I_1(u(t_1^-)) - I_1(u^*(t_1^-))]\|_{\mathbb{H}}^2 \\ &\quad + 4E\|\int_0^t T_{\alpha}(t-s)s^n[f(s, u_s) - f(s, u_s^*)]ds\|_{\mathbb{H}}^2 \\ &\quad + 4E\|\int_0^t T_{\alpha}(t-s)s^n[g(s, u_s) - g(s, u_s^*)]dw(s)\|_{\mathbb{H}}^2, \\ &\leq [4\widetilde{M}_S^2 L_h + 4\widetilde{M}_S^2 L_I + 4\widetilde{M}_T^2 \frac{T^{2\alpha+n}}{\alpha} \frac{\Gamma(\alpha)\Gamma(n+1)}{\Gamma(\alpha+n+1)} L_f \\ &\quad + 4\widetilde{M}_T^2 T^{2\alpha-1+n} \frac{\Gamma(2\alpha-1)\Gamma(n+1)}{\Gamma(2\alpha+n)} L_g] \|u - u^*\|_{PC_{\mathcal{L}}}^2. \end{aligned}$$

Similarly, for general  $t \in (t_i, t_{i+1}]$ ,  $i = 2, \dots, m$ , we obtain

$$\begin{aligned} E\|(Pu)(t) - (Pu^*)(t)\|_{\mathbb{H}}^2 &\leq [4\widetilde{M}_S^2 L_h + 4k\widetilde{M}_S^2 L_I + 4\widetilde{M}_T^2 \frac{T^{2\alpha+n}}{\alpha} \frac{\Gamma(\alpha)\Gamma(n+1)}{\Gamma(\alpha+n+1)} L_f \\ &\quad + 4\widetilde{M}_T^2 T^{2\alpha-1+n} \frac{\Gamma(2\alpha-1)\Gamma(n+1)}{\Gamma(2\alpha+n)} L_g] \|u - u^*\|_{PC_{\mathcal{L}}}^2. \end{aligned}$$

Thus for all  $t \in [0, T]$ , we have

$$\begin{aligned} E\|(Pu)(t) - (Pu^*)(t)\|_{PC_{\mathcal{L}}}^2 &\leq [4\widetilde{M}_S^2 L_h + 4m\widetilde{M}_S^2 L_I + 4\widetilde{M}_T^2 \frac{T^{2\alpha+n}}{\alpha} \frac{\Gamma(\alpha)\Gamma(n+1)}{\Gamma(\alpha+n+1)} L_f \\ &\quad + 4\widetilde{M}_T^2 T^{2\alpha-1+n} \frac{\Gamma(2\alpha-1)\Gamma(n+1)}{\Gamma(2\alpha+n)} L_g] \|u - u^*\|_{PC_{\mathcal{L}}}^2, \\ &\leq \Theta \|u - u^*\|_{PC_{\mathcal{L}}}^2. \end{aligned}$$

Since  $\Theta < 1$  implies that the map  $P$  is a contraction map, it has a unique fixed point  $u \in PC_{\mathcal{L}}$  which is the unique mild solution of the problem (1)-(3) on  $J$ . This completes the proof of the theorem.

Now, to prove the next result, we use Schaefer's fixed point theorem [35] and assume the following conditions:

(H2) Functions  $f$  and  $g$  are continuous and there exist continuous functions  $\widetilde{L}_f, \widetilde{L}_g : J \rightarrow (0, \infty)$  such that

$$\begin{aligned} E\|f(t, u_t)\|_{\mathbb{H}}^2 &\leq \widetilde{L}_f(t) \psi(E\|u\|_{\mathbb{H}}^2), \\ E\|g(t, u_t)\|_{\mathcal{L}_2^0}^2 &\leq \widetilde{L}_g(t) \varphi(E\|u\|_{\mathbb{H}}^2), \end{aligned}$$

for all  $\phi, \varphi \in PC_{\mathcal{L}}^0$ .

(H3) Functions  $h$  and  $I_k$  are continuous and there exist positive constant  $M_1$  and  $\Delta$  such that

$$E\|(h(u_{t_1}, u_{t_2}, \dots, u_{t_p}))(t)\|_{\mathbb{H}}^2 \leq M_1; \max_{1 \leq k \leq m} \{E\|I_k(u)\|_{\mathbb{H}}^2\} = \Delta,$$

for all  $u, v \in \mathbb{H}$ .

**Theorem 3.2** *Let the assumptions (H2) and (H3) hold with*

$$\int_0^T \eta(s) ds \leq \int_c^\infty \frac{ds}{\psi(s) + \varphi(s)}, \quad (4)$$

where

$$\begin{aligned} \eta(t) &= \max\{5\widetilde{M}_T^2 \frac{T^\alpha}{\alpha} (t)^{\alpha-1} t^n \widetilde{L}_f(t), 5\widetilde{M}_T^2 (t)^{2(\alpha-1)} t^n \widetilde{L}_g(t)\}, \\ c &= 5\widetilde{M}_S^2 [E\|\phi(0)\|_{\mathbb{H}}^2 + M_1], \end{aligned}$$

then the equation (1)-(3) has at least one mild solution on  $J$ .

**Proof.** Consider the closed subspace  $H_2 = \{u : u \in PC_{\mathcal{L}}\}$  of all continuous processes  $u$ , which are  $F_t$ -adapted measurable processes such that the  $F_0$ -adapted processes  $u(0)$  are endowed with a norm defined by

$$\|u\|_{H_2} = (\sup_{t \in J} \|u(t)\|_{\mathcal{L}^2}^2)^{\frac{1}{2}}.$$

Now, we define the operator  $N : H_2 \rightarrow H_2$  in the same way as in Theorem 3.1. Now, we have to prove that the operator  $N$  has at least one fixed point for general interval  $t \in (t_k, t_{k+1}]$ ,  $k = 0, 1, \dots, m$ .

With this in mind, consider a sequence  $\{u^n\}_{n=0}^\infty$  such that  $u^n \rightarrow u$  in  $H_2$ . Then for  $t \in (t_k, t_{k+1}]$ ,  $k = 0, 1, \dots, m$ , we have

$$\begin{aligned} E\|(Nu^n)(t) - (Nu)(t)\|_{\mathbb{H}}^2 &\leq 4E\|S_\alpha(t)[(h(u_{t_1}^n, u_{t_2}^n, \dots, u_{t_p}^n))(0) \\ &\quad - (h(u_{t_1}, u_{t_2}, \dots, u_{t_p}))(0)]\|_{\mathbb{H}}^2 \\ &\quad + 4kE\|S_\alpha(t - t_k)[I_k(u^n(t_k^-)) - I_k(u(t_k^-))]\|_{\mathbb{H}}^2 \\ &\quad + 4E\|\int_0^t T_\alpha(t-s)s^n[f(s, u_s^n) - f(s, u_s)]ds\|_{\mathbb{H}}^2 \\ &\quad + 4E\|\int_0^t T_\alpha(t-s)s^n[g(s, u_s^n) - g(s, u_s)]dw(s)\|_{\mathbb{H}}^2, \end{aligned}$$

since the functions  $f, g, h$  and  $I_k$ ,  $k = 1, 2, \dots, m$ , are continuous, we get

$$\lim_{n \rightarrow \infty} E\|Nu^n - Nu\|_{\mathbb{H}}^2 = 0,$$

which implies that the operator  $N$  is continuous on  $H_2$ .

Now, we show that  $N$  maps bounded sets into bounded sets in  $H_2$ . Consider

$$B_r = \{u \in H_2 : E\|u\|_{\mathbb{H}}^2 \leq r\} \text{ for } r > 0, \exists \xi > 0, \text{ such that } E\|(Nu)(t)\|_{\mathbb{H}}^2 \leq \xi.$$

It is clear that  $B_r$  is a closed bounded convex subset of  $H_2$ . Let  $u \in B_r$ . Then, we have

$$\begin{aligned} E\|(Nu)(t)\|_{\mathbb{H}}^2 &\leq 5E\|S_\alpha(t)\|^2[\|\phi(0)\|_{\mathbb{H}}^2 + \|(h(u_{t_1}, u_{t_2}, \dots, u_{t_p}))(0)\|_{\mathbb{H}}^2] \\ &\quad + 5E\|\sum_{i=1}^m S_\alpha(t - t_i)I_i(u(t_i^-))\|_{\mathbb{H}}^2 \\ &\quad + 5E\|\int_0^t T_\alpha(t-s)s^n f(s, u_s)ds\|_{\mathbb{H}}^2 \\ &\quad + 5E\|\int_0^t T_\alpha(t-s)s^n g(s, u_s)dw(s)\|_{\mathbb{H}}^2, \\ &\leq 5\widetilde{M}_S^2[\|\phi(0)\|_{\mathbb{H}}^2 + M_1] + 5m\widetilde{M}_S^2\Delta \\ &\quad + 5\widetilde{M}_T^2 \frac{T^\alpha}{\alpha} \psi(r) \int_0^t (t-s)^{\alpha-1} s^n \widetilde{L}_f(s)ds \\ &\quad + 5\widetilde{M}_T^2 \varphi(r) \int_0^t (t-s)^{2(\alpha-1)} s^n \widetilde{L}_g(s)ds, \\ &= \xi. \end{aligned}$$

Next, we prove that  $N$  maps bounded sets into equicontinuous sets of  $B_r$ . Let  $t_k < x < y \leq t_{k+1}$ , for each  $u \in B_r$ , we have

$$\begin{aligned} E\|(Nu)(x) - (Nu)(y)\|_{\mathbb{H}}^2 &\leq 5\|S_\alpha(x) - S_\alpha(y)\|^2[E\|\phi(0)\|_{\mathbb{H}}^2 \\ &\quad + E\|(h(u_{t_1}, u_{t_2}, \dots, u_{t_p}))(0)\|_{\mathbb{H}}^2] \\ &\quad + 5\sum_{i=1}^m \|S_\alpha(x - t_i) - S_\alpha(y - t_i)\|^2 E\|I_i(u(t_i^-))\|_{\mathbb{H}}^2 \\ &\quad + 5E\left\|\int_0^t [T_\alpha(x - s) - T_\alpha(y - s)] \times s^n f(s, u_s) ds\right\|_{\mathbb{H}}^2 \\ &\quad + 5E\left\|\int_0^t [T_\alpha(x - s) - T_\alpha(y - s)] s^n g(s, u_s) dw(s)\right\|_{\mathbb{H}}^2. \end{aligned}$$

Since  $T_\alpha(t)$  and  $S_\alpha(t)$  are strongly continuous,  $\|S_\alpha(x) - S_\alpha(y)\| \rightarrow 0$ ;  $\|S_\alpha(x - t_i) - S_\alpha(y - t_i)\| \rightarrow 0$  and  $\|T_\alpha(x - s) - T_\alpha(y - s)\| \rightarrow 0$  as  $x \rightarrow y$ . Therefore, from the above inequality, we get  $\lim_{x \rightarrow y} E\|(Nu)(x) - (Nu)(y)\|_{\mathbb{H}}^2 = 0$ . Hence, the set  $\{Nu, u \in B_r\}$  is equicontinuous. Now by Arzela-Ascoli's theorem, we conclude that the operator  $N$  is compact.

Finally, we will prove that the set

$$R = \{u \in H_2 \text{ such that } u = qNu(t) \text{ for some } 0 < q < 1\}$$

is bounded. Let  $u \in R$ , then  $u(t) = qNu(t)$  for some  $0 < q < 1$ . Therefore for each  $t \in J$ , we have

$$\begin{aligned} u(t) &= q(S_\alpha(t)[\phi(0) - (h(u_{t_1}, u_{t_2}, \dots, u_{t_p}))(0)] + \sum_{i=1}^m S_\alpha(t - t_i)I_i(u(t_i^-)) \\ &\quad + \int_0^t T_\alpha(t - s)s^n f(s, u_s) ds + \int_0^t T_\alpha(t - s)s^n g(s, u_s) dw(s)), \end{aligned}$$

which shows that

$$\begin{aligned} E\|u(t)\|_{\mathbb{H}}^2 &\leq 5E\|S_\alpha(t)[\phi(0) + (h(u_{t_1}, u_{t_2}, \dots, u_{t_p}))(0)]\|_{\mathbb{H}}^2 \\ &\quad + 5E\left\|\sum_{i=1}^m S_\alpha(t - t_i)I_i(u(t_i^-))\right\|_{\mathbb{H}}^2 \\ &\quad + 5E\left\|\int_0^t T_\alpha(t - s)s^n f(s, u_s) ds\right\|_{\mathbb{H}}^2 \\ &\quad + 5E\left\|\int_0^t T_\alpha(t - s)s^n g(s, u_s) dw(s)\right\|_{\mathbb{H}}^2, \\ &\leq 5\widetilde{M}_S^2[E\|\phi(0)\|_{\mathbb{H}}^2 + M_1] + 5m\widetilde{M}_S^2\Delta \\ &\quad + 5\widetilde{M}_T^2 \frac{T^\alpha}{\alpha} \int_0^t (t - s)^{\alpha-1} s^n \widetilde{L}_f(s) \psi(E\|u(s)\|_{\mathbb{H}}^2) ds \\ &\quad + 5\widetilde{M}_T^2 \int_0^t (t - s)^{2(\alpha-1)} s^n \widetilde{L}_g(s) \varphi(E\|u(s)\|_{\mathbb{H}}^2) ds. \end{aligned}$$

Let the function  $\lambda(t)$  be defined as

$$\begin{aligned}\lambda(t) &= \sup\{E\|u(s)\|_{\mathbb{H}}^2, 0 \leq s \leq t\}, \quad 0 \leq t \leq T, \\ \lambda(t) &\leq 5\widetilde{M}_S^2[E\|\phi(0)\|_{\mathbb{H}}^2 + M_1] + 5m\widetilde{M}_S^2\Delta \\ &\quad + 5\widetilde{M}_T^2 \frac{T^\alpha}{\alpha} \int_0^t (t-s)^{\alpha-1} s^n \widetilde{L}_f(s) \psi(\lambda(s)) ds \\ &\quad + 5\widetilde{M}_T^2 \int_0^t (t-s)^{2(\alpha-1)} s^n \widetilde{L}_g(s) \varphi(\lambda(s)) ds.\end{aligned}$$

The last inequality in the right-hand side is denoted by  $\mu(t)$ , then we have

$$\mu(0) = c = 5\widetilde{M}_S^2[E\|\phi(0)\|_{\mathbb{H}}^2 + M_1], \quad \lambda(t) \leq \mu(t).$$

On the other hand

$$\begin{aligned}\mu'(t) &= 5\widetilde{M}_T^2 \frac{T^\alpha}{\alpha} (t)^{\alpha-1} t^n \widetilde{L}_f(t) \psi(\lambda(t)) + 5\widetilde{M}_T^2 (t)^{2(\alpha-1)} t^n \widetilde{L}_g(t) \varphi(\lambda(t)). \\ &\leq 5\widetilde{M}_T^2 \frac{T^\alpha}{\alpha} (t)^{\alpha-1} t^n \widetilde{L}_f(t) \psi(\mu(t)) + 5\widetilde{M}_T^2 (t)^{2(\alpha-1)} t^n \widetilde{L}_g(t) \varphi(\mu(t)),\end{aligned}$$

or by equation (4) we have

$$\int_{\mu(0)}^{\mu(t)} \frac{ds}{\psi(s) + \varphi(s)} \leq \int_0^T \eta(s) ds < \int_c^\infty \frac{ds}{\psi(s) + \varphi(s)}.$$

This inequality shows that there is a constant  $\mathbb{C}$  such that  $\mu(t) \leq \mathbb{C}, t \in J$ , and hence,  $\lambda(t) \leq \mathbb{C}$ , for every  $t \in J$ . Further, we get  $\|u(t)\| \leq \lambda(t) \leq \mu(t) \leq \mathbb{C}, t \in J$ . As the consequence of Schaefer's fixed point theorem, we deduce that  $N$  has a fixed point on  $J$  which is a solution to (1)-(3). This completes the proof of the theorem.

#### 4 Application

Consider the following nonlocal impulsive fractional partial differential equation of the form

$$\begin{aligned}\frac{\partial^\alpha}{\partial t^\alpha} u(t, x) &= \frac{\partial^2}{\partial y^2} u(t, x) + t \frac{\|u(s-d, x)\|}{36 + \|u(s-d, x)\|} + t \frac{\|u(s-d, x)\|}{49 + \|u(s-d, x)\|} \frac{dw(t)}{dt}, \\ u(t, 0) &= u(t, \pi) = 0, \quad t \geq 0,\end{aligned}\tag{5}$$

$$u(t, x) + \sum_{i=0}^n \int_0^\pi k(x, y) u_{t_i}(t, y) dy = (\phi(t))(x), \quad t \in [-d, 0], x \in [0, \pi],\tag{6}$$

$$\Delta u(t_i)(x) = \int_{-\infty}^{t_i} q_i(t_i - s) u(s, x) ds, \quad x \in [0, \pi],\tag{7}$$

where  $\frac{\partial^\alpha}{\partial t^\alpha}$  is Caputo's fractional derivative of order  $\alpha \in (0, 1)$ ,  $0 < t_1 < 1$  are prefixed numbers and  $\phi \in PC_{\mathcal{L}^2}$ . Let  $\mathbb{H} = L^2[0, \pi]$  and define the operator  $A : D(A) \subset \mathbb{H} \rightarrow \mathbb{H}$  by  $A\omega = \omega''$  with the domain  $D(A) := \{\omega \in X : \omega, \omega' \text{ are absolutely continuous, } \omega'' \in \mathbb{H}, \omega(0) = 0 = \omega(\pi)\}$ . Then

$A\omega = \sum_{n=1}^{\infty} n^2(\omega, \omega_n)\omega_n, \omega \in D(A)$ , where  $\omega_n(x) = \sqrt{\frac{2}{\pi}}\sin(nx), n \in \mathbb{N}$  is the orthogonal set of eigenvectors of  $A$ . It is well known that  $A$  is the infinitesimal generator of an analytic semigroup  $(T(t))_{t \geq 0}$  in  $\mathbb{H}$  and is given by

$$T(t)\omega = \sum_{n=1}^{\infty} e^{-n^2 t}(\omega, \omega_n)\omega_n, \text{ for all } \omega \in \mathbb{H}, \text{ and every } t > 0.$$

The subordination principle of solution operator (Theorem 3.1 in [6]) implies that  $A$  is the infinitesimal generator of a solution operator  $\{S_{\alpha}(t)\}_{t \geq 0}$ . Since  $S_{\alpha}(t)$  is strongly continuous on  $[0, \infty)$ , by uniformly bounded theorem, there exists a constant  $M > 0$ , such that  $\|S_{\alpha}(t)\|_{L(\mathbb{H})} \leq M$  for  $t \in [0, 1]$ .

Furthermore, we can see

$$E\|f(t, x_t) - f(t, y_t)\|_{\mathbb{H}}^2 \leq \frac{1}{36}E\|x - y\|_{\mathbb{H}}^2.$$

Hence the function  $f$  satisfies (H1). Similarly, we can show that the functions  $g, I_k, h$  satisfy (H1). Furthermore, we have

$$L_f = \frac{1}{36}, L_g = \frac{1}{49}, L_h = L_I = \frac{1}{25}, \widetilde{M}_S = \widetilde{M}_T = 1, \alpha = \frac{3}{4}, n = 1.$$

It can be calculated that  $\Theta = .37 < 1$ . Hence the condition of Theorem 3.1 is fulfilled, so we deduce that the system (5)-(7) has a unique mild solution on  $[0, 1]$ .

## 5 Conclusion

Fractional order stochastic differential equation is an equation in which randomness is included. In this paper, we established the sufficient conditions for the existence results for a class of impulsive fractional functional stochastic differential equations with nonlocal initial condition. To prove the stated theorems we utilized the well known fixed point theorems with suitable setting of abstract spaces. In our subsequent study, we will try to address the existence and uniqueness issue for the class of stochastic fractional neutral integro-differential equation with non-instantaneous impulsive conditions.

## References

- [1] Agarwal, R. P., Andrade, B. D. and Siracusa G. On fractional integro-differential equations with state-dependent delay. *Computers & Mathematics with Applications* **62**(3) (2011) 1143–1149.
- [2] Agarwal, R. P., Benchohra, M. and Slimani, B. A. Existence results for differential equation with fractional order and impulses. *Memoirs on Differential Equations and Mathematical Physics* **44**(2) (2008) 1–21.
- [3] Ahmad, B. and Nieto J. J. Existence results for a coupled system of nonlinear fractional differential equations with three-point boundary conditions. *Computers & Mathematics Applications* **58**(9) (2009) 1838–1843.
- [4] Araya, D. and Lizama, C. Almost automorphic mild solutions to fractional differential equations. *Nonlinear Analysis: Theory, Methods and Applications* **69**(11) (2009) 3692–3705.

- [5] Balasubramaniam, P., Park, J. Y. and Kumar, A. V. A. Existence of solutions for semilinear neutral stochastic functional differential equations with nonlocal conditions. *Nonlinear Analysis: Theory, Methods and Applications* **71** (3-4) (2009) 1049–1058.
- [6] Bazhlekova, E. *Fractional Evolution Equations in Banach Spaces*. University Press Facilities, Eindhoven University of Technology, 2001.
- [7] Benaabane, A. and Sakthivel, R. Sobolev-type fractional stochastic differential equations with non-Lipschitz coefficients. *Journal of Computational and Applied Mathematics* <http://dx.doi.org/10.1016/j.cam.2015.12.020> (2015).
- [8] Byszewski, L. Theorems about existence and uniqueness of solutions of a semi-linear evolution nonlocal Cauchy problem. *Journal of Mathematical Analysis and Applications* **162** (2) (1991) 494–505.
- [9] Byszewski, L. and Lakshmikantham, V. Theorem about the existence and uniqueness of a solution of a nonlocal abstract Cauchy problem in a Banach space. *Applicable Analysis* **40** (1) (1991) 11–19.
- [10] Chadha, A. and Pandey, D.N. Approximations of solutions for a sobolev type fractional order differential equation. *Nonlinear Dynamics and Systems Theory* **14** (1) (2014) 11–29.
- [11] Chadha, A. and Pandey, D. N. Existence results for an impulsive neutral stochastic fractional integro-differential equation with infinite delay. *Nonlinear Analysis: Theory, Methods & Applications* **128** (2015) 149–175.
- [12] Chauhan, A. and Dabas, J. Local and global existence of mild solution to an impulsive fractional functional integro-differential equation with nonlocal condition. *Communications in Nonlinear Science and Numerical Simulation* **19** (4) (2014) 821–829.
- [13] Dabas, J., Chauhan, A. and Kumar, M. Existence of the mild solutions for impulsive fractional equations with infinite delay. *International Journal of Differential Equations* **2011** (2011) Article ID 793023, 20 pages, doi:10.1155/2011/793023.
- [14] Dabas, J. and Gautam, G. R. Impulsive neutral fractional integro-differential equations with state dependent delays and integral conditions. *Electronic Journal of Differential Equations* **2013** (273) (2013) 1–13.
- [15] Das, S., Pandey, D. N. and Sukavanam, N. Approximate of solutions of a stochastic fractional differential equation with deviating argument. *Journal of Fractional Calculus and Applications* **6** (2) (2015) 160–170.
- [16] Feckan, M., Zhou, Y. and Wang, J. On the concept and existence of solution for impulsive fractional differential equations. *Communications in Nonlinear Science and Numerical Simulation* **17** (2012) 3050–3060.
- [17] Gautam, G. R. and Dabas, J. Mild solutions for class of neutral fractional functional differential equations with not instantaneous impulses. *Applied Mathematics and Computation* **259** (2015) 480–489.
- [18] Guendouzi, T. Existence and controllability results for fractional stochastic semilinear differential inclusions. *Differential Equations and Dynamical Systems* **23** (3) (2015) 225–240.
- [19] Guendouzi, T. and Hamada, I. Existence and controllability result for fractional neutral stochastic integro-differential equations with infinite delay. *Advanced Modeling and Optimization* **2013**(15) (2013) 281–300.
- [20] Gupta, V. and Dabas, J. Existence results for a fractional integro-differential equation with nonlocal boundary conditions and fractional impulsive conditions. *Nonlinear Dynamics and Systems Theory* **15** (4) (2015) 370–382.
- [21] Haase, M. *The Functional Calculus for Sectorial Operators, Operator Theory, Advances and Applications*. Birkhauser-Verlag, Basel, 169, 2006.



- [22] Hale, J. K. and Lunel, S. M. V. *Introduction to Functional Differential Equations*. Springer-Verlag, New York, 1993.
- [23] Hernandez, E., Pierri, M. and Goncalves, G. Existence results for an impulsive abstract partial differential equation with state-dependent delay. *Computers & Mathematics with Applications* **52** (3-4) (2006) 411–420.
- [24] Herzallah, M. A .E. Mild and strong solution to few types of fractional order nonlinear equations with periodic boundary conditions. *Indian Journal of Pure and Applied Mathematics* **43** (6) (2012) 619–635.
- [25] Kamaljeet and Bahuguna, D. Extremal mild solutions for finite delay differential equations of fractional order in banach spaces. *Nonlinear Dynamics and Systems Theory* **14** (4) (2014) 371–382.
- [26] Kilbas, A. A., Srivastava, H. M. and Trujillo, J. J. *Theory and Applications of Fractional Differential Equations*. Elsevier Science, Amsterdam, 2006.
- [27] Kolmanovskii, V. and Myshkis, A. *Applied Theory of Functional Differential Equations*. Kluwer Academic Publishers, Dordrecht, 1992.
- [28] Lakshmikantham, V., Bainov, D. and Simeonov, P. S. *Theory of Impulsive Differential Equations*. In: Series in Modern Applied Mathematics, World Scientific Publishing Co., Inc., Teaneck, NJ, 1989.
- [29] Mao, X. R. *Stochastic Differential Equations and Applications*. Horwood, Chichester, UK, 1997.
- [30] Miller, K. S., and Ross, B. *An Introduction to the Fractional Calculus and Fractional Differential Equations*. John Wiley and Sons, Inc., New York, 1993.
- [31] Oksendal, B. *Stochastic Differential Equations*. fifth ed., Springer, Berlin, Germany, 2002.
- [32] Podlubny, I. *Fractional Differential Equations*. in: Mathematics in Science and Engineering, 198, Academic Press, San Diego, 1999.
- [33] Prato, G. D. and Zabczyk, J. *Stochastic Equations in Infinite Dimensions*. 44, Encyclopedia of Mathematics and its Applications, Cambridge University Press, Cambridge, Mass, USA, 1992.
- [34] Revathi, P., Sakthivel, R., Ren, Y. and Anthoni, S. M. Existence of almost automorphic mild solutions to non-autonomous neutral stochastic differential equations. *Applied Mathematics and Computaion* **230** (2014) 639–649.
- [35] Sakthivel, R., Revathi, P. and Ren Y. Existence of solutions for nonlinear fractional stochastic differential equations. *Nonlinear Analysis* **81** (2013) 70–86.
- [36] Slama, A. and Boudaoui, A. Existence of solutions for nonlinear fractional impulsive stochastic differential equations with nonlocal conditions and infinite delay. *International Journal of Differential Equations and Applipations* **13**(4) (2014) 185–201.
- [37] Vinodkumar, A. Existence uniqueness and stability results of impulsive stochastic semilinear functional differential equations with infinite delay. *Journal of Nonlinear Sciences and Applications* **4**(4) (2011) 236–246.
- [38] Wang, J., Feckan, M. and Zhou, Y. On the new concept of solutions and existence results for impulsive fractional evolution equations. *Dynamics of Partial Differential Equations* **8** (4) (2011) 345–361.
- [39] Wang, J. R., Wei, W. and Yang, Y. L. On some impulsive fractional differential equations in Banach space. *Opuscula Mathematica* **30** (4) (2010).
- [40] Yan, Z. and Jia, X. Impulsive problems for fractional partial neutral functional integro-differential inclusions with infinite delay and analytic resolvent operators. *Mediterranean Journal of Mathematics* **11** (2014) 393–428.
- [41] Zhou, Y. and Jiao, F. Existence of mild solutions for fractional neutral evolution equations. *Computers & Mathematics with Applications* **59** (3) (2010) 1063–1077.



## Refinements of Some Pachpatte and Bihari Inequalities on Time Scales

B. Ben Nasser<sup>1</sup>, K. Boukerrioua<sup>2</sup>, M. Defoort<sup>3</sup>, M. Djemai<sup>3\*</sup>,  
and M.A. Hammami<sup>4</sup>

<sup>1</sup> *Computer, Electrical and Mathematical Sciences and Engineering Division (CEMSE), King Abdullah University of Science and Technology (KAUST), Thuwal 23955-6900, Kingdom of Saudi Arabia*

<sup>2</sup> *Department of Mathematics, University of Badji Mokhtar, Annaba, Algeria*

<sup>3</sup> *LAMIH, CNRS, UMR-8201, UVHC, F-59313 Valenciennes, France*

<sup>4</sup> *Department of Mathematics, Faculty of Sciences of Sfax, University of Sfax, P.O. Box 1171, Sfax 3000, Tunisia*

Received: May 6, 2017;    Revised: October 26, 2017

**Abstract:** In this study, some generalizations and refinements of some inequalities of Pachpatte and Bellman-Bihari types are established on arbitrary time domains using the time scale theory. The obtained results unify continuous and discrete inequalities and extend some results known in the literature. The paper ends up with two illustrative examples to highlight the utility of our results.

**Keywords:** *dynamic equations; time scale, Gronwall-Bellman inequality; Bellman-Bihari inequality.*

**Mathematics Subject Classification (2010):** 26D15, 26D20, 39A12.

### 1 Introduction

The Gronwall-Bellman and Bellman-Bihari integral inequalities play important roles in the study of qualitative and quantitative properties of differential equations [1–6]. Similarly, discrete Gronwall and Bihari inequalities have been developed for the analysis of difference equations [7]. New classes of differential and integral equations have been studied using Gronwall-Bellman-Pachpatte inequalities [5, 8, 9]. Recently, the time scale theory, which was introduced in [10], gives a promising direction that unifies continuous

---

\* Corresponding author: [mohamed.djemai@univ-valenciennes.fr](mailto:mohamed.djemai@univ-valenciennes.fr)

and discrete analysis in a consistent way. Using this theory, many works (see, for instance, [11–16]) have investigated new Patchpatte-type and Gronwall-Bihari inequalities for dynamic equations defined on arbitrary time scales.

The aim of this paper is to extend some results on Patchpatte-type and Gronwall-Bihari inequalities for dynamic equations defined on time scales. On the one hand, some Pachpatte-type inequalities, containing in the right-hand side two nonlinear integral terms involving Lipschitz kind functions, are studied. Using elementary analytic methods, we investigate extensions of some continuous and discrete inequalities appearing in [5, 8, 9, 17, 18] to an arbitrary time scale and refine some Pachpatte-type inequalities given in [14, 19–23]. On the other hand, some Bellman-Bihari inequalities on time scale, including two nonlinear integral terms using class  $\mathcal{S}$  or  $\mathcal{T}$  functions, are introduced. Some similar inequalities have been studied for the continuous-time case in [3, 4, 6]. However, there are very few results for Bellman-Bihari inequalities on arbitrary time scales involving class  $\mathcal{S}$  functions (see [24–26]). These inequalities can be applied to analyze qualitative and quantitative properties of integro-differential equations on time scales.

The rest of this paper is as follows. In Section 2, some basics on the time scale theory are recalled. In Section 3, some generalizations of Pachpatte-type inequalities on arbitrary time scale are presented. In Section 4, some new Bellman-Bihari inequalities on time scales are given. In the last section, some illustrative examples are presented to highlight the utility of our results.

## 2 Preliminaries on Time Scale

Let us consider the time scale  $\mathbb{T}$  which is an arbitrary non-empty closed subset of  $\mathbb{R}$ . For  $t \in \mathbb{T}$ , we can define

- the forward jump operator  $\sigma : \mathbb{T} \rightarrow \mathbb{T}$  by  $\sigma(t) := \inf\{s \in \mathbb{T}; s > t\}$ ,
- the backward jump operator  $\rho : \mathbb{T} \rightarrow \mathbb{T}$  by  $\rho(t) := \sup\{s \in \mathbb{T}; s < t\}$ ,
- the graininess function  $\mu : \mathbb{T} \rightarrow \mathbb{R}_+$  by  $\mu(t) := \sigma(t) - t$ .

An element  $t \in \mathbb{T}$  is said to be right-dense if  $\sigma(t) = t$ , left-dense if  $\rho(t) = t$ , left-scattered if  $\rho(t) < t$ . If  $\mathbb{T}$  has a left-scattered maximal element  $m$ , then  $\mathbb{T}^\kappa = \mathbb{T} - \{m\}$  or else  $\mathbb{T}^\kappa = \mathbb{T}$ . For a function  $f : \mathbb{T} \rightarrow \mathbb{R}$  and  $t \in \mathbb{T}^\kappa$ , we define the delta derivative  $f^\Delta(t)$  at  $t$  (provided it exists) such that

$$f^\Delta(t) := \lim_{s \rightarrow t, s \neq \sigma(t)} \frac{f(\sigma(t)) - f(s)}{\sigma(t) - s}.$$

Clearly, it becomes the usual derivative when  $\mathbb{T} = \mathbb{R}$ , i.e.  $f^\Delta(t) = f'(t)$  and the usual forward difference operator  $f^\Delta(t) = \Delta f(t)$ , if  $\mathbb{T} = \mathbb{Z}$ . A function  $F : \mathbb{T} \rightarrow \mathbb{R}$  is called an antiderivative of  $f : \mathbb{T} \rightarrow \mathbb{R}$  provided  $F^\Delta(t) = f(t)$  holds for all  $t \in \mathbb{T}^\kappa$ . The Cauchy integral of  $f$  is defined by

$$\int_a^t f(s) \Delta s = F(t) - F(a) \quad \text{for all } a, t \in \mathbb{T}.$$

A function  $f : \mathbb{T} \rightarrow \mathbb{R}$  is said to be rd-continuous (denoted by  $f \in \mathcal{C}_{rd} := \mathcal{C}_{rd}(\mathbb{T}, \mathbb{R})$ ), provided  $f$  is continuous at every right-dense points in  $\mathbb{T}$  and  $\lim_{s \rightarrow t^-} f(s)$  exists and

is finite at every left-dense point  $t \in \mathbb{T}$ . A function  $f : \mathbb{T} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  is said to be rd-continuous if  $h$  defined by  $h(t) = f(t, x(t))$  is rd-continuous for any continuous function  $x : \mathbb{T} \rightarrow \mathbb{R}^n$ . A rd-continuous function  $f : \mathbb{T} \rightarrow \mathbb{R}$  is said to be regressive (denoted by  $f \in \mathcal{R} := \mathcal{R}(\mathbb{T}, \mathbb{R})$ ) if  $1 + \mu(t)f(t) \neq 0$  for all  $t \in \mathbb{T}^\kappa$ .  $\mathcal{R}^+ := \{f \in \mathcal{R} : 1 + \mu(t)f(t) > 0 \text{ for all } t \in \mathbb{T}\}$  is the set of all positively regressive elements of  $\mathcal{R}$ .

**Theorem 2.1** ([27, 28]) Let  $t_0 \in \mathbb{T}$ ,  $p \in \mathcal{R}$ . The first order linear dynamic equation

$$x^\Delta(t) = p(t)x, \quad x(t_0) = 1$$

has a unique solution on  $\mathbb{T}$  called the exponential function, denoted by  $e_p(t, t_0)$ .

To derive our main results, one must recall the Gronwall's inequality on time scale.

**Theorem 2.2** ([29, Theorem 5.4]). Let  $t_0 \in \mathbb{T}$ ,  $x, f \in \mathcal{C}_{rd}$  and  $p \in \mathcal{R}^+$ . Then

$$x^\Delta(t) \leq p(t)x(t) + f(t) \quad \text{for all } t \in \mathbb{T}^\kappa$$

implies

$$x(t) \leq x(t_0)e_p(t, t_0) + \int_{t_0}^t e_p(t, \sigma(\tau))f(\tau)\Delta\tau \quad \text{for all } t \in \mathbb{T}.$$

**Lemma 2.1** ([27, Theorem 1.117]) Let  $t_0 \in \mathbb{T}^\kappa$  and assume that  $L : \mathbb{T} \times \mathbb{T}^\kappa \rightarrow \mathbb{R}$  is continuous at  $(t, t)$ , where  $t \in \mathbb{T}^\kappa$  with  $t > t_0$ . Also assume that  $L^{\Delta_t}(t, \cdot)$  is rd-continuous on  $[t_0, \sigma(t)]$ .

Suppose that for each  $\varepsilon > 0$  there exists a neighborhood  $U$  of  $t$ , independent of  $\tau \in [t_0, \sigma(t)]$ , such that

$$|L(\sigma(t), \tau) - L(s, \tau) - L^{\Delta_t}(t, \tau)(\sigma(t) - s)| \leq \varepsilon |\sigma(t) - s| \quad \text{for all } s \in U,$$

where  $L^{\Delta_t}$  denotes the derivative of  $L$  with respect to the first variable. Then

$$g(t) := \int_{t_0}^t L(t, \tau)\Delta\tau$$

implies

$$g^\Delta(t) = L(\sigma(t), t) + \int_{t_0}^t L^{\Delta_t}(t, \tau)\Delta\tau.$$

**Lemma 2.2** ([24, Lemma 2.1]) Let  $a, b \in \mathbb{T}$ , and a delta differentiable function  $r : [a, b]_{\mathbb{T}} \rightarrow ]0, \infty[$  with  $r^\Delta(t) \geq 0$  on  $[a, b] \cap \mathbb{T}^\kappa$ . Define

$$G(x) = \int_{x_0}^x \frac{ds}{g(s)}, \quad x > 0, x_0 > 0, \quad (1)$$

where  $g \in C(\mathbb{R}_+, \mathbb{R}_+)$  is positive and nondecreasing on  $]0, \infty[$ . Then, for each  $t \in [a, b]_{\mathbb{T}}$  one has

$$G(r(t)) \leq G(r(a)) + \int_a^t \frac{r^\Delta(\tau)}{g(r(\tau))}\Delta\tau.$$

**Lemma 2.3** ([30, Lemma 2.1]) Assume that  $a \geq 0$ ,  $p \geq q \geq 0$  and  $p \neq 0$ , then

$$a^{\frac{q}{p}} \leq \frac{q}{p} K^{\frac{q-p}{p}} a + \frac{p-q}{p} K^{\frac{q}{p}}, \quad \text{for any } K > 0.$$

**Definition 2.1**

- A nondecreasing continuous function  $g : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is said to belong to class  $\mathcal{F}$  (see [3, Section 3]) if it satisfies the following conditions:
  - ★  $g(x)$  is positive for  $x \geq 0$ ;
  - ★★  $(1/y)g(x) \leq g(x/y)$ , for  $x \geq 0$  and  $y > 0$  (or  $(1/y)g(x) \leq g(x/y)$ , for  $x > 0$  and  $y \geq 1$  (see [1, Section 5] as an equivalent characterization)).
- A nondecreasing continuous function  $g : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is said to belong to class  $\mathcal{S}$  (see [24]) if it satisfies the following conditions:
  - ★  $g(x)$  is positive for  $x > 0$ ;
  - ★★  $(1/y)g(x) \leq g(x/y)$  for  $x \geq 0$  and  $y \geq 1$ .
- A strictly increasing continuous function  $g : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is said to belong to class  $\mathcal{T}$  if it satisfies the following conditions:
  - ★  $g(x)$  is positive for  $x > 0$ ;
  - ★★  $(1/y)g(x) \geq g(x/y)$  for  $x \geq 0$  and  $y \geq 1$ .

**Remark 2.1**

- Any function of class  $\mathcal{F}$  is of class  $\mathcal{S}$ . The converse is not true. For example,  $f(x) = x^\alpha$ ,  $x \in \mathbb{R}_+$ ,  $\alpha \in [0, 1]$ , is of class  $\mathcal{S}$  but is not of class  $\mathcal{F}$ .
- In [25], the authors introduce the class  $\mathcal{F}$  as similar to class  $\mathcal{S}$ , without distinguishing slight difference between these two classes. In [26] class  $\mathcal{S}$  functions are designed by the class  $\mathcal{S}^*$ .

**3 Pachpatte-type Inequalities**

In this section, we derive some new results on Pachpatte-type inequalities, which can be used in the analysis of differential equations on arbitrary time scales. We suppose that  $t \geq t_0$ ,  $t \in \mathbb{T}^\kappa$ .

**Theorem 3.1** Assume that  $u, f \in C_{rd}(\mathbb{T}, \mathbb{R}_+)$  and  $S : \mathbb{T} \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is a rd-continuous function which satisfies

$$0 \leq S(t, x) - S(t, y) \leq R(t, y)(x - y) \quad (2)$$

for  $t \in \mathbb{T}$ ,  $x \geq y \geq 0$  and

$$S^{\Delta_t}(t, 0) \geq 0, \quad R^{\Delta_t}(t, 0) \geq 0, \quad (3)$$

for  $t \in \mathbb{T}^\kappa$ , where  $R : \mathbb{T} \times \mathbb{R}_+ \rightarrow \mathbb{R}_+^*$  is a rd-continuous function. If  $L(t, s)$  is defined as in Lemma 2.1 so that  $L(t, s) \geq 0$  and  $L^{\Delta_t}(t, s) \geq 0$  for  $t, s \in \mathbb{T}$  with  $s \leq t$ , then

$$u(t) \leq c + \int_{t_0}^t f(\eta) \left( S(\eta, u(\eta)) + \int_{t_0}^\eta L(\eta, \tau) S(\tau, u(\tau)) \Delta\tau \right) \Delta\eta, \quad (4)$$

with  $c \geq 0$ , for all  $t \in \mathbb{T}^\kappa$ , implies

$$u(t) \leq c + \int_{t_0}^t f(\eta) \left( \left[ S(t_0, 0) + R(t_0, 0)c \right] e_{A_1^*}(\eta, t_0) + \int_{t_0}^\eta e_{A_1^*}(\eta, \sigma(\tau)) B_1^*(\tau) \Delta\tau \right) \Delta\eta, \quad (5)$$

for all  $t \in \mathbb{T}^\kappa$  with

$$A_1^*(t) = R(\sigma(t), 0)f(t) + \frac{R^{\Delta_t}(t, 0)}{R(t, 0)} + L(\sigma(t), t) + \int_{t_0}^t L^{\Delta_t}(t, \tau)\Delta\tau, \quad t \in \mathbb{T}^\kappa,$$

and

$$B_1^*(t) = S^{\Delta_t}(t, 0) + L(\sigma(t), t)S(t, 0) + \int_{t_0}^t L^{\Delta_t}(t, \tau)S(\tau, 0)\Delta\tau, \quad t \in \mathbb{T}^\kappa.$$

**Proof.** Let us set function  $z(t)$  by the right-hand side of (4). The Delta-derivative of  $z$  satisfies the following inequality

$$z^\Delta(t) \leq f(t)v(t)$$

with

$$v(t) = S(t, 0) + R(t, 0)z(t) + \int_{t_0}^t L(t, \tau)[S(\tau, 0) + R(\tau, 0)z(\tau)]\Delta\tau.$$

Using Lemma 2.1, one can easily obtain

$$\begin{aligned} v^\Delta(t) &= S^{\Delta_t}(t, 0) + R(\sigma(t), 0)z^\Delta(t) + R^{\Delta_t}(t, 0)z(t) + \\ &\quad L(\sigma(t), t)[S(t, 0) + R(t, 0)z(t)] + \int_{t_0}^t L^{\Delta_t}(t, \tau)[S(\tau, 0) + R(\tau, 0)z(\tau)]\Delta\tau. \end{aligned}$$

It is easy to see that  $v(t)$  is nonnegative nondecreasing function. Further, one gets

$$v^\Delta(t) \leq A_1^*(t)v(t) + B_1^*(t).$$

Theorem 2.2 yields the following inequality

$$v(t) \leq [S(t_0, 0) + R(t_0, 0)c]e_{A_1^*}(t, t_0) + \int_{t_0}^t e_{A_1^*}(t, \sigma(\tau))B_1^*(\tau)\Delta\tau.$$

Hence, one can deduce inequality (5).

For special forms of function  $S$  one can see that the proposed Pachpatte-type inequality is a generalization of some existing results.

**Remark 3.1** Assume that  $S(t, u(t)) = u(t)$ .

- Let  $t_0 = 0$ . If  $\mathbb{T} = \mathbb{R}$ , Theorem 3.1 implies Theorem 2.1( $a_1$ ) of [9]. If  $\mathbb{T} = \mathbb{N}_0$ , Theorem 3.1 reduces to Theorem 2.5( $c_1$ ) in [9]. If  $\mathbb{T} = \mathbb{N}_0$  and  $L(t, s) = l(s)$  with  $l(\cdot)$  being a nonnegative function, then Theorem 3.1 validates Theorem 1.4.1 of [8].
- For an arbitrary time scale  $\mathbb{T}$ , one can easily obtain [21, Theorem 3.1]. Moreover, when  $\mathbb{T} = \mathbb{T}^\kappa$ , Theorem 3.1 includes [14, Lemma] and [19, Corrolary 4.9], if  $L(t, s) = l(s)$  with  $l(\cdot)$  being a nonnegative rd-continuous function.
- The inequality given in Theorem 3.1, when  $\mathbb{T} = \mathbb{T}^\kappa$ , solves the integral approximation of Theorem 3.2 in [22] satisfied by an rd-continuous nonnegative function  $u(\cdot)$  designated to bound the solution of the considered nonlinear integro-differential equation. In this case,  $L(t, \cdot) = f(\cdot)$  is Lebesgue  $\Delta$ -integrable function ( $f \in L_{\mathbb{T}}^1 := L_{\mathbb{T}}^1(\mathbb{T}, \mathbb{R}_+)$ ), for more information about the Lebesgue  $\Delta$ -integration see [31]), we obtain  $u(t) \leq Mc$ , for all  $t \geq t_0$  with  $M = \frac{1}{2}(1 + \exp(2\|f\|_{L_{\mathbb{T}}^1}))$ .

- Suppose that  $L(t, \cdot) = g(\cdot)$ , where  $g$  is a nonnegative rd-continuous function, if  $\mathbb{T}$  is an arbitrary time scale, with Theorem 3.1 one can obtain the inequality proved in [26, Theorem 1].

As an extension of Theorem 3.1, one can derive an integral inequality involving positive real powers. In the following, it is supposed that  $p \neq 0, p, q, r$  are real constants such that  $0 \leq q, r \leq p$ .

**Proposition 3.1** *Assume that all conditions of Theorem 3.1 are satisfied except for inequalities (3). Then*

$$u^p(t) \leq c + \int_{t_0}^t f(\eta) \left( u^q(\eta) + \int_{t_0}^\eta L(\eta, \tau) S(\tau, u^r(\tau)) \Delta\tau \right) \Delta\eta, \quad (6)$$

for all  $t \in \mathbb{T}^\kappa$ , implies

$$u(t) \leq \left( c + \int_{t_0}^t f(\eta) \left[ \left( \frac{q}{p} K^{\frac{q-p}{p}} c + \frac{p-q}{p} K^{\frac{q}{p}} \right) e_{A_2^*}(\eta, t_0) + \int_{t_0}^\eta e_{A_2^*}(\eta, \sigma(\tau)) B_2^*(\tau) \Delta\tau \right] \Delta\eta \right)^{\frac{1}{p}},$$

for all  $t \in \mathbb{T}^\kappa$ , for any  $K > 0$  with

$$\begin{aligned} A_2^*(t) &= \frac{q}{p} K^{\frac{q-p}{p}} f(t) + \frac{r}{q} K^{\frac{r-q}{p}} \left( L(\sigma(t), t) R\left(t, \frac{p-r}{p} K^{\frac{r}{p}}\right) \right. \\ &\quad \left. + \int_{t_0}^t L^{\Delta_t}(t, \tau) R\left(\tau, \frac{p-r}{p} K^{\frac{r}{p}}\right) \Delta\tau \right), \quad t \in \mathbb{T}^\kappa \end{aligned}$$

and

$$B_2^*(t) = L(\sigma(t), t) S\left(t, \frac{p-r}{p} K^{\frac{r}{p}}\right) + \int_{t_0}^t L^{\Delta_t}(t, \tau) S\left(\tau, \frac{p-r}{p} K^{\frac{r}{p}}\right) \Delta\tau, \quad t \in \mathbb{T}^\kappa.$$

**Proof.** The proof is similar to the proof of Theorem 3.1. Hence, the details are omitted.

One can highlight that the last result generalizes some existing works as follows.

### Remark 3.2

- The result of Proposition 3.1 holds for any arbitrary time scales. Setting  $S(t, u(t)) = u(t)$ , we see that the obtained inequality is as seen in Theorem 3.1 in [20].
- Proposition 3.1 can be viewed as a generalization of some results on some particular time scales. For example, letting  $S(t, u(t)) = u(t)$  and  $\mathbb{T} = \mathbb{R}$ . If  $K = 1$ , one can easily derive Theorem 3.1 in [17].

One can extend the result of Theorem 3.1, changing the nonnegative constant  $c$  on the right-side of (4) by an increasing positive function.

**Proposition 3.2** *Assume that all conditions of Theorem 3.1 are satisfied except for inequalities (3). Let  $a(\cdot)$  be a nondecreasing function in  $C_{rd}(\mathbb{T}, \mathbb{R}_+^*)$ . Suppose that*

$$S^{\Delta_t}(t, 0) a(t) \geq S(t, 0) a^{\Delta}(t), \quad R^{\Delta_t}(t, 0) \geq 0 \quad \text{for all } \mathbb{T}^\kappa.$$

Then,

$$u(t) \leq a(t) + \int_{t_0}^t f(\eta) \left( S(\eta, u(\eta)) + \int_{t_0}^{\eta} L(\eta, \tau) S(\tau, u(\tau)) \Delta\tau \right) \Delta\eta, \quad (7)$$

for all  $t \in \mathbb{T}^{\kappa}$ , implies

$$u(t) \leq a(t) \left( 1 + \int_{t_0}^t f(\eta) \left[ [R(t_0, 0) + M(t_0, 0)] e_{A_1^*}(\eta, t_0) + \int_{t_0}^{\eta} e_{A_1^*}(\eta, \sigma(\tau)) B_3^*(\tau) \Delta\tau \right] \Delta\eta \right),$$

for all  $t \in \mathbb{T}^{\kappa}$ , with

$$B_3^*(t) = M^{\Delta_i}(t, 0) + L(\sigma(t), t) M(t, 0) + \int_{t_0}^t L^{\Delta_i}(t, \tau) M(\tau, 0) \Delta\tau, \quad t \in \mathbb{T}^{\kappa},$$

and

$$M(t, z) = \frac{1}{a(t)} S(t, a(t)z), \quad z \geq 0, \quad t \in \mathbb{T}.$$

**Proof.** Setting  $w(t) = \frac{u(t)}{a(t)}$ , one can reformulate (7) as

$$w(t) \leq 1 + \int_{t_0}^t f(\eta) \left( M(\eta, w(\eta)) + \int_{t_0}^{\eta} L(\eta, \tau) M(\tau, w(\tau)) \Delta\tau \right) \Delta\eta.$$

Clearly,  $M$  verifies relation (2), i.e.

$$M(t, x) - M(t, y) \leq R_1(t, y)(x - y), \quad x \geq y \geq 0, \quad t \in \mathbb{T},$$

where  $R_1(t, y) = R(t, a(t)y)$ . From our hypothesis we see that  $S$  and  $R_1$  verify relation (3). Using Theorem 3.1, it yields

$$w(t) \leq 1 + \int_{t_0}^t f(\eta) \left[ [R(t_0, 0) + M(t_0, 0)] e_{A_1^*}(\eta, t_0) + \int_{t_0}^{\eta} e_{A_1^*}(\eta, \sigma(\tau)) B_3^*(\tau) \Delta\tau \right] \Delta\eta.$$

This concludes the proof.

This proposition generalizes some well known inequalities.

**Remark 3.3** Assume that  $S(t, u(t)) = u(t)$ .

- Take a special case in references [8, 21, 26] with  $a(\cdot) \neq 0$ .  
For an arbitrary time scale  $\mathbb{T}$ , inequalities in Proposition 3.2 reduce to [21, Theorem 3.2]. If  $L(t, s) = g(s)$  with  $g \in \mathcal{C}_{rd}(\mathbb{T}, \mathbb{R}_+)$ , then Proposition 3.2 includes [26, Theorem 2 (a)]. For  $\mathbb{T} = \mathbb{N}_0$  and  $t_0 = 0$ , the result [8, Theorem 1.4.2] is a particular case of Proposition 3.2 where  $L(t, s) = c(s)$  with  $c(\cdot)$  being a nonnegative function defined on  $\mathbb{N}_0$ .
- If  $\mathbb{T} = \mathbb{R}_+$  and  $t_0 = 0$ , then Proposition 3.2 generalizes [5, Theorem 1.7.4] when  $L(t, s) = g(s)$  with  $g(\cdot)$  being a nonnegative continuous function on  $\mathbb{R}_+$ .



#### 4 Bihari-type Inequalities

In this part, some new Gronwall-Bellman-Bihari type inequalities, containing in the right-hand side two nonlinear integral terms involving class  $\mathcal{S}$  or  $\mathcal{T}$  functions, are introduced. These inequalities can be applied to analyze qualitative and quantitative properties of integro-differential equations on time scales. In this section it is assumed that  $t \geq t_0$ ,  $t \in \mathbb{T}$ .

Let us begin with the following inequality which will be used in the proof of the next results.

**Theorem 4.1** *Let us consider  $u, f \in C_{rd}(\mathbb{T}, \mathbb{R}_+)$  and  $c$  is a positive constant. Let  $g : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  be a continuous function which is nondecreasing positive on  $]0, +\infty[$ ,  $L : \mathbb{T} \times \mathbb{T} \rightarrow \mathbb{R}_+$  be a rd-continuous function and  $G$  be given by (1). If*

$$u(t) \leq c + \int_{t_0}^t f(\eta) \left( g(u(\eta)) + \int_{t_0}^{\eta} L(\eta, \tau) g(u(\tau)) \Delta\tau \right) \Delta\eta, \quad (8)$$

for  $t \in \mathbb{T}$ , then for all  $t \in \mathbb{T}$  satisfying

$$G(c) + \int_{t_0}^t f(\eta) \left[ 1 + \int_{t_0}^{\eta} L(\eta, \tau) \Delta\tau \right] \Delta\eta \in \text{Dom}(G^{-1}),$$

we have

$$u(t) \leq G^{-1} \left( G(c) + \int_{t_0}^t f(\eta) \left[ 1 + \int_{t_0}^{\eta} L(\eta, \tau) \Delta\tau \right] \Delta\eta \right),$$

where  $G^{-1}$  is the inverse function of  $G$ .

**Proof.** Let us define function  $z(t)$  by the right-hand side of (8). Then, we have  $z(t_0) = c$  and

$$u(t) \leq z(t).$$

As  $g$  is a nondecreasing function, the Delta-derivative of  $z(t)$  satisfies the following inequality

$$z^\Delta(t) \leq f(t)g(z(t)) \left[ 1 + \int_{t_0}^t L(t, \tau) \Delta\tau \right].$$

Dividing both sides by  $g(z(t))$ , one can get

$$\frac{z^\Delta(t)}{g(z(t))} \leq f(t) \left[ 1 + \int_{t_0}^t L(t, \tau) \Delta\tau \right].$$

Since  $z(t)$  is nondecreasing, from Lemma 2.2, one can obtain

$$G(z(t)) \leq G(c) + \int_{t_0}^t f(\eta) \left[ 1 + \int_{t_0}^{\eta} L(\eta, \tau) \Delta\tau \right] \Delta\eta.$$

Further

$$z(t) \leq G^{-1} \left( G(c) + \int_{t_0}^t f(\eta) \left[ 1 + \int_{t_0}^{\eta} L(\eta, \tau) \Delta\tau \right] \Delta\eta \right).$$

This concludes the proof.

In the following, some results which can be considered as some extensions of Theorem 4.1 are investigated. The next corollary allows us to get a relaxed integral bound of an unknown function using the image of a continuous increasing function.

**Corollary 4.1** *Let us consider  $u, f \in \mathcal{C}_{rd}(\mathbb{T}, \mathbb{R}_+)$  and  $c$  is a positive constant. Let  $g : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  be a continuous strictly increasing function with  $g([0, +\infty[) = [0, +\infty[$  and  $L : \mathbb{T} \times \mathbb{T} \rightarrow \mathbb{R}_+$  be a rd-continuous function. Define*

$$F(y) = \int_{y_0}^y \frac{1}{g^{-1}(s)} ds, \quad y > 0, \quad y_0 > 0, \quad (9)$$

where  $g^{-1}$  is the inverse function of  $g$ . If

$$g(u(t)) \leq c + \int_{t_0}^t f(\eta) \left( u(\eta) + \int_{t_0}^{\eta} L(\eta, \tau) u(\tau) \Delta\tau \right) \Delta\eta, \quad (10)$$

for  $t \in \mathbb{T}$ , then for all  $t \in \mathbb{T}$  satisfying

$$F(c) + \int_{t_0}^t f(\eta) \left( 1 + \int_{t_0}^{\eta} L(\eta, \tau) \Delta\tau \right) \Delta\eta \in \text{Dom}(F^{-1}),$$

and

$$F^{-1} \left[ F(c) + \int_{t_0}^t f(\eta) \left( 1 + \int_{t_0}^{\eta} L(\eta, \tau) \Delta\tau \right) \Delta\eta \right] \in \text{Dom}(g^{-1}),$$

we get

$$u(t) \leq g^{-1} \left( F^{-1} \left[ F(c) + \int_{t_0}^t f(\eta) \left( 1 + \int_{t_0}^{\eta} L(\eta, \tau) \Delta\tau \right) \Delta\eta \right] \right),$$

where  $F^{-1}$  is the inverse function of  $F$ .

**Proof.** Let us define function  $z(t)$  by the right-hand side of (10). Using the properties of  $g$ , one can get

$$z(t) \leq c + \int_{t_0}^t f(\eta) \left( g^{-1}(z(\eta)) + \int_{t_0}^{\eta} L(\eta, \tau) g^{-1}(z(\tau)) \Delta\tau \right) \Delta\eta.$$

Applying Theorem 4.1, one can obtain

$$z(t) \leq F^{-1} \left[ F(c) + \int_{t_0}^t f(\eta) \left( 1 + \int_{t_0}^{\eta} L(\eta, \tau) \Delta\tau \right) \Delta\eta \right].$$

Inspired by the concept of inequality in [26, Theorem 7], one can derive a Bihari type bound of an integral inequality in the next corollary using functions of class  $\mathcal{S}$  (introduced in Section 2).

**Corollary 4.2** *Assume that  $u, f \in \mathcal{C}_{rd}(\mathbb{T}, \mathbb{R}_+)$ ,  $a : \mathbb{T} \rightarrow \mathbb{R}_+$  is a rd-continuous non-decreasing function. Let  $g : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  be of class  $\mathcal{S}$ ,  $L : \mathbb{T} \times \mathbb{T} \rightarrow \mathbb{R}_+$  be a rd-continuous function and  $G$  be defined by (1). If*

$$u(t) \leq a(t) + \int_{t_0}^t f(\eta) \left( g(u(\eta)) + \int_{t_0}^{\eta} L(\eta, \tau) g(u(\tau)) \Delta\tau \right) \Delta\eta, \quad (11)$$

for  $t \in \mathbb{T}$ , then for all  $t \in \mathbb{T}$  satisfying

$$G(1) + \int_{t_0}^t f(\eta) \left[ 1 + \int_{t_0}^{\eta} L(\eta, \tau) \Delta\tau \right] \Delta\eta \in \text{Dom}(G^{-1}),$$

we obtain

$$u(t) \leq \max(a(t), 1)G^{-1} \left( G(1) + \int_{t_0}^t f(\eta) \left[ 1 + \int_{t_0}^{\eta} L(\eta, \tau) \Delta \tau \right] \Delta \eta \right).$$

**Proof.** Define function  $b$  by  $b(t) = \max(a(t), 1)$ . Then, from (11) we get

$$\frac{u(t)}{b(t)} \leq 1 + \int_{t_0}^t \frac{f(\eta)}{b(\eta)} \left( g(u(\eta)) + \int_{t_0}^{\eta} L(\eta, \tau) g(u(\tau)) \Delta \tau \right) \Delta \eta.$$

Set  $w(t) := \frac{u(t)}{b(t)}$ . As  $g \in \mathcal{S}$ , we deduce the following inequality

$$w(t) \leq 1 + \int_{t_0}^t f(\eta) \left( g(w(\eta)) + \int_{t_0}^{\eta} L(\eta, \tau) g(w(\tau)) \Delta \tau \right) \Delta \eta.$$

A suitable application of Theorem 4.1 gives

$$w(t) \leq G^{-1} \left( G(1) + \int_{t_0}^t f(\eta) \left[ 1 + \int_{t_0}^{\eta} L(\eta, \tau) \Delta \tau \right] \Delta \eta \right).$$

It is equivalent to the desired inequality, in view of the fact that  $u(t) = w(t)b(t)$ .

By a similar reasoning as in Corollary 4.1, an integral approximation of an unknown function using its image by a function of class  $\mathcal{T}$  is derived in the next corollary.

**Corollary 4.3** *Let  $u, a, f, L$  be as defined in Corollary 4.2 and  $F$  be as given in (9). Suppose that  $g \in \mathcal{T}$  with  $g([0, +\infty]) = [0, +\infty]$ . If*

$$g(u(t)) \leq a(t) + \int_{t_0}^t f(\eta) \left( u(\eta) + \int_{t_0}^{\eta} L(\eta, \tau) u(\tau) \Delta \tau \right) \Delta \eta, \quad (12)$$

then for all  $t \in \mathbb{T}$  satisfying

$$F(1) + \int_{t_0}^t f(\eta) \left( 1 + \int_{t_0}^{\eta} L(\eta, \tau) \Delta s \right) \Delta \eta \in \text{Dom}(F^{-1}),$$

and

$$F^{-1} \left[ F(1) + \int_{t_0}^t f(\eta) \left( 1 + \int_{t_0}^{\eta} L(\eta, \tau) \Delta \tau \right) \Delta \eta \right] \in \text{Dom}(g^{-1}),$$

we have

$$u(t) \leq \max(a(t), 1)g^{-1} \left( F^{-1} \left[ F(1) + \int_{t_0}^t f(\eta) \left( 1 + \int_{t_0}^{\eta} L(\eta, \tau) \Delta \tau \right) \Delta \eta \right] \right).$$

**Proof.** Take  $b(t) = \max(a(t), 1)$ , then (12) can be written as

$$\frac{g(u(t))}{b(t)} \leq 1 + \int_{t_0}^t \frac{f(\eta)}{b(\eta)} \left( u(\eta) + \int_{t_0}^{\eta} L(\eta, \tau) u(\tau) \Delta s \right) \Delta \eta. \quad (13)$$

Set  $w(t) = \frac{u(t)}{b(t)}$  on  $\mathbb{T}$ . Taking into account the fact that  $g$  is a nondecreasing function of class  $\mathcal{T}$  and  $g([0, +\infty]) = [0, +\infty]$ , from (13), one can get

$$g(w(t)) \leq 1 + \int_{t_0}^t f(\eta) \left( w(\eta) + \int_{t_0}^{\eta} L(\eta, \tau) w(\tau) \Delta \tau \right) \Delta \eta.$$

Applying Corollary 4.1 the requested inequality is obtained.

## 5 Illustrative Examples

In this section, we apply some inequalities obtained in the previous sections to investigate certain properties of the solutions of dynamic equations on arbitrary time scales.

**Example 5.1** Using a straightforward extension of Theorem 3.1, let us discuss the boundedness behavior of the solution of the nonlinear dynamic equation defined as:

$$\begin{cases} (x^p(t))^\Delta = P\left(t, S(t, x^q(\sigma(t))), \int_{t_0}^t H(t, s, S(s, x^r(s))) \Delta s\right), & t_0, t \in \mathbb{T}^\kappa, \\ x^p(t_0) = c, \end{cases}$$

where  $t \geq t_0$ ,  $c \neq 0$  a real constant,  $P : \mathbb{T} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ ,  $H : \mathbb{T} \times \mathbb{T} \times \mathbb{R} \rightarrow \mathbb{R}$  are rd-continuous functions and  $p, q$  and  $r$  are real constants such that  $p \neq 0$ ,  $0 \leq q, r \leq p$ . We shall assume that the proposed initial value problem has a unique solution  $x(t)$ . We also consider that functions  $P$  and  $H$  satisfy

$$|P(t, U, V)| \leq f(t)(|U| + |V|), \quad t \in \mathbb{T}, \quad U, V \in \mathbb{R}, \quad (14)$$

$$|H(t, s, U)| \leq L(t, s)|U|, \quad t, s \in \mathbb{T}, \quad U \in \mathbb{R}, \quad (15)$$

where  $f$  and  $L$  are as mentioned in Theorem 3.1. Let us assume that functions  $S, R : \mathbb{T} \times \mathbb{R} \rightarrow \mathbb{R}_+$  satisfy the following properties

$$S(t, y) \leq S(t, x), \quad t \in \mathbb{T}, \quad y \leq x, \quad x, y \in \mathbb{R}, \quad (16)$$

$$S(t, x) - S(t, y) \leq R(t, y)(x - y), \quad t \in \mathbb{T}, \quad 0 \leq y \leq x. \quad (17)$$

Suppose that there exists  $K > 0$  such that

$$R^{\Delta_t}\left(t, \frac{p-q}{p}K^{\frac{q}{p}}\right) \geq 0, \quad S^{\Delta_t}\left(t, \frac{p-q}{p}K^{\frac{q}{p}}\right) \geq 0, \quad (18)$$

for all  $t \in \mathbb{T}^\kappa$  and  $-\frac{q}{p}K^{\frac{q-p}{p}}R\left(., \frac{p-q}{p}K^{\frac{q}{p}}\right)f(.) \in \mathcal{R}^+$ .

Clearly, the solution  $x(t)$  of the system under consideration satisfies the following integral equation

$$x^p(t) = c + \int_{t_0}^t P\left(\eta, S(\eta, x^q(\sigma(\eta))), \int_{t_0}^\eta H(\eta, \tau, S(\tau, x^r(\tau))) \Delta \tau\right) \Delta \eta. \quad (19)$$

It follows from relations (14)-(19) that

$$|x(t)|^p \leq |c| + \int_{t_0}^t f(\eta) \left( S(\eta, |x(\sigma(\eta))|^q) + \int_{t_0}^\eta L(\eta, \tau) S(\tau, |x(\tau)|^r) \Delta \tau \right) \Delta \eta.$$

Then, following a similar approach as in Theorem 3.1, one can easily obtain

$$\begin{aligned} |x(t)| &\leq \left( |c| + \int_{t_0}^t f_\mu(\eta) \left[ \left( \frac{q}{p} K^{\frac{q-p}{p}} R\left(t_0, \frac{p-q}{p} K^{\frac{q}{p}}\right) |c| \right. \right. \right. \\ &\quad \left. \left. \left. + S\left(t_0, \frac{p-q}{p} K^{\frac{q}{p}}\right) \right) e_{A_4^*}(\eta, t_0) + \int_{t_0}^\eta e_{A_4^*}(\eta, \sigma(\tau)) B_4^*(\tau) \Delta \tau \right] \Delta \eta \right)^{\frac{1}{p}}, \end{aligned}$$

where

$$\begin{aligned} f_{\mu}(t) &= \frac{f(t)}{1 - \mu(t) \frac{q}{p} K^{\frac{q-p}{p}} R\left(t, \frac{p-q}{p} K^{\frac{q}{p}}\right) f(t)}, \\ A_4^*(t) &= \frac{q}{p} K^{\frac{q-p}{p}} R\left(\sigma(t), \frac{p-q}{p} K^{\frac{q}{p}}\right) f_{\mu}(t) + \frac{R^{\Delta_t}\left(t, \frac{p-q}{p} K^{\frac{q}{p}}\right)}{R\left(t, \frac{p-q}{p} K^{\frac{q}{p}}\right)} \\ &\quad + \frac{r}{q} K^{\frac{r-q}{p}} \left( L(\sigma(t), t) \frac{R\left(t, \frac{p-r}{p} K^{\frac{r}{p}}\right)}{R\left(t, \frac{p-q}{p} K^{\frac{q}{p}}\right)} + \int_{t_0}^t L^{\Delta_t}(t, \tau) \frac{R\left(\tau, \frac{p-r}{p} K^{\frac{r}{p}}\right)}{R\left(\tau, \frac{p-q}{p} K^{\frac{q}{p}}\right)} \Delta\tau \right) \end{aligned}$$

and

$$B_4^*(t) = S^{\Delta_t}\left(t, \frac{p-q}{p} K^{\frac{q}{p}}\right) + S\left(t, \frac{p-r}{p} K^{\frac{r}{p}}\right) L(\sigma(t), t) + \int_{t_0}^t L^{\Delta_t}(t, \tau) S\left(\tau, \frac{p-r}{p} K^{\frac{r}{p}}\right) \Delta\tau.$$

In the following example, applying Theorem 4.1, an integral approximation of the solution of a dynamical system is presented below.

**Example 5.2** Let us consider the following initial value problem on an arbitrary time scale

$$\begin{cases} x^{\Delta}(t) = f(t) \left( g(x(t)) + \int_{t_0}^t L(t, \tau) g(x(\tau)) \Delta\tau \right) & t_0, t \in \mathbb{T}, \\ x(t_0) = c, \end{cases}$$

where  $t \geq t_0$ ,  $c \neq 0$  a real constant,  $f, L$  are as defined in Theorem 4.1 and  $g: \mathbb{R} \rightarrow \mathbb{R}_+$  a continuous function and nondecreasing positive on  $\mathbb{R}^*$ . Assume that  $x(t)$  is the unique solution of the system under investigation, then it can be expressed as

$$x(t) = c + \int_{t_0}^t f(\eta) \left( g(x(\eta)) + \int_{t_0}^{\eta} L(\eta, \tau) g(x(\tau)) \Delta\tau \right) \Delta\eta.$$

Further,

$$|x(t)| \leq |c| + \int_{t_0}^t f(\eta) \left( g(|x(\eta)|) + \int_{t_0}^{\eta} L(\eta, \tau) g(|x(\tau)|) \Delta\tau \right) \Delta\eta.$$

Applying Theorem 4.1, one can obtain

$$|x(t)| \leq G^{-1} \left( G(|c|) + \int_{t_0}^t f(\eta) \left( 1 + \int_{t_0}^{\eta} L(\eta, \tau) \Delta\tau \right) \Delta\eta \right).$$

where  $G$ , as given by (1), is such that

$$G(|c|) + \int_{t_0}^t f(\eta) \left( 1 + \int_{t_0}^{\eta} L(\eta, \tau) \Delta\tau \right) \Delta\eta \in \text{Dom}(G^{-1}).$$

## 6 Conclusion

In this work, some new inequalities of Pachpatte and Bellman-Bihari types were derived on arbitrary time scales. As discussed in the paper, they can be thought of as generalizations and refinements of many existing results. These inequalities help us in the study of some classes of integral and integro-differential equations. They can be used in the stability analysis of some classes of dynamical systems on time scales.

## Acknowledgment

This work has been supported by the European Community, the Regional Delegation for Research and Technology, the Haut de France Region and the National Center for Scientific Research under the projects ELSAT VUMOPE and the UVHC BI-CFNes.

## References

- [1] Beesack, P. R. On Lakshmikantham's comparison method for Gronwall inequalities. *Ann. Polon. Math.* **35** (2) (1977/78), 187–222.
- [2] Bihari, I. A generalization of a lemma of Bellman and its application to uniqueness problems of differential equations. *Acta Math. Acad. Sci. Hungar.* **7** (1956) 81–94.
- [3] Dhongade, U. D. and Deo, S. G. Some generalizations of Bellman-Bihari integral inequalities. *J. Math. Anal. Appl.* **44** (1973) 218–226.
- [4] Oguntuase, J. A. On integral inequalities of Gronwall-Bellman-Bihari type in several variables. *JIPAM. J. Inequal. Pure Appl. Math.* **1** (2) (2000) Article 20, 1–7.
- [5] Pachpatte, B. G. *Inequalities for Differential and Integral Equations*. Mathematics in Science and Engineering, 197. Academic Press, Inc., San Diego, CA, 1998.
- [6] Beesack, P. R. On some Gronwall-type integral inequalities in  $n$  independent variables. *J. Math. Anal. Appl.* **100** (2) (1984) 393–408.
- [7] Qin, Y. *Integral and Discrete Inequalities and Their Applications*, Vol. 2. Springer International Publishing AG, Birkhäuser, 2016.
- [8] Pachpatte, B. G. *Inequalities for Finite Difference Equations*. Monographs and Textbooks in Pure and Applied Mathematics, 247. Marcel Dekker, Inc., New York, 2002.
- [9] Pachpatte, B. G. Bounds on certain integral inequalities. *JIPAM. J. Inequal. Pure Appl. Math.* **3** (3) (2002), Article 47, 1–10.
- [10] Aulbach, B. and Hilger, S. A unified approach to continuous and discrete dynamics. *Qualitative theory of differential equations (Szeged, 1988)*, *Colloq. Math. Soc. János Bolyai*, **53**, North-Holland, Amsterdam, 1990, 37–56.
- [11] Ben Nasser, B., Boukerrioua, K. and Hammami, M. A. On Stability and Stabilization of Perturbed Time Scale Systems with Gronwall Inequalities. *Journal of Mathematical Physics, Analysis, Geometry* **11** (3) (2015) 207–235.
- [12] Du, L. and Xu, R. Some new Pachpatte type inequalities on time scales and their applications. *J. Math. Inequal.* **6** (2) (2012) 229–240.
- [13] Li, W. N. Some Pachpatte type inequalities on time scales. *Comput. Math. Appl.* **57** (2) (2009) 275–282.
- [14] Pachpatte, D. P. On a nonlinear dynamic integrodifferential equation on time scales. *J. Appl. Anal.* **16** (2) (2010) 279–294.
- [15] Saker, S. H. Some nonlinear dynamic inequalities on time scales. *Math. Inequal. Appl.* **14** (3) (2011) 633–645.
- [16] Sun, Y. and Hassan, T. Some nonlinear dynamic integral inequalities on time scales. *Appl. Math. Comput.* **220** (2013) 221–225.
- [17] Boukerrioua, K. Note on some nonlinear integral inequalities and applications to differential equations. *Int. J. Differ. Equ.* (2011) Art. ID 456216, 1–15.
- [18] Boukerrioua, K. and Guezane-Lakoud, A. Some nonlinear integral inequalities arising in differential equations. *Electron. J. Differential Equations* (2008) Article 80, 1–6.

- [19] Akin-Bohner, E., Bohner, M. and Akin, F. Pachpatte inequalities on time scales. *JIPAM. J. Inequal. Pure Appl. Math.* **6**(1) (2005) Article 6, 1–23.
- [20] Boukerrioua, K. Note on some nonlinear integral inequalities on time scales and applications to dynamic equations. *J. Adv. Res. Appl. Math.* **5**(2) (2013) 1–12.
- [21] Li, W. N. and Sheng, W. Some Gronwall type inequalities on time scales. *J. Math. Inequal.* **4**(1) (2010) 67–76.
- [22] Liu, G., Xiang, X. and Peng, Y. Nonlinear integro-differential equations and optimal control problems on time scales. *Comput. Math. Appl.* **61**(2) (2011) 155–169.
- [23] Pachpatte, D. B. On approximate solutions of a Volterra type integrodifferential equation on time scales. *Int. J. Math. Anal.* **4**(34) (2010) 1651–1659.
- [24] Ferreira, R. A. C. and Torres, D. F. M. Generalizations of Gronwall-Bihari inequalities on time scales. *J. Difference Equ. Appl.* **15**(6) (2009) 529–539.
- [25] Ma, Q.-H., Wang, J.-W., Ke, X.-H. and Pecaric, J. On the boundedness of a class of nonlinear dynamic equations of second order. *Appl. Math. Lett.* **26**(11) (2013) 1099–1105.
- [26] Wong, F.-H., Yeh, C.-C. and Hong, C.-H. Gronwall inequalities on time scales. *Math. Inequal. Appl.* **9**(1) (2006) 75–86.
- [27] Bohner, M. and Peterson, A. *Dynamic Equations on Time Scales: An Introduction with Applications*. Birkhäuser Boston, Inc., Boston, MA, 2001.
- [28] Bohner, M. and Peterson, A. *Advances in dynamic equations on time scales*. Birkhäuser Boston Inc., Boston, MA, 2003.
- [29] Agarwal, R., Bohner, M. and Peterson, A. Inequalities on time scales: a survey. *Math. Inequal. Appl.* **4**(4) (2001) 535–557.
- [30] Jiang, F. and Meng, F. Explicit bounds on some new nonlinear integral inequalities with delay. *J. Comput. Appl. Math.* **205**(1) (2007) 479–486.
- [31] Cabada, A. and Vivero, D. R. Expression of the Lebesgue  $\Delta$ -integral on time scales as a usual Lebesgue integral: application to the calculus of  $\Delta$ -antiderivatives. *Math. Comput. Modelling* **43**(1,2) (2006) 194–207.



# Boundedness of the New Modified Hyperchaotic Pan System

Samia Rezzag<sup>\*</sup>

*Department of Mathematics and Informatics,  
University Larbi Ben M'hidi, 04000, Oum-El-Bouaghi, Algeria*

Received: May 25, 2016;    Revised: October 27, 2017

**Abstract:** In this paper, we attempt to investigate the ultimate bound and positively invariant set for the new modified hyperchaotic Pan system using a technique combining the generalized Lyapunov function theory and optimization. For this system, we derive a four-dimensional ellipsoidal ultimate bound and positively invariant set. Furthermore, the two-dimensional parabolic ultimate bound with respect to  $x - z$  is established. Finally, a numerical example is provided to illustrate the main result.

**Keywords:** *Pan system; upper bounds.*

**Mathematics Subject Classification (2010):** 65P20, 65P30, 65P40.

## 1 Introduction

In the last four decades, chaos as a very interesting nonlinear phenomenon has been intensively studied. Hyperchaotic system is usually defined as a chaotic system with more than one positive Lyapunov exponent. It is even more complicated than chaotic systems and has more unstable manifolds. At the same time, due to its theoretical and practical applications in technological fields, such as secure communications, lasers, nonlinear circuits, control, synchronization, hyperchaos has recently become a central topic in the research of nonlinear sciences.

In particular, the ultimate boundedness is very important for the study of the qualitative behavior of a chaotic system. If one can show that a chaotic or a hyperchaotic system under consideration has a globally attractive set, one knows that the system cannot have the equilibrium points, periodic or quasi-periodic solutions, or other chaotic or hyperchaotic attractors existing outside the attractive set. This greatly simplifies the analysis

---

<sup>\*</sup> Corresponding author: <mailto:rezzag.samia@gmail.com>



of dynamics of the system of a chaotic or hyperchaotic system [7]. The boundedness of a chaotic system also plays an important role in chaos control and chaos synchronization.

Such an estimation is quite difficult to achieve technically, however, several works on this topic were realized for some 3D and 4D dynamical systems [2, 4–6, 8–15].

Furthermore, there are no unified methods for constructing the Lyapunov functions to study the boundedness of the chaotic systems. Therefore, it is necessary to study the boundedness of the hyperchaotic systems. In the present paper, we investigate the ultimate bound and positively invariant set for the new modified hyperchaotic Pan system using a technique combining the generalized Lyapunov function theory and optimization. First, we derive an ellipsoidal ultimate bound and positively invariant set. Then we obtain a two-dimensional parabolic ultimate bound with respect to  $x - z$ . Finally, a numerical example is provided to illustrate the main result.

## 2 The Ultimate Bound and Positively Invariant Set for the New Modified Hyperchaotic Pan System

- Consider the system

$$\dot{X} = f(X), \quad (1)$$

where  $X \in \mathbb{R}^n$ ,  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ ,  $X = (x_1, x_2, \dots, x_n)^T$ ,  $t_0 \geq 0$  is the initial time, and  $X(t, t_0, X_0)$  is a solution to system (1) satisfying  $X(t_0, t_0, X_0) = X_0$ , which for simplicity is denoted by  $X(t)$ . Assume  $\Omega \in \mathbb{R}^n$  is a compact set.

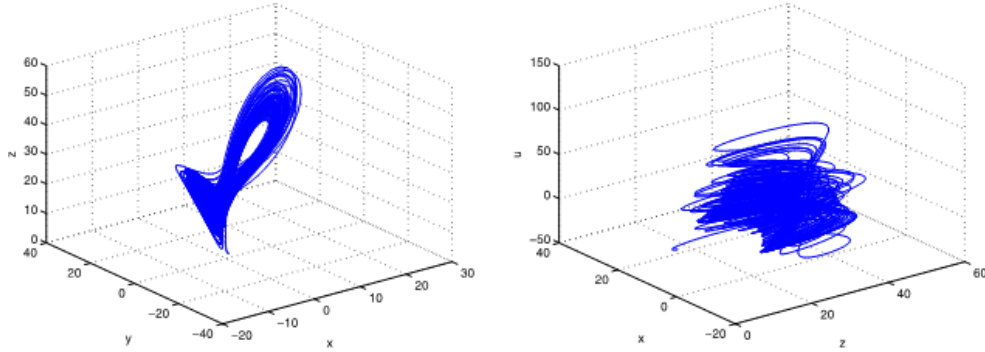
- Define the distance between the solution  $X(t, t_0, X_0)$  and the set  $\Omega$  by  $\rho(X(t, t_0, X_0), \Omega) = \inf_{Y \in \Omega} \|X(t, t_0, X_0) - Y\|$ , and denote  $\Omega_\varepsilon = \{X / \rho(X, \Omega) < \varepsilon\}$ . Clearly,  $\Omega \subset \Omega_\varepsilon$ .

**Definition 2.1** Suppose that there is a compact set  $\Omega \subset \mathbb{R}^n$ . If, for every  $x_0 \in \mathbb{R}^n / \Omega$ ,  $\lim_{t \rightarrow \infty} \rho(x(t), \Omega) = 0$ , that is, for any  $\varepsilon > 0$ , there is a  $T > t_0$ , such that for  $t \geq T$ ,  $x(t, t_0, x_0) \subset \Omega_\varepsilon$ , then the set  $\Omega$  is called an ultimate bound for system (1). If, for any  $x_0 \in \Omega$  and all  $t \geq t_0$ ,  $x(t, t_0, x_0) \subset \Omega$ , then  $\Omega$  is called the positively invariant set for system (1).

Consider the new modified hyperchaotic Pan system [1] :

$$\begin{cases} x' = ay - ax, \\ y' = cx - xz + u, \\ z' = xy - bz, \\ u' = -dy, \end{cases} \quad (2)$$

where  $a, b, c, d$  are real parameters. System (2) displays a typical hyperchaotic attractor when  $(a, b, c, d) = \left(10, \frac{8}{3}, 28, 10\right)$ , the corresponding three-dimensional phase diagrams in  $(x - y - z)$ ,  $(x - z - u)$  spaces are shown in Fig. 1.



**Figure 1:** Hyperchaotic attractor of the new modified hyperchaotic Pan system (2) with  $(a, b, c, d) = \left(10, \frac{8}{3}, 28, 10\right)$  and the initial value  $(x_0, y_0, z_0, w_0) = (1, 1, 1, 1)$ .

Some basic dynamical properties of the new modified hyperchaotic Pan system (2) were studied in [1]. But many properties of the system (2) remain to be uncovered. In the following, we will discuss the boundedness of the new modified hyperchaotic Pan system (2).

**Theorem 2.1** Denote

$$\Omega = \left\{ (x, y, z, u) / x^2 + dy^2 + d \left( z - c - \frac{a}{d} \right)^2 + u^2 \leq R^2 \right\}, \quad (3)$$

where

$$R^2 = \begin{cases} \frac{b^2 (dc + a)^2}{4ad(b - a)}, & \text{si } b \geq 2a \\ \frac{(cd + a)^2}{d}, & \text{si } b < 2a. \end{cases} \quad (4)$$

If  $a > 0$ ,  $b > 0$ ,  $c > 0$  and  $d > 0$ , then all orbits of system (2), including hyperchaotic attractors, are trapped into a bounded region, and so the hyperellipsoid  $\Omega$  is an ultimate bound and positively invariant set for system (2).

**Proof.** Define the following Lyapunov function

$$V = x^2 + dy^2 + d \left( z - c - \frac{a}{d} \right)^2 + u^2. \quad (5)$$

Then, its time derivative along the orbits of system (2) is

$$\frac{1}{2} \dot{V} = -ax^2 - dbz^2 + b(cd + a)z = 0. \quad (6)$$

That is to say, for  $a > 0$ ,  $b > 0$ ,  $d > 0$ , the surface, defined by

$$\Gamma = \left\{ (x, y, z, u) / ax^2 + db \left( z - \frac{cd + a}{2d} \right)^2 = \frac{b(cd + a)^2}{4d}, \right\} \quad (7)$$

is an ellipsoid in  $4D$  space for certain values of  $a, b, c$  and  $d$ . Outside  $\Gamma$ , we have  $\dot{V} < 0$ , while inside  $\Gamma$ , we have  $\dot{V} > 0$ . Since the function  $V = x^2 + dy^2 + d\left(z - c - \frac{a}{d}\right)^2 + u^2$  is continuous on the closed set  $\Gamma$ ,  $V$  can reach its maximum on the surface  $\Gamma$ . Denote the maximum value of  $V$  by  $R^2$ , that is  $R^2 = \max_{(x,y,z,u) \in \Gamma} V$ . Next, we use the Lagrange multiplier method to obtain the optimal value of  $V$  on  $\Gamma$ . Define

$$F = x^2 + dy^2 + d\left(z - c - \frac{a}{d}\right)^2 + u^2 + \lambda \left[ ax^2 + db\left(z - \frac{cd+a}{2d}\right)^2 - \frac{b(cd+a)^2}{4d} \right] \quad (8)$$

and let

$$\begin{cases} \frac{\partial F(x,y,z,u)}{\partial x} = 2x + 2\lambda ax = 0, \\ \frac{\partial F(x,y,z,u)}{\partial y} = 2dy = 0, \\ \frac{\partial F(x,y,z,u)}{\partial z} = 2d\left(z - c - \frac{a}{d}\right) + 2\lambda db\left(z - \frac{cd+a}{2d}\right) = 0, \\ \frac{\partial F(x,y,z,u)}{\partial u} = 2u = 0, \\ \frac{\partial F(x,y,z,u)}{\partial \lambda} = ax^2 + db\left(z - \frac{cd+a}{2d}\right)^2 - \frac{b(cd+a)^2}{4d} = 0. \end{cases} \quad (9)$$

Thus,

(i) When  $\lambda \neq \frac{-1}{a}$ , we have  $(x, y, z, u) = (0, 0, 0, 0)$  or  $(x, y, z, u) = \left(0, 0, \frac{cd+a}{d}, 0\right)$  and  $R^2 = \frac{(cd+a)^2}{d}$  or  $R^2 = 0$  correspondingly.

(ii) When  $\lambda = \frac{-1}{a}$ , and  $b \geq 2a$ , we have  $(x, y, z, u) = \left(\pm \frac{b(cd+a)\sqrt{b-2a}}{2\sqrt{ad}(a-b)}, 0, \frac{(cd+a)(2a-b)}{2d(a-b)}, 0\right)$  and  $R^2 = \frac{b^2(dc+a)^2}{4ad(b-a)}$ . Summarizing (i)–(ii) above, we have

$$R^2 = \begin{cases} \frac{b^2(dc+a)^2}{4ad(b-a)}, & \text{if } b \geq 2a, \\ \frac{(cd+a)^2}{d}, & \text{if } b < 2a. \end{cases} \quad (10)$$

For the set  $\Omega$ , as shown in (3), we have  $\Gamma \subset \Omega$ . Next, we will show

$$\lim_{t \rightarrow \infty} \rho(X(t), \Omega) = 0, \quad (11)$$

using the reduction to absurdity, where  $X(t) = (x(t), y(t), z(t), u(t))$ . If (11) does not hold, we can conclude that the orbits of system (2) are outside  $\Omega$  permanently, thus  $\dot{V} < 0$ . Therefore,  $V(X(t))$  monotonously decreases outside  $\Omega$ , which leads to the following result  $\lim_{t \rightarrow \infty} V(X(t)) = v^* > l$ . Let  $s = \inf_{X \in D} (-\dot{V}(X(t)))$  where  $D = \{X(t)/V^* \leq V(X(t)) \leq V(X(t_0))\}$ , while  $t_0$  is the initial time. Consequently, we have that  $s, V^*$  are positive constants, and  $\frac{dV(X(t))}{dt} \leq -s$ . As  $t \rightarrow \infty$ , we have  $0 \leq V(X(t)) \leq$

$V(X(t_0)) - s(t - t_0) \rightarrow -\infty$  this is inconsistent. Therefore (11) actually holds, that is to say,  $\Omega$  is the ultimate bound of system (2). Finally, to see that  $\Omega$  is also the positively invariant set, reason as follows. Suppose  $V$  attains its maximum value on surface  $\Gamma$  at point  $P_0(\hat{x}_0, \hat{y}_0, \hat{z}_0, \hat{u}_0)$ . Since  $\Gamma \subset \Omega$ , for any point  $X(t)$  on  $\Omega$  and  $X(t) \neq P_0$ , we have  $\dot{V}(X) < 0$ , thus, any orbit  $X(t)$  ( $X(t) \neq P_0$ ) of system (2) will go into  $\Omega$ . When  $X(t) = P_0$ , by the continuation theorem [3],  $X(t)$  will also go into  $\Omega$ . Summarizing the above, we conclude that  $\Omega$  is the positively invariant set of system (2).

**Corollary 2.1** *For  $a > 0$ ,  $b > 0$ ,  $c > 0$  and  $d > 0$ , the solution of the system (2) is bounded by the conditions*

$$\left\{ \begin{array}{l} |x| \leq R, \\ |y| \leq \frac{R}{\sqrt{d}}, \\ \frac{cd+a}{d} - \frac{R}{\sqrt{d}} \leq z \leq \frac{R}{\sqrt{d}} + \frac{cd+a}{d}, \\ |u| \leq R. \end{array} \right. \quad (12)$$

**Proof.** Direct consequence of the previous theorem.

### 3 The Estimation of the Two-Dimensional Parabolic Ultimate Bound with Respect to $x$ - $z$

**Theorem 3.1** *When  $b < 2a$ , the system (2) has the following two-dimensional parabolic ultimate bound*

$$z \geq \frac{x^2}{2a}. \quad (13)$$

**Proof.** Define

$$V(t) = \frac{1}{2a}x^2(t) - z(t).$$

Then, its time derivative along the orbits of system (2) is

$$\dot{V} = \frac{1}{a}xx - \dot{z} = -x^2 + bz.$$

Thus,

$$\dot{V} + bV = -x^2 + bz + \frac{b}{2a}x^2 - bz = \left(\frac{b}{2a} - 1\right)x^2.$$

When  $b < 2a$ , we have

$$\dot{V} + bV \leq 0.$$

For any initial value  $V(t_0) = V_0$ , according to the comparison theorem, we have

$$V(t) \leq V_0 e^{-b(t-t_0)} \rightarrow 0 \quad (t \rightarrow \infty)$$

Thus

$$\lim_{t \rightarrow \infty} V(t) = \lim_{t \rightarrow \infty} \left[ \frac{1}{2a}x^2(t) - z(t) \right] \leq 0.$$

So, we get that system orbits satisfy the parabolic ultimate bound

$$z \geq \frac{x^2}{2a}.$$

This completes the proof.

#### 4 Example

Consider the system (2), when  $a = 10$ ,  $b = \frac{8}{3}$ ,  $c = 28$  and  $d = 10$ .

We have

$$V(x, y, z, u) = x^2 + 10y^2 + 10(z - 29)^2 + u^2,$$

$$\Gamma = \left\{ (x, y, z, u) / 10x^2 + \frac{80}{3} \left( z - \frac{29}{2} \right)^2 = \frac{20}{3} \times 29^2 \right\}$$

and

$$R^2 = \max_{(x,y,z,u) \in \Gamma} V_{(x,y,z,u)} = \frac{(cd + a)^2}{d} = 10 \times 29^2.$$

Therefore, the estimate of ultimate bound for system (2) is

$$\Omega = \left\{ (x, y, z, u) / x^2 + 10y^2 + 10(z - 29)^2 + u^2 \leq 10 \times 29^2 \right\}.$$

Consequently, we have

$$\begin{cases} |x| \leq 29 \times \sqrt{10}, \\ |y| \leq 29, \\ 0 \leq z \leq 58, \\ |u| \leq 29 \times \sqrt{10}. \end{cases}.$$

It is obvious that the orbits of system (2) locate in the section where  $z \geq 0$ .

#### 5 Conclusion

In this paper, we have investigated the ultimate bound and positively invariant set for the new modified hyperchaotic Pan system. We have first derived a four-dimensional ellipsoidal ultimate bound and positively invariant set. Then, we have obtained a two-dimensional parabolic bound with respect to  $x - z$ , which shows that, in the four-dimensional space, the orbits of the system are located inside the parabolic cylinder  $z \geq \frac{x^2}{2a}$ , accordingly, we have also got  $z \geq 0$ . Finally, a numerical example is provided to illustrate the main result.

#### References

- [1] Alazzawi, S. F. Study of dynamical properties and effective of a state  $u$  for hyperchaotic Pan systems. *Al-Rafiden J. Comput. Sci. Math.* **10** (2013) 89–99.
- [2] Elhadj, Z. and Sprott, J. C. About the boundedness of 3D continuous time quadratic systems. *Nonlinear oscillations* **13** (2-3) (2010) 515–521.
- [3] Lefschetz, S. *Differential Equations: Geometric Theory*. New York: Interscience Publishers, 1963.
- [4] Leonov, G., Bunin, A. and Koksche, N. Attractor localisation of the Lorenz system. *Zeitschrift für Angewandte Mathematik und Mechanik* **67** (1987) 649–656.
- [5] Li, D., Lu, J.A., Wu, X. and Chen, G. Estimating the bounds for the Lorenz family of chaotic systems. *Chaos, Solitons & Fractals* **23** (2005) 529–534.

- [6] Li, D., Wu, X. and Lu, J. Estimating the ultimating bound and positively invariant set for the hyperchaotic Lorenz-Haken system. *Chaos, Solitons & Fractals* **39** (2009) 1290–1296.
- [7] Liao, X., Fu, Y., Xie, S., Yu, P. Globally exponentially attractive sets of the family of Lorenz systems. *Sci. China, Ser. F* **51** (2008) 283–292.
- [8] Pogromsky, A. Y., Santoboni, G. and Nijmeijer, H. An ultimate bound on the trajectories of the Lorenz systems and its applications. *Non-linearity* **16** (2003) 1597–1605.
- [9] Rezzag, S. Zehrou, O. and Aliouche, A. Estimating the bounds for the general 4-D hyper-chaotic system. *Nonlinear studies* **22** (1) (2015) 41–48.
- [10] Rezzag, S. Zehrou, O. and Aliouche, A. Estimating the Bounds for the General 4-D Continuous-Time Autonomous System. *Nonlinear Dyn. Syst. Theory* **15** (3) (2015) 313–320.
- [11] Sun, Y. J. Solution bounds of generalized Lorenz chaotic system. *Chaos, Solitons & Fractals* **40** (2009) 691–696.
- [12] Wang, P., Li, D. and Hu, Q. Bounds of the hyper-chotic Lorenz-Stenflo system. *Commun Nonlinear Sci. Numer. Simulat.* **15** (2010) 2514–2520.
- [13] Zehrou, O. and Elhadj, Z. Boundedness of the generalized 4-D hyper-chaotic model containing Lorenz-Stenflo and Lorenz-Haken systems. *Nonlinear studies* **19** (4) (2012) 1–7.
- [14] Zehrou, O. and Elhadj, Z. Ellipsoidal chaos. *Nonlinear studies* **19** (1) (2012) 71–77.
- [15] Zhang, F., Li, Y. and Mu, C. Bounds of Solutions of a Kind of Hyper-Chaotic Systems and Application. *Journal of Mathematical Research with Applications* **33** (3) (2013) 345–352.



# Pseudo Almost Automorphic Mild Solutions to Some Fractional Differential Equations with Stepanov-like Pseudo Almost Automorphic Forcing Term

Vikram Singh<sup>1\*</sup> and Dwijendra N. Pandey<sup>1</sup>

<sup>1</sup> *Department of Mathematics, Indian Institute of Technology Roorkee  
Roorkee-247667, India*

Received: January 12, 2017; Revised: October 23, 2017

**Abstract:** In this paper, we show the existence of  $(\mu, \nu)$ -pseudo almost automorphic mild solutions to some fractional differential equations in light of measure theory in a Banach space with new concept of Stepanov-like  $(\mu, \nu)$ -pseudo almost automorphy by virtue of Leray-Schauder alternate theorem.

**Keywords:** *almost automorphic and  $(\mu, \nu)$ -pseudo almost automorphic functions; Stepanov-like  $(\mu, \nu)$ -pseudo almost automorphic functions; mild solution; sectorial operator; solution operator; fractional differential equation.*

**Mathematics Subject Classification (2010):** 43A60, 26A33, 34C27, 28D05.

## 1 Introduction

In recent years, fractional differential equations with almost automorphic solutions have gained considerable interest. This is due to the fact that fractional differential equations are powerful tools to describe the hereditary properties and memory of various materials. Fractional differential equations have great applications in nonlinear oscillations of earthquakes, fractal theory, diffusion in porous media, viscoelastic panel in super sonic gas flow. For more details, we refer to the papers [2, 3, 8, 9, 18] and references therein.

The concept of almost automorphy was first introduced by Bochner [6]. Afterwards, being a most attractive topic in qualitative theory of differential equations, the theory of classical almost automorphy has been studied extensively by numerous authors and

---

\* Corresponding author: <mailto:vikramiitr1@gmail.com>

generalized further in different ways using measure theory and weighted functions, see [4, 5, 14–16].

More recently, a new concept of the so-called  $(\mu, \nu)$ -pseudo almost automorphy was introduced by Diagana et. al. [10] and Abdelkarim et. al. [1], which is an interesting generalization of both  $\mu$ -pseudo almost automorphy and weighted pseudo almost automorphy. Further, Chang et. al. [8] proposed the concept of Stepanov-like  $\mu$ -pseudo almost automorphic mild solutions to semilinear functional differential equations. In this paper, stimulated by [1, 4, 8, 10], we will introduce the concept of Stepanov-like  $(\mu, \nu)$ -almost automorphic functions.

In this paper, we investigate the existence of  $(\mu, \nu)$ -pseudo almost automorphic mild solutions to the following fractional differential equation of order  $1 < \eta < 2$ ,

$$D_t^\eta y(t) = Ay(t) + D_t^{\eta-1} \mathcal{F}\left(t, y(t), \int_{-\infty}^t \mathcal{K}(t-s)h(s, y(s))ds\right), \quad t \in \mathbb{R}, \quad (1)$$

where  $A : D(A) \subseteq E \rightarrow E$  is a densely defined linear operator of sectorial type  $\omega < 0$  on a complex Banach space  $E$ . The functions  $h, \mathcal{F}$  are Stepanov-like  $(\mu, \nu)$ -pseudo almost automorphic. Here the derivative is taken in Riemann-Liouville sense and  $\mathcal{K} \in L^1(\mathbb{R})$  with  $|\mathcal{K}(t)| \leq C_K e^{-bt}, b > 0$ .

The rest of this paper is organized as follows: Section 2 provides some basic definitions, lemmas and theorems. In Section 3, we obtain main results by using Leray-Schauder alternate theorem fixed point theorem.

## 2 Preliminaries

Let  $(E, \|\cdot\|)$  be a Banach space and  $\mathbb{C}, \mathbb{R}$ , and  $\mathbb{N}$  stand for complex number, real number and natural numbers respectively.  $C(\mathbb{R}, E)$  and  $BC(\mathbb{R}, E)$  represent the sets of continuous functions and bounded continuous functions, respectively. For a linear operator  $A$  on  $E$ , let  $\varrho(A), \rho(A), \mathcal{D}(A)$  and  $\mathcal{R}(A)$  stand for the spectrum, the resolvent set, the domain and the range of  $A$ , respectively.

Now, we recall some definitions on fraction calculus (for more details, see [18]).

**Definition 2.1** The fractional integral of a function  $\phi : \mathbb{R}^+ \rightarrow E$  with the lower limit zero of order  $\eta > 0$  is given by

$$I^\eta \phi(t) = \frac{1}{\Gamma(\eta)} \int_0^t (t-\tau)^{\eta-1} \phi(\tau) d\tau,$$

where  $\Gamma(\cdot)$  denotes the Gamma function.

**Definition 2.2** The Riemann-Liouville fractional derivative of a function  $\phi : \mathbb{R}^+ \rightarrow E$  with the lower limit zero of order  $\eta > 0$  is given by

$$D^\eta \phi(t) = \frac{1}{\Gamma(n-\eta)} \frac{d^n}{dt^n} \int_0^t (t-\tau)^{n-\eta-1} \phi(\tau) d\tau, \quad n-1 < \eta < n, n \in \mathbb{N}.$$

**Definition 2.3** A densely defined closed linear operator  $A$  with domain  $\mathcal{D}(A)$  in a Banach space  $E$  is said to be sectorial of type  $\omega$  and angle  $\theta$  if there exists

$$\theta \in (0, \frac{\pi}{2}), \quad \mathcal{M} > 0, \quad \omega \in \mathbb{R},$$



such that its resolvent exists outside the sector  $\omega + \Sigma_\theta := \{\omega + \lambda : \lambda \in \mathbb{C}, |\arg(-\lambda)| < \theta\}$ , and

$$\|(\lambda - A)^{-1}\| \leq \frac{\mathcal{M}}{|\lambda - \omega|}, \quad \lambda \notin \omega + \Sigma_\theta.$$

It is easy to verify that an operator  $A$  is sectorial of type  $\omega$  if and only if  $\omega I - A$  is sectorial of type 0. For more details on sectorial operators see [13].

**Definition 2.4** Let  $1 < \eta < 2$  and  $A$  be a closed linear operator defined on the domain  $\mathcal{D}(A)$  in a Banach space  $E$ . Then we say  $A$  is the generator of solution operator if there exists a  $\omega \in \mathbb{R}$  and a strongly continuous function  $\mathcal{S}_\eta : \mathbb{R}^+ \rightarrow \mathcal{L}(E)$  such that  $\{\lambda^\eta : \operatorname{Re} \lambda > \omega\} \subset \rho(A)$  and

$$\lambda^{\eta-1}(\lambda^\eta - A)^{-1}y = \int_0^\infty e^{-\lambda t} \mathcal{S}_\eta(t) y dt, \quad \operatorname{Re} \lambda > \omega, \quad y \in E.$$

In this case,  $\mathcal{S}_\eta(t)$  is called the solution operator generated by  $A$  and one can deduce that if  $A$  is sectorial of type  $\omega$  with  $0 < \theta < \pi(1 - \frac{\eta}{2})$ , then  $A$  generates the solution operator given by

$$\mathcal{S}_\eta(t)y = \frac{1}{2\pi i} \int_\Gamma e^{-\lambda t} \lambda^{\eta-1} (\lambda^\eta - A)^{-1} y dt, \quad (2)$$

where  $\Gamma$  is a suitable path lying outside the sector  $\omega + \Sigma_\theta$  (see [9]).

Recently, Cuesta in [9] has shown that if  $A$  is a sectorial operator of type  $\omega$  for some  $\mathcal{M} > 0$  and  $0 < \theta < \pi(1 - \frac{\eta}{2})$ , then there exists a constant  $\mathcal{C} > 0$  depending solely on  $\theta$  and  $\eta$  such that

$$\|\mathcal{S}_\eta(t)\|_{\mathcal{L}(E)} \leq \frac{\mathcal{C}\mathcal{M}}{1 + |\omega|t^\eta}, \quad t \geq 0.$$

In boundary case, when  $\eta = 1$ , this is analogous to the statement that  $A$  is the generator of exponentially stable  $C_0$ -semigroup. Next, if  $\eta > 1$ , then solution family  $\mathcal{S}_\eta(t)$  decays  $t^{-\eta}$ , in fact,  $\mathcal{S}_\eta(t)$  is integrable on  $(0, \infty)$  i.e.

$$\int_0^\infty \frac{1}{1 + |\omega|s^\eta} ds = \frac{|\omega|^{-\frac{1}{\eta}} \pi}{\eta \sin(\frac{\pi}{\eta})}, \quad 1 < \eta < 2. \quad (3)$$

**Definition 2.5** A continuous function  $f : \mathbb{R} \rightarrow E$  is almost automorphic (in Bochner's sense) if for each sequence of real numbers  $\{\tau'_n\}$ , there exist a subsequence  $\{\tau_n\}$  and a function  $g : \mathbb{R} \rightarrow E$  such that

$$g(t) = \lim_{n \rightarrow \infty} f(t + \tau_n), \quad \text{is well defined for each } t \in \mathbb{R}, \text{ and } f(t) = \lim_{n \rightarrow \infty} g(t - \tau_n).$$

The set of all almost automorphic functions is denoted by  $AA(E)$  and constitutes a Banach space endowed with the supnorm.

**Definition 2.6** A function  $f : \mathbb{R} \times E \rightarrow E$  is said to be almost automorphic if  $f(\cdot, x) \in AA(\mathbb{R}, E)$  for all  $x \in E$ , and  $f$  is uniformly continuous in second variable on each compact set  $K$  of  $E$ . The set of all such functions is denoted by  $AA(\mathbb{R} \times E, E)$ .

Next we recall some definitions and basic results on Stepanov-like almost automorphic functions (for more details, see [8, 11, 20]).

**Definition 2.7** The Bochner transform  $f^b(t, s)$ ,  $s \in [0, 1], t \in \mathbb{R}$ , of a function  $f : \mathbb{R} \rightarrow E$  is defined by  $f^b(t, s) = f(t + s)$ .

**Definition 2.8** The space of all Stepanov-like bounded functions denoted by  $BS^p(\mathbb{R}, E)$  consists of all measurable functions  $f : \mathbb{R} \rightarrow E$ , with exponent  $p \in [1, \infty)$  such that  $f^b \in L^\infty(\mathbb{R}, L^p([0, 1], E))$  and constitutes a Banach space with the norm

$$\|f\|_{S^p} = \|f^b\|_{L^\infty(\mathbb{R}, L^p)} = \sup_{t \in \mathbb{R}} \left( \int_t^{t+1} \|f(\xi)\|^p d\xi \right)^{\frac{1}{p}}.$$

**Definition 2.9** The space of Stepanov-like almost automorphic functions denoted by  $S^pAA(\mathbb{R}, E)$  consists of all  $f \in BS^p(\mathbb{R}, E)$  such that

$$f^b \in AA(\mathbb{R}, L^p([0, 1], E)).$$

In other words, a function  $f \in L^p_{loc}(\mathbb{R}, E)$  is a Stepanov-like almost automorphic function if its Bochner transform  $f^b : \mathbb{R} \rightarrow L^p([0, 1], E)$  is almost automorphic in the sense that every sequence  $\{\tau'_n\}$  of real numbers contains a subsequence  $\{\tau_n\}$  and a function  $g \in L^p_{loc}([0, 1], E)$  such that

$$\lim_{n \rightarrow \infty} \left[ \int_t^{t+1} \|f(s + \tau_n) - g(s)\|^p ds \right]^{\frac{1}{p}} \rightarrow 0, \quad \text{and} \quad \lim_{n \rightarrow \infty} \left[ \int_t^{t+1} \|g(s - \tau_n) - f(s)\|^p ds \right]^{\frac{1}{p}} \rightarrow 0,$$

for all  $t \in \mathbb{R}$ .

**Definition 2.10** A function  $f : \mathbb{R} \times E \rightarrow E$ , with  $f(\cdot, y) \in L^p(\mathbb{R}, E)$  for each  $y \in K$  is said to be Stepanov-like almost automorphic function in  $t \in \mathbb{R}$ , uniformly for  $y \in K$ , if  $t \rightarrow f(t, y)$  is Stepanov-like almost automorphic for each  $y \in K$ .

**Remark 2.1** [7] It can be observed that if  $f$  is almost automorphic, then  $f$  is Stepanov-like almost automorphic, i.e.  $AA(\mathbb{R}, E) \subset S^pAA(\mathbb{R}, E)$  [1]. Moreover, let  $1 \leq p \leq q < \infty$ , if  $f \in S^qAA(\mathbb{R}, E)$  implies that  $f \in S^pAA(\mathbb{R}, E)$ .

Throughout this paper, we denote the Lebesgue  $\sigma$ -field of  $\mathbb{R}$  by  $\mathfrak{B}$ , and the set of all positive measures  $\mu$  on  $\mathfrak{B}$  by  $\mathfrak{M}$  satisfying  $\mu(\mathbb{R}) = \infty$  and  $\mu([a, b]) < \infty$ , for all  $a, b \in \mathbb{R} (a \leq b)$ .

Next, we define new ergodic space and the notion of Stepanov-like  $(\mu, \nu)$ -pseudo almost automorphic functions with positive measures  $\mu, \nu \in \mathfrak{M}$ .

**Definition 2.11** [10] Let  $\mu, \nu \in \mathfrak{M}$  and  $p \in [1, \infty)$ . A function  $\psi \in BS^p(\mathbb{R}, E)$  is said to be  $(\mu, \nu)$ -ergodic if

$$\lim_{\gamma \rightarrow \infty} \frac{1}{\nu(\mathcal{Q}_\gamma)} \int_{\mathcal{Q}_\gamma} \left( \int_t^{t+1} \|\psi(s)\|^p ds \right)^{\frac{1}{p}} d\mu(t) = 0,$$

where  $\mathcal{Q}_\gamma = [-\gamma, \gamma]$  and  $\mu(\mathcal{Q}_\gamma) = \int_{\mathcal{Q}_\gamma} d\mu(t)$ . We denote all such functions by  $\mathcal{E}^p(\mathbb{R}, E, \mu, \nu)$ .

**Definition 2.12** Let  $\mu, \nu \in \mathfrak{M}$ . A function  $f \in C(\mathbb{R}, E)$  is said to be  $(\mu, \nu)$ -pseudo almost automorphic function, if it can be decomposed as  $f = \phi + \psi$ , where  $\phi \in AA(\mathbb{R}, E)$  and  $\psi \in \mathcal{E}^1(\mathbb{R}, E, \mu, \nu)$ . The collection of all such functions by  $PAA(\mathbb{R}, E, \mu, \nu)$  is a Banach space equipped with sup norm.

**Definition 2.13** Let  $\mu, \nu \in \mathfrak{M}$ . A function  $f \in BS^p(\mathbb{R}, E)$  is said to be Stepanov-like  $(\mu, \nu)$ -pseudo almost automorphic function, if it can be decomposed as  $f = \phi + \psi$ , where  $\phi \in S^pAA(\mathbb{R}, E)$  and  $\psi \in \mathcal{E}^p(\mathbb{R}, E, \mu, \nu)$ . We denote the collection of all such functions by  $S^pPAA(\mathbb{R}, E, \mu, \nu)$ .

**Definition 2.14** [10] A continuous function  $f : \mathbb{R} \times E \rightarrow E$  is said to be  $(\mu, \nu)$ -ergodic in  $t \in \mathbb{R}$  uniformly with respect to  $y \in E$ , if the following conditions are true:

- (i)  $f(\cdot, y) \in \mathcal{E}^p(\mathbb{R} \times E, E, \mu, \nu)$ , for all  $y \in E$ ,
- (ii) The function  $f(\cdot, y)$  is uniformly continuous with the second variable in a compact set  $K$  in  $E$ .

We denote the collection of all such functions by  $\mathcal{E}^pU(\mathbb{R} \times E, E, \mu, \nu)$ .

**Definition 2.15** The function  $f \in BS^p(\mathbb{R} \times E, E)$  is said to be Stepanov-like  $(\mu, \nu)$ -pseudo almost automorphic, if it has decomposition of the form  $f = \phi + \psi$ , where  $\phi \in S^pAAU(\mathbb{R} \times E, E)$  and  $\psi \in \mathcal{E}^pU(\mathbb{R} \times E, E, \mu, \nu)$ . We denote the set of all such functions by  $S^pPAAU(\mathbb{R} \times E, E, \mu, \nu)$ .

We assume the following:

(M<sub>1</sub>) Let  $\mu, \nu \in \mathfrak{M}$ , then  $\lim_{\gamma \rightarrow \infty} \frac{\mu(Q_\gamma)}{\nu(Q_\gamma)} < \infty$ .

(M<sub>2</sub>) For all  $s \in \mathbb{R}$  and  $\nu \in \mathfrak{M}$ , there exist a bounded interval  $I$  and  $\alpha > 0$  such that  $\mu(\{a + s, a \in D\}) \leq \alpha \mu(D)$  if  $D \in \mathfrak{B}$  satisfies  $D \cap I = \emptyset$ .

**Theorem 2.1** [10] Assume that  $\mu, \nu \in \mathfrak{M}$  and (M<sub>1</sub>) – (M<sub>2</sub>) hold. Then  $S^pPAA(\mathbb{R}, E, \mu, \nu)$  is translation invariant and the set  $(S^pPAA(\mathbb{R}, E, \mu, \nu), \|\cdot\|_{S^p})$  is the Banach space.

**Theorem 2.2** Let  $\mu, \nu \in \mathfrak{M}$ ,  $f = \phi + \psi \in S^pPAAU(\mathbb{R} \times E \times E, E, \mu, \nu)$  with  $\phi \in S^pAAU(\mathbb{R} \times E \times E, E)$ ,  $\psi \in \mathcal{E}^pU(\mathbb{R} \times E \times E, E, \mu, \nu)$ . Suppose that the following conditions hold:

- (i)  $\phi$  is uniformly continuous on a bounded subset  $\Omega \subset E \times E$  for all  $t \in \mathbb{R}$ .
- (ii)  $f$  is uniformly continuous on a bounded subset  $\Omega \subset E \times E$  for all  $t \in \mathbb{R}$ .
- (iii)  $\xi = \alpha + \beta, \chi = u + v \in S^pPAA(\mathbb{R}, E, \mu, \nu)$  with  $\alpha, u \in S^pAA(\mathbb{R}, E)$  and  $\beta, v \in \mathcal{E}^p(\mathbb{R}, E, \mu, \nu)$  and  $\{\alpha(t) \in \mathbb{R}\}, \{u(t) \in \mathbb{R}\}$  are compact in  $E$ .

Then  $t \mapsto f(t, \xi(t), \chi(t)) \in S^pPAA(\mathbb{R}, E, \mu, \nu)$ .

**Proof.** The proof is similar to the proof of Theorem 3.2 in [21] and hence the details are omitted here.

**Lemma 2.1** Let  $y = y_1 + y_2 \in S^pPAA(\mathbb{R}, E, \mu, \nu)$  and  $\mathcal{R}y = \overline{\{y_1(t) : t \in \mathbb{R}\}}$  be a compact set in  $E$ . Suppose that  $h = \phi + \psi \in S^pPAAU(\mathbb{R} \times E, E, \mu, \nu)$ , with  $\phi \in S^pAAU(\mathbb{R} \times E, E)$ ,  $\psi \in \mathcal{E}^pU(\mathbb{R} \times E, E, \mu, \nu)$  satisfying

$$\|h(t, y) - h(t, z)\| \leq L_h \|y - z\| \quad \text{and} \quad \|\phi(t, y) - \phi(t, z)\| \leq L_\phi \|y - z\|, \quad y, z \in E, \quad t \in \mathbb{R},$$

where  $L_\phi, L_h > 0$  are constants. Then

$$\Psi_h(t) := \int_{-\infty}^t \mathcal{K}(t-s)h(s, y(s))ds \in S^pPAA(\mathbb{R}, E, \mu, \nu). \quad (4)$$

**Proof.** The proof is similar to the proof of Lemma 3.2 in [19] and hence the details are omitted here.

**Lemma 2.2** *Let  $(M_1)$  and  $(M_2)$  hold and let  $f \in S^pPAA(\mathbb{R}, E, \mu, \nu)$ . Then the function is defined by*

$$\Lambda_f(t) = \int_{-\infty}^t \mathcal{S}_\eta(t-s)f(s)ds \in PAA(\mathbb{R}, E, \mu, \nu).$$

**Proof.** Since  $f \in S^pPAA(\mathbb{R}, E, \mu, \nu)$ , there exist  $\phi \in S^pAA(\mathbb{R}, E)$  and  $\psi \in \mathcal{E}^p(\mathbb{R}, E, \mu, \nu)$ , such that  $f(t) = \phi(t) + \psi(t)$ . Now consider

$$\Lambda_f(t) = \int_{-\infty}^t \mathcal{S}_\eta(t-s)f(s)ds = \Lambda_\phi(t) + \Lambda_\psi(t),$$

where

$$\Lambda_\phi(t) = \int_{-\infty}^t \mathcal{S}_\eta(t-s)\phi(s)ds \quad \text{and} \quad \Lambda_\psi(t) = \int_{-\infty}^t \mathcal{S}_\eta(t-s)\psi(s)ds.$$

First, we show  $\Lambda_\phi \in AA(\mathbb{R}, E)$ . Define a sequence of integral operators for  $n = 1, 2, 3, \dots$ ,

$$\Lambda_\phi^n(t) = \int_{t-n}^{t-n+1} \mathcal{S}_\eta(t-s)\phi(s)ds.$$

Using Holder's inequality, we have  $\|\Lambda_\phi^n(t)\| < \infty$ . Now by Weierstrass' theorem, the series  $\Lambda_\phi(t) = \sum_{n=1}^{\infty} \Lambda_\phi^n(t) = \int_{-\infty}^t \mathcal{S}_\eta(t-s)\phi(s)ds$  converges uniformly on  $\mathbb{R}$ . Moreover,

$$\|\Lambda_\phi(t)\| \leq \sum_{n=1}^{\infty} \|\Lambda_\phi^n(t)\| \leq \|\phi_n\|_{S^p\mathcal{CM}} \sum_{n=1}^{\infty} \left( \frac{1}{1+|\omega|(n-1)^\eta} \right) < \infty \Rightarrow \Lambda_\phi \in C(\mathbb{R}, E).$$

Further, for  $n = 1, 2, 3, \dots$  we show that  $\Lambda_\phi^n \in AA(\mathbb{R}, E)$ . Since  $\phi \in S^pAA(\mathbb{R}, E)$ , this implies that every sequence  $\{\tau'_n\}$  of real numbers contains a subsequence  $\{\tau_n\}$  and a function  $\tilde{\phi} \in L^p_{loc}([0, 1], E)$  such that

$$\left[ \int_t^{t+1} \|\phi(s + \tau_n) - \tilde{\phi}(s)\|^p ds \right]^{\frac{1}{p}} \rightarrow 0, \quad \text{and} \quad \left[ \int_t^{t+1} \|\tilde{\phi}(s - \tau_n) - \phi(s)\|^p ds \right]^{\frac{1}{p}} \rightarrow 0, \quad (5)$$

as  $n \rightarrow \infty$  and  $t \in \mathbb{R}$ . Consider

$$\begin{aligned} \|\Lambda_\phi^n(t + \tau_n) - \Lambda_\phi^n(t)\| &\leq \int_{n-1}^n \|\mathcal{S}_\eta(s)[\phi(t + \tau_n - s) - \tilde{\phi}(t - s)]\| ds \\ &\leq \left( \int_{n-1}^n \|\mathcal{S}_\eta(s)\|^q ds \right)^{\frac{1}{q}} \left( \int_{n-1}^n \|\phi(t + \tau_n - s) - \tilde{\phi}(t - s)\|^p ds \right)^{\frac{1}{p}} \\ &\leq \mathcal{CM} \left( \frac{1}{1+|\omega|(n-1)^\eta} \right) \left( \int_{n-1}^n \|\phi(t + \tau_n - s) - \tilde{\phi}(t - s)\|^p ds \right)^{\frac{1}{p}}. \end{aligned}$$

It is obvious from (5), that the last inequality goes to 0 as  $n \rightarrow \infty$  on  $\mathbb{R}$ . Similarly one can show that

$$\|\Lambda_{\tilde{\phi}}(s - \tau_n) - \Lambda_{\phi}(s)\| \rightarrow 0, \quad (6)$$

as  $n \rightarrow \infty$  on  $\mathbb{R}$ . Thus we conclude that  $\Lambda_{\phi} \in S^pAA(\mathbb{R}, E)$ .

Next, we show that  $\Lambda_{\psi} \in \mathcal{E}(\mathbb{R}, E, \mu, \nu)$ . To complete this task we consider the integral operator for  $n = 1, 2, 3, \dots$

$$\Lambda_{\psi}^n(t) = \int_{t-n}^{t-n+1} \mathcal{S}_{\eta}(t-s)\psi(s)ds = \int_{n-1}^n \mathcal{S}_{\eta}(s)\psi(t-s)ds.$$

Now, we get

$$\begin{aligned} \|\Lambda_{\psi}^n(t)\| &\leq \left( \int_{n-1}^n \|\mathcal{S}_{\eta}(s)\|^q ds \right)^{\frac{1}{q}} \left( \int_{n-1}^n \|\psi(t-s)\|^p ds \right)^{\frac{1}{p}} \\ &\leq \|\psi\|_{S^p \mathcal{CM}} \left[ \int_{n-1}^n \left( \frac{1}{1+|\omega|(s)^{\eta}} \right)^q ds \right]^{\frac{1}{q}} \\ &\leq \|\psi\|_{S^p \mathcal{CM}} \left[ \frac{1}{1+|\omega|(n-1)^{\eta}} \right] \\ &< \infty, \end{aligned}$$

where  $q = p/(p-1)$ . Further, for  $\gamma > 0$ ,

$$\begin{aligned} \lim_{\gamma \rightarrow \infty} \frac{1}{\nu(\mathcal{Q}_{\gamma})} \int_{\mathcal{Q}_{\gamma}} \|\Lambda_{\psi}^n(t)\| d\mu(t) \\ \leq \frac{\mathcal{CM}}{1+|\omega|(n-1)^{\eta}} \lim_{\gamma \rightarrow \infty} \frac{1}{\nu(\mathcal{Q}_{\gamma})} \int_{\mathcal{Q}_{\gamma}} \left( \int_{t-n}^{t-n+1} \|\psi(s)\|^p ds \right)^{\frac{1}{p}} d\mu(t). \end{aligned}$$

Since  $\psi \in \mathcal{E}^p(\mathbb{R}, E, \mu, \nu)$ , the above estimation leads to  $\Lambda_{\psi}^n \in \mathcal{E}^p(\mathbb{R}, E, \mu, \nu)$  for  $n = 1, 2, 3, \dots$ . The above inequality also implies that the series  $\mathcal{CM} \sum_{n=1}^{\infty} \left[ \frac{1}{1+|\omega|(n-1)^{\eta}} \right]$  is convergent, then we deduce in view of Weierstrass test that the series  $\sum_{n=1}^{\infty} \Lambda_{\psi}^n(t)$  converges uniformly on  $\mathbb{R}$  and

$$\Lambda_{\psi}(t) = \sum_{n=1}^{\infty} \Lambda_{\psi}^n(t) = \int_{-\infty}^t \mathcal{S}_{\eta}(t-s)\psi(s)ds.$$

Further, from  $\Lambda_{\psi}^n \in \mathcal{E}^p(\mathbb{R}, E, \mu, \nu)$  and

$$\begin{aligned} \frac{1}{\nu(\mathcal{Q}_{\gamma})} \int_{\mathcal{Q}_{\gamma}} \|\Lambda(t)\| d\mu(t) &\leq \frac{\mathcal{CM}}{1+|\omega|(n-1)^{\eta}} \frac{1}{\nu(\mathcal{Q}_{\gamma})} \int_{\mathcal{Q}_{\gamma}} \left\| \Lambda_{\psi}(s) - \sum_{n=1}^N \Lambda_{\psi}^n(s) \right\| d\mu(s) \\ &\quad + \sum_{n=1}^N \frac{\mathcal{CM}}{1+|\omega|(n-1)^{\eta}} \frac{1}{\nu(\mathcal{Q}_{\gamma})} \int_{\mathcal{Q}_{\gamma}} \|\Lambda_{\psi}^n(s)\| d\mu(s), \end{aligned}$$

it follows that uniform limit  $\Lambda(t) = \sum_{n=1}^{\infty} \Lambda_{\psi}^n(t) \in \mathcal{E}(\mathbb{R}, E, \mu, \nu)$ .

Now, before moving further we briefly describe compactness criteria and the Leray-Schauder alternate theorem. Let  $\mathcal{H} : \mathbb{R} \rightarrow \mathbb{R}$  be continuous such that  $\mathcal{H}(t) \rightarrow \infty$  as  $|t| \rightarrow \infty$  and  $\mathcal{H}(t) \geq 1$  for all  $t \in \mathbb{R}$ . We define a Banach space

$$C_{\mathcal{H}}(\mathbb{R}, E) = \{v \in C(\mathbb{R}, E) : \lim_{|t| \rightarrow \infty} v(t)/\mathcal{H}(t) = 0\},$$

equipped with the norm  $\|v\|_{\mathcal{H}} = \sup_{t \in \mathbb{R}} (\|v(t)\|/\mathcal{H}(t))$ .

**Lemma 2.3** [17] *A set  $K \subseteq C_{\mathcal{H}}(\mathbb{R}, E)$  is relative compact in  $C_{\mathcal{H}}(\mathbb{R}, E)$ , if the following conditions hold:*

- (a<sub>1</sub>) *The set  $K(t) = \{v(t) : v \in K, t \in \mathbb{R}\}$  is relative compact in  $E$ .*
- (a<sub>2</sub>) *The set  $K$  is equicontinuous.*
- (a<sub>3</sub>) *For each  $\epsilon > 0$ , there exists a constant  $L > 0$  such that  $\|v(t)\|_{\mathcal{H}} \leq \epsilon \mathcal{H}(t)$  for all  $|t| > L$  and  $u \in K$ .*

**Lemma 2.4** ([12] *Leray-Schauder Alternate Theorem*) *Let  $\mathcal{D}$  be a closed convex subset of a Banach space  $E$  such that  $0 \in \mathcal{D}$ . Let  $f : \mathcal{D} \rightarrow \mathcal{D}$  be a completely continuous map. Then the set  $\{y \in \mathcal{D} : y = \lambda f(y), 0 < \lambda < 1\}$  is unbounded or the map  $f$  has a fixed point in  $\mathcal{D}$ .*

### 3 Main Results

In this section, we investigate the existence of  $(\mu, \nu)$ -pseudo almost automorphic mild solutions to (1).

**Definition 3.1** [2] A function  $y \in C(\mathbb{R}, E)$  is said to be a mild solution of (1) if the function  $s \mapsto \mathcal{S}_{\eta}(s)\mathcal{F}(s, y(s), \Psi y(s))$  is integrable on  $(-\infty, s)$  for each  $s \in \mathbb{R}$  and

$$y(t) = \int_{-\infty}^t \mathcal{S}_{\eta}(t-s)\mathcal{F}(s, y(s), \Psi_h y(s))ds,$$

where  $\mathcal{S}_{\eta}(t)$  is a solution operator and  $\Psi_h$  is defined by  $\Psi_h y(t) = \int_{-\infty}^t \mathcal{K}(t-s)h(s, y(s))ds$ .

To establish the existence results, we consider the following assumptions:

- (L<sub>1</sub>) Suppose that  $\mathcal{F} = \phi + \psi \in S^p PAAU(\mathbb{R} \times E \times E, E, \mu, \nu)$  with  $\phi \in S^p AAU(\mathbb{R} \times E \times E, E)$ ,  $\psi \in \mathcal{E}^p U(\mathbb{R} \times E \times E, E, \mu, \nu)$  is uniformly continuous on a bounded set  $V \subset X \times X$  for all  $t \in \mathbb{R}$  and  $\{\mathcal{F}(t, y, z) : y, z \in V\}$  is bounded in  $S^p PAAU(\mathbb{R} \times E \times E, E, \mu, \nu)$ .
- (L<sub>2</sub>) There exist a nondecreasing continuous function  $\mathcal{W} : [0, \infty) \rightarrow [0, \infty)$  such that

$$\|\mathcal{F}(t, y, z)\| \leq \mathcal{W}(\|y\| + \|z\|), \quad \text{for each } t \in \mathbb{R}, \quad y, z \in E.$$

**Theorem 3.1** *Let  $A$  be a sectorial operator of type  $\omega < 0$  and  $(M_1)$  and  $(M_2)$  hold. Assume that  $\mathcal{F} : \mathbb{R} \times E \times E \rightarrow E$  is a function satisfying  $(L_1)$  and  $(L_2)$  and the following additional conditions hold:*

$(L_3)$  For  $k, a \geq 0$ ,

$$\lim_{|t| \rightarrow \infty} \int_{-\infty}^t \frac{\mathcal{W}((1+k)a\mathcal{H}(s))}{1 + |\omega|(t-s)^\eta} ds = 0,$$

where  $\mathcal{H}$  is defined in Lemma 2.3. We set

$$\beta(a) := \mathcal{CM} \left\| \int_{-\infty}^t \frac{\mathcal{W}((1+k)a\mathcal{H}(s))}{1 + |\omega|(t-s)^\eta} ds \right\|.$$

$(L_4)$  For every  $y, z \in C_{\mathcal{H}}(\mathbb{R}, E)$  and each  $\epsilon > 0$  there exists a  $\delta > 0$  such that  $\|y - z\| \leq \delta$  implies that

$$\mathcal{CM} \int_{-\infty}^t \frac{\|\mathcal{F}(s, y(s), \Psi_h y(s)) - \mathcal{F}(s, z(s), \Psi_h z(s))\|}{1 + |\omega|(t-s)^\eta} ds \leq \epsilon.$$

$(L_5)$   $\liminf_{s \rightarrow \infty} \frac{s}{\beta(s)} > 1$ .

$(L_6)$  The set  $\{f(s, y(s), \Psi_h y(s)) : c \leq s \leq d, y \in C_{\mathcal{H}}, \|y\|_{\mathcal{H}} \leq \lambda\}$  is relatively compact in  $E$  for  $c, d \in \mathbb{R}, c < d$  and  $\lambda > 0$ .

Then equation (1) admits a  $(\mu, \nu)$ -pseudo almost automorphic mild solution.

**Proof.** Let us define an operator  $\Lambda_{\mathcal{F}} : C_{\mathcal{H}}(\mathbb{R}, E) \rightarrow C_{\mathcal{H}}(\mathbb{R}, E)$  by

$$\Lambda_{\mathcal{F}} y(t) = \int_{-\infty}^t \mathcal{S}_{\eta}(t-s) \mathcal{F}(s, y(s), \Psi_h y(s)) ds.$$

Now, we need only to show that  $\Lambda_{\mathcal{F}}$  has a fixed point in  $PAA(\mathbb{R}, E, \mu, \nu)$ . For the sake of convenience, we provide the proof in several steps.

**Step 1 :**  $\Lambda_{\mathcal{F}}$  is well defined.

For  $y \in C_{\mathcal{H}}(\mathbb{R}, E)$  with  $(L_1)$  we have

$$\begin{aligned} \|\Lambda_{\mathcal{F}} y(t)\| &\leq \mathcal{CM} \int_{-\infty}^t \frac{\mathcal{W}(\|y(s)\| + \|\Psi_h y(s)\|)}{1 + |\omega|(t-s)^\eta} ds \\ &\leq \mathcal{CM} \int_{-\infty}^t \frac{\mathcal{W}[(1 + \|\Psi_h\|)\|y\|_{\mathcal{H}} \mathcal{H}(s)]}{1 + |\omega|(t-s)^\eta} ds. \end{aligned}$$

Hence by  $(L_3)$   $\Lambda_{\mathcal{F}}$  is well defined.

**Step 2 :** The operator  $\Lambda_{\mathcal{F}}$  is continuous. In fact, let  $y, z \in C_{\mathcal{H}}(\mathbb{R}, E)$ . For any  $\epsilon > 0$  we take  $\delta > 0$  such that  $\|y - z\| \leq \delta$ , then

$$\|\Lambda_{\mathcal{F}} y(t) - \Lambda_{\mathcal{F}} z(t)\| \leq \mathcal{CM} \int_{-\infty}^t \frac{\|\mathcal{F}(s, y(s), \Psi_h y(s)) - \mathcal{F}(s, z(s), \Psi_h z(s))\|}{1 + |\omega|(t-s)^\eta} ds \leq \epsilon,$$

which shows the assertion.

**Step 3 :** Next, we show that  $\Lambda_{\mathcal{F}}$  is completely continuous. Let  $B_{\lambda}(E)$  denote a closed

ball in a space  $E$  with radius  $\lambda$  and center at 0. Let us denote  $U = \Lambda_{\mathcal{F}}(B_{\lambda}(C_{\mathcal{H}}(E)))$  and  $w = \Lambda_{\mathcal{F}}(v)$  for  $v \in B_{\lambda}(C_{\mathcal{H}}(E))$ . Now, we show that  $U$  is a relative compact subset of  $E$ . The condition  $(L_3)$  implies that  $\frac{\mathcal{W}((1+\|\Psi_h\|)\lambda\mathcal{H}(t-s))}{1+|\omega|(s)^\eta}$  is integrable on  $[0, \infty)$ . Hence, for  $\epsilon > 0$ , we can chose  $\alpha \geq 0$  such that  $\mathcal{CM} \int_0^\infty \frac{\mathcal{W}((1+\|\Psi_h\|)\lambda\mathcal{H}(t-s))}{1+|\omega|(s)^\eta} ds \leq \epsilon$ .

Since

$$\begin{aligned} w(t) &= \int_0^\alpha \mathcal{S}_\eta(s) \mathcal{F}(t-s, y(t-s), \Psi_h y(t-s)) ds \\ &\quad + \int_\alpha^\infty \mathcal{S}_\eta(s) \mathcal{F}(t-s, y(t-s), \Psi_h y(t-s)) ds, \end{aligned}$$

and

$$\left\| \int_0^\alpha \mathcal{S}_\eta(s) \mathcal{F}(t-s, y(t-s), \Psi_h y(t-s)) ds \right\| \leq \mathcal{CM} \int_\alpha^\infty \frac{\mathcal{W}((1+\|\Psi_h\|)\lambda\mathcal{H}(t-s))}{1+|\omega|(s)^\eta} ds \leq \epsilon,$$

we deduce that  $w(t) \in \overline{\alpha C_0(M)} + B_\epsilon(E)$ , where  $C_0(M)$  denotes the convex hull of  $M$  and

$$M = \{\mathcal{S}_\eta(s) f(\xi, (\xi)y, \Psi_h(\xi)y) : 0 \leq s \leq \alpha, t-\alpha \leq \xi \leq t, \|y\|_{\mathcal{H}} \leq \lambda\}.$$

By the strong continuity of  $\mathcal{S}_\eta$  and  $(L_6)$  we deduce that  $M$  is relatively compact set and  $U \in \overline{\alpha C_0(M)} + B_\epsilon(E)$  which establishes the assertion.

Further, we show that  $U$  is equicontinuous. In fact, we can decompose

$$\begin{aligned} w(t+h) - w(t) &= \int_0^h \mathcal{S}_\eta(s) \mathcal{F}(t+h-s, y(t+h-s), \Psi_h y(t+h-s)) ds \\ &\quad + \int_0^\alpha [\mathcal{S}_\eta(h+s) - \mathcal{S}_\eta(s)] \mathcal{F}(t-s, y(t-s), \Psi_h y(t-s)) ds \\ &\quad + \int_\alpha^\infty [\mathcal{S}_\eta(h+s) - \mathcal{S}_\eta(s)] \mathcal{F}(t-s, y(t-s), \Psi_h y(t-s)) ds. \end{aligned}$$

For each  $\epsilon > 0$ , we can take  $\alpha > 0$  and  $\delta_1$  such that

$$\begin{aligned} &\left\| \int_0^h \mathcal{S}_\eta(s) \mathcal{F}(t+h-s, y(t+h-s), \Psi_h y(t+h-s)) ds \right. \\ &\quad \left. + \int_\alpha^\infty [\mathcal{S}_\eta(h+s) - \mathcal{S}_\eta(s)] \mathcal{F}(t-s, y(t-s), \Psi_h y(t-s)) ds \right\| \\ &\leq \mathcal{CM} \left[ \int_0^s \frac{\mathcal{W}((1+\|\Psi_h\|)\lambda\mathcal{H}(t+h-s))}{1+|\omega|(s)^\eta} ds \right. \\ &\quad \left. + \int_\alpha^\infty \frac{\mathcal{W}((1+\|\Psi_h\|)\lambda\mathcal{H}(t-s))}{1+|\omega|(s)^\eta} ds \right] \leq \frac{\epsilon}{2}, \end{aligned}$$

for  $h \leq \delta_1$ . Moreover, since  $\mathcal{S}_\eta$  is strongly continuous and  $\{\mathcal{F}(t-s, y(t-s), \Psi_h y(t-s)) : 0 \leq s \leq \alpha, y \in (B_\lambda(C_{\mathcal{H}}(E)))\}$  is relative compact, we can take  $\delta_2 > 0$  such that

$$\|[\mathcal{S}_\eta(h+s) - \mathcal{S}_\eta(s)] \mathcal{F}(t-s, y(t-s), \Psi_h y(t-s))\| \leq \frac{\epsilon}{2\alpha},$$

for  $h \leq \delta_2$ . We have from the above estimation that  $\|w(t+h) - w(t)\| \leq \epsilon$  for small  $\epsilon$  and is independent of  $y \in B_\lambda(C_{\mathcal{H}}(E))$ .



Finally, from  $(L_3)$  we deduce

$$\frac{\|w(t)\|}{\mathcal{H}(t)} \leq \frac{\mathcal{CM}}{\mathcal{H}(t)} \int_0^\infty \frac{\mathcal{W}((1 + \|\Psi_h\|)\lambda\mathcal{H}(s))}{1 + |\omega|(t-s)^\eta} ds \rightarrow 0, \quad \text{as } |t| \rightarrow \infty,$$

uniformly and is independent of  $y \in B_\lambda(C_{\mathcal{H}}(E))$ . Thus, by Lemma 2.3,  $U$  is a relatively compact set in  $C_{\mathcal{H}}(E)$ .

**Step 4 :** Let for some  $0 < \tau < 1$ ,  $y^\tau(\cdot)$  be a solution of the equation  $y = \tau\Lambda_{\mathcal{F}}(y^\tau)$ .

Then, we have the estimate

$$\begin{aligned} \|y^\tau(t)\| &\leq \tau \int_{-\infty}^t \|\mathcal{S}_\eta(t-s)\mathcal{F}(s, y^\tau(s), \Psi_h y^\tau(s))\| ds \\ &\leq \mathcal{CM} \int_{-\infty}^t \frac{\mathcal{W}[(1 + \|\Psi_h\|)\|y^\tau\|_{\mathcal{H}}\mathcal{H}(s)]}{1 + |\omega|(t-s)^\eta} ds \\ &\leq \beta(\|y^\tau\|_{\mathcal{H}})\mathcal{H}(t). \end{aligned}$$

It leads to

$$\frac{\|y^\tau(t)\|}{\beta(\|y^\tau\|_{\mathcal{H}})} < 1.$$

We deduce from the above relation and  $(L_5)$  that the set  $\{y^\tau : y^\tau = \tau\Lambda_{\mathcal{F}}(y^\tau), 0 < \tau < 1\}$  is a bounded set.

**Step 5 :** We deduce from Remark 2.1,  $(L_1)$  and Theorem 2.2 that the function  $t \mapsto \mathcal{F}(t, y(t), \Psi_h y(t)) \in S^pPAA(\mathbb{R}, E, \mu, \nu)$ , whenever  $y \in PAA(\mathbb{R}, E, \mu, \nu) \subset S^pPAA(\mathbb{R}, E, \mu, \nu)$ . Further, by Lemma 2.2, we get  $\Lambda_{\mathcal{F}}(PAA(\mathbb{R}, E, \mu, \nu)) \subset PAA(\mathbb{R}, E, \mu, \nu)$  and notice that  $PAA(\mathbb{R}, E, \mu, \nu)$  is a closed subspace of  $C_{\mathcal{H}}(\mathbb{R}, E)$ . Now, using the Steps 1 – 4, we obtain that the map  $\Lambda_{\mathcal{F}}$  is completely continuous. Applying Lemma 2.4, we infer that mapping  $\Lambda_{\mathcal{F}}$  has a fixed point in  $PAA(\mathbb{R}, E, \mu, \nu)$ .

## Acknowledgment

The authors would like to express their sincere gratitude to the reviewers for their helpful and valuable comments in improving the presentation and quality of the paper. This research was completed with the support of “Ministry of Human Resource and Development, India under grant number: MHR-02-23-200-44”.

## References

- [1] Abdelkarim, N.A., Khalil, E. and Souden, L. Pseudo almost periodic and pseudo almost automorphic mild solutions to nonautonomous partial evolution equation. *Nonauton. dyn. syst.* (2015) 12–30.
- [2] Araya, D. and Lizama, C. Almost automorphic mild solutions to fractional differential equations. *Nonlinear Anal.* **69** (2008) 3692–3705.
- [3] Bing, H., Junfei, C. and Bicheng, Y. Weighted Stepanov-like pseudo-almost automorphic mild solutions for semilinear fractional differential equations. *Adv. diff. eqn.* (2015) 1–36.
- [4] Blot, J., Cieutat, P. and Ezzinbi, K. Measure theory and pseudo almost automorphic functions: New developments and applications. *Nonlinear Anal.* **75**(2012) 2426–2447.

- [5] Blot, J., Cieutat, P. and Ezzinbi, K. New approach for weighted pseudo almost periodic functions under the light of measure theory, basic results and applications. *Appl. Anal.* **92** (2013) 493–526.
- [6] Bouchner, S. Continuous mapping of almost automorphic and almost periodic functions. *Proc. Natl. Acad. Sci. USA* **52**(1964) 907–910.
- [7] Casarino, V. Characterizations of almost automorphic groups and semigroups. *Rend. Accad. Naz. Sci. XL Mem. Appl.* **5**(2000) 219–235.
- [8] Chang, Y.K., N'Guérékata, G.M. and Zhang, R. Stepanov like weighted pseudo almost automorphic functions via measure theory. *Bull. Malaya. Math. Sci. Soc.* DOI 10.1007/s40840-015-026-1. (2014).
- [9] Cuesta, E. Asymptotic behaviour of the solutions of fractional integrodifferential equations and some time discretizations. *Discrete Cont. Dyn. Syst.* (2007) 277–285.
- [10] Diagana, T., Ezzinbi, K. and Miraoui, M. Pseudo-almost periodic and pseudo-almost automorphic solutions to some evolution equations involving theoretical measure theory. *Cubo* **16** (2014) 1–31.
- [11] Diagana, T. and N'Guérékata, G.M. Stepanov-like almost automorphic functions and applications to some semilinear equations. *Appl. Anal.* **86**(2007) 723–733.
- [12] Granas, A. and Dugundji, J. *Fixed Point Theory*. Springer-Verlag, New York, 2003.
- [13] Haase, M. *The Functional Calculus for Sectorial Operators. Operator Theory: Advances and Applications*, vol. 169, Birkhuser, Basel, 2006.
- [14] Liang, J., Zhang, J. and Xiao, T.J. Composition of pseudo almost automorphic and asymptotically almost automorphic functions. *J. Math. Anal. Appl.* **340** (2008) 1493–1499.
- [15] N'Guérékata, G.M. Sur les solutions presque-automorphes d'équations différentielles abstraites. *Ann. Sci. Math. Québec* **5** (1981) 69–79.
- [16] N'Guérékata, G.M. *Topics in Almost Automorphy*. Springer-Verlag, New York, 2005.
- [17] Henriquez, H.R. and Lizama, C. Compact almost automorphic solutions to integral equations with infinite delay. *Nonlinear Anal.* **71**(2009) 6029–6037.
- [18] Podlubny, I. *Fractional Differential Equations*. Academic Press, New York, 1999.
- [19] Syed, A., Kavitha, V. and Murugesu, R. Stepanov-like weighted pseudo almost automorphic solutions to fractional order abstract integro-differential equations. *Proc. Indian Acad. Sci. Math. Sci.* **125** (2015) 323–351.
- [20] Xia, Z. and Fan, M. Weighted Stepanov-like pseudo almost automorphy and applications. *Nonlinear Anal. Theory Methods. Appl.* **75** (2012) 2378–2397.
- [21] Zhang, R., Chang, Y.K. and NGuerekata, G.M. New composition theorems of Stepanov-like weighted pseudo almost automorphic functions and applications to nonautonomous evolution equations. *Nonlinear Anal.: Real World Appl.* **13** (2012) 2866–2879.



# Nonlinear Parabolic Equations with Singular Coefficient and Diffuse Data

K. Zaki<sup>1</sup> and H. Redwane<sup>2\*</sup>

<sup>1</sup> *Laboratoire MISI, FST Settat, Université Hassan 1, 26000 Settat, Morocco.*

<sup>2</sup> *Faculté des Sciences Juridiques, Économiques et Sociales, Université Hassan 1, B.P. 764. Settat. Morocco.*

Received: January 7, 2017; Revised: October 22, 2017

**Abstract:** In this paper we introduce a notion of renormalized solution for nonlinear parabolic problems whose model is  $\frac{\partial b(u)}{\partial t} - \Delta A(u) - \operatorname{div}(\Phi(x, t, u)Du) = \mu$  in  $Q$ , where  $b$  is a strictly increasing  $C^1$ -function defined on  $\mathbb{R}$ , and  $A(z) = \int_0^z a(s)ds$ . The function  $a(s)$  is continuous on an interval  $] -\infty, m[$  of  $\mathbb{R}$  such that  $a(u)$  blows up for a finite value  $m$  of the unknown  $u$ ,  $\Phi$  is a Carathéodory function and  $\mu$  is a diffuse measure.

**Keywords:** *nonlinear parabolic equations; renormalized solutions; soft measure.*

**Mathematics Subject Classification (2010):** Primary 47A15, Secondary 46A32, 47D20.

## 1 Introduction

Let  $\Omega$  be a bounded open set of  $\mathbb{R}^N$  ( $N \geq 1$ ),  $T$  be a positive real number, and  $Q = \Omega \times (0, T)$ .

In this paper we deal with the existence of a renormalized solution for a class of nonlinear parabolic equations of the type

$$\frac{\partial b(u)}{\partial t} - \Delta A(u) - \operatorname{div}(\Phi(x, t, u)Du) = \mu \text{ in } Q, \quad (1)$$

$$b(u(t=0)) = b(u_0) \text{ in } \Omega, \quad (2)$$

$$u = 0 \text{ on } \partial\Omega \times (0, T). \quad (3)$$

---

\* Corresponding author: [mailto:redwane\\_hicham@yahoo.fr](mailto:redwane_hicham@yahoo.fr)

In problem (1)-(3), the function  $b$  is assumed in  $C^1(\mathbb{R})$ , such that it is strictly increasing, and  $A(z) = \int_0^z a(s)ds$ , where the function  $a \in C^0(]-\infty, m[, \mathbb{R}^+)$  ( $m$  is a positive real number) such that  $\lim_{s \rightarrow m^-} a(s) = +\infty$ . The function  $\Phi$  is Carathéodory on  $Q \times \mathbb{R}$  with values in  $\mathbb{R}^+$  and  $u_0 \in L^1(\Omega)$  such that  $u_0 \leq m$  a.e. in  $\Omega$ .

We study problem (1)-(3) in the presence of diffuse measure data  $\mu$ . We call a finite measure  $\mu$  diffuse if it does not charge sets of zero 2-capacity and  $\mathcal{M}_0(Q)$  will denote the set of all diffuse measures in  $Q$  (see, [14]). In [9] the authors proved that for every  $\mu \in \mathcal{M}_0(Q)$  there exist  $f \in L^1(Q)$ ,  $g \in L^2(0, T; H_0^1(\Omega))$  and  $G \in L^2(0, T; H^{-1}(\Omega))$  such that  $\mu = f + G + g_t$  in  $\mathcal{D}'(Q)$ . For  $v = b(u) - g$ , equation (1) is equivalent in  $\mathcal{D}'(Q)$  to  $\frac{\partial v}{\partial t} - \operatorname{div}(a(b^{-1}(v+g))D(b^{-1}(v+g))) - \operatorname{div}(\Phi(x, t, b^{-1}(v+g))D(b^{-1}(v+g))) = f + G$  with  $f + G \in L^1(Q) + L^2(0, T; H^{-1}(\Omega))$ . The first difficulty in solving this equation is defining the field  $a(b^{-1}(v+g))D(b^{-1}(v+g))$  on the subset  $\{(x, t); v+g = b(m)\}$  of  $Q$ , since on this set,  $a(b^{-1}(v+g)) = +\infty$ . In addition, the field  $\Phi(x, t, b^{-1}(v+g))D(b^{-1}(v+g)) \notin \mathcal{D}'(Q)$  in general, since  $g \notin L^\infty(Q)$  in general.

The second difficulty is represented here by the presence of the measure data  $\mu$  and the nonlinear term  $b(u)$ . To overcome these difficulties, we use in this paper the framework of renormalized solutions. A large number of papers was then devoted to the study of renormalized (or entropy) solution of parabolic problems with rough data under various assumptions and in different contexts: in addition to the references already mentioned, see, e.g., [1, 3, 6–8, 10, 11].

The existence of a renormalized solution of (1)-(3) has been proved in [2] in the stationary case where  $\Phi(x, t, u) = 0$  and  $\mu \in L^2(\Omega)$ .

The existence and uniqueness of renormalized solution of (1)-(3) have been proved in [9], in the case where  $u_0 \in L^1(\Omega)$  and  $\Delta A(u)$  is replaced by  $p$ -Laplacian operator  $\Delta_p u$ ,  $\Phi(x, t, u) = 0$  and for every measure  $\mu \in \mathcal{M}_0(Q)$ . In the case where  $b(u) = u$ ,  $\Delta A(u)$  is replaced by  $-\operatorname{div}(a(t, x, u, \nabla u))$ ,  $\Phi(x, t, u) = \Phi(u)$  and  $\mu = f + \operatorname{div} g$  where  $f \in L^1(Q)$  and  $g \in (L^{p'}(Q))^N$ , the existence of renormalized solution has been proved in [5].

When  $b$  is assumed to satisfy  $0 < b_0 \leq b'(r) \leq b_1$ ,  $\forall r \in \mathbb{R}$ , and  $\Delta A(u)$  is replaced by  $\operatorname{div}(a(x, t, \nabla u))$ ,  $\Phi(x, t, u) = 0$  and  $\mu \in \mathcal{M}_0(Q)$ , the existence and uniqueness of renormalized solution have been established in [4].

In the stationary and evolution cases of  $u_t - \operatorname{div}(A(x, t, u)\nabla u) = f$  in  $Q$ , where the matrix  $A(x, t, s)$  blows up (uniformly with respect to  $(x, t)$ ) as  $s \rightarrow m^-$  and where  $f \in L^1(Q)$ , the existence of renormalized solution has been proved in [3].

In the case of  $u_t - \operatorname{div}(d(u)Du) = \mu$ , where the coefficients  $d(s) = (d_i(s))$  are continuous on an interval  $]-\infty, m[$  of  $\mathbb{R}$  (with  $m > 0$ ) with value in  $\mathbb{R}^+$ ,  $u_0 \in L^1(\Omega)$  and  $\mu \in \mathcal{M}_0(Q)$ , the existence of renormalized solution has been proved in [15]. Our goal is to extend the approach from [15].

The organization of the paper is the following. In Section 2, we give some preliminaries on the concept of  $p$ -capacity and set out the main notation we will use throughout the paper. Section 3 will be devoted to the exposition of our main assumptions and to the definition of renormalized solution of (1)-(3). In Section 4 (Theorem 4.1) we establish the existence of such a solution. In Section 5 (Appendix), we provide the proof of Theorem 2.2. Section 6 is devoted to an example which illustrates our abstract result.

## 2 Preliminaries on Parabolic Capacity and Measures

For every open subset  $U \subset Q$  the 2-parabolic capacity of  $U$  is given by (for further details see, [9, 14]):  $\text{cap}_2(U) = \inf \left\{ \|u\|_W : u \in W, u \geq \chi_U \text{ a.e. in } Q \right\}$ , where  $W = \left\{ u \in L^2(0, T; H_0^1(\Omega)), u_t \in L^2(0, T; H^{-1}(\Omega)) \right\}$ , endowed with the norm  $\|u\|_W = \|u\|_{L^2(0, T; H_0^1(\Omega))} + \|u_t\|_{L^2(0, T; H^{-1}(\Omega))}$ . The 2-parabolic capacity is then extended to arbitrary Borel set  $B \subseteq Q$  as  $\text{cap}_2(B) = \inf \left\{ \text{cap}_2(U) : U \text{ open set of } Q, B \subseteq U \right\}$ . We will denote by  $\mathcal{M}(Q)$  the set of all Radon measures with bounded variation on  $Q$ , while, as we have already mentioned,  $\mathcal{M}_0(Q)$  will denote the set of all measures with bounded variation over  $Q$  that do not charge the sets of zero 2-capacity, that is: if  $\mu \in \mathcal{M}_0(Q)$  then  $\mu(E) = 0$  for all  $E \subseteq Q$  such that  $\text{cap}_2(E) = 0$ .

In [9] the authors proved the following decomposition theorem:

**Theorem 2.1** *Let  $\mu$  be a bounded measure on  $Q$ . If  $\mu \in \mathcal{M}_0(Q)$ , then there exists  $(f, G, g)$  such that  $f \in L^1(Q)$ ,  $G \in L^2(0, T; H^{-1}(\Omega))$ ,  $g \in L^2(0, T; H_0^1(\Omega))$  and*

$$\int_Q \phi d\mu = \int_Q f \phi dx dt + \int_0^T \langle G, \phi \rangle dt - \int_0^T \langle \phi_t, g \rangle dt \quad \phi \in C_c^\infty(\Omega \times [0, T]).$$

Such a triplet  $(f, G, g)$  will be called a decomposition of  $\mu$ .

Note that the decomposition of  $\mu$  is not uniquely determined.

The following theorem will be a key point in the existence result given in the next section. The proof follows the arguments in Theorem 1.2 in [13].

**Theorem 2.2** *Let  $a \in C^0(\mathbb{R}) \cap L^\infty(\mathbb{R})$ ,  $b \in C^1(\mathbb{R})$  with  $0 < \beta \leq b' \leq \gamma$ ,  $\Phi$  be a Carathéodory function such that  $\Phi \in L^\infty(Q \times \mathbb{R})$ ,  $\mu \in \mathcal{M}_0(Q) \cap L^2(0, T; H^{-1}(\Omega))$  and  $u_0 \in L^2(\Omega)$ , let  $u \in W$  be the (unique) weak solution of*

$$\begin{cases} \frac{\partial b(u)}{\partial t} - \Delta A(u) - \text{div}(\Phi(x, t, u)Du) = \mu & \text{in } Q, \\ b(u(t=0)) = b(u_0) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \times (0, T). \end{cases} \quad (4)$$

Then,  $\text{cap}_2\{|u| > K\} \leq \frac{C}{\sqrt{K}} \quad \forall K \geq 1$ , where  $C > 0$  is a constant depending on  $\|\mu\|_{\mathcal{M}(Q)}$  and  $\|u_0\|_{L^2(\Omega)}$ .

**Proof.** The proof of Theorem 2.2 is postponed to the Appendix in Section 5.  $\square$

**Definition 2.1** A sequence of measures  $(\mu_n)$  in  $Q$  is equidiffuse if for every  $\eta > 0$  there exists  $\delta > 0$  such that  $\text{cap}_2(E) < \delta \implies |\mu_n|(E) < \eta \quad \forall n \geq 1$ .

The following result is proved in [13]:

**Lemma 2.1** *Let  $\rho_n$  be a sequence of mollifiers on  $Q$ . If  $\mu \in \mathcal{M}_0(Q)$ , then the sequence  $(\rho_n * \mu_n)$  is equidiffuse.*

Here are some notations we will use throughout the paper. For any nonnegative real number  $K$  we denote by  $T_K(r) = \min(K, \max(r, -K))$  the truncation function at level  $K$  for every  $r \in \mathbb{R}$ . We consider the following smooth approximation of  $T_K(s)$ : for  $m > 0$ ,  $\eta \in ]0, 1[$  and  $\sigma \in ]0, 1[$ , we define  $S_{K,\sigma}^m, T_K^m : \mathbb{R} \rightarrow \mathbb{R}$  by

$$S_{K,\sigma}^{m,\eta}(s) = \begin{cases} 1 & \text{if } -K \leq s \leq m - \sigma, \\ 0 & \text{if } s \leq -K - \eta \text{ or } s \geq m, \text{ and } T_K^m(s) = \begin{cases} s & \text{if } -K \leq s \leq m, \\ -K & \text{if } s \leq -K, \\ m & \text{if } s \geq m, \end{cases} \\ \text{affine} & \text{otherwise,} \end{cases} \quad (5)$$

and let us denote  $T_{K,\sigma}^{m,\eta}(z) = \int_0^z S_{K,\sigma}^{m,\eta}(s) ds$ .

### 3 Main Assumptions and Definition of Renormalized Solution

Throughout the paper, we assume that the following assumptions hold true:  $\Omega$  is a bounded open set on  $\mathbb{R}^N$  ( $N \geq 2$ ),  $T > 0$  is given and we set  $Q = \Omega \times (0, T)$ .

$b : \mathbb{R} \rightarrow \mathbb{R}$  is a strictly increasing  $\mathcal{C}^1$ -function such that  $0 < \beta \leq b'$  and  $b(0) = 0$ , (6)

$$a \in C^0([-\infty, m[, \mathbb{R}^+) \text{ with } a(s) < +\infty \quad \forall s < m, \quad (7)$$

$$\exists \alpha > 0 \text{ such that } : a(s) \geq \alpha, \quad \forall s \in ]-\infty, m[, \quad (8)$$

$$\lim_{s \rightarrow m^-} a(s) = +\infty \text{ and } \int_0^m a(s) ds < +\infty, \quad (9)$$

$$\Phi : Q \times \mathbb{R} \rightarrow \mathbb{R}^+ \text{ is a Carathéodory function such that } \Phi(x, t, 0) = 0, \quad (10)$$

$$\max_{\{|r| < K\}} |\Phi(x, t, r)| \in L^\infty(Q) \quad \text{for all } K > 0, \quad (11)$$

$$\mu \in \mathcal{M}_0(Q), \quad (12)$$

$$u_0 \in L^1(\Omega) \text{ such that } u_0 \leq m \text{ a.e. in } \Omega. \quad (13)$$

We now give the definition of a renormalized solution of problem (1)-(3).

**Definition 3.1** A function  $u \in L^1(Q)$  is a renormalized solution of problem (1)-(3) if

$$u \leq m \text{ a.e. in } Q \text{ and } T_K(u) \in L^2(0, T; H_0^1(\Omega)) \quad \forall K > 0, \quad (14)$$

$$a(u)DT_K^m(u)\chi_{\{u < m\}} \in (L^2(Q))^N \quad \forall K > 0, \quad (15)$$

if there exist a sequence of nonnegative measures  $\Lambda_K \in \mathcal{M}(Q)$  and a nonnegative measure  $\Gamma_m \in \mathcal{M}(Q)$  such that

$$\lim_{K \rightarrow +\infty} \|\Lambda_K\|_{\mathcal{M}(Q)} = 0, \quad (16)$$

$$\int_Q \varphi d\Gamma_m = 0 \quad \forall \varphi \in \mathcal{C}_0^1([0, T]), \quad (17)$$

and if, for every  $K > 0$

$$\begin{aligned} \frac{\partial B_K^m(u)}{\partial t} - \operatorname{div} \left( a(u) D T_K^m(u) \chi_{\{u < m\}} \right) - \operatorname{div} \left( \Phi(x, t, T_K^m(u)) D T_K^m(u) \right) \\ = \mu + \Lambda_K + \Gamma_m \quad \text{in } \mathcal{D}'(Q), \end{aligned} \quad (18)$$

where  $B_K^m(z) = \int_0^z b'(s) (T_K^m)'(s) ds$ .

**Remark 3.1** 1/ Note that, in view of (14), (15) and (16), all terms in (18) are well defined. 2/ Let us point out that, in (17), the function  $\varphi \in \mathcal{C}_0^1([0, T])$  does not depend on the variable  $x$ , we are not able to prove (17) with any function  $\varphi \in L^2(0, T; H^1(\Omega)) \cap L^\infty(Q)$  such that  $D\varphi = 0$  a.e. in  $\{(x, t) ; u(x, t) = m\}$  because of a lack of regularity on  $u$  with respect to  $t$  in the parabolic case.

#### 4 Existence of a Renormalized Solution

This section is devoted to establishing the following existence theorem.

**Theorem 4.1** *Under assumptions (6)-(13) there exists at least one renormalized solution of problem (1)-(3) in the sense of Definition 3.1.*

**Proof.** The proof is divided into 4 steps. At Step 1, we introduce an approximate problem. Step 2 is devoted to establishing a few *a priori* estimates and we prove that  $u$  satisfies (14) and (15) of Definition 3.1. At last, Step 3 and Step 4 are aimed to prove that  $u$  satisfies (16), (17) and (18) of Definition 3.1.  $\square$

★ **Step 1.** A regularized problem.

Let us introduce the following regularization of the data: for  $n \geq 1$  fixed

$$b_n(s) = b(T_n(s)) + \frac{1}{n}s \quad \text{and} \quad a^n(s) = a\left(T_{\frac{1}{n}}^{m-\frac{1}{n}}(s)\right) \quad \forall s \in \mathbb{R}, \quad (19)$$

$$u_0^n \in C_c^\infty(\Omega) : b_n(u_0^n) \rightarrow b(u_0) \text{ strongly in } L^1(\Omega) \text{ as } n \text{ tends to } +\infty, \quad (20)$$

$$\Phi_n(x, t, s) = \Phi(x, t, T_n(s)) \quad \forall s \in \mathbb{R}. \quad (21)$$

We consider a sequence of mollifiers  $(\rho_n)$ , and we define the convolution  $\rho_n * \mu$  for every  $(x, t) \in Q$  by  $\mu^n(x, t) = \rho_n * \mu(x, t) = \int_Q \rho_n(x - y, t - s) d\mu(y, s)$ . Let us now consider the following regularized problem

$$\frac{\partial b_n(u^n)}{\partial t} - \Delta A^n(u^n) - \operatorname{div}(\Phi_n(x, t, u^n) Du^n) = \mu^n \quad \text{in } Q, \quad (22)$$

$$b_n(u^n(t = 0)) = b_n(u_0^n) \quad \text{in } \Omega, \quad (23)$$

$$u^n = 0 \quad \text{on } \partial\Omega \times (0, T). \quad (24)$$

As a consequence, proving existence of a weak solution  $u^n \in L^2(0, T; H_0^1(\Omega))$  of (22)-(24) is an easy task (see e.g. [12]).

★ **Step 2.** A priori estimates. Taking  $T_K(u^n)$  as a test function in (22) gives

$$\int_\Omega B_K^n(u^n)(T) dx + \int_Q D A^n(u^n) D T_K(u^n) dx dt \quad (25)$$

$$+ \int_Q \Phi_n(x, t, u^n) Du^n DT_K(u^n) dx dt = \int_Q \mu^n T_K(u^n) dx dt + \int_\Omega B_K^n(u_0^n) dx,$$

where  $B_K^n(z) = \int_0^z b'_n(s) T_K(s) ds$ . We deduce

$$\int_\Omega B_K^n(u^n)(T) dx + \int_Q (a^n(u^n) + \Phi_n(x, t, u^n)) |DT_K(u^n)|^2 dx dt \leq CK \quad (26)$$

since  $\|\mu^n\|_{L^1(Q)}$  and  $\|b_n(u_0^n)\|_{L^1(\Omega)}$  are bounded. We deduce for any  $K \geq 0$

$$T_K(u^n) \text{ is bounded in } L^2(0, T; H_0^1(\Omega)), \quad (27)$$

and

$$a^n(u^n)^{\frac{1}{2}} DT_K(u^n) \text{ is bounded in } (L^2(Q))^N. \quad (28)$$

Now, using  $\frac{1}{r} T_r(u^n) \chi_{(0,t)}$  as a test function in (22) we obtain

$$\int_\Omega \frac{1}{r} B_r^n(u^n) dx + \frac{1}{r} \int_0^t \int_\Omega (a^n(u^n) + \Phi_n(x, t, u^n)) |DT_r(u^n)|^2 dx dt \leq C, \quad (29)$$

where  $B_r^n(z) = \int_0^z b'_n(s) T_r(s) ds$ . The second term in the left-hand side of the above inequality is nonnegative. Taking the limit in (29) as  $r$  tends to 0 we obtain  $b_n(u^n)$  is bounded in  $L^\infty(0, T; L^1(\Omega))$ . According to (7)-(9), we have for any  $K \geq 0$ ,  $\left| \int_0^{u^n} a^n(s) \chi_{\{-K \leq s \leq m\}} dx \right| \leq \int_{-K}^m a(s) ds \equiv C_K < +\infty$ , then we can use  $\int_0^{u^n} a^n(s) \chi_{\{-K \leq s \leq m\}} ds$  in  $L^2(0, T; H_0^1(\Omega)) \cap L^\infty(Q)$  as a test function in (22), we have

$$\beta \int_\Omega \int_0^{u^n} \int_0^z a^n(s) \chi_{\{-K \leq s \leq m\}} ds dz dx \quad (30)$$

$$+ \int_Q \left( (a^n(u^n))^2 + \Phi_n(x, t, u^n) a^n(u^n) \right) |DT_K^m(u^n)|^2 \leq (\|\mu^n\|_{L^1} + \|b_n(u_0^n)\|_{L^1}) \int_{-K}^m a(s) ds.$$

Since  $\int_\Omega \int_0^{u^n} \int_0^z a^n(s) ds dz dx$  and  $\int_Q \Phi_n(x, t, u^n) a^n(u^n) |DT_K^m(u^n)|^2 dx dt$  are positives,  $\|\mu^n\|_{L^1(Q)}$  and  $\|b_n(u_0^n)\|_{L^1(\Omega)}$  are bounded, we deduce from (30) that

$$a^n(u^n) DT_K^m(u^n) \text{ is bounded in } (L^2(Q))^N. \quad (31)$$

For any integer  $M \geq 1$ , let  $S_M$  be an increasing function of  $C^\infty(\mathbb{R})$  and such  $S_M(r) = r$  for  $|r| \leq \frac{M}{2}$  and  $S_M(r) = M \operatorname{sg}(r)$  for  $|r| \geq M$ . Note that for any  $M$ ,  $\operatorname{supp} S'_M \subset [-M, M]$ . We will show that for any fixed integer  $M$  the sequence  $S_M(b_n(u^n))$  satisfies

$$S_M(b_n(u^n)) \text{ is bounded in } L^2(0, T; H_0^1(\Omega)), \quad (32)$$

and

$$\frac{\partial S_M(b_n(u^n))}{\partial t} \text{ is bounded in } L^1(Q) + L^2(0, T; H^{-1}(\Omega)) \quad (33)$$



independently of  $n$ . Due to the definition of  $b_n$ , it is clear that for  $|b_n(u^n)| \leq M$  we have  $|b(T_n(u^n))| \leq M$  and  $|u^n| < K_M$  as soon as  $n > K_M$  and where  $K_M = \max\{b^{-1}(M), |b^{-1}(-M)|\}$ . As a first consequence we obtain  $DS_M(b_n(u^n)) = S'_M(b_n(u^n))b'_n(u^n)DT_{K_M}(u^n)$  as soon as  $n > K_M$ , since  $S'_M(b_n(u^n)) = 0$  on the set  $\{|b_n(u^n)| > M\}$ , and  $K_M = \max\{-b^{-1}(M), |b^{-1}(-M)|\}$ . Secondly, the following estimate holds true  $\|S'_M(b_n(u^n))b'_n(u^n)\|_{L^\infty(Q)} \leq \|S'_M\|_{L^\infty(\mathbb{R})} \left( \max_{|r| \leq K_M} |b'(r)| + 1 \right)$  as soon as  $n > K_M$ . Since  $b'$  is continuous on  $\mathbb{R}$ , it follows that for any integer  $M$ ,  $S'_M(b_n(u^n))b'_n(u^n)$  is bounded in  $L^\infty(Q)$  independently of  $n$  as soon as  $n > K_M$ . As a consequence of (27) we then obtain (32).

To show that (33) holds true, we multiply the equation (22) by  $S'_M(b_n(u^n))$  to obtain

$$\frac{\partial S_M(b_n(u^n))}{\partial t} = \operatorname{div} \left( S'_M(b_n(u^n))a^n(u^n)Du^n \right) - S''_M(b_n(u^n))b'_n(u^n)a^n(u^n)|Du^n|^2 \quad (34)$$

$$+ \operatorname{div} \left( S'_M(b_n(u^n))\Phi_n(x, t, u^n)Du^n \right) - S''_M(b_n(u^n))b'_n(u^n)\Phi_n(x, t, u^n)|Du^n|^2 + \mu^n S'_M(b_n(u^n))$$

in  $\mathcal{D}'(Q)$ . Each term in the right-hand side of (34) is bounded either in  $L^2(0, T; H^{-1}(\Omega))$  or in  $L^1(Q)$ . Indeed, since  $\operatorname{supp} S'_M$  and  $\operatorname{supp} S''_M$  are both included in  $[-M, M]$ ,  $u^n$  may be replaced by  $T_{K_M}(u^n)$  in each of these terms.

Proceeding as in [5] we see that estimates (32) and (33) imply that, for a subsequence still indexed by  $n$ ,  $b_n(u^n) \rightarrow \chi$  almost everywhere in  $Q$ . Since  $b^{-1}$  is continuous on  $\mathbb{R}$ ,  $b_n^{-1}$  converges everywhere to  $b^{-1}$  when  $n$  goes to  $\infty$ , so that  $u^n \rightarrow u = b^{-1}(\chi)$  a.e. in  $Q$  and using (27), (28) and (31), we obtain

$$b_n(u^n) \rightarrow b(u) \text{ almost everywhere in } Q, \quad (35)$$

$$T_K(u^n) \rightharpoonup T_K(u) \text{ weakly in } L^2(0, T; H_0^1(\Omega)), \quad (36)$$

$$(a^n(u^n))^{\frac{1}{2}}DT_K(u^n) \rightharpoonup X_K \text{ weakly in } (L^2(Q))^N, \quad (37)$$

$$a^n(u^n)DT_K^m(u^n) \rightharpoonup Y_K \text{ weakly in } (L^2(Q))^N. \quad (38)$$

By using the admissible test function  $T_{2m}^{n+}(u^n) - T_m^{n+}(u^n)$  in (22) we have

$$\int_Q (a^n(u^n) + \Phi_n(x, t, u^n)) |D(T_{2m}^{n+}(u^n) - T_m^{n+}(u^n))|^2 dx dt \leq Cm. \quad (39)$$

Now, since  $\Phi_n(x, t, u^n) \geq 0$ , and in view of (19) and the Poincaré inequality we deduce

$$a(m - \frac{1}{n}) \int_Q |T_{2m}^{n+}(u^n) - T_m^{n+}(u^n)|^2 dx dt \leq Cm. \quad (40)$$

According to (9) and (20) (since  $d_p(m - \frac{1}{n}) \rightarrow +\infty$  as  $n$  tends to  $+\infty$ ) passing to the limit in (40) as  $n$  tends to  $+\infty$ , we deduce that  $T_{2m}^+(u) - T_m^+(u) = 0$  a.e. in  $Q$ , hence

$$u \leq m \quad \text{a.e. in } Q. \quad (41)$$

In view of (37), (38) and (41) we deduce for any  $K \geq 0$

$$X_K = (a(u))^{\frac{1}{2}}DT_K(u) \text{ and } Y_K = a(u)DT_K^m(u) \text{ a.e. in } \{(x, t) \in Q / u(x, t) < m\}. \quad (42)$$

We define, for any fixed  $K \geq 1$ ,  $0 < \eta < 1$  and  $0 < \sigma < 1$ , the functions  $H_{K,\eta}$  and  $Z_{m,\sigma}$  by

$$H_{K,\eta}(s) = \begin{cases} -1, & \text{if } s \leq -K - \eta, \\ 0, & \text{if } s \geq -K, \\ \text{affine,} & \text{otherwise,} \end{cases} \quad \text{and} \quad Z_{m,\sigma}(s) = \begin{cases} 0, & \text{if } s \leq m - \sigma, \\ 1, & \text{if } s \geq m, \\ \text{affine,} & \text{otherwise.} \end{cases} \quad (43)$$

We use the admissible test functions  $H_{K,\eta}(u^n)$  and  $Z_{m,\sigma}(u^n)$  in (22) to get

$$\int_{\Omega} \bar{H}_{K,\eta}(u^n)(T) dx + \int_Q DA^n(u^n) DH_{K,\eta}(u^n) dx dt \quad (44)$$

$$+ \int_Q \Phi_n(x, t, u^n) Du^n DH_{K,\eta}(u^n) dx dt = \int_Q H_{K,\eta}(u^n) \mu^n dx dt + \int_{\Omega} \bar{H}_{K,\eta}(u_0^n) dx,$$

and

$$\int_{\Omega} \bar{Z}_{m,\sigma}(u^n)(T) dx + \int_Q DA^n(u^n) DZ_{m,\sigma}(u^n) dx dt \quad (45)$$

$$+ \int_Q \Phi_n(x, t, u^n) Du^n DZ_{m,\sigma}(u^n) dx dt = \int_Q Z_{m,\sigma}(u^n) \mu^n dx dt + \int_{\Omega} \bar{Z}_{m,\sigma}(u_0^n) dx,$$

where  $\bar{H}_{K,\eta}(r) = \int_0^r b'_n(s) H_{K,\eta}(s) ds \geq 0$  for  $r \leq 0$  and  $\bar{Z}_{m,\sigma}(r) = \int_0^r b'_n(s) Z_{m,\sigma}(s) ds \geq 0$  for  $r \geq 0$ . Hence, using (43) and dropping a nonnegative term, we obtain

$$\begin{aligned} & \frac{1}{\eta} \int_{\{-K-\eta \leq u^n \leq -K\}} (a^n(u^n) + \Phi_n(x, t, u^n)) |Du^n|^2 dx dt \\ & \leq \int_{\{u^n \leq -K\}} |\mu^n| dx dt + \int_{\{u_0^n \leq -K\}} |b_n(u_0^n)| dx \leq C_1, \end{aligned} \quad (46)$$

and

$$\frac{1}{\sigma} \int_{\{m-\sigma \leq u^n \leq m\}} (a^n(u^n) + \Phi_n(x, t, u^n)) |Du^n|^2 dx dt \leq \|\mu^n\|_{L^1(Q)} + \|b_n(u_0^n)\|_{L^1(\Omega)} \leq C_2. \quad (47)$$

Thus, there exists a bounded Radon measure  $\Psi_K^n$ , as  $\eta$  tends to zero

$$\Psi_{K,\eta}^n \equiv \frac{1}{\eta} (a^n(u^n) + \Phi_n(x, t, u^n)) |Du^n|^2 \chi_{\{-K-\eta \leq u^n \leq -K\}} \rightharpoonup \Psi_K^n * - \text{ weakly in } \mathcal{M}(Q). \quad (48)$$

★ **Step 3.** At this step we prove that  $u$  satisfies (18). Let  $S_{K,\sigma}^{m,\eta}$  be the function defined by (5) for all real numbers  $\sigma > 0$ ,  $\eta > 0$  and  $K > 0$ . Since  $\text{supp}(S_{K,\sigma}^{m,\eta})' \subset [-K - \eta, -K] \cup [m - \sigma, m]$ , we multiply the equation (22) by  $S_{K,\sigma}^{m,\eta}(u^n)$  to get

$$\frac{\partial B_{K,\sigma}^{n,m,\eta}(u^n)}{\partial t} - \text{div} \left( DA^n(u^n) S_{K,\sigma}^{m,\eta}(u^n) \right) + DA^n(u^n) DS_{K,\sigma}^{m,\eta}(u^n) \quad (49)$$

$$- \text{div} \left( \Phi_n(x, t, u^n) Du^n S_{K,\sigma}^{m,\eta}(u^n) \right) + \Phi_n(x, t, u^n) Du^n DS_{K,\sigma}^{m,\eta}(u^n) = \mu^n S_{K,\sigma}^{m,\eta}(u^n) \text{ in } \mathcal{D}'(Q),$$

where  $B_{K,\sigma}^{n,m,\eta}(z) = \int_0^z b'_n(s) S_{K,\sigma}^{m,\eta}(s) ds$ . Let

$$\lambda_{m,\sigma}^n \equiv \frac{1}{\eta} (a^n(u^n) + \Phi_n(x, t, u^n)) |Du^n|^2 \chi_{\{m-\sigma \leq u^n \leq m\}}. \quad (50)$$

From (48), (50) and (49), we deduce that

$$\begin{aligned} \frac{\partial B_{K,\sigma}^{n,m,\eta}(u^n)}{\partial t} - \operatorname{div} \left( DA^n(u^n) S_{K,\sigma}^{m,\eta}(u^n) \right) - \operatorname{div} \left( \Phi_n(x, t, u^n) DT_{K,\sigma}^{m,\eta}(u^n) \right) \\ = \mu^n + \left( S_{K,\sigma}^{m,\eta}(u^n) - 1 \right) \mu^n - \Psi_{K,\eta}^n + \lambda_{m,\sigma}^n \quad \text{in } \mathcal{D}'(Q). \end{aligned} \quad (51)$$

Passing to the limit in (51) as  $\eta$  tends to zero, we deduce

$$\begin{aligned} \frac{\partial B_{K,\sigma}^{n,m}(u^n)}{\partial t} - \operatorname{div} \left( DA^n(u^n) S_{K,\sigma}^m(u^n) \right) - \operatorname{div} \left( \Phi_n(x, t, u^n) DT_{K,\sigma}^m(u^n) \right) \\ = \mu^n - \mu^n \chi_{\{u^n < -K\}} - Z_{m,\sigma}(u^n) \mu^n - \Psi_K^n + \lambda_{m,\sigma}^n \quad \text{in } \mathcal{D}'(Q). \end{aligned} \quad (52)$$

We define the measures  $\Lambda_K^n = -\mu^n \chi_{\{u^n < -K\}} - \Psi_K^n$  and  $\Gamma_{m,\sigma}^n = -Z_{m,\sigma}(u^n) \mu^n + \lambda_{m,\sigma}^n$ . Now, using the properties of convolution  $\mu_n = \rho_n * \mu$  and in view of (46), (47), (48) and (50), we deduce that  $\Lambda_K^n$  and  $\Gamma_{m,\sigma}^n$  are bounded in  $L^1(Q)$  independently of  $n$ , so that there exist bounded measures  $\Lambda_K$  and  $\Gamma_{m,\sigma}$  such that  $\Lambda_K^n \rightharpoonup \Lambda_K$   $*$ -weakly in  $\mathcal{M}(Q)$  and  $\Gamma_{m,\sigma}^n \rightharpoonup \Gamma_{m,\sigma}$   $*$ -weakly in  $\mathcal{M}(Q)$ . We deduce from (35), (36), (38), (41) (42) and (52) that  $u$  satisfies

$$\begin{aligned} B_{K,\sigma}^m(u)_t - \operatorname{div} \left( a(u) DT_K^m(u) S_{K,\sigma}^m(u) \chi_{\{u < m\}} \right) \\ - \operatorname{div} \left( \Phi(x, t, T_K^m(u)) DT_{K,\sigma}^m(u) \right) = \mu + \Lambda_K + \Gamma_{m,\sigma} \quad \text{in } \mathcal{D}'(Q). \end{aligned} \quad (53)$$

To end the proof of (18), we use

$$\int_Q |\Gamma_{m,\sigma}| \, dx \, dt \leq \liminf_{n \rightarrow +\infty} \int_Q |\Gamma_{m,\sigma}^n| \, dx \, dt \leq 2 \|\mu\|_{\mathcal{M}(Q)} + \|b(u_0)\|_{L^1(\Omega)}$$

so that there exists a bounded measure  $\Gamma_m$  such that  $\Gamma_{m,\sigma}$  converges to  $\Gamma_m$   $*$ -weakly in  $\mathcal{M}(Q)$ . Therefore, as  $\sigma$  tends to zero in (53), we obtain in  $\mathcal{D}'(Q)$

$$\frac{\partial B_K^m(u)}{\partial t} - \operatorname{div} \left( a(u) DT_K^m(u) \chi_{\{u < m\}} \right) - \operatorname{div} \left( \Phi(x, t, T_K^m(u)) DT_K^m(u) \right) = \mu + \Lambda_K + \Gamma_m, \quad (54)$$

where  $B_K^m(z) = \int_0^z b'(s) (T_K^m)'(s) ds$ , and (18) is then established.

★ **Step 4.** At this step we prove that  $\Lambda_K$  and  $\Gamma_m$  satisfy (16) and (17). From (46) and (48), it follows that

$$\|\Lambda_K^n\|_{L^1(Q)} = \left\| -\mu^n \chi_{\{u^n < -K\}} + \Psi_K^n \right\|_{L^1(Q)} \leq 2 \int_{\{u^n < -K\}} |\mu^n| \, dx \, dt + \int_{\{u_0^n < -K\}} |b_n(u_0^n)| \, dx. \quad (55)$$

Since  $\|\Lambda_K\|_{\mathcal{M}(Q)} \leq \liminf_{n \rightarrow +\infty} \|\mu^n \chi_{\{u^n < -K\}} + \Psi_K^n\|_{\mathcal{M}(Q)}$ , the sequence  $(\mu^n)$  is equidiffuse, and the function  $b_n(u_0^n)$  converges to  $b(u_0)$  strongly in  $L^1(\Omega)$ , we deduce from theorem

2.2 and (55) that  $\|\Lambda_K\|_{\mathcal{M}(Q)}$  tends to zero as  $K$  tends to infinity, then we obtain (16).

To prove (17), we can write for all  $\varphi \in C_0^1([0, T])$

$$\int_Q \varphi d\Gamma_m = \lim_{\sigma \rightarrow 0} \int_Q \varphi d\Gamma_{m\sigma} = \lim_{\sigma \rightarrow 0} \lim_{n \rightarrow +\infty} \int_Q \varphi \Gamma_\sigma^n dx dt, \quad (56)$$

where  $\Gamma_{m,\sigma}^n = \lambda_{m,\sigma}^n - Z_{m,\sigma}(u^n)\mu^n$ . Taking the admissible test function  $Z_{m,\sigma}(u^n)\varphi$  in (22), we have

$$\begin{aligned} & - \int_Q \bar{Z}_{m,\sigma}(u^n)\varphi_t dx dt - \int_\Omega \bar{Z}_{m,\sigma}(u_0^n)\varphi(0) dx + \int_Q DA^n(u^n)D(Z_{m,\sigma}(u^n)\varphi) dx dt \\ & + \int_Q \Phi(x, t, u^n)D(Z_{m,\sigma}(u^n)\varphi) dx dt = \int_Q Z_{m,\sigma}(u^n)\mu^n \varphi dx dt, \end{aligned} \quad (57)$$

where  $\bar{Z}_{m,\sigma}(r) = \int_0^r b'_n(s)Z_{m,\sigma}(s)ds$ . We deduce from (57) that

$$\begin{aligned} & - \int_Q \bar{Z}_{m,\sigma}(u^n)\varphi_t dx dt - \int_\Omega \bar{Z}_{m,\sigma}(u_0^n)\varphi(0) dx \\ & = \int_{\{m-\sigma \leq u^n \leq m\}} \frac{1}{\sigma} (a^n(u^n) + \Phi_n(x, t, u^n)) |Du^n|^2 \varphi dx dt - \int_Q Z_{m,\sigma}(u^n)\mu^n \varphi dx dt. \end{aligned} \quad (58)$$

In the sequel we pass to the limit in (58) when  $n$  tends to infinity and then  $\sigma$  tends to zero. Note that  $\bar{Z}_{m,\sigma}(u^n)$  converges to  $\bar{Z}_{m,\sigma}(u)$  strongly in  $L^1(Q)$  and  $\bar{Z}_{m,\sigma}(u_0^n)$  converges to  $\bar{Z}_{m,\sigma}(u_0)$  strongly in  $L^1(\Omega)$  as  $n$  tends to infinity. Moreover, since  $\bar{Z}_{m,\sigma}(u)$  converges to  $(b(u) - b(m))^+$  as  $\sigma$  tends to zero,  $u \leq m$  and  $u_0 \leq m$  almost everywhere, then it is easy to see that

$$\lim_{\sigma \rightarrow 0} \lim_{n \rightarrow +\infty} \int_Q \bar{Z}_{m,\sigma}(u^n)\varphi_t dx dt = 0 \text{ and } \lim_{\sigma \rightarrow 0} \lim_{n \rightarrow +\infty} \int_\Omega \bar{Z}_{m,\sigma}(u_0^n)\varphi(0) dx = 0. \quad (59)$$

Then, from (56), (58) and (59) we deduce (17).

As a conclusion of step 1 to step 4, the proof of Theorem 4.1 is complete.

## 5 Appendix

Here we prove Theorem 2.2.

**Proof.** Let  $b(u) = v$ , then equation (4) is equivalent to

$$\begin{cases} v_t - \operatorname{div}(G(x, t, v)Dv) = \mu & \text{in } Q, \\ v(x, 0) = b(u(x, 0)) & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega \times (0, T), \end{cases} \quad (60)$$

where  $G(x, t, v) = \frac{a(b^{-1}(v)) + \Phi(x, t, b^{-1}(v))}{b'(b^{-1}(v))}$ . For simplicity we assume that  $\mu \geq 0$  and  $u_0 \geq 0$ . We use the admissible test function  $T_K(u)$  in (60) to get

$$\int_\Omega \bar{T}_K(v) dx + \int_Q \left| (G(x, t, v))^{\frac{1}{2}} DT_K(v) \right|^2 dx dt \leq K \left( \|\mu\|_{\mathcal{M}(Q)} + \|b(u_0)\|_{L^1(\Omega)} \right) \equiv KM, \quad (61)$$

where  $\bar{T}_K(r) = \int_0^r T_K(s)ds$ . Since  $\frac{1}{2}T_K^2(r) \leq \bar{T}_K(r) \leq Kr$ ,  $\beta \leq \beta' \leq \gamma$  and  $G(x, t, v) \geq \frac{\alpha}{\gamma}$ , we deduce that  $\max \left\{ \|T_K(v)\|_{L^\infty(0,T;L^2(\Omega))}^2; \|G(x, t, v)^{\frac{1}{2}}DT_K(v)\|_{L^2(Q)}^2 \right\} \leq KM$  and  $\|T_K(v)\|_{L^2(0,T;H_0^1(\Omega))}^2 \leq \gamma \frac{KM}{\alpha}$ . Let  $z \in W$  be the solution of

$$\begin{cases} -z_t - \operatorname{div}(G(x, t, v)Dz) = -2 \operatorname{div}(G(x, t, v)DT_K(v)) & \text{in } Q, \\ z = 0 & \text{on } (0, T) \times \partial\Omega, \\ z(t = T) = T_K(v(t = T)) & \text{in } \Omega. \end{cases} \quad (62)$$

Taking the admissible test function  $z$  in (62) and integrating between  $\tau$  and  $T$ , we have by Young's inequality that  $\max \left\{ \|z\|_{L^\infty(0,T;L^2(\Omega))}^2; \|Dz\|_{L^2(Q)}^2 \right\} \leq CKM$ . Moreover, the equation (62) implies that  $\|z_t\|_{L^2(0,T;H^{-1}(\Omega))} \leq C \left( \|z\|_{L^2(0,T;H_0^1(\Omega))} + \|T_K(v)\|_{L^2(0,T;H_0^1(\Omega))} \right)$ . Hence we deduce that  $\|z\|_W \leq C\sqrt{K}$ . Since  $\mu \geq 0$ ,  $b(u_0) \geq 0$  and  $G(x, t, v) \geq 0$ , we have  $v_t - \operatorname{div}(G(x, t, v)Dv) \geq 0$  and  $v \geq 0$  in  $Q$ , and by a nonlinear version of Kato's inequality for parabolic equations (see [13]), we deduce that  $T_K(v)_t - \operatorname{div}(G(x, t, v)DT_K(v)) \geq 0$ . Then we conclude that  $-z_t - \operatorname{div}(G(x, t, v)Dz) \geq -T_K(v)_t - \operatorname{div}(G(x, t, v)DT_K(v))$  in  $\mathcal{D}'(Q)$ . Now, using the standard comparison argument, we easily see that  $z \geq T_K(v)$  a.e. in  $Q$ , hence  $z \geq K$  a.e. on  $\{v > K\}$ , and we conclude that  $\operatorname{cap}_2\{v > K\} \leq \left\| \frac{z}{K} \right\|_W \leq \frac{C}{\sqrt{K}}$ , the proof of Theorem 2.2 is complete.  $\square$

## 6 Example

Let us consider the following special case:  $b(s) = s(e^s + 1)$ ,  $a(s) = \frac{1}{(m-s)^{\frac{1}{3}}}$  for  $s < m$  and  $\Phi(x, t, s) = L(x, t)e^{s^2}$ , where  $L(x, t) \in L^\infty(Q)$ . Note that  $A(s) = \int_0^s a(r)dr = \frac{3}{2}(m^{\frac{2}{3}} - (m-s)^{\frac{2}{3}})$  and  $A(m) = \frac{3}{2}m^{\frac{2}{3}} < +\infty$ . Finally, it is easy to show that the hypotheses of Theorem 4.1 are satisfied. Therefore, for all  $\mu \in \mathcal{M}_0(Q)$  and  $u_0 \in L^1(\Omega)$  with  $u_0 \leq m$ , there exists at least one renormalized solution of problem (1)-(3), and then  $u$  satisfies

$$u \in L^1(Q), \quad u \leq m \text{ a.e. in } Q \text{ and } T_K(u) \in L^2(0, T; H_0^1(\Omega)) \quad \forall K > 0, \quad (63)$$

$$\frac{1}{(m-u)^{\frac{1}{3}}}DT_K^m(u)\chi_{\{u < m\}} \in (L^2(Q))^N \quad \forall K > 0. \quad (64)$$

There exist a sequence of nonnegative measures  $\Lambda_K \in \mathcal{M}(Q)$  and a nonnegative measure  $\Gamma_m \in \mathcal{M}(Q)$  such that

$$\lim_{K \rightarrow +\infty} \|\Lambda_K\|_{\mathcal{M}(Q)} = 0 \text{ and } \int_Q \varphi d\Gamma_m = 0 \quad \forall \varphi \in \mathcal{C}_0^1([0, T]), \quad (65)$$

and for every  $K > 0$

$$\frac{\partial B_K^m(u)}{\partial t} - \operatorname{div} \left( \frac{1}{(m-u)^{\frac{1}{3}}}DT_K^m(u)\chi_{\{u < m\}} \right) - \operatorname{div} \left( L(x, t)e^{(T_K^m(u))^2}DT_K^m(u) \right) \quad (66)$$

$$= \mu + \Lambda_K + \Gamma_m \quad \text{in } \mathcal{D}'(Q),$$

where  $B_K^m(z) = \int_0^z (1 + e^s + se^s)(T_K^m)'(s) ds$ .

## Acknowledgment

The first author has been supported by Ministry of Higher Education and Scientific Research-Yemen.

## References

- [1] Bénilan, P., Boccardo, L., Gallouët, T., Gariepy, R., Pierre, M. and Vazquez, J.L. An  $L^1$ -theory of existence and uniqueness of solutions of nonlinear elliptic equations. *Annali della Scuola Normale Superiore di Pisa* **22** (4) (1995) 241–273.
- [2] Blanchard, D. and Redwane, H. Quasilinear diffusion problems with singular coefficients with respect to the unknown. *Proceedings of the Royal Society of Edinburgh. Section A.* **132** (5) (2002) 1105–1132.
- [3] Blanchard, D., Guibé, O. and Redwane, H. Nonlinear equations with unbounded heat conduction and integrable data. *Annali di Matematica Pura ed Applicata* **187** (4) (2008) 405–433.
- [4] Blanchard, D., Petitta, F. and Redwane, H. Renormalized solutions of nonlinear parabolic equations with diffuse measure data. *Manuscripta Mathematica* **141** (3-4) (2013) 601–635.
- [5] Blanchard, D., Murat, F. and Redwane, H. Existence and uniqueness of a renormalized solution for a fairly general class of nonlinear parabolic problems. *Journal of Differential Equations* **177** (2) (2001) 331–374.
- [6] Boccardo, L., Dall’Aglia, A., Gallouët, T. and Orsina, L. Nonlinear parabolic equations with measure data. *Journal of Functional Analysis* **147** (1997) 237–258.
- [7] Boccardo, L., Giachetti, D., Diaz, J.I. and Murat, F. Existence and regularity of renormalized solutions for some elliptic problems involving derivatives of nonlinear terms. *Journal of Differential Equations* **106** (1993) 215–237.
- [8] DiPerna, R.J. and Lions, P.L. On the Cauchy problem for Boltzmann equations: global existence and weak stability. *Annals of Mathematics* **130** (2) (1989) 321–366.
- [9] Droniou, J., Porretta, A. and Prignet, A. Parabolic capacity and soft measures for nonlinear equations. *Potential Analysis* **19** (2003) 99–161.
- [10] Petitta, F., Ponce, A.C. and Porretta, A. Approximation of diffuse measures for parabolic capacities. *Comptes Rendus de l’Académie des Sciences* **346** (2008) 161–166.
- [11] Orsina, L. Existence results for some elliptic equations with unbounded coefficients. *Asymptotic Analysis* **34** (2003) 187–198.
- [12] Lions, J.L. *Quelques méthodes de résolution des problèmes aux limites non linéaire*. Dunod et Gauthier-Villars, Paris, 1969. [French]
- [13] Petitta, F., Ponce, A.C. and Porretta, A. Diffuse measures and nonlinear parabolic equations. *Journal of Evolution Equations* **11** (4) (2011) 861–905.
- [14] Pierre, M. Parabolic capacity and Sobolev spaces. *SIAM Journal on Mathematical Analysis* **14** (1983) 522–533.
- [15] Zaki, K. and Redwane, H. Nonlinear parabolic equations with blowing-up coefficients with respect to the unknown and with soft measure data. *Electronic Journal of Differential Equations* **327** (2016) 1–12.
- [16] Simon, J. Compact sets in the space  $L^p(0, T; B)$ . *Annali di Matematica Pura ed Applicata* **146** (1987) 65–96.



## Contents of Volume 17, 2017

**Volume 17** **Number 1** **March 2017**

### PERSONAGE IN SCIENCE

On the Occasion of the 100th Birthday of Academician Yu. A. Mitropolsky .....	1
Approximate Controllability of Non-densely Defined Semilinear Control System with Non Local Conditions .....	5
<i>Urvashi Arora and N. Sukavanam</i>	
Integro-differential Equations, Compact Maps, Positive Kernels, and Schaefer's Fixed Point Theorem .....	19
<i>T. A. Burton</i>	
Analysis of a Set of Trajectories of Generalized Standard Systems: Averaging Technique .....	29
<i>A. A. Martynyuk</i>	
Robust Output Feedback Stabilization and Optimization of Control Systems .....	42
<i>A. G. Mazko</i>	
Relation Between Fuzzy Semigroups and Fuzzy Dynamical Systems .....	60
<i>S. Melliani, A. El Allaoui and L. S. Chadli</i>	
Shunt Active Power Filter Based Harmonics Compensation of a Low-Voltage Network Using Fuzzy Logic System .....	70
<i>A. Morsli, A. Tlemçani and M. S. Boucherit</i>	
A Simple Approach for Q-S Synchronization of Chaotic Dynamical Systems in Continuous-Time ...	86
<i>Adel Ouannas</i>	
Extending the Property of a System to Admit a Family of Oscillations to Coupled Systems .....	95
<i>V. N. Tkhai and I. N. Barabanov</i>	

**Volume 17** **Number 2** **June 2017**

On the Global Asymptotic Stability of a Class of Nonlinear Switched Systems .....	107
<i>A. Yu. Aleksandrov, E.B. Aleksandrova, A. V. Platonov and M. V. Voloshin</i>	
Global Existence of Weak Solutions to a Fractional Landau-Lifshitz-Gilbert Equation .....	121
<i>Chahid Ayouch, El-Hassan Essoufi and Mouhcine Tilioua</i>	
Exponential Domination and Bondage Numbers in Some Graceful Cyclic Structure .....	139
<i>V. Aytaç and T. Turac</i>	
On the Hyers-Ulam Stability of Certain Partial Differential Equations of Second Order .....	150
<i>Emel Biçer and Cemil Tunç</i>	
Multiplicity of Periodic Solutions for a Class of Second Order Hamiltonian Systems .....	158
<i>K. Fathi</i>	
Capacity, Theorem of H. Brezis and F.E. Browder Type in Musielak–Orlicz–Sobolev Spaces and Application .....	175
<i>M.C. Hassib, Y. Akdim, A. Benkirane and N. Aissaoui</i>	
Function Projective Dual Synchronization of Chaotic Systems with Uncertain Parameters .....	193
<i>A. Almatroud Othman, M.S.M. Noorani and M. Mossa Al-Sawalha</i>	
Global Dynamics of a Cooperative and Supportive Network System with Subnetwork Deactivation	205
<i>P. Raja Sekhara Rao, K. Venkata Ratnam, P. Lalitha and Dipak Kumar Satpathi</i>	

**Volume 17** **Number 3** **September 2017**

Existence Result for Nonlinear Degenerated Parabolic Systems .....	217
<i>A. Aberqi, J. Bennouna and M. Hammoumi</i>	
Existence of Positive Periodic Solutions for a Second-Order Nonlinear Neutral Differential Equation by the Krasnoselskiis Fixed Point Theorem .....	230
<i>A. Bouakkaz, A. Ardjouni and A. Djoudi</i>	
Maximal Regularity of Non-autonomous Forms with Bounded Variation .....	239
<i>M. Boukdir</i>	
A NASC for Equicontinuous Maps for Integral Equations .....	247
<i>T.A. Burton and Bo Zhang</i>	
A New Hyper Chaotic System and Study of Hybrid Projective Synchronization Behavior .....	266
<i>A. Khan, N. Aneja, P. Tripathi and J. Biswas</i>	
Different Types of Synchronization Between Different Fractional Order Chaotic Systems .....	279
<i>A. Khan, M. Budhrraja and A. Ibraheem</i>	
Controllability of Neutral Functional Differential Equations Driven by Fractional Brownian Motion with Infinite Delay .....	291
<i>El Hassan Lakhel</i>	
Approximation of the Optimal Control Problem on an Interval with a Family of Optimization Problems on Time Scales .....	303
<i>O. Lavrova, V. Moglyova, O. Stanzhytskyi and O. Misiats</i>	
Stability Analysis of Impulsive Hopfield-Type Neuron System on Time Scale .....	315
<i>T.A. Lukyanova and A.A. Martynyuk</i>	

**Volume 17** **Number 4** **December 2017**

Random Impulsive Partial Hyperbolic Fractional Differential Equations .....	327
<i>Mouffak Benchohra and Amel Heris</i>	
On the Hyers-Ulam Stability of Laguerre and Bessel Equations by Laplace Transform Method ....	340
<i>E. Bicer and C. Tunc</i>	
Output Tracking of Some Class Non-minimum Phase Nonlinear Uncertain Systems .....	347
<i>Firman, Janson Naiborhu, Roberd Saragih and Sapto Wahyu Indratno</i>	
Passivity Based Control of Continuous Bioreactors .....	357
<i>M. Makkar and J.-Y. Dieulot</i>	
Mathematical Models of Nonlinear Oscillations of Mechanical Systems with Several Degrees of Freedom .....	369
<i>G.I. Melnikov, N.A. Dudarenko, K.S. Malykh, L.N. Ivanova and V.G. Melnikov</i>	
Existence Results for Mild Solution for a Class of Impulsive Fractional Stochastic Problems with Nonlocal Conditions .....	376
<i>Mohd Nadeem and Jaydev Dabas</i>	
Refinements of Some Pachpatte and Bihari Inequalities on Time Scales .....	388
<i>B. Ben Nasser, K. Boukerrioua, M. Defoort, M. Djemai and M.A. Hammami</i>	
Boundedness of the New Modified Hyperchaotic Pan System .....	402
<i>Samia Rezzag</i>	
Pseudo Almost Automorphic Mild Solutions to Some Fractional Differential Equations with Stepanov-like Pseudo Almost Automorphic Forcing Term .....	409
<i>Vikram Singh and Dwijendra N. Pandey</i>	
Nonlinear Parabolic Equations with Singular Coefficient and Diffuse Data .....	421
<i>K. Zaki and H. Redwane</i>	
Contents of Volume 17, 2017 .....	433



CAMBRIDGE SCIENTIFIC PUBLISHERS

AN INTERNATIONAL BOOK SERIES  
STABILITY OSCILLATIONS AND OPTIMIZATION OF SYSTEMS

**Stability Analysis of Nonlinear Systems under Structural Perturbations**

*Stability, Oscillations and Optimization of Systems: 2014, Volume 8, 252 pp.*

ISBN 978-1-908106-37-7 £55 /\$100 /€ 80

**A.A. Martynyuk**

*Institute of Mechanics, National Academy of Sciences of Ukraine, Kyiv, Ukraine*

**V.G. Miladzhanov**

*Andizhan University, Andizhan, O'zbekiston*

This book focuses on some problems of stability theory of nonlinear large-scale systems. The purpose of this book is to describe some new applications of Liapunov matrix-valued functions method to the stability of evolution problems governed by nonlinear continuous systems, discrete-time systems, impulsive systems and singularly perturbed systems under nonclassical structural perturbations. The authors take a challenging and original approach based on concept of structural perturbations combined with direct Liapunov's method. This new approach will lead to results that cannot be obtained by standard theories of stability in the field.

The **Stability Analysis of Nonlinear Systems under Structural Perturbations** addresses to specialists in dynamical systems, applied differential equations, and the stability theory. It may be useful for graduated students in mathematics, control theory, and mechanical engineering.

**CONTENTS**

**Preface • Generalities • Continuous Systems under Nonclassical Structural Perturbations • Discrete-Time Systems under Nonclassical Structural Perturbations • Impulsive Systems under Nonclassical Structural Perturbations • Singularly Perturbed Systems under Nonclassical Structural Perturbations • References • Subject Index**

---

Please send order form to:

**Cambridge Scientific Publishers**

PO Box 806, Cottenham, Cambridge CB4 8RT Telephone: +44 (0) 1954 251283

Fax: +44 (0) 1954 252517 Email: [janie.wardle@cambridgescientificpublishers.com](mailto:janie.wardle@cambridgescientificpublishers.com)

Or buy direct from our secure website: [www.cambridgescientificpublishers.com](http://www.cambridgescientificpublishers.com)

---