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Closed-Form Solution of European Option under Fractional Heston Model

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Abstract: In this paper, we give a closed-form solution of a European option generated by the fractional Heston stochastic volatility model based on the Adomian decomposition method.

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1 Introduction

The valuation of options is one of the most popular problems in financial mathematical literature. This problem is of interest for both academics and traders. As compared to the case of the Black and Scholes model, where the volatility is constant, the Heston model is more common since the volatility is stochastic, inasmuch as the dynamics of the volatility is fundamental to elaborate strategies for hedging and for arbitrage and a model based on a constant volatility cannot explain the reality of the financial markets. So, the pricing of option under stochastic volatility model is then very important and required.

During the last few decades, several papers studied the existence of closed-form solution of the European option using many methods and generated by different models, for example, the Black and Scholes case [3–5], the Hull and White model [14], the Heston model [6, 12] and recently, Jerbi has given a new closed-form solution for the European option [15] based on a new stochastic process.

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The fractional calculus is used in several research axes. Recently, it has been introduced in the mathematical finance field [9–11, 16], and especially to generate the underlying asset price in order to give a closed form solution for the evaluation of a European option problem [10, 11, 16, 18, 20]. Many methods are proposed in order to resolve linear and nonlinear fractional differential equations, see, for example, [2, 19]. In this work we use the Adomian decomposition method [1, 7, 8].

In the following, we shall need to introduce the dynamic of the Heston model. Let S_t and V_t represent two stochastic processes so that X_t is generated by the following process :

$$dS_t = rS_t dt + S_t \sqrt{V_t} dW_t^S \tag{1}$$

and V_t follows a mean reversion and a square-root diffusion process given by:

$$dV_t = k_V(\theta_V - V_t)dt + \sigma_V \sqrt{V_t} dW_t^V, \tag{2}$$

where r is supposed to be constant, W_t^S and W_t^V are two correlated standard Brownian motions, i.e. $W_t^S = \sqrt{1-\rho^2}B_t^1 + \rho B_t^2$ and $W_t^V = B_t^2$, where B is a standard 2-dimensional Brownian motion and $\rho \in]-1,1[$. The parameters θ_V, k_V and σ_V are respectively, the long-term mean, the rate of mean reversion, and the volatility of the stochastic process V_t. We assume that the volatility process V_t is strictly positive.
So, based on the Heston stochastic volatility model a two dimensional parabolic partial differential equation can be derived for the value of the European option, see, for instance, [13].

2 Preliminaries

In what follows, we give some definitions related to the fractional calculus which constitute the basis of our work. For an organic presentation of the fractional theory, we can refer the readers to Podlubny's book [17].

Definition 2.1 The Riemann-Liouville fractional integral of order $\alpha > 0$ is defined as

$$I_{t_0}^{\alpha} x(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-\tau)^{\alpha-1} x(\tau) d\tau,$$

where $\Gamma(\alpha) = \displaystyle\int_0^{+\infty} e^{-t} t^{\alpha-1} dt$.

Definition 2.2 The Caputo fractional derivative is defined as

$$D_{t_0,t}^{\alpha} x(t) = \frac{1}{\Gamma(m-\alpha)} \int_{t_0}^{t} (t-\tau)^{m-\alpha-1} \frac{d^m}{d\tau^m} x(\tau) d\tau, \ (m-1 < \alpha < m).$$

When $0 < \alpha < 1$, then the Caputo fractional derivative of order α of f reduces to

$$D_{t_0,t}^{\alpha} x(t) = \frac{1}{\Gamma(1-\alpha)} \int_{t_0}^{t} (t-\tau)^{-\alpha} \frac{d}{d\tau} x(\tau) d\tau. \tag{3}$$

Note that the relation between the Riemann-Liouville operator and the Caputo fractional differential operator is given by the following equality:

$$I_{t_0}^{\alpha} D_{t_0,t}^{\alpha} f(t) = D_{t_0,t}^{-\alpha} D_{t_0,t}^{\alpha} f(t) = f(t) - \sum_{k=0}^{m-1} \frac{t^k}{k!} f^k(0), \qquad m-1 < \alpha \leq m. \tag{4}$$

Similarly to the exponential function used in the solutions of integer-order differential systems, the Mittag-Leffler function is frequently used in the solutions of fractional-order differential systems.

Definition 2.3 The Mittag-Leffler function with two parameters is defined as

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{+\infty} \frac{z^k}{\Gamma(k\alpha + \beta)},$$

where $\alpha > 0$, $\beta > 0$, $z \in C$.

When $\beta = 1$, we have $E_\alpha(z) = E_{\alpha,1}(z)$, furthermore, $E_{1,1}(z) = e^z$.

3 Main Results

When the volatility is stochastic, the value $P(S_t, V_t)$ of a European option is given by the following nonlinear fractional differential equation

$$D_t^\alpha P(S_t, V_t) + A[P](S_t, V_t) = 0, \qquad 0 < \alpha \leq 1, \tag{5}$$

in the unbounded domain $\{(S_t, V_t) | S_t \geq 0, V_t \geq 0 \ and \ t \in [0, T]\}$ with the initial value

$$P(S_0, V_0). \tag{6}$$

For the boundary conditions, in the case of a call option, at maturity T with an exercise price K, the payoff function is

$$\max(S_T - K, 0) \tag{7}$$

and for the put option the payoff function is equal to

$$\max(K - S_T, 0), \tag{8}$$

where $D_t^\alpha = \frac{\partial^\alpha}{\partial t}$ and

$$A[P] = rS\frac{\partial P}{\partial S} + k(\theta - V)\frac{\partial P}{\partial V} + \frac{1}{2}VS^2\frac{\partial^2 P}{\partial S^2} + \rho\sigma VS\frac{\partial^2 P}{\partial S \partial V} - \frac{1}{2}\sigma V\frac{\partial^2 P}{\partial V^2} - rP.$$

Theorem 3.1 *Let $(P_t)_{t\geq 0}$ be the European option price, a function of the underlying asset price and the volatility. Under the same hypotheses of the Heston model, the price of the European option is given by the following formula:*

$$P(S_t, V_t) = E_\alpha(-t^\alpha A[P(S_0, V_0)]),$$

where $0 < \alpha \leq 1$, E_α *is the Mittag-Leffler function and* $A[P] = rS\frac{\partial P}{\partial S} + k(\theta - V)\frac{\partial P}{\partial V} + \frac{1}{2}VS^2\frac{\partial^2 P}{\partial S^2} + \rho\sigma VS\frac{\partial^2 P}{\partial S \partial V} - \frac{1}{2}\sigma V\frac{\partial^2 P}{\partial V^2} - rP.$

Proof. Multiplying equation (5)by the operator $D_t^{-\alpha}$ and on taking into account (4), we get

$$P(S_t, V_t) = P(S_0, V_0) + D_t^{-\alpha}(-A[P](S_t, V_t)), \tag{9}$$

so, using the Adomian decomposition method we get the solution in the following form

$$P(S_t, V_t) = P_0(S_t, V_t) + \sum_{k=1}^{\infty} P_k(S_t, V_t), \tag{10}$$

by substituting (10) into (5), we have:

$$P_{n+1}(S_t,V_t) = D_t^{-\alpha}(-A[P_n](S_t,V_t))$$

$$= -A[P(S_0,V_0)]^n D_t^{-\alpha}\Big(\frac{t^{n\alpha}}{\Gamma(1+n\alpha)}\Big) \tag{11}$$

with $P_0(S_t,V_t) = P(S_0,V_0)$, we get:

$$P(S_t,V_t) = \sum_{k=0}^{\infty}(-1)^k \frac{t^{k\alpha}}{\Gamma(1+k\alpha)} A[P(S_0,V_0)]^k$$

$$= E_\alpha(-t^\alpha A[P(S_0,V_0)]). \tag{12}$$

The convergence of the power series of the fractional Heston model is guaranteed for a real and positive α.

4 Conclusion

In this paper, we have elaborated a new closed-form solution of a European option generated by the fractional Heston stochastic volatility model. In this work, we have performed two extensions: when we take $\alpha = 1$, we return to the standard Heston model and for a constant volatility, we have the fractional Black-Scholes model.

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