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NONLINEAR DYNAMICS AND SYSTEMS THEORY

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Entropy Solutions of Nonlinear $p(x)$-Parabolic Inequalities

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Abstract: In this paper we prove the existence of entropy solutions for weighted $p(x)$-parabolic problem associated with the equation:

$$\frac{\partial u}{\partial t} + Au = g(u)\omega(x)\Big|\nabla u\Big|^{p(x)} + f \quad \text{in} \quad \Omega \times (0, T),$$

where the operator $Au = -\text{div}\Big(\omega(x)\Big|\nabla u\Big|^{p(x)-2}\nabla u\Big)$ and on the right-hand side f belongs to $L^1(\Omega \times (0, T))$ and $\omega(x)$ is a weight function.

Keywords: *parabolic problems; entropy solutions; Sobolev space with variable exponent; penalized equations.*

Mathematics Subject Classification (2010): 47A15, 46A32, 47D20.

1 Introduction

Let Ω be an open bounded subset of $\mathbb{R}^N$, $N \geq 2$, T be a positive real number and $Q = \Omega \times (0, T)$, while the variable exponent $p : \quad \bar{\Omega} \to (1, \infty)$ is a continuous function, the data $f \in L^1(Q)$ and $u_0 \in L^1(\Omega)$. The objective of this paper is to study the existence of an entropy solution for the obstacle parabolic problems of the type:

$$\begin{cases} u \geq \psi, & \text{a.e. in } \Omega \times (0, T), \\ \frac{\partial u}{\partial t} - \text{div}\Big(\omega(x)\Big|\nabla u\Big|^{p(x)-2}\nabla u\Big) = \omega(x)g(u)\Big|\nabla u\Big|^{p(x)} + f, & \text{in } \Omega \times (0, T), \\ u(x, 0) = u_0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega \times (0, T). \end{cases} \tag{1}$$

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The operator $-div\left(\omega(x)\left|\nabla u\right|^{p(x)-2}\nabla u\right)$ is a Leray-Lions operator defined on $L^{p^-}\left(0,T;W_0^{1,p(x)}(\Omega,\omega)\right)$ which is coercive.

In recent years, the study of partial differential equations and variational problems with variable exponent has received considerable attention in many models coming from various branches of mathematical physics, such as elastic mechanics,electro-rheological fluid dynamics and image processing, etc. We refer the readers to [12, 22]. Degenerate phenomena appear in the area of oceanography, turbulent fluid flows, induction heating and electrochemical problems.The notion of entropy solutions has been proposed by Bènilan et al. in [8] for the nonlinear elliptic problems.

Recently, when $\omega(x) \equiv 1$, the existence and uniqueness of entropy solutions of $p(x)$-Laplace equation with L^1 data were proved in [24]by Sanchón and Urbano. This notion was adapted to the study of the entropy solutions for nonlinear elliptic equations with variable exponents by Chao Zhang in [26] and the existence of solutions of some unilateral problems in the framework of Orlicz spaces has been established by M. Kbiri Alaoui, D. Meskine, A. Souissi in [17] in terms of the penalization method. E. Azroul, H. Redwane and M. Rhoudaf [5] have proved the existence of renormalized solution in Orlicz spaces in the case where $b(u) = u$. Fortunately, Kim, Wang and Zhang [18]have shown good properties of a function space and the so-called weighted variable exponent Lebesque-Sobolev spaces, and the existence and some properties of solutions for degenerate elliptic equations with exponent variable have been proved by Ky Ho, Inbo Sim [16].Other work in this direction can be found in [4] by Y. Akdim, C. Allalou, N. El gorch.

Now we review some definitions and basic properties of the weighted variable exponent Lebesgue spaces $L^{p(x)}(\Omega,\omega)$ and the weighted variable exponent Sobolev spaces $W^{1,p(x)}(\Omega,\omega)$.

Let ω be a measurable positive and a.e. finite function in $\mathbb{R}^N$. Set

$$C_+(\overline{\Omega}) = \{h \in C(\overline{\Omega}) : \min_{x\in\overline{\Omega}} h(x) > 1\}.$$

For any $h \in C_+(\overline{\Omega})$, we define $\quad h^+ = \max_{x\in\overline{\Omega}} h(x), \quad h^- = \min_{x\in\overline{\Omega}} h(x).$

For any $p \in C_+(\overline{\Omega})$, we introduce the weighted variable exponent Lebesgue space $L^{p(x)}(\Omega,\omega)$ which consists of all measurable real-valued functions u such that

$$\int_{\Omega}\left|u(x)\right|^{p(x)}\omega(x)dx < \infty,$$

endowed with the Luxemburg norm

$$||u||_{L^{p(x)}(\Omega,\omega)} = \inf\left\{\lambda > 0 : \int_{\Omega}\left|\frac{u(x)}{\lambda}\right|^{p(x)}\omega(x)dx \leq 1\right\}$$

becomes a normed space. When $\omega(x) \equiv 1$, we have $L^{p(x)}(\Omega,\omega) \equiv L^{p(x)}(\Omega)$ and we use the notation $||u||_{L^{p(x)}(\Omega)}$ instead of $||u||_{L^{p(x)}(\Omega,w)}$. The following Hölder type inequality is useful for the next sections. The weighted variable exponent Sobolev space $W^{1,p(x)}(\Omega,\omega)$ is defined by

$$W^{1,p(x)}(\Omega,\omega) = \left\{u \in L^{p(x)}(\Omega); \left|\nabla u\right| \in L^{p(x)}(\Omega,\omega)\right\}$$

with the norm

$$||u||_{W^{1,p(x)}(\Omega,\omega)} = ||u||_{L^{p(x)}(\Omega)} + ||\nabla u||_{L^{p(x)}(\Omega,\omega)} \tag{2}$$

or, equivalently

$$\|u\|_{W^{1,p(x)}(\Omega,\omega)} = \inf \left\{ \lambda > 0 : \int_{\Omega} \left| \frac{u(x)}{\lambda} \right|^{p(x)} + \omega(x) \left| \frac{\nabla u(x)}{\lambda} \right|^{p(x)} dx \leq 1 \right\}$$

for all $u \in W^{1,p(x)}(\Omega,\omega)$.

It is significant that smooth functions are not dense in $W^{1,p(x)}(\Omega)$ without additional assumptions on the exponent $p(x)$. This feature was observed by Zhikov [27] in connection with the Lavrentiev phenomenon. However, when the exponent $p(x)$ is log-Hölder continuous, i.e., there is a constant C such that

$$|p(x) - p(y)| \leq \frac{C}{-\log|x-y|} \tag{3}$$

for every x, y with $|x - y| \leq \frac{1}{2}$, then smooth functions are dense in variable exponent Sobolev spaces and there is no confusion in defining the Sobolev space with zero boundary values, $W_0^{1,p(x)}(\Omega)$, as the completion of $C_0^\infty(\Omega)$ with respect to the norm $\|u\|_{W^{1,p(x)}(\Omega)}$ (see [15]). $W_0^{1,p(x)}(\Omega,\omega)$ is defined as the completion of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega,\omega)$ with respect to the norm $\|u\|_{W^{1,p(x)}(\Omega,\omega)}$. Throughout the paper, we assume that $p \in C_+(\overline{\Omega})$ and ω is a measurable positive and a.e. finite function in Ω.

The plan of the paper is as follows. Section 2 presents the mathematical preliminaries. In Section 3 we make precise all the assumptions on A, g, f and u_0, and give the definition of an entropy solution of $(\mathcal{P})$. In Section 4 we establish the existence of such a solution in Theorem 4.1.

2 Preliminaries

In this section, we state some elementary properties for the weighted variable exponent Lebesgue-Sobolev spaces which will be used in the next sections. The basic properties of the variable exponent Lebesgue-Sobolev spaces, that is when $\omega(x) \equiv 1$, can be found in [13, 19].

Lemma 2.1 *(See [13, 19])(Generalised Hölder inequality).*

i) *For any functions $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$, we have*
$$\left| \int_{\Omega} uv dx \right| \leq \left(\frac{1}{p^-} + \frac{1}{p'^-} \right) \|u\|_{p(x)} \|v\|_{p'(x)} \leq 2\|u\|_{p(x)} \|v\|_{p'(x)}.$$

ii) *For all $p, q \in C_+(\overline{\Omega})$ such that $p(x) \leq q(x)$ a.e. in Ω, we have $L^{q(.)} \hookrightarrow L^{p(x)}$ and the embedding is continuous.*

Lemma 2.2 *(See [18]) Denote $\rho(u) = \int_{\Omega} \omega(x) |u(x)|^{p(x)} dx$ for all $u \in L^{p(x)}(\Omega,\omega)$. Then*

$$\|u\|_{L^{p(x)}(\Omega,\omega)} < 1(= 1; > 1) \ if \ and \ only \ if \rho(u) < 1(= 1; > 1), \tag{4}$$

$$if \ \|u\|_{L^{p(x)}(\Omega,\omega)} > 1 \ then \ \|u\|_{L^{p(x)}(\Omega,\omega)}^{p^-} \leq \rho(u) \leq \|u\|_{L^{p(x)}(\Omega,\omega)}^{p^+}, \tag{5}$$

$$if \ \|u\|_{L^{p(x)}(\Omega,\omega)} < 1 \ then \ \|u\|_{L^{p(x)}(\Omega,\omega)}^{p^+} \leq \rho(u) \leq \|u\|_{L^{p(x)}(\Omega,\omega)}^{p^-}. \tag{6}$$

Remark 2.1 ([23].) We set

$$I(u) = \int_\Omega \left| u(x) \right|^{p(x)} + \omega(x)|\nabla u(x)|^{p(x)} dx.$$

Then, following the same argumen, we have
$$\min\left\{ \|u\|_{W^{1,p(x)}(\Omega,\omega)}^{p^-}, \|u\|_{W^{1,p(x)}(\Omega,\omega)}^{p^+} \right\} \le I(u) \le \max\left\{ \|u\|_{W^{1,p(x)}(\Omega,\omega)}^{p^-}, \|u\|_{W^{1,p(x)}(\Omega,\omega)}^{p^+} \right\}.$$

Throughout the paper, we assume that ω is a measurable positive and a.e. finite function in Ω satisfying the following relations:
$(\mathbf{W}_1)$ $\omega \in L^1_{loc(\Omega)}$ and $\omega^{-\frac{1}{(p(x)-1)}} \in L^1_{loc}(\Omega)$;
$(\mathbf{W}_2)$ $\omega^{-s(x)} \in L^1(\Omega)$ with $s(x) \in (\frac{N}{p(x)}, \infty) \cap [\frac{1}{p(x)-1}, \infty)$.
The reasons why we assume $(\mathbf{W}_1)$ and $(\mathbf{W}_2)$ can be found in [18].

Remark 2.2 ([18].) (i) If ω is a positive measurable and finite function, then $L^{p(x)}(\Omega, \omega)$ is a reflexive Banach space.
(ii) Moreover, if $(\mathbf{W}_1)$ holds, then $W^{1,p(x)}(\Omega, \omega)$ is a reflexive Banach space.

For $p, s \in C_+(\overline{\Omega})$, denote

$$p_s(x) = \frac{p(x)s(x)}{s(x)+1} < p(x),$$

where $s(x)$ is given in $(\mathbf{W}_2)$. Assume that we fix the variable exponent restrictions

$$p_s^*(x) = \begin{cases} \frac{p(x)s(x)N}{(s(x)+1)N - p(x)s(x)}, & \text{if } N > p_s(x), \\ \text{arbitrary}, & \text{if } N \le p_s(x), \end{cases}$$

for almost all $x \in \Omega$. These definitions play a key role in our paper. We shall frequently make use of the following (compact) imbedding lemma for the weighted variable exponent Lebesgue-Sobolev space in the next sections.

Lemma 2.3 ([18].) *Let $p, s \in C_+(\overline{\Omega})$ satisfy the log-Hölder continuity condition (3), and let $(\mathbf{W}_1)$ and $(\mathbf{W}_2)$ be satisfied. If $r \in C_+(\overline{\Omega})$) and $1 < r(x) \le p_s^*$, then we obtain the continuous imbedding*

$$W^{1,p(x)}(\Omega, \omega) \hookrightarrow L^{r(x)}(\Omega).$$

Moreover, we have the compact imbedding

$$W^{1,p(x)}(\Omega, \omega) \hookrightarrow L^{r(x)}(\Omega),$$

provided that $1 < r(x) < p_{s(x)}^*$ for all $x \in \overline{\Omega}$.

From Lemma 2.3, we have Poincaré-type inequality immediately.

Corollary 2.1 ([18].) *Let $p \in C_+(\overline{\Omega})$ satisfy the log-Hölder continuity condition (3). If $(\mathbf{W}_1)$ and $(\mathbf{W}_2)$ hold, then the estimate*

$$\|u\|_{L^{p(x)}(\Omega)} \le C\|\nabla u\|_{L^{p(x)}(\Omega,\omega)}$$

holds for every $u \in C_0^\infty(\Omega)$ with a positive constant C independent of u.

Throughout this paper, let $p \in C_+(\overline{\Omega})$ satisfy the log-Hölder continuity condition (3) and $X := W_0^{1,p(x)}(\Omega, \omega)$ be the weighted variable exponent Sobolev space that consists of all real valued functions u from $W^{1,p(x)}(\Omega, \omega)$ which vanish on the boundary $\partial\Omega$, endowed with the norm

$$\|u\|_X = \inf\left\{\lambda > 0 : \int_\Omega \left|\frac{\nabla u(x)}{\lambda}\right|^{p(x)} \omega(x)dx \le 1\right\},$$

which is equivalent to the norm (2) due to Corollary 2.1. The following proposition gives the characterization of the dual space $(W_0^{k,p(x)}(\Omega, \omega))^*$, which is analogous to [[19], Theorem 3.16]. We recall that the dual space of weighted Sobolev spaces $W_0^{1,p(x)}(\Omega, \omega)$ is equivalent to $W^{-1,p'(x)}(\Omega, \omega)$, where $\omega^* = \omega^{1-p'(x)}$.

We will also use the standard notation for Bochner spaces, i.e., if $q \ge 1$ and X is a Banach space, then $L^q(0, T; X)$ denotes the space of strongly measurable function $u : (0, T) \to X$ for which $t \to \|u(t)\|_X \in L^q(0, T)$ Moreover, $C([0; T]; X)$ denotes the space of continuous function $u : [0; T] \to X$ endowed with the norm $\|u\|_{C([0;T];X)} = max_{t \in [0;T]}\|u\|_X$,

$$L^{p^-}(0, T; W_0^{1,p(x)}(\Omega, \omega)) = \left\{u : (0, T) \to W_0^{1,p(x)}(\Omega, \omega) \text{ measurable};\right.$$

$$\left.\left(\int_0^T \|u(t)\|_{W_0^{1,p(x)}(\Omega,\omega)}^{p^-}\right)^{1/p^-} < \infty\right\}$$

and we define the space

$$L^\infty(0, T; X) = \left\{u : (0, T) \to X \text{ measurable}; \exists C > 0/\|u(t)\|_X \le C \text{ a.e.}\right\},$$

where the norm is defined by:

$$\|u\|_{L^\infty(0,T;X)} = \inf\left\{C > 0; \|u(t)\|_X \le C \text{ a.e.}\right\}.$$

We introduce the functional space (see [6])

$$V = \left\{f \in L^{p^-}(0, T; W_0^{1,p(x)}(\Omega, \omega)); |\nabla f| \in L^{p(x)}(Q, \omega)\right\}, \tag{7}$$

endowed with the norm

$$\|f\|_V = \|\nabla f\|_{L^{p(x)}(Q,\omega)}$$

or the equivalent norm

$$\|\|f\|\|_V = \|f\|_{L^{p^-}(0,T;W_0^{1,p(x)}(\Omega,\omega))} + \|\nabla f\|_{L^{p(x)}(Q,\omega)},$$

which is a separable and reflexive Banach space. The equivalence of the two norms is an easy consequence of the continuous embedding $L^{p(x)}(Q) \hookrightarrow L^{p^-}(0, T; L^{p(x)}(\Omega))$ and the Poincaré inequality. We state some further properties of V in the following lemma.

Lemma 2.4 *Let V be defined as in (7) and its dual space be denoted by V^*. Then*
i) We have the following continuous dense embeddings:

$$L^{p^+}(0, T; W_0^{1,p(x)}(\Omega, \omega)) \hookrightarrow V \hookrightarrow L^{p^-}(0, T; W_0^{1,p(x)}(\Omega, \omega)).$$

In particular, since $D(Q)$ is dense in $L^{p^+}(0, T; W_0^{1,p(x)}(\Omega, \omega)$, it is dense in V and for the corresponding dual spaces, we have

$$L^{(p^-)'}(0, T; (W_0^{1,p(x)}(\Omega, \omega))^*) \hookrightarrow V^* \hookrightarrow L^{(p^+)'}(0, T; W_0^{1,p(x)}(\Omega, \omega))^*).$$

Note that we have the following continuous dense embeddings

$$L^{p^+}(0, T; L^{p(x)}(\Omega, \omega)) \hookrightarrow L^{p(x)}(Q, \omega) \hookrightarrow L^{p^-}(0, T; L^{p(x)}(\Omega, \omega)).$$

ii) One can represent the elements of V^* as follows: if $T \in V^*$, then there exists $F = (f_1, ..., f_N) \in (L^{p'(x)}(Q))^N$ such that $T = div_X F$ and

$$\langle T, \xi \rangle_{V^*, V} = \int_0^T \int_\Omega F \cdot \nabla \xi dx dt$$

for any $\xi \in V$. Moreover, we have

$$\|T\|_{V^*} = \max \left\{ \|f_i\|_{L^{p(\cdot)}(Q, \omega)}, \ i = 1, ..., n \right\}.$$

Remark 2.3 The space $V \cap L^\infty(Q)$, endowed with the norm

$$\|v\|_{V \cap L^\infty(Q)} = \max \left\{ \|v\|_V, \|v\|_{L^\infty(Q)} \right\}, \ v \in V \cap L^\infty(Q),$$

is a Banach space. In fact, it is the dual space of the Banach space $V + L^1(Q)$ endowed with the norm

$$\|v\|_{V^* + L^1(Q)} := \inf \left\{ \|v_1\|_{V^*} + \|v_2\|_{L^1(Q)} \right\}; \ v = v_1 + v_2, \ v_1 \in V^*, v_2 \in L^1(Q).$$

2.1   Some technical results

This subsection introduces some basic technical lemmas and results that will be needed throughout this paper. For some details concerning the related issues, the reader can consult papers [7, 9].

Lemma 2.5 *(see [3]) Assume (9) and let $(u_n)_n$ be a sequence in $L^{p^-}(0, T, W_0^{1,p(x)}(\Omega, \omega))$ such that $u_n \rightharpoonup u$ weakly in $L^{p^-}(0, T, W_0^{1,p(x)}(\Omega, \omega))$ and*

$$\int_Q \Big(a(x, t, u_n, \nabla u_n) - a(x, t, u_n, \nabla u) \Big) \cdot \nabla (u_n - u) dx dt \to 0. \tag{8}$$

Then $u_n \to u$ strongly in $L^{p^-}(0, T, W_0^{1,p(x)}(\Omega, \omega))$.

Besides, $a(x, t, u, \nabla u) = \left(\left| \nabla u \right|^{p(x)-2} \nabla u \right)$ in our case.

Lemma 2.6 *([6]) Let $g \in L^{p(x)}(Q, \omega)$ and let $g_n \in L^{p(x)}(Q, \omega)$, with $\|g_n\|_{L^{p(x)}(Q, \omega)} \leq c$, $1 < r(x) < \infty$. If $g_n(x) \to g(x)$ a.e. in Q, then $g_n \rightharpoonup g$ in $L^{p(\cdot)}(Q, \omega)$, where $\rightharpoonup$ denotes weak convergence and ω is a weight function on Q.*

Lemma 2.7 *(See [23]) $W := \left\{ u \in V; u_t \in V^* + L^1(Q) \right\} \hookrightarrow C([0, T]; L^1(\Omega))$ and $W \cap L^\infty(Q) \hookrightarrow C([0, T]; L^2(\Omega))$.*

3 Assumptions and Definition

Throughout this paper, we assume that the following assumptions hold true.

3.1 Basic assumptions

Let Ω be an open bounded subset of $\mathbb{R}^N, N \geq 2$, $T > 0$ be a positive real number and let us set $Q = \Omega \times (0,T)$ and let $p \in C_+(\bar{\Omega})$ and assume that $p(x)$ satisfies the log-Hölder condition (3) with $1 < p^- \leq p(x) \leq p^+ < \infty$. The differential operator $A : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ defined by

$$Au = -\mathrm{div}\Big(\omega(x)\big|\nabla u\big|^{p(x)-2}\nabla u\Big), \tag{9}$$

is a Leray-Lions operator which is coercive and

$$g: \quad \mathbb{R} \to \mathbb{R}^+ \tag{10}$$

is a bounded and continuous positive function that belongs to $L^\infty(\mathbb{R})$,

$$f \text{ is an element of } L^1(Q), u_0 \in L^1(\Omega), \quad u_0 \geq 0 \text{ and } p \in C_+(\bar{\Omega}). \tag{11}$$

Let ψ be a measurable function with values in $\overline{\mathbb{R}}$ such that $\psi \in W_0^{1,p(x)}(\Omega,\omega) \cap L^\infty(\Omega)$, (see [17]), K is defined by: $K = \Big\{u \in W_0^{1,p(x)}(\Omega,\omega); \;\; u(x) \geq \psi(x) \;\; a.e. \;\; in \;\; \Omega\Big\}$ and consider the convex set

$$K_\psi = \Big\{u \in V, \; u(t) \in K\Big\}.$$

We recall that, for $k > 0$ and $s \in \mathbb{R}$, the truncation function $T_k(.)$ is defined by

$$T_k(s) = \begin{cases} s, & if \quad |s| \leq k, \\ k\frac{s}{|s|}, & if \quad |s| \geq k. \end{cases}$$

3.2 Definition of entropy solution

Definition 3.1 Let $f \in L^1(Q)$ and $u_0 \in L^1(\Omega)$. A measurable function u defined on Q is a unilateral entropy solution of problem $(\mathcal{P})$ if

$$u \geq \psi \text{ a.e. in } Q, \tag{12}$$

$$T_k(u) \in L^{p^-}(0,T; W_0^{1,p(x)}(\Omega,\omega)), \; for \; all \; k \geq 0 \; and \quad u \in C(0,T;L^1(\Omega)), \tag{13}$$

$$\int_\Omega \Big[S_k(u-v)\Big]_0^T dx + \int_Q \frac{\partial v}{\partial t} T_k(u-v) dx dt$$

$$+ \int_Q \omega(x)\big|\nabla u\big|^{p(x)-2}\nabla u \nabla T_k(u-v) dx dt$$

$$\leq \int_Q \omega(x)g(u)\big|\nabla u\big|^{p(x)} T_k(u-v) dx dt \tag{14}$$

$$+ \int_Q f T_k(u-v) dx dt,$$

for all $v \in K_\psi \cap L^\infty(Q), \frac{\partial v}{\partial t} \in L^{(p^-)'}(0,T;(W_0^{1,p(x)}(\Omega,\omega))^*)$, where $S_k(s) = \int_0^s T_k(r)dr \quad \forall k > 0$.

4   The Principal Result

The aim of the present work is to prove the following result.

Theorem 4.1 *Under assumptions* (9)-(11)*, there exists at least one unilateral entropy solution of problem* (1)*.*

Proof of Theoreme 4.1. Existence of entropy solutions.

We first introduce the approximate problems. Find two sequences of functions $\{f_n\} \subset L^{p'(x)}(Q)$ and $\{u_{0n}\} \subset D(\Omega)$ strongly converging with respct to f in $L^1(Q)$ and to u_0 in $L^1(\Omega)$ such that

$$\|f_n\|_{L^1(Q)} \leq \|f\|_{L^1(Q)} \quad and \quad \|u_{0n}\|_{L^1(\Omega)} \leq \|u_0\|_{L^1(\Omega)}. \tag{15}$$

Then, we consider the approximate problem of

$$
\begin{cases}
\dfrac{\partial u_n}{\partial t} - \operatorname{div}\Big(\omega(x)\big|\nabla u_n\big|^{p(x)-2}\nabla u_n\Big) - nT_n\Big((u_n - \psi)^-\Big) \\
\qquad\qquad = \omega(x)g(u_n)\big|\nabla u_n\big|^{p(x)} + f_n, & \text{in } D'(Q), \\
u_n = 0, & \text{on } \partial\Omega \times (0,T), \\
u_n(t=0) = u_{0n}, & \text{in } \Omega.
\end{cases}
\tag{16}
$$

Moreover, since $f_n \in L^{(p^-)'}(0,T;(W_0^{1,p(x)}(\Omega,\omega))^*)$, proving the existence of weak solution $u_n \in L^{p^-}(0,T;W_0^{1,p(.)}(\Omega))$ of (16) is an easy task (see [4]).

Our aim is to prove that a subsequence of these approximate solution u_n converges to a measurable function u, which is an entropy solution of the problem.

Step 1: A priori estimates. The estimate derived in this step relies on standard techniques for problems of the type (16).

Proposition 4.1 *Assume that* (9)-(11) *hold true and let* u_n *be a solution of the approximate problem* (16)*. Then for all* $k > 0$*, we have*

$$\|T_k(u_n)\|_{L^{p^-}(0,T,W_0^{1,p(x)}(\Omega,\omega))} \leq C\,k \quad for\ all\ n \in \mathbb{N},$$

where C *is a constant independent of* n*.*

Proof.   Let $h > k > 0$ and consider the test function $\varphi = T_h\Big(u_n - T_k(u_n)\Big)\exp(G(u_n)) \in L^{p^-}(0,T;W_0^{1,p(x)}(\Omega,\omega)) \cup L^\infty(Q)$ in the approximate problem (16), where $G(s) = \displaystyle\int_0^s g(r)dr$, we have

$$\Big\langle\!\Big\langle \frac{\partial u_n}{\partial t}, T_h\Big(u_n - T_k(u_n)\Big)\exp(G(u_n))\Big\rangle\!\Big\rangle$$

$$+ \int_{\{k \leq |u_n| \leq k+h\}} \Big(\omega(x)\big|\nabla u_n\big|^{p(x)-2}\nabla u_n\Big)\nabla u_n \exp(G(u_n))dxdt$$

$$+ \int_Q \Big(\omega(x)\big|\nabla u_n\big|^{p(x)-2}\nabla u_n\Big)\nabla u_n T_h\Big(u_n - T_k(u_n)\Big)g(u_n)\exp(G(u_n))dxdt$$

$$- \int_Q nT_n\Big((u_n - \psi)^-\Big)T_h\Big(u_n - T_k(u_n)\Big)\exp(G(u_n))dxdt$$

$$= \int_Q \omega(x)g(u_n)\Big|\nabla u_n\Big|^{p(x)} T_h\Big(u_n - T_k(u_n)\Big)\exp(G(u_n))dxdt$$

$$+ \int_Q f_n T_h\Big(u_n - T_k(u_n)\Big)\exp(G(u_n))dxdt,$$

then

$$\Big\langle\Big\langle\frac{\partial u_n}{\partial t}, T_h(u_n - T_k(u_n))\exp(G(u_n))\Big\rangle\Big\rangle + \int_{\{k\leq|u_n|\leq k+h\}} \omega(x)\Big|\nabla u_n\Big|^{p(x)}\exp(G(u_n))dxdt$$

$$- \int_Q nT_n\Big((u_n - \psi)^-\Big)T_h(u_n - T_k(u_n))\exp(G(u_n))dxdt$$

$$= \int_Q f_n T_h\Big(u_n - T_k(u_n)\Big)\exp(G(u_n))dxdt.$$

On the one hand, we have

$$\Big\langle\Big\langle\frac{\partial u_n}{\partial t}, T_h\Big(u_n - T_k(u_n)\Big)\exp(G(u_n))\Big\rangle\Big\rangle = \int_\Omega S_h^k(u_n(T))dx - \int_\Omega S_h^k(u_{0n})dx,$$

where $S_h^k(s) = \int_0^s T_h\Big(q - T_k(q)\Big)\exp(G(q))dq$, and by using the fact that

$\int_\Omega S_h^k(u_n(T))dx \geq 0$ and $\int_\Omega S_h^k(u_{0n})dx \leq h\exp(\|g\|_{L^1(\mathbb{R})})\|u_{0n}\|_{L^1(\Omega)}$, we get

$$\int_{\{k\leq|u_n|\leq k+h\}} \omega(x)\Big|\nabla u_n\Big|^{p(x)}\exp(G(u_n))dxdt$$

$$- \int_Q nT_n\Big((u_n - \psi)^-\Big)T_h\Big(u_n - T_k(u_n)\Big)\exp(G(u_n))dxdt$$

$$\leq h\exp(\|g\|_{L^1(\mathbb{R})})\Big[\|f_n\|_{L^1(Q)} + \|u_{0n}\|_{L^1(\Omega)}\Big].$$

We have

$$\int_{\{k\leq|u_n|\leq k+h\}} \omega(x)\Big|\nabla u_n\Big|^{p(x)}\exp(G(u_n))dxdt$$

$$- \int_Q nT_n\Big((u_n - \psi)^-\Big)T_h\Big(u_n - T_k(u_n)\Big)\exp(G(u_n))dxdt$$

$$\leq C_1 h. \tag{17}$$

We obtain

$$\int_{\{k\leq|u_n|\leq k+h\}} \omega(x)\Big|\nabla u_n\Big|^{p(x)}\exp(G(u_n))dxdt$$

$$\leq C_1 h + (h+k)\|g\|_\infty \exp(\|g\|_{L^1(\mathbb{R})})\int_Q nT_n\Big((u_n - \psi)^-\Big)dxdt,$$

then

$$\int_{\{k\leq|u_n|\leq k+h\}} \omega(x)\Big|\nabla u_n\Big|^{p(x)}\exp(G(u_n))dxdt \leq C_2 h\int_Q nT_n\Big((u_n - \psi)^-\Big)dxdt.$$

Let us take $\rho_1(u_n) = \int_0^{u_n} g(s)\chi_{\{|s|\leq k\}} ds \exp(G(u_n))$ as a test function of (16) , we obtain

$$\left[\int_\Omega \varphi_2(u_n) dx \right]_0^T + \int_Q \omega(x) \big|\nabla u_n\big|^{p(x)} g(u_n) \exp(G(u_n)) \chi_{\{|u_n|\leq k\}} dxdt$$

$$- \int_Q nT_n\Big((u_n - \psi)^-\Big) \rho_1(u_n) dxdt$$

$$\leq \Big(\int_0^\infty g(s) ds \Big) \exp \Big(\|g\|_{L^1(\mathbb{R})} \Big) \|f_n\|_{L^1(Q)},$$

where $\varphi_2(r) = \int_0^r \rho_1(s) ds$, which implies, in view of $\varphi_2(r) \geq 0$, that

$$\int_{\{|u_n|\leq k\}} \omega(x) \big|\nabla u_n\big|^{p(x)} g(u_n) \exp(G(u_n)) dxdt$$

$$\leq \|g\|_\infty \exp\Big(\|g\|_{L^1(\mathbb{R})} \Big) \Big[\|f_n\|_{L^1(Q)} + \|u_0\|_{L^1(\Omega)} \Big]$$

$$+ (h+k)\|g\|_\infty \exp(\|g\|_{L^1(\mathbb{R})}) \int_Q nT_n\Big((u_n - \psi)^-\Big) dxdt.$$

Then

$$\int_{\{|u_n|\leq k\}} \omega(x) \big|\nabla u_n\big|^{p(x)} g(u_n) \exp(G(u_n)) dxdt$$

$$\leq hC_3 \int_Q nT_n\Big((u_n - \psi)^-\Big) dxdt.$$

Similarly, taking $\rho_2 = \int_0^{u_n} g(s)\chi_{\{|s|\geq k+h\}} ds \exp(G(u_n))$ as a test function of (16) , we conclude that

$$\int_{\{|u_n|\geq k+h\}} \big|\nabla u_n\big|^{p(x)} g(u_n) \exp(G(u_n)) dxdt \leq hC_4 \int_Q nT_n((u_n - \psi)^-) dxdt.$$

Consequently, we have :

$$\begin{cases} \int_Q \omega(x) \big|\nabla u_n\big|^{p(x)} g(u_n) \exp(G(u_n)) dxdt \\ \leq \int_{\{|u_n|\geq k+h\}} \omega(x) \big|\nabla u_n\big|^{p(x)} g(u_n) \exp(G(u_n)) dxdt \\ + \int_{\{|u_n|\leq k\}} \omega(x) \big|\nabla u_n\big|^{p(x)} g(u_n) \exp(G(u_n)) dxdt \\ + \int_{\{k\leq|u_n|\leq k+h\}} \omega(x) \big|\nabla u_n\big|^{p(x)} g(u_n) \exp(G(u_n)) dxdt \\ \leq hC_5 \int_Q nT_n((u_n - \psi)^-) dxdt, \quad \text{where } C_5 = Max(C_2, C_3, C_4). \end{cases}$$

Using (17), we have

$$\int_Q \omega(x) \big|\nabla u_n\big|^{p(x)} g(u_n) \exp(G(u_n)) dxdt - \int_Q nT_n\Big((u_n - \psi)^-\Big) T_h\Big(u_n - T_k(u_n)\Big) \exp(G(u_n)) dxdt$$

$$\leq hC_5 \int_Q nT_n\Big((u_n - \psi)^-\Big) dxdt,$$

we obtain

$$-\int_Q nT_n\Big((u_n-\psi)^-\Big)T_h\Big(u_n-T_k(u_n)\Big)\exp(G(u_n))dxdt \le hC_5\int_Q nT_n\Big((u_n-\psi)^-\Big)dxdt$$

so that

$$-\int_Q nT_n\Big((u_n-\psi)^-\Big)\frac{T_h\Big(u_n-T_k(u_n)\Big)}{h}\exp(G(u_n))dxdt \le C_5\int_Q nT_n\Big((u_n-\psi)^-\Big)dxdt.$$

Let us now fix $k > \|\psi\|_\infty$, by the fact that $nT_n((u_n-\psi)(u_n-k)\chi_{\{u_n\le\psi;k\le u_n\le k+h\}} \ge 0$ and letting $h \to 0$, one has

$$\int_Q nT_n\Big((u_n-\psi)^-\Big)dxdt \le C_6. \tag{18}$$

Let use $v = T_k(u_n)\exp(G(u_n))\chi(0,\tau)$ as a test function of (16)

$$\Big[\int_\Omega \varphi_3(u_n)dx\Big]_0^T + \int_{Q^\tau}\omega(x)\Big|\nabla u_n\Big|^{p(x)-2}\nabla u_n\nabla T_k(u_n)\exp(G(u_n))dxdt$$

$$+ \int_{Q^\tau}\omega(x)\Big|\nabla u_n\Big|^{p(x)}T_k(u_n)g(u_n)\exp(G(u_n))dxdt$$

$$- \int_{Q^\tau} nT_n\Big((u_n-\psi)^-\Big)T_k(u_n)\exp(G(u_n))dxdt$$

$$= \int_{Q^\tau}\omega(x)\Big|\nabla u_n\Big|^{p(x)}g(u_n)T_k(u_n)\exp(G(u_n))dxdt$$

$$+ \int_{Q^\tau} f_nT_k(u_n)\exp(G(u_n))dxdt,$$

where $\varphi_3(r) = \int_0^r T_k(s)\exp(G(s))ds$. Due to the definition of φ_3 and the fact that $|G(u_n)| \le \exp(\|g\|_{L^1(\mathbb{R})}\|u_{0n}\|_{L^1(\Omega)})$, we have $0 \le \int_\Omega \varphi_3(u_{0n})dx \le k\exp(\|g\|_{L^1(\mathbb{R})}\|u_{0n}\|_{L^1(\Omega)})$, and by using (18) we arrive at

$$\int_{Q^\tau}\omega(x)\Big|\nabla T_k(u_n)\Big|^{p(x)}\exp(G(u_n))dxdt$$

$$\le k\exp(\|g\|_{L^1(\mathbb{R})})\Big[\|f_n\|_{L^1(Q)} + \|u_0\|_{L^1(\Omega)+C_7}\Big].$$

Let us take $\rho_4(u_n) = \int_0^{u_n} g(s)\chi_{\{s\ge0\}}ds\exp(G(u_n))$ as a test function of 16, we obtain

$$\Big[\int_\Omega \varphi_4(u_n)dx\Big]_0^T + \int_Q \omega(x)\Big|\nabla u_n\Big|^{p(x)}g(u_n)\exp(G(u_n))\chi_{\{u_n\ge0\}}dxdt$$

$$- \int_Q nT_n\Big((u_n-\psi)^-\Big)\rho_4(u_n)dxdt$$

$$\le \Big(\int_0^\infty g(s)ds\Big)\exp\Big(\|g\|_{L^1(\mathbb{R})}\Big)\|f_n\|_{L^1(Q)},$$

where $\varphi_4(r) = \int_0^r \rho_4(s)ds$, which implies, in view of $\varphi_4(r) \geq 0$, that

$$\int_{\{u_n \geq 0\}} \omega(x)\Big|\nabla u_n\Big|^{p(x)} g(u_n) \exp(G(u_n))dxdt$$

$$\leq \|g\|_\infty \exp\Big(\|g\|_{L^1(\mathbb{R})}\Big)\Big[\|f_n\|_{L^1(Q)} + \|u_0\|_{L^1(\Omega)}\Big] + C_8,$$

then

$$\int_{\{u_n \geq 0\}} \omega(x)\Big|\nabla u_n\Big|^{p(x)} g(u_n) \exp(G(u_n))dxdt \leq C_9.$$

Similarly, taking $\rho_5 = \int_{u_n}^0 g(s)\chi_{\{s \leq 0\}} ds \exp(G(u_n))$ as a test function of (16), we conclude that

$$\int_{\{u_n \leq 0\}} \omega(x)\Big|\nabla u_n\Big|^{p(x)} g(u_n) \exp(G(u_n))dxdt \leq C_{10}.$$

Consequently,

$$\int_Q \omega(x)\Big|\nabla u_n\Big|^{p(x)} g(u_n) \exp(G(u_n))dxdt \leq C_{11}.$$

As $C_1, ..., C_{11}$ are constants independent of n, we deduce that

$$\int_Q \omega(x)\Big|\nabla T_k(u_n)\Big|^{p(x)}dxdt \leq k\, C_{12}$$

$$\Rightarrow \|T_k(u_n)\|_{L^{p^-}(0,T,W_0^{1,p(x)}(\Omega,\omega))} \leq C_{13}\, k. \tag{19}$$

Then, we conclude that $T_k(u_n)$ is bounded in $L^{p^-}(0,T;W_0^{1,p(x)}(\Omega,\omega))$, independently of n for any $k > 0$.

Now we turn to proving that $(u_n)_n$ is a Cauchy sequence in measures. Let $k > 0$ be large enough and B_R be a ball of Ω. Using (19) and applying Hölder's inequality and Poincarè's inequality, we obtain that

$$k\ meas\Big(\{\Big|u_n\Big| > k\} \cap B_R \times [0,T]\Big) = \int_0^T \int_{\{|u_n|>k\}\cap B_R} \Big|T_k(u_n)\Big|dxdt$$

$$\leq \int_0^T \int_{B_R} \Big|T_k(u_n)\Big|dxdt$$

$$\leq C\|\nabla T_k(u_n)\|_{L^{p(x)}(\Omega,\omega)}$$

$$\leq C\left(\int_Q \Big|\nabla T_k(u_n)\Big|^{p(x)} wdxdt\right)^{\frac{1}{\theta}}$$

$$\leq Ck^{\frac{1}{\theta}},$$

where

$$\theta = \begin{cases} p^-, & if \quad \|T_k(u_n)\|_{L^{p(x)}(\Omega,\omega)} \leq 1 \\ p^+, & if \quad \|T_k(u_n)\|_{L^{p(x)}(\Omega,\omega)} > 1. \end{cases}$$

This implies that

$$meas\Big(\{|u_n| > k\} \cap (B_R \times [0,T])\Big) \leq \frac{c_1}{k^{1-\frac{1}{\theta}}}, \quad \forall k \geq 1. \tag{20}$$

So, we have

$$\lim_{k \to +\infty} \Big(meas\Big(\{|u_n| > k\} \cap (B_R \times [0,T])\Big)\Big) = 0.$$

Then, we obtain for all $\delta > 0$

$$meas\Big(\{|u_n - u_m| > \delta\} \cap (B_R \times [0,T])\Big) \leq meas\Big(\{|u_n| > k\} \cap (B_R \times [0,T])\Big)$$
$$+ meas\Big(\{|u_m| > k\} \cap (B_R \times [0,T])\Big)$$
$$+ meas\Big(\{|T_k(u_n) - T_k(u_m)| > \delta\}\Big).$$

Since $T_k(u_n)$ is bounded in $L^{p^-}(0,T; W_0^{1,p(x)}(\Omega, \omega))$, it is clear that $T_k(u_n) \to v_k$ strongly in $L^{p(x)}(Q,\omega)$ and almost everywhere in Q. Hence $(T_k(u_n))_n$ is a Cauchy sequence in measure in Q.

Let $\epsilon > 0$, then by (20)), there exists a $k(\epsilon) > 0$ such that

$$meas\Big(\{|u_n - u_m| > \delta\} \cap (B_R \times [0,T])\Big) < \epsilon \quad \forall n,m \geq n_0(k(\epsilon), \delta, R).$$

This proves that $(u_n))_n$ is a Cauchy sequence in measures in B_R.

Consider a non-decreasing function $g_k \in C^2(\mathbb{R})$ such that

$$g_k(s) = \begin{cases} s, & if \ |s| \leq \frac{k}{2}, \\ k, & if \ |s| \geq k. \end{cases}$$

Multiplying the approximate equation by $g_k'(u_n)$, we get

$$\frac{\partial g_k(u_n)}{\partial t} - \text{div}\Big(\omega(x)\big|\nabla u_n\big|^{p(x)-2}\nabla u_n)g_k'(u_n)\Big) + \omega(x)\big|\nabla u_n\big|^{p(x)}g_k''(u_n)$$
$$- nT_n\Big((u_n - \psi)^-\Big)g_k'(u_n) = \omega(x)g(u_n)\big|\nabla u_n\big|^{p(x)}g_k'(u_n) + f_n g_k'(u_n) \tag{21}$$

in the sense of distributions. This implies, thanks to the fact that g_k' has compact support, that $g_k(u_n)$ is bounded in $L^{p^-}(0,T; W_0^{1,p(x)}(\Omega, \omega))$, while its time derivative $\frac{\partial g_k(u_n)}{\partial t}$ is bounded in $L^1(Q) + V^*$ Due to the choice of g_k, we conclude that for each k, the sequence $T_k(u_n)$ converges almost everywhere in Q, which implies that the sequence u_n converges almost everywhere to some measurable function v in Q. Thus, by using the same argument as in [9], [10], [11], we can show the following lemma.

Lemma 4.1 *Let u_n be a solution of (16). Then*

$$u_n \to u \quad a.e. \quad in \quad Q. \tag{22}$$

We can deduce from (19) that

$$T_k(u_n) \rightharpoonup T_k(u) \quad in \quad L^{p^-}(0,T; W_0^{1,p(x)}(\Omega, \omega)). \tag{23}$$

Lemma 4.2 *[2] Let u_n be a solution of (16). Then*

$$\lim_{m\to\infty} \limsup_{n\to\infty} \int_{\{m\le |u_n|\le m+1\}} a(x,t,u_n,\nabla(u_n))\nabla u_n dx dt = 0. \tag{24}$$

Step 3: **Almost everywhere convergence of the gradients.** This step is devoted to introducing, for a fixed $k \ge 0$, a time regularization of the function $T_k(u)$ in order to apply the monotonicity method. This specific time regularization of $T_k(u)$ (for fixed $k \ge 0$) is defined as follows. Let $(v_0^\mu)_\mu$ be a sequence of functions defined on Ω such that

$$v_0^\mu \in L^\infty(\Omega) \cap W_0^{1,p(x)}(\Omega,\omega) \quad for\ all\ \mu > 0, \tag{25}$$

$$\|v_0^\mu\|_{L^\infty(\Omega)} \le k \quad for\ all\ \mu > 0, \tag{26}$$

$$v_0^\mu \to T_k(u_0)\ a.e.\ in\ \Omega\ and\ \frac{1}{\mu}\|v_0^\mu\|_{L^{p(x)}(\Omega,\omega)} \to 0,\ as\ \mu \to \infty. \tag{27}$$

For fixed $k,\ \mu > 0$, let us consider the unique solution $(T_k(u))_\mu \in L^\infty(Q) \cap L^{p^-}(0,T;W_0^{1,p(x)}(\Omega,\omega))$ of the monotone problem:

$$\frac{\partial(T_k(u))_\mu}{\partial t} + \mu\Big((T_k(u))_\mu - T_k(u)\Big) = 0 \quad in\ D'(Q), \tag{28}$$

$$(T_k(u))_\mu(t=0) = v_0^\mu \quad in \quad \Omega. \tag{29}$$

Note that due to (28), we have for $\mu > 0$ and $k \ge 0$

$$\frac{\partial(T_k(u))_\mu}{\partial t} \in L^{p^-}(0,T;W_0^{1,p(x)}(\Omega,\omega)). \tag{30}$$

We just recall here that (28)–(29) imply that

$$(T_k(u))_\mu \to T_k(u)\ a.e.\ in\ Q, \tag{31}$$

as well as weakly in $L^\infty(Q)$ and strongly in $L^{p^-}(0,T;W_0^{1,p(x)}(\Omega,\omega))$ as $\mu \to \infty$. Note that for any μ and any $k \ge 0$, we have

$$\|(T_k(u))_\mu\|_{L^\infty(Q,\omega)} \le \max\Big(\|T_k(u)\|_{L^\infty(Q,\omega)};\ \|v_0^\mu\|_{L^\infty(\Omega,\omega)}\Big) \le k. \tag{32}$$

We introduce a sequence of increasing $C^\infty(\mathbb{R})$–functions S_m such that

$$S_m(r) = r\ for\ |r| \le m,\ supp(S_m') \subset [-(m+1), m+1], \|S_m''\|_{L^\infty(\mathbb{R})} \le 1,$$

for any $m \ge 1$, and we denote by $\omega(n,\mu,\eta,m)$ the quantities such that

$$\lim_{m\to\infty} \lim_{\eta\to 0} \lim_{\mu\to\infty} \lim_{n\to\infty} \omega(n,\mu,\eta,m) = 0.$$

Lemma 4.3 *([2, 11]). We have*

$$\int_0^T \left\langle \frac{\partial u_n}{\partial t}, T_\eta\Big(u_n - (T_k(u))_\mu\Big)^+ \exp(G(u_n))S_m'(u_n) \right\rangle dt \ge \omega(n,\mu,\eta)\ \forall m \ge 1. \tag{33}$$

Taking now $v = T_\eta\Big(u_n - (T_k(u))_\mu\Big)^+ S'_m(u_n)\exp(G(u_n))$ of (16), we get

$$\int_0^T \left\langle \frac{\partial u_n}{\partial t}, T_\eta\Big(u_n - (T_k(u))_\mu\Big)^+ \exp(G(u_n))S'_m(u_n)\right\rangle dt$$

$$+ \int_Q \Big(\omega(x)\big|\nabla u_n\big|^{p(x)-2}\nabla u_n\Big)\nabla\Big(T_\eta\Big(u_n - (T_k(u))_\mu\Big)^+\Big)\exp(G(u_n))S'_m(u_n)dxdt$$

$$+ \int_{\{m\leq|u_n|\leq m+1\}} \Big(\omega(x)\big|\nabla u_n\big|^{p(x)-2}\nabla u_n\Big)T_\eta(u_n - (T_k(u))_\mu)^+\exp(G(u_n))S''_m(u_n)\nabla u_ndxdt$$

$$+ \int_Q \Big(\omega(x)\big|\nabla u_n\big|^{p(x)-2}\nabla u_n\Big)g(u_n)\nabla u_n\exp(G(u_n))T_\eta\Big(u_n - (T_k(u))_\mu\Big)^+ S'_m(u_n)dxdt$$

$$- n\int_Q T_n\Big((u_n - \psi)^-\Big)T_\eta\Big(u_n - (T_k(u))_\mu\Big)^+\exp(G(u_n))S'_m(u_n)dxdt \qquad (34)$$

$$= \int_Q \omega(x)g(u_n)\big|\nabla u_n\big|^{p(x)}T_\eta\Big(u_n - (T_k(u))_\mu\Big)^+ S'_m(u_n)\exp(G(u_n))dxdt$$

$$+ \int_Q f_nT_\eta\Big(u_n - (T_k(u))_\mu\Big)^+ S'_m(u_n)\exp(G(u_n))dxdt.$$

From (19),(24),(33),(34) it follows that

$$\int_Q \Big(\omega(x)\big|\nabla u_n\big|^{p(x)-2}\nabla u_n\Big)\nabla\Big(T_\eta\Big(u_n - (T_k(u))_\mu\Big)^+\Big)\exp(G(u_n))S'_m(u_n)dxdt$$

$$\leq C\eta + \omega(n,\mu,\eta,m), \qquad (35)$$

where C is a constant independent of n and m. On the other hand, let
$A = \{0 \leq T_k(u_n) - (T_k(u))_\mu < \eta\}$ and $B = \{0 \leq u_n - (T_k(u))_\mu < \eta\}$. Then, we have

$$\int_Q \Big(\omega(x)\big|\nabla u_n\big|^{p(x)-2}\nabla u_n\Big)\nabla\Big(T_\eta\Big(u_n - (T_k(u))_\mu\Big)^+\Big)\exp(G(u_n))S'_m(u_n)dxdt$$

$$= \int_B \Big(\omega(x)\big|\nabla u_n\big|^{p(x)-2}\nabla u_n\Big)\Big(\nabla u_n - \nabla(T_k(u))_\mu\Big)\exp(G(u_n))S'_m(u_n)dxdt \qquad (36)$$

$$= \int_A \Big(\omega(x)\big|\nabla u_n\big|^{p(x)-2}\nabla u_n\Big)\Big)\Big(\nabla T_k(u_n) - \nabla(T_k(u))_\mu\Big)\exp(G(u_n))S'_m(u_n)dxdt$$

$$+ \int_{\{|u_n|>k\}\cap B} \Big(\omega(x)\big|\nabla u_n\big|^{p(x)-2}\nabla u_n\Big)\Big(\nabla u_n - \nabla(T_k(u))_\mu\Big)\exp(G(u_n))S'_m(u_n)dxdt.$$

Given the definition of $S'_m[S'_m(u_n) = 1$ a.e. in $\{|u_n| \leq k\}$ if $k \leq m]$, it is possible to obtain from (35) and (36), that

$$\int_A \Big(\omega(x)\big|\nabla T_k(u_n)\big|^{p(x)-2}\nabla T_k(u_n))\Big)\Big(\nabla T_k(u_n) - \nabla(T_k(u))_\mu\Big)\exp(G(u_n)S'_m(u_n)dxdt$$

$$\leq \int_{\{|u_n|>k\}\cap B} \Big(\omega(x)\big|\nabla u_n\big|^{p(x)-2}\nabla u_n\Big)\nabla(T_k(u))_\mu\exp(G(u_n))S'_m(u_n)dxdt$$

$$+ C\eta + \omega(n,\mu,\eta,m). \qquad (37)$$

Since $\nabla T_{k+\eta}(u_n)$ is bounded in $(L^{p'(x)}(Q,\omega))^N$ and $u_n \to u$ a.e. in Q, one has $\nabla T_{k+\eta}(u_n) \rightharpoonup \nabla T_{k+\eta}(u)$ weakly in $(L^{p'(x)}(Q,\omega))^N$. Consequently,

$$\int_{\{|u_n|>k\}\cap B} \omega(x)\big|\nabla T_{k+\eta}(u_n)\big|^{p(x)-2}\nabla T_{k+\eta}(u_n)|\nabla(T_k(u))_\mu|\exp(G(u_n))S'_m(u_n)dxdt$$

$$= \int_{\{|u|>k\}\cap\{0\leq u-(T_k(u))_\mu<\eta\}} \omega(x)\big|\nabla T_{k+\eta}(u_n)\big|^{p(x)-2}\nabla(T_k(u))_\mu\exp(G(u))S'_m(u)dxdt + \omega(n).$$

Thanks to (31) one easily has

$$\int_{\{|u|>k\}\cap\{0\leq u-(T_k(u))_\mu<\eta\}} \omega(x)\Big|\nabla T_{k+\eta}(u)\Big|^{p(x)-2}\nabla(T_k(u))_\mu \exp(G(u))S'_m(u)dxdt = \omega(\mu).$$

Hence,

$$\int_A \Big(\omega(x)\Big|\nabla u_n\Big|^{p(x)-2}\nabla u_n\Big)\nabla\Big(T_\eta\Big(u_n - (T_k(u))_\mu\Big)^+\Big)\exp(G(u_n))S'_m(u_n)dxdt$$
$$\leq C\eta + \omega(n,\mu,\eta,m). \tag{38}$$

On the other hand, note that

$$\int_A \Big(\omega(x)\Big|\nabla T_k(u_n)\Big|^{p(x)-2}\nabla T_k(u_n)\Big)\Big(\nabla T_k(u_n) - \nabla(T_k(u))_\mu\Big)\exp(G(u_n))dxdt$$

$$= \int_A \Big(\omega(x)|\nabla T_k(u_n)|^{p(x)-2}\nabla T_k(u_n)\Big)\Big(\nabla T_k(u_n) - \nabla T_k(u)\Big)\exp(G(u_n))dxdt$$

$$+ \int_A \Big(\omega(x)\Big|\nabla T_k(u_n)\Big|^{p(x)-2}\nabla T_k(u_n)\Big)\Big(\nabla T_k(u) - \nabla(T_k(u))_\mu\Big)\exp(G(u_n))dxdt, \tag{39}$$

and the last integral tends to 0 as $n \to \infty$ and $\mu \to \infty$. Indeed, we have that

$$\int_A \Big(\omega(x)\Big|\nabla T_k(u_n)\Big|^{p(x)-2}\nabla T_k(u_n))\Big(\nabla T_k(u) - \nabla(T_k(u))_\mu\Big)\exp(G(u_n))dxdt$$

$$\to \int_{\{0\leq T_k(u)-(T_k(u))_\mu<\eta\}} \Big(\omega(x)\Big|\nabla T_k(u)\Big|^{p(x)-2}\nabla T_k(u))\Big(\nabla T_k(u) - \nabla(T_k(u))_\mu\Big)\exp(G(u))dxdt$$

as $n \to \infty$.
Using (31) and Lebesgue's theorem, we have

$$\int_{\{0\leq T_k(u)-(T_k(u))_\mu<\eta\}} \Big(\omega(x)\Big|\nabla T_k(u)\Big|^{p(x)-2}\nabla T_k(u))\Big(\nabla T_k(u) - \nabla(T_k(u))_\mu\Big)\exp(G(u))dxdt \to 0$$

as $\mu \to \infty$. We deduce then that

$$\int_A \Big(\omega(x)\Big|\nabla T_k(u_n)\Big|^{p(x)-2}\nabla T_k(u_n))\Big(\nabla T_k(u_n) - \nabla T_k(u)\Big)\exp(G(u_n)dxdt$$
$$\leq C\eta + \omega(n,\mu,\eta,m). \tag{40}$$

Let $M_n = \Big(\Big[\Big(\omega(x)\Big|\nabla T_k(u_n)\Big|^{p(x)-2}\nabla T_k(u_n)\Big]\Big[\nabla T_k(u_n) - \nabla T_k(u)\Big]\Big) \times \Big(\exp(G(u_n))\Big).$
Then, for any $0 < \theta < 1$, we write

$$I_n = \int_{\{|u_n-(T_k(u))_\mu|\geq 0\}} M_n^\theta dxdt = \int_{\{|T_k(u_n)-(T_k(u))_\mu|\leq\eta,\ u_n-T_k(u)_\mu\geq 0\}} M_n^\theta dxdt$$

$$+ \int_{\{|T_k(u_n)-(T_k(u))_\mu|>\eta,\ u_n-(T_k(u))_\mu\geq 0\}} M_n^\theta dxdt.$$

Since $\nabla T_k(u_n)$ is bounded in $(L^{p(x)}(Q,\omega))^N$, we obtain by applying Hölder's inequality that

$$I_n \leq C_1\Big(\int_{\{0\leq T_k(u_n)-(T_k(u))_\mu<\eta\}} M_n dxdt\Big)^\theta \tag{41}$$

$$+ C_2 \operatorname{meas}\Big\{(x,t)\in Q : \Big|T_k(u_n) - (T_k(u))_\mu\Big| > \eta ,\ u_n - (T_k(u))_\mu \geq 0\Big\}^{1-\theta}.$$

On the other hand, we have

$$
\int_{\{0\leq T_k(u_n)-(T_k(u))_\mu)<\eta\}} M_n \, dx \, dt
$$
$$
= \int_{\{0\leq T_k(u_n)-(T_k(u))_\mu)<\eta\}} \left(\omega(x)\Big|\nabla T_k(u_n)\Big|^{p(x)-2}\nabla T_k(u_n)\right)\times
$$
$$
\times \Big(\nabla T_k(u_n)-\nabla T_k(u)\Big)\exp(G(u_n)) \, dx \, dt \tag{42}
$$
$$
- \int_{\{0\leq T_k(u_n)-(T_k(u))_\mu)<\eta\}} \left(\omega(x)\Big|\nabla T_k(u_n)\Big|^{p(x)-2}\nabla T_k(u)\right)\times
$$
$$
\times \Big(\nabla T_k(u_n)-\nabla T_k(u)\Big)\exp(G(u_n)) \, dx \, dt
$$
$$
= I_n^1 + I_n^2.
$$

Using (40), we have

$$
I_n^1 \leq C \, \eta + w(n,\mu,\eta,m). \tag{43}
$$

Concerning I_n^2, that is the second term of the right-hand side of the (42), it is easy to see that

$$
I_n^2 = w(n,\mu). \tag{44}
$$

Therefore, for all $i = 1, ..., N$, we have $\frac{\partial T_k(u_n)}{\partial x_i} \rightharpoonup \frac{\partial T_k(u)}{\partial x_i}$ in $L^{p(x)}(Q,\omega)$. Combining (41), (42), (43) and (44), we get

$$
I_n \leq C_1 \left(C \, \eta + w(n,\mu,\eta,m)\right)^\theta + C_2 \left(w(n,\mu)\right)^{1-\theta}
$$

and by passing to the limit sup over n, μ and η

$$
\int_{\{u_n-(T_k(u))_\mu\geq 0\}} \left(\omega(x)\Big[\Big|\nabla T_k(u_n)\Big|^{p(x)-2}\nabla T_k(u_n))-\Big|\nabla T_k(u_n)\Big|^{p(x)-2}\nabla T_k(u))\Big]\times\right.
$$
$$
\left.\times \Big[\nabla T_k(u_n)-\nabla T_k(u)\Big]\right)^\theta dx \, dt = w(n). \tag{45}
$$

On the other hand, we choose $v = T_\eta\Big(u_n-(T_k(u))_\mu\Big)^- \exp(-G(u_n))$ in (16) and obtain:

$$
\int_{\{u_n-T_k(u)_\mu\leq 0\}} \left(\Big[\omega(x)\Big|\nabla T_k(u_n)\Big|^{p(x)-2}\nabla T_k(u_n))-\Big|\nabla T_k(u_n)\Big|^{p(x)-2}\nabla T_k(u))\Big]\times\right.
$$
$$
\left.\times \Big[\nabla T_k(u_n)-\nabla T_k(u)\Big]\right)^\theta dx \, dt = w(n). \tag{46}
$$

Moreover, (45) and (46) imply that

$$
\int_Q \left(\omega(x)\Big[\Big|\nabla T_k(u_n)\Big|^{p(x)-2}\nabla T_k(u_n))-\Big|\nabla T_k(u_n)\Big|^{p(x)-2}\nabla T_k(u))\Big]\times\right.
$$
$$
\left.\times \Big[\nabla T_k(u_n)-\nabla T_k(u)\Big]\right)^\theta dx \, dt = w(n), \tag{47}
$$

which implies that

$$
T_k(u_n) \to T_k(u) \quad in \quad L^{p^-}(0,T;W_0^{1,p(x)}(\Omega,\omega)) \, \forall \, k \geq 0. \tag{48}
$$

According to [9,10], there exists a subsequence also denoted by u_n such that

$$
\nabla u_n \to \nabla u \quad \text{a.e. in } Q. \tag{49}
$$

Proposition 4.2 *Let u_n be a solution of* (16). *Then* $\quad u \geq \psi \quad$ *a.e. in Q.*

Proof. Thanks to (18), we can write $\int_Q T_n\left((u_n - \psi)^-\right)dxdt \leq \frac{C}{n}$. So, by using Fatou's lemma as $n \to \infty$, we infer that $\int_Q (u-\psi)^- dxdt = 0$, which implies that $(u-\psi)^- = 0$ a.e. in Q. Consequently, we conclude that $u \geq \psi$ a.e. in Q.

Step 4: **Passing to the limit**

a) we claim that $u \in C(0,T; L^1(\Omega))$ .We will show that

$$u_n \to u \quad in \quad C(0,T; L^1(\Omega)).$$

Since $T_k(u) \in K_\psi$, for every $k \geq \|\psi\|_{L^\infty}$ there exists a sequence $v_j \in K_\psi \cap D(\bar{Q})$ such that

$$v_j \to T_k(u) \quad in \quad L^{p^-}(0,T; W_0^{1,p(x)}(\Omega, \omega))$$

for the modular convergence.

Let $\omega_{j,\mu}^{i,l} = (T_l(v_j))_\mu + e^{-\mu t}T_l(\eta_i)$ with $\eta_i \geq 0$ converge to u_0 in $L^1(\Omega)$, where $(T_l(v_j))_\mu$ is the mollification of $T_l(v_j)$ with respect to time. Note that $\omega_{j,\mu}^{i,l}$ is a smooth function having the following properties:

$$\frac{\partial \omega_{j,\mu}^{i,l}}{\partial t} = \mu(T_l(v_j) - \omega_{j,\mu}^{i,l}), \quad \omega_{j,\mu}^{i,l}(0) = T_l(\eta_i), \quad |\omega_{j,\mu}^{i,l}| \leq l, \tag{50}$$

$$\omega_{j,\mu}^{i,l} \to T_l(v_j) \quad in \ L^{p^-}(0,T; W_0^{1,p(x)}(\Omega, \omega)) \quad as \ \mu \to \infty. \tag{51}$$

Choosing now $v = T_k(u_n - \omega_{j,\mu}^{i,l})\chi_{(0,\tau)}$ as a test function of (16), we get

$$\left\langle \frac{\partial u_n}{\partial t}, T_k(u_n - \omega_{j,\mu}^{i,l}) \right\rangle_{Q^\tau} + \int_{Q^\tau} \omega(x)\left|\nabla u_n\right|^{p(x)-2}\nabla u_n \nabla T_k(u_n - \omega_{j,\mu}^{i,l})dxdt$$

$$- \int_{Q^\tau} nT_n\left((u_n - \psi)^-\right)T_k(u_n - \omega_{j,\mu}^{i,l})dxdt \tag{52}$$

$$= \int_{Q^\tau} \omega(x)g(u_n)\left|\nabla u_n\right|^{p(x)}T_k(u_n - \omega_{j,\mu}^{i,l})dxdt + \int_{Q^\tau} f_n T_k(u_n - \omega_{j,\mu}^{i,l})dxdt.$$

By using the fact that $-\int_{Q^\tau} nT_n(u_n - \psi)^- T_k(u_n - \omega_{j,\mu}^{i,l})dxdt \geq 0$, we deduce that:

$$\left\langle \frac{\partial u_n}{\partial t}, T_k(u_n - \omega_{j,\mu}^{i,l}) \right\rangle_{Q^\tau} + \int_{Q^\tau} \omega(x)\left|\nabla u_n\right|^{p(x)-2}\nabla u_n \nabla T_k(u_n - \omega_{j,\mu}^{i,l})dxdt$$

$$= \int_{Q^\tau} \omega(x)g(u_n)\left|\nabla u_n\right|^{p(x)}T_k(u_n - \omega_{j,\mu}^{i,l})dxdt + \int_{Q^\tau} f_n T_k(u_n - \omega_{j,\mu}^{i,l})dxdt.$$

• On the one hand, we have

$$I = \int_{Q^\tau} \omega(x)\left|\nabla u_n\right|^{p(x)-2}\nabla u_n \nabla T_k(u_n - \omega_{j,\mu}^{i,l})dxdt$$

$$= \int_{\{|T_k(u_n)-\omega_{j,\mu}^{i,l}|\leq k\}} \omega(x)\left|\nabla u_n\right|^{p(x)-2}\nabla u_n[\nabla T_k(u_n) - \nabla\omega_{j,\mu}^{i,l}]dxdt. \tag{53}$$

In the following, we pass to the limit in (53): By letting n and μ to infinity and by using Lebesgue theorem, we have

$$I = \int_{\{|T_k(u) - T_l(v_j)| \leq k\}} \omega(x) \Big| \nabla T_k(u) \Big|^{p(x)-2} \nabla T_k(u) \Big[\nabla T_k(u) - \nabla T_l(v_j) \Big] dx dt + \epsilon(n, \mu),$$

consequently, by taking the limit as $j \to \infty$, we deduce that

$$I = \epsilon(n, \mu, j, l).$$

• On the other hand, we have

$$J = \int_{Q^\tau} \omega(x) g(u_n) \Big| \nabla u_n \Big|^{p(x)} T_k(u_n - \omega_{j,\mu}^{i,l}) dx dt. \tag{54}$$

In the following, we pass to the limit in (54): Taking the limit as $n \to \infty$ in (54) and since $\omega(x) g(u_n) |\nabla u_n|^{p(x)} \to \omega(x) g(u) |\nabla u|^{p(x)}$ in $L^1(Q)$, and by using Lebesque theorem, we obtain $J = \int_{Q^\tau} g(u) |\nabla u|^{p(x)} T_k(u - \omega_{j,\mu}^{i,l}) dx dt + \epsilon(n)$ and by letting μ and j to infinity, we have

$$J = \epsilon(n, \mu, j, l).$$

• Due to (15) , $u_n \to u_0$ and letting n , μ and j to infinity, we have

$$\int_{Q^\tau} f_n \Big[T_k(u_n - \omega_{j,\mu}^{i,l}) \Big] dx dt = \epsilon(n, \mu, j, l)$$

and by using Vitali's theorem, we get

$$\limsup_{k \to \infty} \limsup_{i \to 0} \limsup_{j \to \infty} \limsup_{\mu \to \infty} \lim_{n \to \infty} \Big\langle \frac{\partial u_n}{\partial t}, T_k(u_n - \omega_{j,\mu}^{i,l}) \Big\rangle_{Q^\tau} \leq 0. \tag{55}$$

We have (see([1]))

$$\Big\langle \frac{\partial \omega_{j,\mu}^{i,l}}{\partial t}, T_k(u_n - \omega_{j,\mu}^{i,l}) \Big\rangle_{Q^\tau} = \mu \int_{Q^\tau} (T_k(v_j) - \omega_{j,\mu}^{i,l})) T_k(u_n - \omega_{j,\mu}^{i,l}) \geq \epsilon(n, j, \mu, l) \tag{56}$$

uniformly on τ. Therefore, by writing

$$\int_\Omega S_k \Big(u_n(\tau) - \omega_{j,\mu}^{i,l}(\tau) \Big) dx = \Big\langle \frac{\partial u_n}{\partial t}, T_k(u_n - \omega_{j,\mu}^{i,l}) \Big\rangle_{Q^\tau}$$

$$- \Big\langle \frac{\partial \omega_{j,\mu}^{i,l}}{\partial t}, T_k(u_n - \omega_{j,\mu}^{i,l}) \Big\rangle_{Q^\tau} + \int_\Omega S_k \Big(u_n(0) - T_l(\eta_i) \Big) dx \tag{57}$$

and using (55) and (56)) and (57), we see that

$$\int_\Omega S_k \Big(u_n(\tau) - \omega_{j,\mu}^{i,l}(\tau) \Big) dx \leq \epsilon(n, j, \mu, l), \tag{58}$$

which implies, by writing

$$\int_\Omega S_k \Big(\frac{u_n(\tau) - u_m(\tau)}{2} \Big) dx \leq \frac{1}{2} \Big(\int_\Omega S_k(u_n(\tau) - \omega_{j,\mu}^{i,l}(\tau)) dx$$

$$+ \int_\Omega S_k \Big(u_m(\tau) - \omega_{j,\mu}^{i,l}(\tau)) dx \Big), \tag{59}$$

that

$$\int_\Omega S_k\Big(\frac{u_n(\tau) - u_m(\tau)}{2}\Big)dx \leq \epsilon_1(n,m).$$

We deduce then that

$$\int_\Omega |u_n(\tau) - u_m(\tau)|dx \leq \epsilon_2(n,m), \text{ independently } \text{ of } \tau \tag{60}$$

and thus (u_n) is a Cauchy sequence in $C(0,T;L^1(\Omega))$, and since $u_n \to u$, a.e. in Q, we deduce that

$$u_n \to u \quad in \quad C(0,T;L^1(\Omega)). \tag{61}$$

b) We prove that u satisfies (14)

Indeed, let $v \in K_\psi \cap L^\infty(Q)$, $\frac{\partial v}{\partial t} \in L^{(p^-)'}(0,T;(W_0^{1,p(x)}(\Omega,\omega))^*)$. By the pointwise multiplication of (16) by $T_k(u_n - v)$, we get

$$\int_\Omega S_k\Big(u_n(T) - v(T)\Big)dx - \int_\Omega S_k\Big(u_{0n} - v(0)\Big)dx$$
$$+ \int_Q \frac{\partial v}{\partial t}T_k(u_n - v)dxdt + \int_Q \Big(\omega(x)\big|\nabla u\big|^{p(x)-2}\nabla u\Big)\nabla T_k(u_n - v)dxdt$$
$$- \int_Q nT_n\Big((u_n - \psi)^-\Big)T_k(u_n - v)dxdt$$
$$= \int_Q \omega(x)g(u_n)\big|\nabla u_n\big|^{p(x)}T_k(u_n - v)dxdt$$
$$+ \int_Q f_nT_k(u_n - v)dxdt,$$

where $S_k(s) = \int_0^s T_k(r)dr$.

Since $v \in K_\psi \cap L^\infty(Q)$, we have $- \int_Q nT_n(u_n - \psi)^- T_k(u_n - v)dxdt \geq 0$, we deduce that

$$\int_\Omega S_k\Big(u_n(T) - v(T)\Big)dx - \int_\Omega S_k\Big(u_{0n} - v(0)\Big)dx + \int_Q \frac{\partial v}{\partial t}T_k(u_n - v)dxdt$$
$$+ \int_Q \Big(\omega(x)\big|\nabla u\big|^{p(x)-2}\nabla u\Big)\nabla T_k(u_n - v)dxdt$$
$$\leq \int_Q \omega(x)g(u_n)\big|\nabla u_n\big|^{p(x)}T_k(u_n - v)dxdt \tag{62}$$
$$+ \int_Q f_nT_k(u_n - v)dxdt.$$

• Let us pass to the limit with $n \to \infty$ in each term in (62). We saw that $u_n \to u$ in $C(0,T,L^1(\Omega))$. Therefore $u_n(t) \to u(t)$ in $L^1(\Omega)$ for all $t \leq T$.

As S_k is Lipschitz of coefficient k, when $n \to \infty$, we have

$$\int_\Omega S_k(u_n - v)(T)dx \to \int_\Omega S_k(u - v)(T)dx$$
$$and \int_\Omega S_k(u_n - v)(0)dx = \int_\Omega S_k(u_{0n} - v(0))dx \to \int_\Omega S_k(u_0 - v(0))dx.$$

- Since $\frac{\partial v}{\partial t} \in L^{(p^-)'}(0,T;(W_0^{1,p(x)}(\Omega,\omega))^*)$, one has

$$\int_0^T \left\langle \frac{\partial v}{\partial t}, T_k(u_n - v) \right\rangle dt \to \int_0^T \left\langle \frac{\partial v}{\partial t} T_k(u - v) \right\rangle dt.$$

- On the other hand, we note $M = \|v\|_\infty$. Then, we get

$$\int_Q \left(\omega(x)\left|\nabla u_n\right|^{p(x)-2} \nabla u_n \right) \nabla T_k(u_n - v) dx dt$$

$$= \int_0^T \int_\Omega \left(\omega(x)\left|\nabla u_n\right|^{p(x)-2} \nabla u_n \right) \nabla T_k \left(T_{k+M}(u_n) - v \right) dx dt$$

$$= \int_0^T \int_\Omega \left(\omega(x)\left|\nabla u_n\right|^{p(x)-2} \nabla u_n \right) \nabla T_{k+M}(u_n)) \mathbf{1}_{\{|T_{k+M}(u_n)-v|\leq k\}} dx dt$$

$$- \int_0^T \int_\Omega \left(\omega(x)\left|\nabla u_n\right|^{p(x)-2} \nabla u_n \right) \nabla v \mathbf{1}_{\{|T_{k+M}(u_n)-v|\leq k\}} dx dt.$$

As $T_{k+M}(u_n)$ is bounded in $L^{p^-}(0,T;W_0^{1,p(x)}(\Omega))$ and $\nabla u_n \to \nabla u$ a.e. in Q, then

$$\nabla T_{k+M}(u_n) \to \nabla T_{k+M}(u) \quad \text{almost everywhere},$$

and by using Lebesgue theorem, we deduce that

$$\int_Q \left(\omega(x)\left|\nabla u_n\right|^{p(x)-2} \nabla u_n \right) \nabla T_{k+M}(u_n)) \mathbf{1}_{\{|T_{k+M}(u_n)-v|\leq k\}} dx dt$$

$$\to \int_Q \left(\omega(x)\left|\nabla u\right|^{p(x)-2} \nabla u \right) \mathbf{1}_{\{|T_{k+M}(u-v)|\leq k\}} dx dt$$

and

$$\int_0^T \int_\Omega \left(\omega(x)\left|\nabla u_n\right|^{p(x)-2} \nabla u_n \right) \nabla v \mathbf{1}_{\{|T_{k+M}(u_n)-v|\leq k\}} dx dt \to$$

$$\int_0^T \int_\Omega \left(\omega(x)\left|\nabla u\right|^{p(x)-2} \nabla u \right) \nabla v \mathbf{1}_{\{|T_{k+M}(u-v)|\leq k\}} dx dt,$$

then

$$\int_Q \left(\omega(x)\left|\nabla u_n\right|^{p(x)-2} \nabla u_n \right) T_k(u_n - v) dx dt \to \int_Q \left(\omega(x)\left|\nabla u\right|^{p(x)-2} \nabla u \right) T_k(u - v) dx dt.$$

- Let us pass to the limit for other term. Due to (15) , $T_k(u_n) \to T_k(u)$ in $V \ \forall \ k \geq 0$ and $u_n \to u$ a.e. in Q, we have

$$f_n T_k(u_n - v) \to f T_k(u - v) \text{ strongly } \text{ in } L^1(Q)$$

and by Lebesgue theorem, we have

$$\int_Q f_n T_k(u_n - v) \to \int_Q f T_k(u - v) \text{ strongly } in \ L^1(Q).$$

- Similarly, since g is a bounded and continuous function belonging to $L^1(\mathbb{R})$ and $u_n \to u$ a.e. in Q, we obtain

$$\int_Q \omega(x)g(u_n)\left|\nabla u_n\right|^{p(x)-2} T_k(u_n - v) \to \int_Q \omega(x)g(u)\left|\nabla u\right|^{p(x)-2} T_k(u - v) \text{ strongly } in \ L^1(Q).$$

Then, we conclude that u satisfies (14).

As a conclusion of Step 1 to Step 4, the proof of Theorem 4.1 is complete. $\square$

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Monotone Method for Finite Systems of Nonlinear Riemann-Liouville Fractional Integro-Differential Equations

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Abstract: In this paper we develop the monotone method for nonlinear finite N-systems of Riemann-Liouville integro-differential equations of order $0 < q < 1$. The iterative technique approximates maximal and minimal coupled quasisolutions to the nonlinear system using sequences of linear systems that are constructed via coupled lower and upper solutions of varying types. Preliminary existence and comparison theorems are presented and proven where appropriate. Finally, we present a numerical example.

Keywords: *monotone method; Riemann-Liouville fractional integro-differential equations; finite systems.*

Mathematics Subject Classification (2010): 26A33, 34A08, 45J05, 34A34, 65L05.

1 Introduction

Fractional differential equations have various applications in widespread fields of science, such as engineering [5], chemistry [14, 15], physics [1, 8], and others [9, 10]. Despite there being a number of existence theorems for nonlinear fractional differential equations, much as in the integer order case, this does not necessarily imply that calculating a solution explicitly will be routine, or even possible. Therefore, it may be necessary to employ an iterative technique to numerically approximate a needed solution. In this paper we

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construct such a method. For some existence results on fractional differential equations we refer the reader to the papers [6, 7] and the books [9, 16] along with references therein.

The iterative technique we construct in this paper is a generalization of the monotone method. Put simply, this method constructs two sequences from upper and lower solutions that converge monotonically and uniformly to maximal and minimal solutions. The advantage of the monotone method is that solutions of nonlinear differential equations are approximated by solutions of linear differential equations. Further, the interval of existence for the solution is guaranteed due to the nature of the upper and lower solutions and the method is valid whether the original DE has a unique solution or not. There are complications that arise when developing the monotone method for Riemann-Liouville equations. A major wrinkle comes from the fact that the constructed sequences do not converge uniformly themselves, but instead the weighted sequences $\{t^{1-q}v_n\}, \{t^{1-q}w_n\}$ converge uniformly to weighted maximal and minimal solutions, where q is the order of the system.

The monotone method has been constructed for various forms of differential equations, in this paper we extend the method to approximate Riemann-Liouville fractional integro-differential systems. Integro-differential equations generalize the problem by incorporating an integral transformation within the forcing function of the problem, e.g. $f(t, x, \int_0^t K(x, s)x(s)ds)$, and therefore generalize the possibilities of models, see [13]. A generalized monotone method for the scalar form of this problem was constructed in [2], and in this paper we extend the problem to an N-system of these equations. Moving to finite systems allows for generalizations that include many combinations of monotone properties along with upper and lower solution constructions. For example, we can reorder the variables within f for each iterate so that it increases in some variables and decreases in others, e.g. $f_i(t, x) = f_i(t, [x]_{s_i}, [x]_{r_i})$ where f_i increases in $[x]_{s_i}$ and decreases in $[x]_{r_i}$. When combined with an integral transformation T we establish the generalized system of the form

$$D^q x_i = f_i(t, x_i, [x]_{r_i}, [x]_{s_i}, [Tx]_{\rho_i}, [Tx]_{\sigma_i}),$$

where f_i is split into components where it is increasing and decreasing respectively.

There is more nuance to these generalizations than described here, and we will go into more detail in Sections 2 and 3. In the final section we will develop numerical examples which exemplify our main results. The monotone method for more standard Riemann-Liouville fractional differential systems and multi-order systems was established in [3, 4], and more information on the monotone method for ordinary differential equations and systems can be found in [11].

2 Preliminary Results

In this section, we will first consider basic results regarding scalar Riemann-Liouville (R-L) differential equations of order q, $0 < q < 1$. We will recall basic definitions and results in this case for simplicity, and we note that many of these results carry over naturally to finite systems. In the next section, we will apply these preliminary results to develop the monotone method for nonlinear fractional integro-differential systems. Note, for simplicity we only consider results on the interval $J = (0, T]$, where $T > 0$. Further, we will let $J_0 = [0, T]$, that is $J_0 = \bar{J}$.

Definition 2.1 Let $p = 1 - q$, a function $\phi(t) \in C(J, R)$ is a C_p continuous function if $t^p\phi(t) \in C(J_0, R)$. The set of C_p continuous functions is denoted $C_p(J, R)$. Further,

given a function $\phi(t) \in C_p(J, R)$, we call the function $t^p \phi(t)$ the continuous extension of $\phi(t)$.

Now we define the R-L integral and derivative of order q on the interval J.

Definition 2.2 Let $\phi \in C_p(J, R)$, then $D_t^q \phi(t)$ is the q-th R-L derivative of ϕ with respect to $t \in J$ defined as

$$D_t^q \phi(t) = \frac{1}{\Gamma(1-q)} \frac{d}{dt} \int_0^t (t-s)^{-q} \phi(s) ds,$$

and $I_t^q \phi(t)$ is the q-th R-L integral of ϕ with respect to $t \in J$ defined as

$$I_t^q \phi(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \phi(s) ds.$$

Note that in cases where the initial value may be different or ambiguous, we will write out the definition explicitly. The next definition is related to the solution of linear R-L fractional differential equations and is also of great importance in the study of the R-L derivative.

Definition 2.3 The Mittag-Leffler function with parameters $\alpha, \beta \in R$, denoted $E_{\alpha,\beta}$, is defined as

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)},$$

which is entire for $\alpha, \beta > 0$.

For fractional differential equations we utilize the weighted C_p version of the Mittag-Leffler function $t^{q-1} E_{q,q}(t^q)$, since it is its own q-th derivative. Further, it attains a convergence result we mention in the following remark.

Remark 2.1 The C_p weighted Mittag-Leffler function

$$t^{q-1} E_{q,q}(\lambda t^q) = \sum_{k=0}^{\infty} \frac{\lambda^k t^{kq+q-1}}{\Gamma(kq + q)},$$

where λ is a constant, converges uniformly on compacta of J. Further

$$D^q \left[t^{q-1} E_{q,q}(\lambda t^q) \right] = \lambda t^{q-1} E_{q,q}(\lambda t^q),$$

and

$$I^q \left[t^{q-1} E_{q,q}(\lambda t^q) \right] = \tfrac{1}{\lambda} t^{q-1} E_{q,q}(\lambda t^q) - \tfrac{1}{\lambda \Gamma(q)} t^{q-1}.$$

The next result gives us that the q-th R-L integral of a C_p continuous function is also a C_p continuous function. This result will give us that the solutions of R-L differential equations are also C_p continuous.

Lemma 2.1 *Let $f \in C_p(J, R)$, then $I^q f(t) \in C_p(J, R)$, i.e. the q-th integral of a C_p continuous function is C_p continuous.*

Note the proof of this theorem for $q \in R^+$ can be found in [4].

Remark 2.2 In [9] and [12] it was proven that if $0 < q < 1$, $G \subset R$ is an open set, and $f : J \times G \to R$ is such that for any $x \in G$, $f \in C_p(J, G)$, then x satisfies the fractional differential equation

$$D^q x = f(t, x), \quad \text{with initial condition} \quad t^p x\big|_{t=0} = x_0, \tag{1}$$

if and only if it satisfies the Volterra fractional integral equation

$$x(t) = x_0 t^{q-1} + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, x) ds. \tag{2}$$

This relationship is especially true if $f : [0, T] \times G \to R$ is continuous.

Now we consider results for the nonhomogeneous linear R-L differential equation,

$$D_t^q x(t) = \lambda x(t) + z(t), \quad t^p x(t)\big|_{t=0} = x^0, \tag{3}$$

where x^0 is a constant and $x, z \in C_p(J, R)$, which has unique solution

$$x(t) = x^0 \Gamma(q) t^{q-1} E_{q,q}(\lambda t^q) + \int_0^t (t-s)^{q-1} E_{q,q}(\lambda(t-s)^q) z(s) \, ds.$$

Next, we recall a result we will utilize extensively in our proceeding comparison and existence results, and likewise in the construction of the monotone method. We note that this result is similar to the well known comparison result found in literature, as in [12], but we do not require the function to be Hölder continuous of order $\lambda > q$.

Lemma 2.2 Let $m \in C_p(J, R)$ be such that for some $t_1 \in J$ we have $m(t_1) = 0$ and $m(t) \le 0$ for $t \in (0, t_1]$. Then
$$D_t^q m(t)\big|_{t=t_1} \ge 0.$$

The proof of this lemma can be found in [4], along with further discussion as to why and how we weaken the Hölder continuous requirement. We use this lemma in the proof of the later main comparison result, which will be critical in the construction of the monotone method.

Now we will consider results for finite N-systems of R-L integro-differential equations. For simplicity, we will henceforth assume that $i \in \{1, 2, 3, \ldots, N\}$, and that for any N element vectors x, y, $x \le y$ implies $x_i \le y_i$ for all i. We extend the concept of C_p continuous functions to R^N in the natural way

$$C_p(J, R^N) = \{\phi \in C(J, R^N) \,|\, t^p \phi_i \in C(J_0, R), \, 1 \le i \le N\}.$$

For simplicity we introduce the following notation for the scalar multiplication form of the continuous extension $x_p(t) = t^p x(t)$, so that $t^p x\big|_{t=0}$ becomes $x_p(0)$. The system we consider is

$$D^q x_i = f_i(t, x, Tx), \quad x_{p_i}(0) = x_i^0, \tag{4}$$

where each x_i^0 is a constant, and Tx is a simplified notation for

$$Tx = \{T_1 x_1, T_2 x_2, T_3 x_3, \ldots T_N x_N\}, \quad T_i x_i = \int_0^t K_i(s, t) x_i(s) ds,$$

and where K_i is continuous and positive on J_0 for each i.

Remark 2.3 Notice that each K_i is bounded on J_0 so letting $\widehat{K}$ be a bound for each K_i, and using Remark 2.1, for each i we have

$$
\begin{aligned}
T_i t^{q-1} E_{q,q}(\lambda t^q) &\leq \frac{\widehat{K}}{\Gamma(q)} \int_0^t \frac{\Gamma(q)(t-s)^p}{(t-s)^p} s^{q-1} E_{q,q}(\lambda s^q) ds \\
&\leq \widehat{K} T^p \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} s^{q-1} E_{q,q}(\lambda s^q) ds < \widehat{K} T^p \Gamma(q) \tfrac{1}{\lambda} t^{q-1} E_{q,q}(\lambda t^q).
\end{aligned}
$$

Now, we introduce the concept of quasimonotonicity, which will be a generalization of monotonicity for our main results.

Definition 2.4 A function $\phi : R^N \to R^N$ is said to be quasimonotone increasing if for each i, $x \leq y$ and $x_i = y_i$ implies $\phi_i(x) \leq \phi_i(y)$. Naturally, ϕ is quasimonotone decreasing if we reverse the inequalities.

From now on, if we wish to designate standard monotonicity we will state that a function increases or decreases traditionally. In our main result, we construct our iterative technique from lower and upper solutions. Further, many of our preliminary results stem from these solutions which we define below.

Definition 2.5 Let $v, w \in C_p(J, R^N)$, then v, w are lower and upper solutions of (4) respectively if

$$
D^q v_i \leq f_i(t, v, Tv), \quad D^q w_i \geq f_i(t, w, Tw), \quad v_{p_i}(0) \leq x_i^0 \leq w_{p_i}(0).
$$

Now we present the main comparison theorem that will form the base of our remaining results. This result gives us conditions for when lower and upper solutions behave in a natural way, i.e. when $v \leq w$ on J. Specifically, if f is quasimonotone in x and monotone in Tx and satisfies a one-sided Lipschitz condition, then $v \leq w$. The result is given below.

Theorem 2.1 *Let $v, w \in C_p(J, R^N)$ be lower and upper solutions of (4). If f is quasimonotone increasing in x and traditionally increasing in Tx, and satisfies the Lipschitz condition:*

$$
f_i(t, x, Tx) - f_i(t, y, Ty) \leq \sum_{k=1}^N L_i(x_k - y_k) + M_i T_k(x_k - y_k),
$$

then $v \leq w$ on J.

Proof. We start by assuming that one of the inequalities is strict, $D^q v_i < f_i(t, v, Tv)$ for each i, and $v_p(0) < w_p(0)$, and we will show that $v < w$ on J. Suppose to the contrary that our claim is not true, then the set

$$
Z = \bigcup_{i=1}^N \{t \in J \,:\, v_i(t) = w_i(t)\}
$$

is nonempty. So let $\tau = \inf Z$, and suppose without loss of generality, via reordering if necessary, that $v_1(\tau) = w_1(\tau)$. Now by the continuity of v_p and w_p on J_0 and since

$v_p(0) < w_p(0)$, we have that $v_p < w_p$ on $[0, \tau)$, and thus giving us that $v \leq w$ on $(0, \tau]$. This also gives us that $T_i v_i(\tau) \leq T_i w_i(\tau)$ for each i.

Letting $m = v - w$ we have by Lemma 2.2 that $D^q m|_{t=\tau} \geq 0$. Now, using this and the quasimonotone and traditional monotone properties of f we obtain:

$$f_1(\tau, v(\tau), Tv(\tau)) > D^q v_j\big|_{t=\tau} \geq D^q w_j\big|_{t=\tau} \geq f_1(\tau, w(\tau), Tw(\tau))$$
$$= f_1(\tau, v_1(\tau), w_2(\tau), w_3(\tau), \dots w_N(\tau), Tw(\tau)) \geq f_1(\tau, v(\tau), Tv(\tau)),$$

which is a contradiction. Therefore, $v < w$ on J.

Now, to prove the theorem as given we will use the strict inequality case. To do so let $\varepsilon > 0$, and construct functions

$$v_{\varepsilon i} = v_i - \varepsilon\varphi, \quad w_{\varepsilon i} = w_i + \varepsilon\varphi,$$

where $\varphi(t) = t^{q-1} E_{q,q}((N+1)\mathcal{L}t^q)$ and $\mathcal{L}$ is defined as

$$\mathcal{L} = \max_{1 \leq i \leq N} \big\{ \widehat{K}T^p\Gamma(q), L_i, M_i \big\},$$

where $\widehat{K}$ is defined as in Remark 2.3. Note that by definition $v_\varepsilon < v$ and $w_\varepsilon > w$ on J since $\varphi > 0$ on J. To start with, note that for each i

$$v_{\varepsilon i}^0 = t^p v_{\varepsilon i}\big|_{t=0} = v_i^0 - \varepsilon E_{q,q}(0) = v_i^0 - \frac{\varepsilon}{\Gamma(q)} < v_i^0,$$

so $v_\varepsilon^0 < v^0$. Then, for each i, we have

$$
\begin{aligned}
D^q v_{\varepsilon i} &\leq f_i(t, v, Tv) - \varepsilon(N+1)\mathcal{L}\varphi \\
&= f_i(t, v_\varepsilon, Tv_\varepsilon) + f_i(t, v, Tv) - f_i(t, v_\varepsilon, Tv_\varepsilon) - \varepsilon(N+1)\mathcal{L}\varphi \\
&\leq f_i(t, v_\varepsilon, Tv_\varepsilon) + \sum_{k=1}^{N} \Big[L_k(v_k - v_{\varepsilon k}) + M_k T(v_k - v_{\varepsilon k}) \Big] - \varepsilon(N+1)\mathcal{L}\varphi \\
&\leq f_i(t, v_\varepsilon, Tv_\varepsilon) + N\mathcal{L}\varepsilon\varphi + N\mathcal{L}T\varepsilon\varphi - \varepsilon(N+1)\mathcal{L}\varphi \\
&< f_i(t, v_\varepsilon, Tv_\varepsilon) + \frac{N\mathcal{L}}{(N+1)}\varepsilon\varphi - \varepsilon\mathcal{L}\varphi < f_i(t, v_\varepsilon, Tv_\varepsilon).
\end{aligned}
$$

We note that the penultimate inequality came from the application of Remark 2.3. Further, we can similarly show that $D^q w_{\varepsilon i} > f_i(t, w_\varepsilon, Tw_\varepsilon)$. Therefore, by the previous work involving strict inequalities we have that $v_\varepsilon < w_\varepsilon$ on J. Then letting $\varepsilon \to 0$ we obtain $v \leq w$ on J, which completes the proof.   This result can be extended to linear systems utilizing the following corrolary, the result follows from the Lipschitz nature of linear systems.

Corollary 2.1 *If g is a continuous function and $v, w \in C_p$ satisfy the following properties*

$$D^q v_i \leq \lambda v_i + g_i(t), \quad D^q w_i \geq \lambda w_i + g_i(t), \quad v_{p_i}(0) \leq w_{p_i}(0),$$

then $v \leq w$ on J.

3   Monotone Method

For this section we expand our general system to cover more cases. To do so we split $\{x_i\}$ and $\{T_i x_i\}$ within each f_i to isolate variables where each f_i is increasing or decreasing in each i. So, for each i, let $r_i, s_i, \rho_i, \sigma_i$ be such that $r_i + s_i = N - 1$ and $\rho_i + \sigma_i = N$. Then, for each i, reorder x and Tx, using the following notation

$$x = \{x_1, x_2, x_3, \ldots, x_N\} = \{x_i, [x]_{r_i}, [x]_{s_i}\},$$
$$Tx = \{T_1 x_1, T_2 x_2, T_3 x_3, \ldots, T_N x_N\} = \{[Tx]_{\rho_i}, [Tx]_{\sigma_i}\}.$$

This reordering allows us to isolate the variables where each f_i increases or decreases, and each $r_i, s_i, \rho_i, \sigma_i$ represents the number of x terms with each monotone property, and yields the following definition regarding f.

Definition 3.1 We say f possesses the mixed quasimonotonicity property if for each i

$$f_i(t, x, Tx) = f_i(t, x_i, [x]_{r_i}, [x]_{s_i}, [Tx]_{\rho_i}, [Tx]_{\sigma_i}),$$

and where f_i is quasimonotone increasing in $[x]_{r_i}$, quasimonotone decreasing in $[x]_{s_i}$, traditionally increasing in $[Tx]_{\rho_i}$, and traditionally decreasing in $[Tx]_{\sigma_i}$.

Remark 3.1 Definition 3.1 generalizes standard monotone cases for system (4), since if $s_i = \sigma_i = 0$, Definition 3.1 reduces down to $f(t, x, Tx)$ which is quasimonotone increasing in x and traditionally increasing in Tx. Similarly, Definition 3.1 reduces to quasimonotone decreasing in x and traditionally decreasing in Tx when $r_i = \rho_i = 0$.

Now, the final fractional integro-differential system we construct the monotone method for is:

$$D^q x_i = f_i(t, x_i, [x]_{r_i}, [x]_{s_i}, [Tx]_{\rho_i}, [Tx]_{\sigma_i}), \quad x_{p_i}(0) = x_i^0, \tag{5}$$

where f has the mixed quasimonotonicity property. This new formulation allows us to define new types of coupled upper and lower quasisolutions. We still have natural upper and lower solutions as defined in Definition 2.5, but in the following definition we introduce coupled, i.e. mixed, forms of the lower and upper solutions.

Definition 3.2 $v, w \in C_p$ are Type I coupled lower and upper quasisolutions of (5) if

$$D^q v_i \leq f_i(t, v_i, [v]_{r_i}, [w]_{s_i}, [Tv]_{\rho_i}, [Tw]_{\sigma_i}), \quad v_{p_i}(0) = v_i^0 \leq x_i^0$$
$$D^q w_i \geq f_i(t, w_i, [w]_{r_i}, [v]_{s_i}, [Tw]_{\rho_i}, [Tv]_{\sigma_i}), \quad w_{p_i}(0) = w_i^0 \geq x_i^0.$$

$v, w \in C_p$ are Type II coupled lower and upper quasisolutions of (5) if

$$D^q v_i \leq f_i(t, w_i, [w]_{r_i}, [v]_{s_i}, [Tw]_{\rho_i}, [Tv]_{\sigma_i}), \quad v_{p_i}(0) = v_i^0 \leq x_i^0$$
$$D^q w_i \geq f_i(t, v_i, [v]_{r_i}, [w]_{s_i}, [Tv]_{\rho_i}, [Tw]_{\sigma_i}), \quad w_{p_i}(0) = w_i^0 \geq x_i^0.$$

If the inequalities in above definitions are replaced with equal signs, then they become coupled Type I or II quasisolutions of (5) and minimal and maximal coupled Type I or II quasisolutions are defined in the natural way given these definitions.

In an effort to simplify our manuscript and make it more readable we introduce the following notation, for each i, let $[v, w]_i$ be such that

$$f_i(t, x_i, [v, w]_i) = f_i(t, x_i, [v]_{r_i}, [w]_{s_i}, [Tv]_{\rho_i}, [Tw]_{\sigma_i}).$$

Thus the first component of $[v, w]_i$ corresponds with the "increasing" portion of f and the second component corresponds with the "decreasing" portion of f. So for example, the above coupled lower and upper quasisolutions can be rewritten as

Type I: $D^q v_i \leq f_i(t, v_i, [v, w]_i)$, $D^q w_i \geq f_i(t, w_i, [w, v]_i)$,

Type II: $D^q v_i \leq f_i(t, w_i, [w, v]_i)$, $D^q w_i \geq f_i(t, v_i, [v, w]_i)$.

Now, if we know of the existence of lower and upper solutions v and w such that $v \leq w$, we can prove the existence of a solution in the set

$$\Omega = \{(t, y) \,:\, v(t) \leq y \leq w(t), \, t \in J\}.$$

We consider this result in the following theorem.

Theorem 3.1 *Let* $v, w \in C_p(J, R^N)$ *be Type I lower and upper solutions of (5) such that* $v \leq w$ *on J and let* $f \in C(\Omega, R^N)$, *where Ω is defined as above. Then there exists a solution* $x \in C_p(J, R^N)$ *of (4) such that* $v \leq x \leq w$ *on J.*

This theorem is proved in the same way as seen in [4], with only minor additions to incorporate the transformation T. In the next theorem we establish our main result. Essentially, if there are Type I lower and upper quasisolutions that satisfy their natural inequalities, that is $v \leq w$ on J, and if f satisfies the described conditions, then we can construct sequences of linear R-L systems, where the C_p continuous extensions converge uniformly and monotonically to maximal and minimal Type I quasisolutions.

Theorem 3.2 *Let f possess the mixed quasimonotone property. Let* $v_0, w_0 \in C_p(J, R^N)$ *be Type I coupled lower and upper quasisolutions of (5) such that* $v_0 \leq w_0$ *on J. For each i suppose f_i satisfies the following one-sided Lipschitz condition in the x_i component:*

$$f_i(t, x_i, [x, x]_i) - f_i(t, y_i, [x, x]_i) \geq -M_i(x_i - y_i),$$

whenever $v_0 \leq x \leq w_0$, *and* $v_{0_i} \leq y_i \leq x_i \leq w_{0_i}$ *on J and $M_i \geq 0$. Then there exist monotone sequences $\{v_n\}$ and $\{w_n\}$ such that*

$$t^p v_n \to t^p v, \quad t^p w_n \to t^p w,$$

uniformly and monotonically on J_0, where v and w are Type I coupled minimal and maximal quasisolutions of (5) on J for solutions $v_0 \leq x \leq w_0$.

Proof. For the construction of the sequences let $\eta, \xi \in C_p(J, R^N)$ with $v_0 \leq \eta, \xi \leq w_0$ on J, then we start by considering the system

$$D^q x_i = f_i(t, \eta_i, [\eta, \xi]_i) - M_i(x_i - \eta_i), \quad x_{p_i}(0) = x_i^0. \tag{6}$$

We note that this system is an uncoupled linear system, therefore for each η, ξ the system has a unique solution x. Thus we can define a transformation A that yields the unique solution of (6) for each η, ξ, that is $A[\eta, \xi] = x$. We will construct our monotone sequences

using this transformation, so we wish to show that A has a mixed monotone property. A is increasing in its first component and decreasing in its second component. To prove this, let $\eta, \widehat{\eta}, \xi \in C_p$ such that $v_0 \leq \eta, \widehat{\eta}, \xi \leq w_0$ and $\eta \geq \widehat{\eta}$ on J. Now suppose $x_a, x_b \in C_p$ such that $A[\eta, \xi] = x_a$ and $A[\widehat{\eta}, \xi] = x_b$.

Now, since $\eta \geq \widehat{\eta}$, we have that $T\eta \geq T\widehat{\eta}$, and then by the mixed quasimonotone property of f we have that

$$f_i(t, \widehat{\eta}_i, [\eta, \xi]_i) \geq f_i(t, \widehat{\eta}_i, [\widehat{\eta}, \xi]_i).$$

So, using this, the definition of x_a and the Lipschitz condition of f we obtain

$$D^q x_{ai} = f_i(t, \widehat{\eta}_i, [\eta, \xi]_i) + f_i(t, \eta_i, [\eta, \xi]_i) - f_i(t, \widehat{\eta}_i, [\eta, \xi]_i) - M_i(x_{ai} - \eta_i)$$
$$\geq f_i(t, \widehat{\eta}_i, [\eta, \xi]_i) - M_i(x_{ai} - \widehat{\eta}_i) \geq f_i(t, \widehat{\eta}_i, [\widehat{\eta}, \xi]_i) - M_i(x_{ai} - \widehat{\eta}_i),$$

and by definition of x_b

$$D^q x_{bi} = f_i(t, \widehat{\eta}_i, [\widehat{\eta}, \xi]_i) - M_i(x_{bi} - \widehat{\eta}_i).$$

Thus, by Theorem 2.1 we have that $x_b \leq x_a$, i.e. $A[\widehat{\eta}, \xi] \leq A[\eta, \xi]$ on J. Since $\eta, \widehat{\eta}$ were arbitrary, we have that A is increasing in its first component. Similarly, we can show that A is decreasing in its second component. Therefore A has a mixed monotone property, and with it we obtain the property that $A[\eta, \xi] \leq A[\xi, \eta]$ when $v_0 \leq \eta \leq \xi \leq w_0$ on J.

The sequences $\{v_n\}$ and $\{w_n\}$ we construct are unique solutions of the fractional systems

$$D^q v_{n+1_i} = f_i(t, v_{ni}, [v_n, w_n]_i) - M_i(v_{n+1_i} - v_{ni}), \quad v_{n+1_i}^0 = x_i^0,$$
$$D^q w_{n+1_i} = f_i(t, w_{ni}, [w_n, v_n]_i) - M_i(w_{n+1_i} - w_{ni}), \quad w_{n+1_i}^0 = x_i^0,$$

where v_0 and w_0 are as defined in the hypothesis. That is, the sequences are defined as

$$v_{n+1} = A[v_n, w_n], \quad w_{n+1} = A[w_n, v_n].$$

With the transformation A it is far more efficient to prove that the sequences are monotonic inductively, since if

$$v_0 \leq v_1 \leq v_2 \leq \cdots \leq v_{k-1} \leq v_k \leq w_k \leq w_{k-1} \leq \cdots \leq w_2 \leq w_1 \leq w_0$$

up to some k, then

$$A[v_{k-1}, w_{k-1}] \leq A[v_k, w_k] \leq A[w_k, v_k] \leq A[w_{k-1}, v_{k-1}],$$

implying $v_k \leq v_{k+1} \leq w_{k+1} \leq w_k$ on J, and giving us the monotonicity of the constructed sequences.

Now we will prove that the weighted sequences $\{t^p v_n\}$, $\{t^p w_n\}$ converge uniformly, to do so we will invoke the Arzela-Ascoli theorem. To begin, note that for all i we have that

$$|t^p(v_i)| \leq |t^p(v_i - v_0)| + |t^p v_0| \leq |t^p(w_0 - v_0)| + |t^p v_0|,$$

giving us that $\{t^p v_n\}$ is uniformly bounded. Now we wish to show that the weighted sequence is uniformly continuous. For simplicity, for each i and n, let

$$F_i(v_{n+1}) = f_i(t, v_{ni}, [v_n, w_n]_i) - M_i(v_{n+1_i} - v_{ni}),$$

then since each v_n is C_p continuous, f is continuous over J_0, and since $\{t^p v_n\}$ is uniformly bounded, we can choose $\mu > 0$ such that $|t^p F_i(v_n)| \le \mu$ for all i and all n. Also, our preceding argument requires analysis of the function

$$\varphi(t) = t^p(t-s)^{-p},$$

for $0 \le s \le t \le T$, specifically we note that φ is decreasing in t, to show why consider

$$\tfrac{d}{dt}\varphi = pt^{p-1}(t-s)^{-p-1}(-s) \le 0.$$

Now, let $\varepsilon > 0$, and let $t, \tau \in (0, T]$ such that, without loss of generality, $0 < t \le \tau$ and $\tau - t < \varepsilon^{1/q}$. Further, suppose ε is sufficiently small enough such that $1 \le \frac{\tau}{t} < 2$. Then via Remark 2.2, utilizing that $\varphi(\tau) \le \varphi(t)$, we have for each i and n,

$$|\tau^p v_{ni}(\tau) - t^p v_{ni}(t)|$$

$$= \left| \frac{1}{\Gamma(q)} \int_0^\tau \varphi(\tau) F_i(v_n) ds - \frac{1}{\Gamma(q)} \int_0^t \varphi(t) F_i(v_n) ds \right|$$

$$\le \frac{1}{\Gamma(q)} \int_t^\tau \varphi(\tau) |F_i(v_n)| ds + \frac{1}{\Gamma(q)} \int_0^t |\varphi(\tau) - \varphi(t)| |F_i(v_n)| ds$$

$$\le \frac{\mu}{\Gamma(q)} \tau^p t^{q-1} \int_t^\tau (\tau - s)^{q-1} ds + \frac{\mu}{\Gamma(q)} \int_0^t (\varphi(t) - \varphi(\tau)) s^{q-1} ds.$$

$$= \frac{\mu}{\Gamma(q+1)} \left(\frac{\tau}{t} \right)^p (\tau - t)^q + \frac{\mu \Gamma(q)}{\Gamma(2q)} t^q - \frac{\mu \tau^p}{\Gamma(q)} \int_0^t (\tau - s)^{q-1} s^{q-1} ds. \qquad (7)$$

From here we will evaluate the third term from (7) individually, and for simplicity without the constant $\frac{\mu}{\Gamma(q)}$. To do so we will use the integral form of the beta funtion $B(q, q)$,

$$B(q, q) = \frac{\Gamma(q)\Gamma(q)}{\Gamma(2q)} = \int_0^1 (1 - \alpha)^{q-1} \alpha^{q-1} d\alpha.$$

Then we will apply the transformation $s = t\alpha$ to obtain

$$-\tau^p \int_0^t (\tau - s)^{q-1} s^{q-1} ds = -\tau^q B(q, q) + \tau^q B(q, q) - \tau^q \int_0^{t/\tau} (1 - \alpha)^{q-1} \alpha^{q-1} d\alpha$$

$$= -\tau^q B(q, q) + \tau^q \int_{t/\tau}^1 (1 - \alpha)^{q-1} \alpha^{q-1} d\alpha$$

$$\le -\tau^q B(q, q) + \tau^q (t/\tau)^{q-1} \int_{t/\tau}^1 (1 - \alpha)^{q-1} d\alpha$$

$$= -\tau^q B(q, q) + \frac{1}{q} \left(\frac{\tau}{t} \right)^p (\tau - t)^q.$$

Putting this result back into (7) we obtain

$$|\tau^p v_{ni}(\tau) - t^p v_{ni}(t)| < \frac{2^{p+1}\mu}{\Gamma(q+1)}(\tau - t)^q + \frac{\mu\Gamma(q)}{\Gamma(2q)}(t^q - \tau^q) < \frac{2^{p+1}\mu}{\Gamma(q+1)} \varepsilon.$$

Thus $t^p v_{ni}(t)$ is continuous at $t > 0$ since the case when $t \ge \tau$ will follow in a similar manner. For the case when $t = 0$, consider

$$|\tau^p v_{ni}(\tau) - x_i^0| \le \frac{\mu\tau^p}{\Gamma(q)} \int_0^\tau (\tau - s)^{q-1} s^{q-1} ds = \frac{\mu\Gamma(q)}{\Gamma(2q)} \tau^q < \frac{\mu\Gamma(q)}{\Gamma(2q)} \varepsilon,$$

so $t^p v_{n_i}(t)$ is continuous at $t = 0$. Further, since ε did not depend on the arbitrary choices of n or i, we have that the weighted sequence $\{t^p v_n\}$ is equicontinuous on J_0.

Now, since $\{t^p v_n\}$ is monotonic, uniformly bounded, and equicontinuous on J_0, by the Arzela-Ascoli theorem we have that $\{t^p v_n\}$ converges uniformly on J_0. Note we can prove the same result for $\{t^p w_n\}$, thus both weighted sequences converge uniformly. Now, suppose that $v, w \in C_p(J, R^N)$ such that $t^p v_n \to t^p v$ and $t^p w_n \to t^p w$ uniformly on J_0. We wish to show that $T v_n \to T v$ and $T w_n \to T w$ uniformly on J_0. To do so, let $\varepsilon > 0$ and choose $\mathcal{M}$ such that for $n \geq \mathcal{M}$, $|t^p(v_n - v)| < \frac{\varepsilon q}{\widehat{K} T^q}$, where $\widehat{K}$ is defined as in Remark 2.3. Then for all $t \in J_0$ and for all $n \geq \mathcal{M}$

$$|T v_n - T v| \leq \widehat{K} \int_0^t |v_n - v| ds < \frac{\varepsilon q}{T^q} \int_0^t s^{q-1} ds = \frac{\varepsilon t^q}{T^q} \leq \varepsilon.$$

Therefore $T v_n \to T v$ uniformly on J_0, similarly $T w_n \to T w$ uniformly on J_0.

Now, due to the fact that f_i is continuous and bounded on J_0 and the nature of C_p continuous functions, for each i there exists a function $\mathcal{F}$ such that

$$\mathcal{F}_i(t, t^p x_i, [t^p x]_{r_i}, [t^p x]_{s_i}, [T x]_{\rho_i}, [T x]_{\sigma_i}) = f_i(t, x_i, [x]_{r_i}, [x]_{s_i}, [T x]_{\rho_i}, [T x]_{\sigma_i}).$$

So, due to all of the convergence properties we have that

$$t^p v_{n+1_i} = \tfrac{1}{\Gamma(q)} + t^p \mathcal{F}_i(t, t^p v_{n_i}, [t^p v_n]_{r_i}, [t^p w_n]_{s_i}, [T v_n]_{\rho_i}, [T w_n]_{\sigma_i}) - M_i t^p(v_{n+1_i} - v_{n_i})$$

converges uniformly to

$$t^p v_i = \tfrac{1}{\Gamma(q)} + t^p \mathcal{F}_i(t, t^p v_i, [t^p v]_{r_i}, [t^p w]_{s_i}, [T v]_{\rho_i}, [T w]_{\sigma_i})$$

on J_0, giving us that

$$v_i = \tfrac{1}{\Gamma(q)} t^{q-1} + f_i(t, v_i, [v]_{r_i}, [w]_{s_i}, [T v]_{\rho_i}, [T w]_{\sigma_i})$$

on J and implying that v is a coupled quasisolution of (5), and we have the similar result for w as well.

Finally, we wish to prove that v, w are minimal and maximal coupled quasisolutions of (5). To do so, let x be any solution of (5) with $v_0 \leq x \leq w_0$, we know such a solution exists thanks to Theorem 3.1. Then note that

$$v_1 = A[v_0, w_0] \leq A[x, x] \leq A[w_0, v_0] = w_1,$$

giving us that $v_1 \leq x \leq w_1$, and continuing this process inductively we can show that $v_n \leq x \leq w_n$ for all n, which implies that $v \leq x \leq w$ on J. Therefore, v, w are minimal and maximal mixed quasisolutions of (5), which completes the proof.

Note that in the case that (5) has a unique solution, e.g. f is fully Lipschitz, then $v = x = w$ on J. Further, this acts as a generalization for the monotone method constructed with natural upper and lower solutions to the system (4) where $f(t, x, Tx)$ is quasimonotone increasing in x and traditionally increasing in Tx. This follows directly from considering the previous theorem where $s_i = \sigma_i = 0$.

We can also construct the monotone method beginning with Type II coupled lower and upper quasisolutions. To do so requires a further assumption that $v_0 \leq w_1$ and $v_1 \leq w_0$, further we get intertwined montone sequences that still converge to minimal and maximal quasisolutions.

Theorem 3.3 *Suppose f satisfies the same properties as in Theorem 3.2. Let $v_0, w_0 \in C_p(J, R^N)$ be Type II lower and upper quasisolutions of (5) such that $v_0 \le w_0$. Let $\{v_n\}$ and $\{w_n\}$ be sequences defined by*

$$D^q v_{n+1_i} = f_i(t, w_{n_i}, [w_n, v_n]_i) - M_i(v_{n+1_i} - w_{n_i}), \quad v_{n+1_i}^0 = x_i^0,$$
$$D^q w_{n+1_i} = f_i(t, v_{n_i}, [v_n, w_n]_i) - M_i(w_{n+1_i} - v_{n_i}), \quad w_{n+1_i}^0 = x_i^0,$$

for $n \ge 1$ and where v_0, w_0 are the given lower and upper solutions. If $v_0 \le w_1$ and $v_1 \le w_0$, then the sequences have the following intertwined monotonic property

$$v_0 \le w_1 \le v_2 \le \dots v_{2n} \le w_{2n+1} \le v_{2n+1} \le w_{2n} \le \dots \le w_2 \le v_1 \le w_0,$$

and the weighted sequences

$$t^p v_{2n}, t^p w_{2n+1} \to t^p \alpha, \quad t^p v_{2n+1}, t^p w_{2n} \to t^p \beta$$

uniformly on J_0, where α and β are Type II coupled minimal and maximal quasisolutions of (5) on J for solutions $v_0 \le x \le w_0$.

The proof of this theorem follows in a similar manner as that of Theorem 3.2, even with the intertwined nature the proof only requires minor adjustments for incorporating the Type II sequences. This theorem is also a generalization for the monotone method constructed with natural upper and lower solutions to the system (4) where $f(t, x, Tx)$ is quasimonotone decreasing in x and traditionally decreasing in Tx. As before, this follows directly from considering $s_i = \sigma_i = 0$.

In the next section we will construct a numerical example that will exemplify our results. In the example we will look at a system when $N = 2$ and $q = 1/2$.

4 Numerical Example

We finish this work by illustrating the result of Theorem 3.2 with an example. Consider the fractional system of the form (5) with $q = \frac{1}{2}$,

$$
\begin{aligned}
D^{1/2} x_1 &= \tfrac{1}{2} + \tfrac{5}{8}t + \tfrac{1}{32}\left(x_1^2 - \tfrac{1}{4}x_2\right) + \tfrac{1}{16}\int_0^t (1+s)x_1 ds, & x_{p_1}(0) &= 0, \\
D^{1/2} x_2 &= \tfrac{1}{6} + \tfrac{1}{5}t + \tfrac{1}{20}\left(x_1 - x_2\right) - \tfrac{1}{20}\int_0^t (1+s)x_2 ds, & x_{p_2}(0) &= 0,
\end{aligned}
\tag{8}
$$

where $p = \frac{1}{2}$, and for simplicity we will consider the same transformation

$$Tx_i(t) = \int_0^t (1+s)x_i(s)ds$$

for $i = 1, 2$, and further for simplicity call

$$f_1(t, x_1, x_2, Tx_1, Tx_2) = \tfrac{1}{2} + \tfrac{5}{8}t + \tfrac{1}{32}x_1^2 + \tfrac{1}{16}Tx_1, \quad f_2(t, x_1, x_2, Tx_1, Tx_2) = \tfrac{1}{6} + \tfrac{1}{5}t + \tfrac{1}{20}x_1,$$

$$g_1(t, x_1, x_2, Tx_1, Tx_2) - \tfrac{1}{128}x_2, \quad g_2(t, x_1, x_2, Tx_1, Tx_2) = -\tfrac{1}{20}x_2 - \tfrac{1}{20}Tx_2.$$

If $J = (0, 1]$ and $J_0 = [0, 1]$, then $f_i(t, x, Tx) + g_i(t, x, Tx)$ together, where $i = 1, 2$, satisfy the mixed quasimonotonicity property. Now let $v_{01}(t) = \frac{\sqrt{t}}{2}$, $v_{02}(t) = 0$, $w_{01}(t) = 3$ and $w_{02}(t) = 3 - t$.

We will illustrate graphically in Figures 1–4 that $v_{0_i}(t)$ and $w_{0_i}(t)$ are Type I coupled lower and upper quasisolutions.

First note,

$$v_{0p_i}(0) = w_{0p_i}(0) = 0 \text{ for } i = 1, 2.$$

Since $D^{1/2}v_{01} = \frac{\sqrt{\pi}}{4}$, we have

$$D^{1/2}v_{01} = \frac{\sqrt{\pi}}{4} \leq \frac{1}{2} + \frac{5}{8}t + \frac{1}{32}\left(v_{01}^2 - \frac{1}{4}w_{02}\right) + \frac{1}{16}Tv_{01} = f_1(t, v_0, Tv_0) + g_1(t, w_0, Tw_0).$$

Similarly,

$$D^{1/2}w_{01} = \frac{3}{\sqrt{\pi t}} \geq \frac{1}{2} + \frac{5}{8}t + \frac{1}{32}\left(w_{01}^2 - \frac{1}{4}v_{02}\right) + \frac{1}{16}Tw_{01} = f_1(t, w_0, Tw_0) + g_1(t, v_0, Tv_0),$$

$$D^{1/2}v_{02} = 0 \leq \frac{1}{6} + \frac{1}{5}t + \frac{1}{20}(v_{01} - w_{02}) - \frac{1}{20}Tw_{02} = f_2(t, v_0, Tv_0) + g_2(t, w_0, Tw_0),$$

$$D^{1/2}w_{02} = \frac{3-2t}{\sqrt{\pi t}} \geq \frac{1}{6} + \frac{1}{5}t + \frac{1}{20}(w_{01} - v_{02}) - \frac{1}{20}Tv_{02} = f_2(t, w_0, Tw_{01}, Tw_0) + g_2(t, v_0, Tv_0).$$

We now show the graphs of these lower and upper quasisolutions in Figures 1–4.
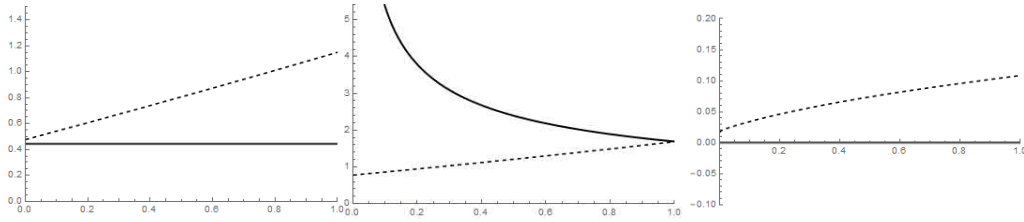
Figure 1: $D^q v_{01} \leq f_1 + g_1$.   **Figure 2**: $D^q w_{01} \geq f_1 + g_1$.   **Figure 3**: $D^q v_{02} \leq f_2 + g_2$.

After verifying that we have indeed Type I coupled lower and upper quasisolutions we computed four iterates of $\{t^{1/2}v_n\}$ and $\{t^{1/2}w_n\}$, for $i = 1, 2$, according to Theorem 3.2 for $t \in J_0 = [0, 1]$. The results are given in Figures 5 and 6 for $0 \leq n \leq 4$.

Finally we show a table of ten values of $\{t^{1/2}v_4\}$ and $\{t^{1/2}w_4\}$, for $i = 1, 2$, on the interval $[0, 1]$.

t	$t^{1/2}v_{41}$	$t^{1/2}w_{41}$	$t^{1/2}v_{42}$	$t^{1/2}w_{42}$	t	$t^{1/2}v_{41}$	$t^{1/2}w_{41}$	$t^{1/2}v_{42}$	$t^{1/2}w_{42}$
0.1	0.0612153	0.0612154	0.0208512	0.0208514	0.2	0.1322313	0.1322315	0.0452015	0.0452027
0.3	0.2132962	0.2132970	0.0729141	0.0729180	0.4	0.3046777	0.3046800	0.1039346	0.1039439
0.5	0.4066788	0.4066841	0.1382209	0.1382401	0.6	0.5196440	0.1757330	0.5196553	0.1757683
0.7	0.6439644	0.6439867	0.2164287	0.5702515	0.8	0.7800823	0.7801239	0.2602622	0.2603606
0.9	0.9284960	0.9285705	0.3071838	0.3073374	1.0	1.0897658	1.0898941	0.3571390	0.3573711

We used Mathematica to compute the iterates, the graphs and the tables.
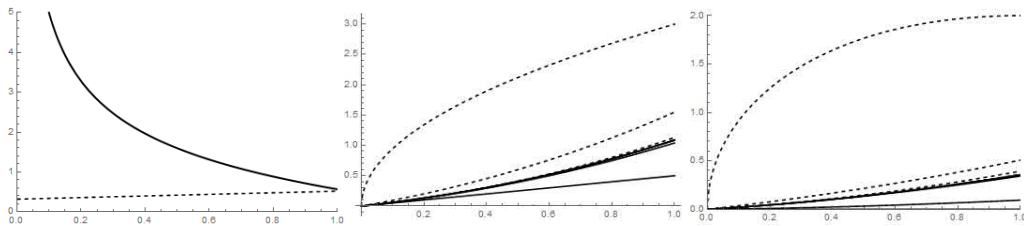
Figure 4: $D^q w_{02} \geq f_2 + g_2$.   **Figure 5**: $t^p v_{n1} \leq t^p w_{n1}$.   **Figure 6**: $t^p v_{n2} \leq t^p w_{n2}$.

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Local Existence and Uniqueness of Solution for Hilfer-Hadamard Fractional Differential Problem

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Abstract: This paper deals with the local existence and uniqueness results for the solution of fractional differential equations involving Hilfer-Hadamard fractional derivative. Using Picard's approximations and generalizing the restrictive conditions imposed on nonlinear function, the iterative scheme for uniformly approximating solution is constructed. An example is given to illustrate the main results.

Keywords: *Picard iterative technique; fractional differential equation; convergence.*

Mathematics Subject Classification (2010): 26A33; 26D10; 34A08; 40A30.

1 Introduction

Fractional differential equations (FDEs) occur in control of dynamical systems, physical and biological sciences, see for details [14, 19, 23] and references therein. Nowadays, many people have given attention to the existence theory of nonlinear FDEs of various types [2–13, 15–18, 21, 22]. Recently, existence and uniqueness of weak solutions for some class of Hilfer-Hadamard and Hilfer fractional differential equations are obtained in [1]. Further, some attractivity and Ulam stability results are obtained [1] by applying the fixed point theory, also one can see [12, 20].

Kassim and Tatar [16] obtained the well-posedness of Cauchy-type problem

$$
\begin{cases}
{}_H\mathcal{D}_{a^+}^{\alpha,\beta} x(t) = f(t, x), & t > a > 0, \\
{}_H\mathcal{J}_{a^+}^{1-\gamma} x(a) = c, & \gamma = \alpha + \beta(1-\alpha),
\end{cases}
\tag{1}
$$

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where $c \in \mathbb{R}$ and $_H\mathcal{D}_{a^+}^{\alpha,\beta}$ is the Hilfer-Hadamard fractional derivative [15] of order $\alpha(0 < \alpha < 1)$ and type $\beta(0 \leq \beta \leq 1)$, in the weighted space of continuous functions $C_{1-\gamma}^{\alpha,\beta}[a,b]$ defined by

$$C_{1-\gamma,\mu}^{\alpha,\beta}[a,b] = \left\{ x \in C_{1-\gamma,\log}[a,b] |_H\mathcal{D}_{a^+}^{\alpha,\beta} x \in C_{\mu,\log}[a,b] \right\}, \quad 0 \leq \mu < 1, \gamma = \alpha + \beta(1-\alpha), \tag{2}$$

where

$$C_{\gamma,log}[a,b] = \left\{ g : (a,b] \to \mathbb{R} | \left(\log \frac{t}{a} \right)^{\gamma} g(t) \in C[a,b] \right\}, \quad 0 \leq \gamma < 1. \tag{3}$$

They obtained the equivalence of initial value problem (IVP) (1) and integral equation

$$x(t) = \frac{c}{\Gamma(\gamma)} \left(\log \frac{t}{a} \right)^{\gamma-1} + \frac{1}{\Gamma(\alpha)} \int_a^t \left(\log \frac{t}{s} \right)^{\alpha-1} f(s,x(s)) \frac{ds}{s}, \quad t > a, c \in \mathbb{R}. \tag{4}$$

Existence result for IVP (1) is proved in [16] using Banach fixed point theorem.

Motivated by these works, to avoid ambiguity of fixed point theory, we adopted the method of successive approximations. In this paper, we study the IVP for fractional differential equation involving Hilfer-Hadamard fractional derivative

$$\begin{cases} _H\mathcal{D}_1^{\alpha,\beta} x(t) = f(t,x), & 0 < \alpha < 1, 0 \leq \beta \leq 1, \\ \lim_{t \to 1} \left(\log t \right)^{1-\gamma} x(t) = x_0, & \gamma = \alpha + \beta(1-\alpha). \end{cases} \tag{5}$$

In this paper we prove the existence and uniqueness results for IVP (5), using some well-known convergence criterion and Picard sequence functions [18, 24]. The computable iterative scheme as well as the uniform convergence criterion for solution are also developed.

The rest of the paper is organised as follows. The next section covers the useful prerequisites which include definitions and lemmas. The main results are proved in Section 3 with the supporting illustrative example.

2   Preliminaries

We need the following basic definitions and properties from fractional calculus [19].

Definition 2.1   [19] Let $(1, b), 1 < b \leq \infty$, be a finite or infinite interval of the half-axis $\mathbb{R}^+$ and let $\alpha > 0$. The left-sided Hadamard fractional integral $_H\mathcal{I}_1^{\alpha} f$ of order $\alpha > 0$ is defined by

$$(_H\mathcal{I}_1^{\alpha} f)(t) = \frac{1}{\Gamma(\alpha)} \int_1^t (\log t)^{\alpha-1} \frac{f(s)ds}{s}, \quad 1 < t < b, \tag{6}$$

provided that the integral exists. When $\alpha = 0$, we set $_H\mathcal{I}_1^0 f = f$.

Definition 2.2   [17, 19] The left-sided Hadamard fractional derivative of order $\alpha(0 \leq \alpha < 1)$ on $(1, b)$ is defined by

$$(_H\mathcal{D}_1^{\alpha} f)(t) = \delta(_H\mathcal{I}_1^{1-\alpha} f)(t), \quad 1 < t < b, \tag{7}$$

where $\delta = t(d/dt)$. In particular, when $\alpha = 0$ we have $_H\mathcal{D}_1^0 f = f$.

Definition 2.3 [16] The left-sided Hilfer-Hadamard fractional derivative of order $\alpha(0 < \alpha < 1)$ and type $\beta(0 \le \beta \le 1)$ with respect to t is defined by

$$(_H\mathcal{D}_1^{\alpha,\beta}f)(t) = (_H\mathcal{J}_1^{\beta(1-\alpha)}{}_H\mathcal{D}_1^{\alpha+\beta(1-\alpha)}f)(t), \tag{8}$$

of function f for which the expression on the right-hand side exists, where $_H\mathcal{D}_1^{\alpha+\beta(1-\alpha)}$ is the Hadamard fractional derivative.

Lemma 2.1 [19] *If $\alpha > 0, \beta > 0$ and $1 < b < \infty$, then*

$$\left(_H\mathcal{J}_1^{\alpha}\left(\log s\right)^{\beta-1}\right)(t) = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}\left(\log t\right)^{\beta+\alpha-1}, \tag{9}$$

$$\left(_H\mathcal{D}_1^{\alpha}\left(\log s\right)^{\beta-1}\right)(t) = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}\left(\log t\right)^{\beta-\alpha-1}. \tag{10}$$

The following lemma plays a vital role in the proof of main results.

Lemma 2.2 [23] *Suppose that $x > 0$. Then $\Gamma(x) = \lim_{m \to +\infty} \frac{m^x m!}{x(x+1)(x+2)\cdots(x+m)}$.*

We denote $D = [1, 1+h], D_h = (1, 1+h], I = (1, 1+l]$ and $J = [1, 1+l]$, for $h > 0$. Here we choose

$$l = \min\left\{h, \left(\frac{b}{M}\frac{\Gamma(\alpha+k+1)}{\Gamma(k+1)}\right)^{\frac{1}{\mu+k}}\right\}, \mu = 1 - \beta(1-\alpha).$$

Further $E = \{x : |x(\log t)^{1-\gamma} - x_0| \le b\}$ for $b > 0$ and $t \in D_h$. A function $x(t)$ is said to be a solution of IVP (5) if there exists $l > 0$ such that $x \in C^0(I)$ satisfies the differential equation $_H\mathcal{D}_1^{\alpha,\beta}x(t) = f(t,x)$ almost everywhere on I along with the condition

$$\lim_{t \to 1}(\log t)^{1-\gamma}x(t) = x_0.$$

To prove our main results, we assume the following hypotheses:

(H1) $(t, x) \to f(t, (\log t)^{\gamma-1}x(t))$ is defined on $D_h \times E$ and satisfies:

  (i) $x \to f(t, (\log t)^{\gamma-1}x(t))$ is continuous on E for all $t \in D_h$,
  $t \to f(t, (\log t)^{\gamma-1}x(t))$ is measurable on D_h for all $x \in E$;

  (ii) there exist $k > (\beta(1-\alpha)-1)$ and $M \ge 0$ such that the relation $|f(t, (\log t)^{\gamma-1}x(t))| \le M(\log t)^k$ holds for all $t \in D_h$ and $x \in E$.

(H2) There exist $A > 0$ and $x_1, x_2 \in E$ such that

$$|f(t, (\log t)^{\gamma-1}x_1(t)) - f(t, (\log t)^{\gamma-1}x_2(t))| \le A(\log t)^k|x_1 - x_2|, \quad \text{for all } t \in I.$$

3 Main Results

In this section, we state and prove the existence and uniqueness results for IVP (5) for Hilfer-Hadamard FDEs. We present the iterative scheme for approximating such a unique solution.

Lemma 3.1 *Suppose that (H1) holds. Then $x : J \to \mathbb{R}$ is the solution of IVP (5) if and only if $x : I \to \mathbb{R}$ is the solution of the Volterra integral equation of second kind:*

$$x(t) = x_0 \left(\log t \right)^{\gamma-1} + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s} \right)^{\alpha-1} f(s, x(s)) \frac{ds}{s}, \quad t > 1. \tag{11}$$

Proof. First we suppose that $x : I \to \mathbb{R}$ is the solution of IVP (5). Then $\left| \left(\log t \right)^{1-\gamma} x(t) - x_0 \right| \le b$ for all $t \in I$. From (H1), there exist a $k > (\beta(1 - \alpha) - 1)$ and $M \ge 0$ such that

$$|f(t, x(t))| = |f(t, (\log t)^{\gamma-1} (\log t)^{1-\gamma} x(t))| \le M(\log t)^k, \qquad \text{for all} \quad t \in I.$$

We have

$$\left| \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s} \right)^{\alpha-1} f(s, x(s)) \frac{ds}{s} \right| \le \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s} \right)^{\alpha-1} M(\log s)^k \frac{ds}{s}$$

$$= M(\log t)^{\alpha+k} \frac{\Gamma(k+1)}{\Gamma(\alpha+k+1)}.$$

Clearly,

$$\lim_{t \to 1} \left(\log t \right)^{1-\gamma} \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s} \right)^{\alpha-1} f(s, x(s)) \frac{ds}{s} = 0.$$

It follows that

$$x(t) = x_0 \left(\log t \right)^{\gamma-1} + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s} \right)^{\alpha-1} f(s, x(s)) \frac{ds}{s}, \quad t \in I.$$

Since $k > (\beta(1 - \alpha) - 1)$, we see that $x \in C^0(I)$ is a solution of integral equation (11).

Conversely, it is easy to see the fact that $x : I \to \mathbb{R}$ is the solution of integral equation (11) implies that x is the solution of IVP (5) defined on J. This completes the proof.

To prove our main results, we choose a Picard function sequence as follows:

$$\phi_0(t) = x_0(\log t)^{\gamma-1}, \qquad t \in I,$$

$$\phi_n(t) = \phi_0(t) + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s} \right)^{\alpha-1} f(s, \phi_{n-1}(s)) \frac{ds}{s}, \quad t \in I, \quad n = 1, 2, \cdots. \tag{12}$$

Lemma 3.2 *Suppose that (H1) holds. Then ϕ_n is continuous on I and satisfies $|(\log t)^{1-\gamma} \phi_n(t) - x_0| \le b$.*

Proof. From (H1), clearly $|f(t, (\log t)^{\gamma-1} x)| \le M(\log t)^k$ for all $t \in D_h$ and $|x(\log t)^{1-\gamma} - x_0| \le b$. For $n = 1$, we have

$$\phi_1(t) = x_0(\log t)^{\gamma-1} + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s} \right)^{\alpha-1} f(s, \phi_0(s)) \frac{ds}{s}. \tag{13}$$

Then

$$\left| \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s} \right)^{\alpha-1} f(s, \phi_0(s)) \frac{ds}{s} \right| \le \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s} \right)^{\alpha-1} M(\log s)^k \frac{ds}{s}$$

$$= M(\log t)^{\alpha+k} \frac{\Gamma(k+1)}{\Gamma(\alpha+k+1)}.$$

This implies $\phi_1 \in C^0(I)$ and from equation (13), we get

$$|(\log t)^{1-\gamma}\phi_1(t) - x_0| \leq (\log t)^{1-\gamma} M(\log t)^{\alpha+k} \frac{\Gamma(k+1)}{\Gamma(\alpha+k+1)}$$

$$\leq Ml^{\alpha+k+1-\gamma} \frac{\Gamma(k+1)}{\Gamma(\alpha+k+1)}. \tag{14}$$

Now by the induction hypothesis for $n = m$, suppose that $\phi_m \in C^0(J)$ and for all $t \in J$, $|(\log t)^{1-\gamma}\phi_m(t) - x_0| \leq b$. We have

$$\phi_{m+1}(t) = x_0(\log t)^{\gamma-1} + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} f(s, \phi_m(s)) \frac{ds}{s}. \tag{15}$$

From the above discussion, we obtain $\phi_{m+1}(t) \in C^0(I)$ and from equation (15), we have

$$|(\log t)^{1-\gamma}\phi_{m+1}(t) - x_0| \leq (\log t)^{1-\gamma} \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} M(\log s)^k \frac{ds}{s}$$

$$= M(\log t)^{\alpha+k+1-\gamma} \frac{\Gamma(k+1)}{\Gamma(\alpha+k+1)}$$

$$\leq Ml^{\alpha+k+1-\gamma} \frac{\Gamma(k+1)}{\Gamma(\alpha+k+1)} \leq b.$$

Thus, the result is true for $n = m + 1$. By the principle of mathematical induction, the result is true for all n. The proof is complete.

Theorem 3.1 *Suppose that (H1) and (H2) hold. Consider the Picard function ϕ_n given in (12). Then the sequence $\{(\log t)^{1-\gamma}\phi_n(t)\}$ is uniformly convergent on J.*

Proof. Consider the series

$$(\log t)^{1-\gamma}\phi_0(t) + (\log t)^{1-\gamma}[\phi_1(t) - \phi_0(t)] + \cdots + (\log t)^{1-\gamma}[\phi_n(t) - \phi_{n-1}(t)] + \cdots, \quad t \in J.$$

By relation (14) driven in the proof of Lemma 3.2 above, we get

$$(\log t)^{1-\gamma}|\phi_1(t) - \phi_0(t)| \leq M(\log t)^{\alpha+k+1-\gamma} \frac{\Gamma(k+1)}{\Gamma(\alpha+k+1)}, \qquad t \in J.$$

From Lemma 3.2, we have

$$(\log t)^{1-\gamma}|\phi_2(t) - \phi_1(t)| \leq (\log t)^{1-\gamma} \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} |f(s, \phi_1(s)) - f(s, \phi_0(s))| \frac{ds}{s}$$

$$= (\log t)^{1-\gamma} \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} |f(s, (\log s)^{\gamma-1}(\log s)^{1-\gamma}\phi_1(s)) -$$

$$f(s, (\log s)^{\gamma-1}(\log s)^{1-\gamma}\phi_0(s))| \frac{ds}{s}$$

$$\leq (\log t)^{1-\gamma} \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} A(\log s)^k |(\log s)^{1-\gamma}\phi_1(s) -$$

$$(\log s)^{1-\gamma}\phi_0(s)| \frac{ds}{s}$$

$$\leq (\log t)^{1-\gamma} \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} A(\log s)^k [(\log s)^{1-\gamma}|\phi_1(s) - \phi_0(s)|] \frac{ds}{s}$$

$$\leq (\log t)^{1-\gamma} \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} A(\log s)^k \left[M(\log s)^{\alpha+k+1-\gamma} \frac{\Gamma(k+1)}{\Gamma(\alpha+k+1)}\right] \frac{ds}{s}.$$

Thus

$$(\log t)^{1-\gamma}|\phi_2(t) - \phi_1(t)| \le AM \frac{\Gamma(k+1)}{\Gamma(\alpha+k+1)} \frac{\Gamma(\alpha+2k+2-\gamma)}{\Gamma(2\alpha+2k+2-\gamma)}(\log t)^{2(\alpha+k+1-\gamma)}.$$

Now suppose that for $n = m$

$$(\log t)^{1-\gamma}|\phi_{m+1}(t) - \phi_m(t)| \le$$

$$A^m M (\log t)^{(m+1)(\alpha+k+1-\gamma)} \prod_{i=0}^{m} \frac{\Gamma((i+1)k + i(\alpha+1-\gamma)+1)}{\Gamma((i+1)(\alpha+k)+i(1-\gamma)+1)}.$$

We have

$$(\log t)^{1-\gamma}|\phi_{m+2}(t) - \phi_{m+1}(t)| \le \frac{(\log t)^{1-\gamma}}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} |f(s, \phi_{m+1}(s)) - f(s, \phi_m(s))| \frac{ds}{s}$$

$$= (\log t)^{1-\gamma} \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \Big| f\big(s, (\log s)^{\gamma-1}(\log s)^{1-\gamma}\phi_{m+1}(s)\big) -$$

$$f\big(s, (\log s)^{\gamma-1}(\log s)^{1-\gamma}\phi_m(s)\big)\Big| \frac{ds}{s}$$

$$\le (\log t)^{1-\gamma} \frac{1}{(\Gamma(\alpha))} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} A(\log s)^k \big[(\log s)^{1-\gamma}|\phi_{m+1}(s) - \phi_m(s)|\big] \frac{ds}{s}$$

$$\le (\log t)^{1-\gamma} \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} A(\log s)^k \Big[A^m M (\log s)^{(m+1)(\alpha+k+1-\gamma)}$$

$$\times \prod_{i=0}^{m} \frac{\Gamma((i+1)k + i(\alpha+1-\gamma)+1)}{\Gamma((i+1)(\alpha+k)+i(1-\gamma)-1)}\Big] \frac{ds}{s}$$

$$= A^{m+1} M (\log t)^{(m+2)(\alpha+k+1-\gamma)} \prod_{i=0}^{m+1} \frac{\Gamma((i+1)k + i(\alpha+1-\gamma)+1)}{\Gamma((i+1)(\alpha+k)+i(1-\gamma)+1)}.$$

Thus

$$(\log t)^{1-\gamma}|\phi_{m+2}(t) - \phi_{m+1}(t)| \le$$

$$A^{m+1} M l^{(m+2)(\alpha+k+1-\gamma)} \prod_{i=0}^{m+1} \frac{\Gamma((i+1)k + i(\alpha+1-\gamma)+1)}{\Gamma((i+1)(\alpha+k)+i(1-\gamma)+1)}.$$

The result is true for $n = m+1$. By the principle of mathematical induction the result is true for all n.

Consider

$$\sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} M A^{n+1} l^{(n+2)(\alpha+k+1-\gamma)} \prod_{i=0}^{n+1} \frac{\Gamma((i+1)k + i(\alpha+1-\gamma)+1)}{\Gamma((i+1)(\alpha+k)+i(1-\gamma)+1)}.$$

We have

$$\frac{u_{n+1}}{u_n} = \frac{M A^{n+2} l^{(n+3)(\alpha+k+1-\gamma)} \prod_{i=0}^{n+2} \frac{\Gamma((i+1)k+i(\alpha+1-\gamma)+1)}{\Gamma((i+1)(\alpha+k)+i(1-\gamma)+1)}}{M A^{n+1} l^{(n+2)(\alpha+k+1-\gamma)} \prod_{i=0}^{n+1} \frac{\Gamma((i+1)k+i(\alpha+1-\gamma)+1)}{\Gamma((i+1)(\alpha+k)+i(1-\gamma)+1)}}$$

$$= A l^{\alpha+k+1-\gamma} \frac{\Gamma((n+3)k + (n+2)(\alpha+1-\gamma)+1)}{\Gamma((n+3)(k+\alpha) + (n+2)(1-\gamma)+1)}.$$

Using Lemma 2.2, we have

$$\frac{u_{n+1}}{u_n} = Al^{\alpha+k+1-\gamma} \frac{\lim_{m\to\infty} \frac{m^{(n+3)k+(n+2)(\alpha+1-\gamma)+1}m!}{((n+3)k+(n+2)(\alpha+1-\gamma)+1)\cdots((n+3)k+(n+2)(\alpha+1-\gamma)+m+1)}}{\lim_{m\to\infty} \frac{m^{(n+3)(k+\alpha)+(n+2)(1-\gamma)+1}m!}{((n+3)(k+\alpha)+(n+2)(1-\gamma)+1)\cdots((n+3)(k+\alpha)+(n+2)(1-\gamma)+m+1)}}$$

$$= Al^{\alpha+k+1-\gamma}[\lim_{m\to\infty} m^{-\alpha} \frac{((n+3)(k+\alpha)+(n+2)(1-\gamma)+1)\cdots((n+3)(k+\alpha)+(n+2)(1-\gamma)+m+1)}{((n+3)k+(n+2)(\alpha+1-\gamma)+1)\cdots((n+3)k+(n+2)(\alpha+1-\gamma)+m+1)}].$$

It is easy to see that

$$\frac{((n+3)(k+\alpha)+(n+2)(1-\gamma)+1)\cdots((n+3)(k+\alpha)+(n+2)(1-\gamma)+m+1)}{((n+3)k+(n+2)(\alpha+1-\gamma)+1)\cdots((n+3)k+(n+2)(\alpha+1-\gamma)+m+1)}$$

is bounded for all m, n. Thus $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = 0$ implies $\sum_{n=1}^{\infty} u_n$ is convergent. Hence

$$(\log t)^{1-\gamma}\phi_0(t) + (\log t)^{1-\gamma}[\phi_1(t) - \phi_0(t)] + \cdots + (\log t)^{1-\gamma}[\phi_n(t) - \phi_{n-1}(t)] + \cdots$$

is uniformly convergent for $t \in J$. Hence $\{(\log t)^{1-\gamma}\phi_n(t)\}$ is uniformly convergent on J.

Theorem 3.2 *Suppose that (H1) and (H2) hold. Then the solution*

$$\phi(t) = (\log t)^{\gamma-1} \lim_{n\to\infty} (\log t)^{1-\gamma}\phi_n(t)$$

is a unique continuous solution of the integral equation (11) *defined on J.*

Proof. Since $\phi(t) = (\log t)^{\gamma-1} \lim_{n\to\infty} (\log t)^{1-\gamma}\phi_n(t)$ on J, and by Lemma 3.2, we have $(\log t)^{1-\gamma}|\phi(t) - x_0| \leq b$. Then

$$|f(t, \phi_n(t)) - f(t, \phi(t))| \leq A(\log t)^k|\phi_n(t) - \phi(t)|, \quad t \in I.$$

Clearly, $(\log t)^{-k}|f(t, \phi_n(t)) - f(t, \phi(t))| \leq A|\phi_n(t) - \phi(t)| \to 0$ uniformly as $n \to \infty$ on I. Therefore

$$(\log t)^{1-\gamma}\phi(t) = \lim_{n\to\infty} \phi_n(t)$$

$$= x_0 + (\log t)^{1-\gamma}\frac{1}{\Gamma(\alpha)} \int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} (\log s)^k \lim_{n\to\infty} \left((\log s)^{-k}f(s, \phi_{n-1}(s))\right)\frac{ds}{s}$$

$$= x_0 + (\log t)^{1-\gamma}\frac{1}{\Gamma(\alpha)} \int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} f(s, \phi(s))\frac{ds}{s}.$$

Then $\phi(t)$ is a continuous solution of integral equation (11) defined on J.

Now we prove uniqueness of solution $\phi(t)$. Suppose that $\psi(t)$ is a solution of integral equation (11). Then $(\log t)^{1-\gamma}|\psi(t)| \leq b$ for all $t \in I$ and

$$\psi(t) = x_0(\log t)^{\gamma-1} + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} f(s, \phi(s))\frac{ds}{s}, \quad t \in I.$$

We prove $\phi(t) \equiv \psi(t)$ on I. From (H1), there exist a $k > (\beta(1-\alpha) - 1)$ and $M \geq 0$ such that

$$|f(t, \psi(t))| = \left|f\left(t, (\log t)^{\gamma-1}(\log t)^{1-\gamma}\psi(t)\right)\right| \leq M(\log t)^k, \quad \text{for all } t \in I.$$

Therefore

$$(\log t)^{1-\gamma}|\phi_0(t) - \psi(t)| = (\log t)^{1-\gamma}\left|\frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} f(s,\psi(s))\frac{ds}{s}\right|$$

$$\leq (\log t)^{1-\gamma}\frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} M(\log s)^k \frac{ds}{s}$$

$$= M(\log t)^{\alpha+k+1-\gamma}\frac{\Gamma(k+1)}{\Gamma(\alpha+k+1)}.$$

Furthermore

$$(\log t)^{1-\gamma}|\phi_1(t) - \psi(t)| = (\log t)^{1-\gamma}\left|\frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}[f(s,\phi_0(s)) - f(s,\psi(s))]\frac{ds}{s}\right|$$

$$\leq AM\frac{\Gamma(k+1)}{\Gamma(\alpha+k+1)}\frac{\Gamma(\alpha+2k+2-\gamma)}{\Gamma(2\alpha+2k+2-\gamma)}(\log t)^{2(\alpha+k+1-\gamma)}.$$

By the induction hypothesis, we suppose that

$$(\log t)^{1-\gamma}|\phi_n(t) - \psi(t)| \leq A^n M(\log t)^{(n+1)(\alpha+k+1-\gamma)}\prod_{i=0}^{n}\frac{\Gamma((i+1)k + i(\alpha+1-\gamma)+1)}{\Gamma((i+1)(\alpha+k) + i(1-\gamma)+1)}.$$

Then

$$(\log t)^{1-\gamma}|\phi_{n+1}(t) - \psi(t)| \leq (\log t)^{1-\gamma}\left|\frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}[f(s,\phi_n(s)) - f(s,\psi(s))]\frac{ds}{s}\right|$$

$$\leq A^{n+1}M(\log t)^{(n+2)(\alpha+k+1-\gamma)}\prod_{i=0}^{n+1}\frac{\Gamma((i+1)k + i(\alpha+1-\gamma)+1)}{\Gamma((i+1)(\alpha+k) + i(1-\gamma)+1)}$$

$$\leq A^{n+1}Ml^{(n+2)(\alpha+k+1-\gamma)}\prod_{i=0}^{n+1}\frac{\Gamma((i+1)k + i(\alpha+1-\gamma)+1)}{\Gamma((i+1)(\alpha+k) + i(1-\gamma)+1)}.$$

Using the same arguments as in Theorem 3.1, we obtain the series

$$\sum_{n=1}^{\infty} A^{n+1}Ml^{(n+2)(\alpha+k+1-\gamma)}\prod_{i=0}^{n+1}\frac{\Gamma((i+1)k + i(\alpha+1-\gamma)+1)}{\Gamma((i+1)(\alpha+k) + i(1-\gamma)+1)},$$

which is convergent. Therefore

$$A^{n+1}Ml^{(n+2)(\alpha+k+1-\gamma)}\prod_{i=0}^{n+1}\frac{\Gamma((i+1)k + i(\alpha+1-\gamma)+1)}{\Gamma((i+1)(\alpha+k) + i(1-\gamma)+1)} \to 0 \quad \text{as} \quad n \to \infty.$$

Also we observe that $\lim_{n\to\infty}(\log t)^{1-\gamma}\phi_n(t) = (\log t)^{1-\gamma}\psi(t)$ uniformly on J. Thus $\phi(t) \equiv \psi(t)$ on I. The proof is complete.

Theorem 3.3 *Suppose that (H1) and (H2) hold. Then the IVP* (5) *has a unique continuous solution* $\phi(t) = (\log t)^{\gamma-1}\lim_{n\to\infty}(\log t)^{1-\gamma}\phi_n(t)$ *on* I.

Proof. From Lemma 3.1 and Theorem 3.1, we can easily obtain that the solution

$$\phi(t) = (\log t)^{\gamma-1}\lim_{n\to\infty}(\log t)^{1-\gamma}\phi_n(t)$$

is a unique continuous solution of IVP (5) defined on I. The proof is complete.

Example 3.1 We consider the Hilfer-Hadamard fractional differential problem

$$\begin{cases} {}_H\mathcal{D}_1^{\frac{1}{2},\frac{1}{2}}x(t) = f(t,x), \quad \alpha = \frac{1}{2}, \beta = \frac{1}{2}, \\ \lim_{t\to 1} \left(\log t\right)^{\frac{1}{4}} x(t) = x_0, \quad \gamma = \frac{3}{4}, \end{cases} \tag{16}$$

where

$$\begin{cases} f(t,x(t)) = \frac{(\log t)^{-\frac{1}{4}}\sin(\log t)}{8(1+\sqrt{(\log t)})(1+|\sin(\log t)|)}, & \text{for } t \in (1,e], \ x \in \mathbb{R}, \\ f(1,x(1)) = 0, & \text{for } x \in \mathbb{R}. \end{cases}$$

It is easy to see that f is singular at $t = 1$, and is a continuous function for $t \in (1,e]$. We choose $\mu = \frac{3}{4}$, $b = 4$, $k = -\frac{1}{4} > -\frac{3}{4}$. Thus $l = \min\left\{1.7182, \left(\frac{4}{M}\frac{\Gamma(\frac{5}{4})}{\Gamma(\frac{3}{4})}\right)^2\right\}$, where

$$M = \max_{t\in[1,e]} \frac{\sin(\log t)}{8(1+\sqrt{\log t})(1+|\sin(\log t)|)} \approx 32$$

with

$$\phi_0(t) = x_0(\log t)^{-\frac{1}{4}}, \qquad t \in (1,e],$$

$$\phi_n(t) = \phi_0(t) + \frac{1}{\Gamma(\frac{1}{2})}\int_1^t \left(\log\frac{t}{s}\right)^{-\frac{1}{2}} f(s,\phi_{n-1}(s))\frac{ds}{s}, \quad n = 1,2,\cdots.$$

Clearly, all the conditions of Theorem 3.3 hold. Therefore IVP (16) has the unique continuous solution

$$\phi(t) = (\log t)^{-\frac{1}{4}} \lim_{n\to\infty} (\log t)^{\frac{1}{4}}\phi_n(t) \quad \text{on } [1,e].$$

Remark 3.1 The initial value considered in IVP (5) is more suitable than that considered in IVP (1) and nonlinear function f may be singular at $t = 1$.

Remark 3.2 In hypothesis (H1), if $(\log t)^{-k}f(t,(\log t)^{\gamma-1}x(t))$ is continuous on $D \times E$, one may choose $M = \max_{t\in J}(\log t)^{-k}f(t,(\log t)^{\gamma-1}x(t))$ continuous on $D_h \times E$ for all $x \in E$.

Acknowledgment

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Different Schemes of Coexistence of Full State Hybrid Function Projective Synchronization and Inverse Full State Hybrid Function Projective Synchronization

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Abstract: This paper presents new synchronization schemes, which assure the co-existence of the full-state hybrid function projective synchronization (FSHFPS) and the inverse full-state hybrid function projective synchronization (IFSHFPS) between wide classes of three-dimensional master systems and four-dimensional slave systems. In order to show the capability of co-existence approaches, numerical examples are reported, which illustrate the co-existence of FSHFPS and IFSHFPS between 3D chaotic system and 4D hyperchaotic system in different dimension.

Keywords: *chaos; full-state hybrid function projective synchronization; inverse full-state hybrid function projective synchronization; co-existence; Lyapunov stability.*

Mathematics Subject Classification (2010): 37B25, 37B55, 93C55, 93D05.

1 Introduction

Synchronization refers to a process wherein two dynamical systems (master and slave systems, respectively) adjust their motion to achieve a common behavior, mainly due to a control input [1]. The issue of synchronization of chaotic dynamical systems was first studied by Pecora and Carroll [2]. By considering the historical timeline of the topic, it can be observed that a large variety of synchronization types has been proposed such as matrix projective synchronization [3], generalized synchronization [4], inverse generalized synchronization [5], $\Lambda - \phi$ generalized synchronization [6,7] and $\Phi - \Theta$ synchronization [8,9] and so on. Among the different types, *full state hybrid projective synchronization* (FSHPS) has been introduced, wherein each slave system variable synchronizes with a linear combination of master system variables [10]. Different types

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of synchronization such as complete synchronization, anti-synchronization, projective synchronization and hybrid synchronization can be achieved from the FSHPS scheme depending on the choice of scaling functions. On the other hand, when the inverted scheme is implemented, i.e., each master system state synchronizes with a linear combination of slave system states, the *inverse full-state hybrid projective synchronization* (IFSHPS) is obtained [11]. Moreover, when the scaling factors are replaced by scaling functions, function-based hybrid synchronization schemes are obtained, i.e., the *full-state hybrid function projective synchronization* (FSHFPS) [12] and the *inverse full-state hybrid function projective synchronization* (IFSHFPS) [13], respectively.

Recently, the topic of coexistence of several synchronization types between chaotic systems has recently started to attract increasing attention. In fact, very recent papers have investigated the co-existence of different types of synchronization when synchronizing two chaotic systems. For example, the approach developed in [14,15] has illustrated a rigorous study to prove the co-existence of some synchronization types between discrete-time chaotic (hyperchaotic) systems. Referring to integer-order chaotic systems, in [16] two synchronization schemes of co-existence have been proposed. The problem of coexistence of some types of synchronization between different dimensional fractional order chaotic systems has been studied [17,18]. New approaches to study the co-existence of some types of synchronization between integer order and fractional order chaotic systems with different dimensions have been introduced in [19]. Meanwhile, to the best of our knowledge, the investigation of coexistence of FSHFPS and IFSHFPS for integer-order differential dynamical systems with different dimensions is not yet explored. The present research work focuses on coexistence of FSHFPS and IFSHFPS between chaotic and hyperchaotic systems.

Based on these considerations, this paper aims to give a further contribution to the topic by considering the co-existence of FSHFPS and IFSHFPS between non-identical and different dimensions chaotic and hyperchaotic systems. Specifically, the paper illustrates new schemes, which prove the co-existence of the full-state hybrid function projective synchronization (FSHFPS) and the inverse full-state hybrid function projective synchronization (IFSHFPS) between a three-dimensional master system and a four-dimensional slave system in 4D and 3D, respectively. These master-slave systems belong to general classes, which include several chaotic (hyperchaotic) systems characterized by different dimensions. The conceived schemes are general approches and the only restriction on the scaling functions is that they must be differentiable and bouned functions.

The paper is organized as follow: Section 2 gives some definitions related to FSHFPS and IFSHFPS. Sections 3 and 4 give the basic mathematical background of the coexistence of FSHFPS and IFSHFPS in 4D and 3D respectively. Section 5 presents some numerical examples of co-existence of synchronization types with the aim to show the effectiveness of the approach developed herein. Section 6 concludes the paper.

2 Definition of FSHFPS and IFSHFPS

We consider the following master and slave systems

$$\dot{X}(t) = F(X(t)), \tag{1}$$

$$\dot{Y}(t) = G(Y(t)) + U, \tag{2}$$

where $X(t) = (x_i(t))_{1 \leq i \leq n}$, $Y(t) = (y_i(t))_{1 \leq i \leq m}$ are the states of the master system and the slave system, respectively, $F : \mathbb{R}^n \to \mathbb{R}^n$, $G : \mathbb{R}^m \to \mathbb{R}^m$ and $U = (u_i)_{1 \leq i \leq m}$ is a

vector controller.

Definition 2.1 The master systems (1) and the slave system (2) are said to be full state hybrid function projective synchronized (FSHFPS), if there exist a controller $U = (u_i)_{1 \le i \le m}$ and differentiable functions $\alpha_{ij}(t) : \mathbb{R}^+ \to \mathbb{R}$, $i = 1, 2, ..., m$; $j = 1, 2, ..., n$, such that the synchronization errors

$$e_i(t) = y_i(t) - \sum_{j=1}^{n} \alpha_{ij}(t) x_j(t), \quad i = 1, 2, ..., m, \tag{3}$$

satisfy $\lim_{t \to \infty} e_i(t) = 0$.

Definition 2.2 The master systems (1) and the slave system (2) are said to be inverse full state hybrid function projective synchronized (IFSHFPS), if there exist a controller $U = (u_i)_{1 \le i \le m}$ and differentiable functions $\beta_{ij}(t) : \mathbb{R}^+ \to \mathbb{R}$, $i = 1, 2, ..., n$; $j = 1, 2, ..., m$, such that the synchronization errors

$$e_i(t) = x_i(t) - \sum_{j=1}^{m} \beta_{ij}(t) y_j(t), \quad i = 1, 2, ..., n, \tag{4}$$

satisfy $\lim_{t \to \infty} e_i(t) = 0$.

3   Scheme 1

Here, we assume that the master system can be considered as

$$\dot{x}_i(t) = f_i(X(t)), \quad i = 1, 2, 3, \tag{5}$$

where $X(t) = (x_i(t))_{1 \le i \le 3}$ is the state vector of the master system (5), $f_i : \mathbb{R}^3 \to \mathbb{R}$, $i = 1, 2, 3$. Also, consider the slave system as

$$\dot{y}_i(t) = \sum_{j=1}^{4} b_{ij} y_j(t) + g_i(Y(t)) + u_i, \quad i = 1, 2, 3, 4, \tag{6}$$

where $Y(t) = (y_i)_{1 \le i \le 4}$ is the state vector of the slave system (6), $(b_{ij}) \in \mathbb{R}^{4 \times 4}$, $g_i : \mathbb{R}^4 \to \mathbb{R}$ are nonlinear functions and u_i, $i = 1, 2, 3, 4$, are controllers to be designed.

Definition 3.1 Let $(\alpha_j(t))_{1 \le j \le 4}$, $(\beta_j(t))_{1 \le j \le 3}$, $(\gamma_j(t))_{1 \le j \le 4}$ and $(\theta_j(t))_{1 \le j \le 3}$ be continuously differentiable and boundary functions, it is said that IFSHFPS and FSHFPS coexist in the synchronization of the master system (5) and the slave system (6), if there exist controllers u_i, $= 1, 2, 3, 4$, such that the synchronization errors

$$
\begin{aligned}
e_1(t) &= x_1(t) - \sum_{j=1}^{4} \alpha_j(t) y_j(t), \\
e_2(t) &= y_2(t) - \sum_{j=1}^{3} \beta_j(t) x_j(t), \\
e_3(t) &= x_3(t) - \sum_{j=1}^{4} \gamma_j(t) y_j(t), \\
e_4(t) &= y_4(t) - \sum_{j=1}^{3} \theta_j(t) x_j(t),
\end{aligned}
\tag{7}
$$

satisfy $\lim_{t \longrightarrow +\infty} e_i(t) = 0, \ \ i = 1, 2, 3, 4.$

Sufficient conditions for co-existence of IFSHFPS and FSHFPS between systems (5) and (6) are given by the following theorem.

Theorem 3.1 *The coexistence of IFSHFPS and FSHFPS between the master system (5) and the slave system (6) will occur if $\alpha_3(t)\gamma_1(t) - \alpha_1(t)\gamma_3(t) \neq 0$ and the control law is designed as follows:*

$$u_1 = \sum_{i=1}^{4} P_i \left(\sum_{j=1}^{4} (b_{ij} - c_{ij}) e_j(t) - R_i \right), \tag{8}$$

$$u_2 = \sum_{j=1}^{4} (b_{2j} - c_{2j}) e_j(t) - R_2,$$

$$u_3 = \sum_{i=1}^{4} Q_i \left(\sum_{j=1}^{4} (b_{ij} - c_{ij}) e_j(t) - R_i \right),$$

$$u_4 = \sum_{j=1}^{4} (b_{4j} - c_{4j}) e_j(t) - R_4,$$

where $(c_{ij})_{4 \times 4}$ *are control constants to be selected and*

$$
\begin{aligned}
P_1 &= \frac{\gamma_3(t)}{\alpha_3(t)\gamma_1(t) - \alpha_1(t)\gamma_3(t)}, \tag{9}\\
P_2 &= \frac{\gamma_3(t)\alpha_2(t) - \alpha_3(t)\gamma_2(t)}{\alpha_3(t)\gamma_1(t) - \alpha_1(t)\gamma_3(t)},\\
P_3 &= \frac{-\alpha_3(t)}{\alpha_3(t)\gamma_1(t) - \alpha_1(t)\gamma_3(t)},\\
P_4 &= \frac{\gamma_3(t)\alpha_4(t) - \alpha_3(t)\gamma_4(t)}{\alpha_3(t)\gamma_1(t) - \alpha_1(t)\gamma_3(t)},\\
Q_1 &= \frac{-\gamma_1(t)}{\alpha_3(t)\gamma_1(t) - \alpha_1(t)\gamma_3(t)},\\
Q_2 &= \frac{\alpha_1(t)\gamma_2(t) - \alpha_2(t)\gamma_1(t)}{\alpha_3(t)\gamma_1(t) - \alpha_1(t)\gamma_3(t)},\\
Q_3 &= \frac{\alpha_1(t)}{\alpha_3(t)\gamma_1(t) - \alpha_1(t)\gamma_3(t)},\\
Q_4 &= \frac{\alpha_1(t)\gamma_4(t) - \alpha_4(t)\gamma_1(t)}{\alpha_3(t)\gamma_1(t) - \alpha_1(t)\gamma_3(t)},
\end{aligned}
$$

and

$$R_1 = f_1(X(t)) - \sum_{j=1}^{4} \dot{\alpha}_j(t) y_j(t) - \sum_{i=1}^{4} \alpha_i(t) \left(\sum_{j=1}^{4} b_{ij} y_j(t) + g_i(Y(t)) \right), \tag{10}$$

$$R_2 = \sum_{j=1}^{4} b_{2j} y_j(t) + g_2(Y(t)) - \sum_{j=1}^{3} \dot{\beta}_j(t) x_j(t) - \sum_{j=1}^{3} \beta_j(t) \dot{x}_j(t), \tag{11}$$

$$R_3 = f_3(X(t)) - \sum_{j=1}^{4} \dot{\gamma}_j(t) y_j(t) - \sum_{i=1}^{4} \gamma_i(t) \left(\sum_{j=1}^{4} b_{ij} y_j(t) + g_i(Y(t)) \right),$$

$$R_4 = \sum_{j=1}^{4} b_{4j} y_j(t) + g_4(Y(t)) - \sum_{j=1}^{3} \dot{\theta}_j(t) x_j(t) - \sum_{j=1}^{3} \theta_j(t) \dot{x}_j(t).$$

Proof. The error system (7) can be differentiated as follows:

$$\dot{e}_1(t) = \dot{x}_1(t) - \sum_{j=1}^{4} \dot{\alpha}_j(t) y_j(t) - \sum_{j=1}^{4} \alpha_j(t) \dot{y}_j(t), \tag{12}$$

$$\dot{e}_2(t) = \dot{y}_2(t) - \sum_{j=1}^{3} \dot{\beta}_j(t) x_j(t) - \sum_{j=1}^{3} \beta_j(t) \dot{x}_j(t),$$

$$\dot{e}_3(t) = \dot{x}_3(t) - \sum_{j=1}^{4} \dot{\gamma}_j(t) y_j(t) - \sum_{j=1}^{4} \gamma_j(t) \dot{y}_j(t),$$

$$\dot{e}_4(t) = \dot{y}_4(t) - \sum_{j=1}^{3} \dot{\theta}_j(t) x_j(t) - \sum_{j=1}^{3} \theta_j(t) \dot{x}_j(t).$$

Furthermore, the error system (12) can be written as

$$\dot{e}_1(t) = \sum_{j=1}^{4} \alpha_j(t) u_j + R_1, \tag{13}$$

$$\dot{e}_2(t) = u_2 + R_2,$$

$$\dot{e}_3(t) = \sum_{j=1}^{4} \gamma_j(t) u_j + R_3,$$

$$\dot{e}_4(t) = u_4 + R_4,$$

where R_i, $i = 1, 2, 3, 4$, were described by (10). By substituting the control law (8) into (13), the error system can be described as

$$\dot{e}_i(t) = \sum_{j=1}^{4} (b_{ij} - c_{ij}) e_j(t), \quad i = 1, 2, 3, 4, \tag{14}$$

or in the compact form

$$\dot{e}(t) = (B - C) e(t), \tag{15}$$

where $B = (b_{ij})_{4 \times 4}$ and $C = (c_{ij})_{4 \times 4}$ is the control matrix. If we select the control matrix C such that all the eigenvalues of $B - C$ are strictly negative, it is immediate that all solutions of the error system (15) go to zero as $t \to \infty$. Therefore, the systems (5) and (6) are globally synchronized in 4D.

4 Scheme 2

Now, the master and the slave systems can be described in the following forms

$$\dot{x}_i(t) = \sum_{j=1}^{3} a_{ij} x_j(t) + f_i(X(t)), \quad i = 1, 2, 3, \tag{16}$$

$$\dot{y}_i(t) = g_i(Y(t)) + u_i, \quad i = 1, 2, 3, 4, \tag{17}$$

where $X(t) = (x_i)_{1 \leq i \leq 3}$, $Y(t) = (y_i)_{1 \leq i \leq 4}$ are the states of the master system and the slave system, respectively, $(a_{ij}) \in \mathbb{R}^{3 \times 3}$, $f_i : \mathbb{R}^3 \to \mathbb{R}$ are nolinear functions, $g_i : \mathbb{R}^4 \to \mathbb{R}$ and $u_i, = 1, 2, 3, 4$, are controllers to be constructed.

Definition 4.1 Let $(\lambda_j(t))_{1 \leq j \leq 3}$, $(\mu_j(t))_{1 \leq j \leq 4}$ and $(\sigma_j(t))_{1 \leq j \leq 3}$ be continuously differentiable and boundary functions, it is said that IFSHFPS and $\overline{\text{FSHFPS}}$ coexist in the synchronization of the master system (16) and the slave system (17), if there exist controllers $u_i, = 1, 2, 3$, such that the synchronization errors

$$
\begin{aligned}
e_1(t) &= y_1(t) - \sum_{j=1}^{3} \lambda_j(t) \, x_j(t), \\
e_2(t) &= x_2(t) - \sum_{j=1}^{4} \mu_j(t) \, y_j(t), \\
e_3(t) &= y_3(t) - \sum_{j=1}^{3} \sigma_j(t) \, x_j(t),
\end{aligned}
\tag{18}
$$

satisfy $\lim_{t \to +\infty} e_i(t) = 0, \ i = 1, 2, 3$.

Hence, we have the following result.

Theorem 4.1 *To achieve the coexistence of IFSHFPS and FSHFPS between the master system (16) and the slave system (17), we assume that $\mu_2(t) \neq 0$ and the control law is constructed as follows:*

$$
\begin{aligned}
u_1 &= \sum_{j=1}^{3} (a_{1j} - l_{1j}) \, e_j(t) - R_1, \tag{19} \\
u_2 &= -\frac{\mu_1(t)}{\mu_2(t)} \left(\sum_{j=1}^{3} (a_{1j} - l_{1j}) \, e_j(t) - R_1 \right) - \frac{1}{\mu_2(t)} \left(\sum_{j=1}^{3} (a_{2j} - l_{2j}) \, e_j(t) - R_2 \right) \\
&\quad - \frac{\mu_3(t)}{\mu_2(t)} \left(\sum_{j=1}^{3} (a_{3j} - l_{3j}) \, e_j(t) - R_3 \right), \\
u_3 &= \sum_{j=1}^{3} (a_{3j} - l_{3j}) \, e_j(t) - R_3, \\
u_4 &= 0,
\end{aligned}
$$

where $(l_{ij})_{3 \times 3}$ are control constants to be determined, whereas R_1, R_2 and R_3 are chosen

as follows

$$R_1 = g_1(Y(t)) - \sum_{j=1}^{3} (a_{1j} - l_{1j}) e_j(t) - \sum_{j=1}^{3} \dot{\lambda}_j(t) x_j(t) \tag{20}$$

$$- \sum_{i=1}^{3} \lambda_i(t) \left(\sum_{j=1}^{3} a_{ij} x_j(t) + f_i(X(t)) \right), \tag{21}$$

$$R_2 = \sum_{j=1}^{3} a_{2j} x_j(t) + f_2(X(t)) - \sum_{j=1}^{3} (a_{2j} - l_{2j}) e_j(t) \tag{22}$$

$$- \sum_{j=1}^{4} \dot{\mu}_j(t) y_j(t) - \sum_{j=1}^{4} \mu_j(t) g_j(Y(t)),$$

$$R_3 = g_3(Y(t)) - \sum_{j=1}^{3} (a_{3j} - l_{3j}) e_j(t) - \sum_{j=1}^{3} \dot{\sigma}_j(t) x_j(t) \tag{23}$$

$$- \sum_{i=1}^{3} \sigma_i(t) \left(\sum_{j=1}^{3} a_{ij} x_j(t) + f_i(X(t)) \right).$$

Proof. Error system (18), between master system (16) and the slave system (17), can be derived as

$$\dot{e}_1(t) = \dot{y}_1(t) - \sum_{j=1}^{3} \dot{\lambda}_j(t) x_j(t) - \sum_{j=1}^{3} \lambda_j(t) \dot{x}_j(t), \tag{24}$$

$$\dot{e}_2(t) = \dot{x}_2(t) - \sum_{j=1}^{4} \dot{\mu}_j(t) y_j(t) - \sum_{j=1}^{4} \mu_j(t) \dot{y}_j(t),$$

$$\dot{e}_3(t) = \dot{y}_3(t) - \sum_{j=1}^{3} \dot{\sigma}_j(t) x_j(t) - \sum_{j=1}^{3} \sigma_j(t) \dot{x}_j(t).$$

Error system (24), after some algebraic manipulations, becomes

$$\dot{e}_1(t) = \sum_{j=1}^{3} (a_{1j} - l_{1j}) e_j(t) + u_1 + R_1, \tag{25}$$

$$\dot{e}_2(t) = \sum_{j=1}^{3} (a_{2j} - l_{2j}) e_j(t) - \sum_{j=1}^{4} \mu_j(t) u_j + R_2,$$

$$\dot{e}_3(t) = \sum_{j=1}^{3} (a_{3j} - l_{3j}) e_j(t) + u_3 + R_3,$$

where R_i, $i = 1, 2, 3$, were given by (21). By considering the control law (19), it follows that the error dynamics between systems (16) and (17) are described by

$$\dot{e}_i(t) = \sum_{j=1}^{4} (b_{ij} - l_{ij}) e_j(t), \quad i = 1, 2, 3, \tag{26}$$

or in the compact form

$$\dot{e}(t) = (A - L)\, e(t), \tag{27}$$

where $e(t) = (e_i(t))_{1 \leq i \leq 3}$, $A = (a_{ij})_{3 \times 3}$, $L = (l_{ij})_{3 \times 3}$. Construct the candidate Lyapunov function in the form: $V(e(t)) = e^T(t)e(t)$, we obtain

$$\begin{aligned}
\dot{V}(e(t)) &= \dot{e}^T(t)e(t) + e^T(t)\dot{e}(t) \\
&= e^T(t)(A - L)^T e(t) + e^T(t)(A - L)\, e(t) \\
&= e^T(t)\left[(A - L)^T + (A - L)\right] e(t).
\end{aligned}$$

If the control matrix L is chosen such that $(A - L)^T + (A - L)$ is a negative definite matrix, we get $\dot{V}(e(t)) < 0$. Thus, from the Lyapunov stability theory, the zero solution of the error system (27) is globally asymptotically stable, i.e,

$$\lim_{t \to \infty} e_i(t) = 0, \quad i = 1, 2, 3. \tag{28}$$

Therefore, systems (16) and (17) are globally synchronized in 3D.

5 Numerical Examples

This section provides several examples of coexistence of FSHFPS and IFSHFPS between 3D chaotic systems and 4D hyperchaotic systems in 4D and 3D, respectively. Each numerical example is related to one of the theorems developed in previous sections.

5.1 Example 1

In this example, the master system is defined by the following new 3D system [20]

$$\begin{aligned}
\dot{x}_1 &= a_1\,(x_2 - x_1), \\
\dot{x}_2 &= x_1 x_3, \\
\dot{x}_3 &= 50 - a_2 x_1^2 - a_3 x_3.
\end{aligned} \tag{29}$$

When $a_1 = 2.9$, $a_2 = 0.7$, $a_3 = 0.6$ and the initial conditions are taken as $(x_1(0), x_2(0), x_3(0)) = (0.6, 0.5, 0.4)$, system (29) exhibits chaotic attractors as shown in Figures 1 and 2.

The salve system is described by

$$\begin{aligned}
\dot{y}_1 &= b_1\,(y_2 - y_1) + y_2 y_3 + y_4 + u_1, \\
\dot{y}_2 &= b_2 y_1 + y_4 - b_3 y_1 y_3 + u_2, \\
\dot{y}_3 &= -b_4 y_3 + b_5 y_1 y_2 + u_3, \\
\dot{y}_4 &= -y_1 - y_2 + u_4.
\end{aligned} \tag{30}$$

When the controllers $u_1 = u_2 = u_3 = u_4 = 0$, $(b_1, b_2, b_3, b_4, b_5) = (18, 40, 5, -3, 4)$ and the initial conditions are given as $(y_1(0), y_2(0), y_3(0), y_4(0)) = (0.5, 0.8, 0.2, 1.3)$, system (30) exhibits hyperchaotic attractors as shown in Figure 2 [21].

Based on the notations used in Section 3, the linear part B and the nonlinear part g of the slave system (30) are given as follows

$$B = \begin{pmatrix} -18 & 18 & 0 & 1 \\ 40 & 0 & 0 & 1 \\ 0 & 0 & -3 & 0 \\ -1 & -1 & 0 & 0 \end{pmatrix} \text{ and } g = \begin{pmatrix} y_2 y_3 \\ -5 y_1 y_3 \\ 4 y_1 y_2 \\ 0 \end{pmatrix}.$$
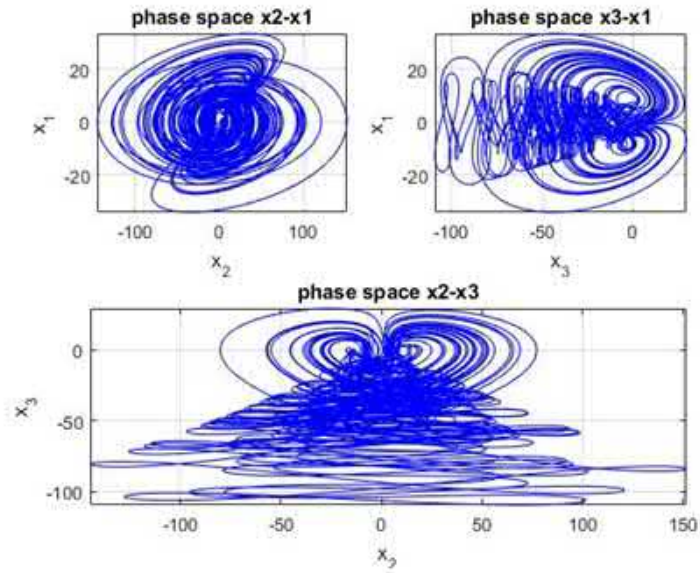
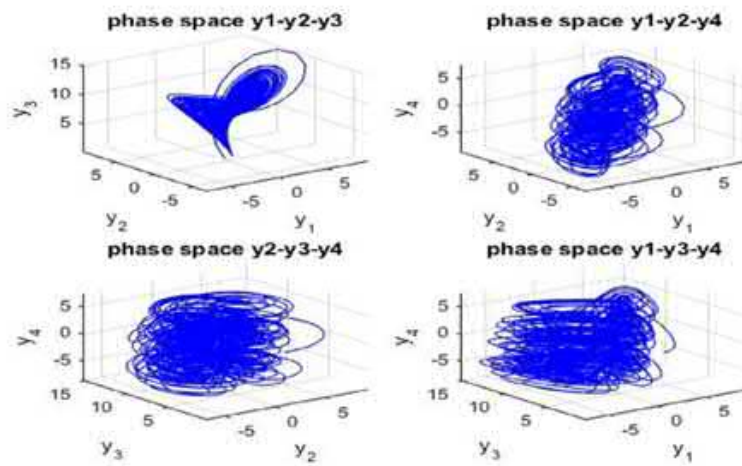**Figure 1**: Phase portraits of the master system (25) in 2D.

Figure 2: Phase portraits of the slave system without control (26) in 3D.

According to the approach developed in Section 3, the synchronization errors between the master system (29) and the slave system (30) are defined as:

$$
\begin{aligned}
e_1 &= x_1 - \alpha_1(t) y_1 - \alpha_2(t) y_2 - \alpha_3(t) y_3 - \alpha_4(t) y_4, \\
e_2 &= y_2 - \beta_1(t) x_1 - \beta_2(t) x_2 - \beta_3(t) x_3, \\
e_3 &= x_3 - \gamma_1(t) y_1 - \gamma_2(t) y_2 - \gamma_3(t) y_3 - \gamma_4(t) y_4, \\
e_4 &= y_4 - \theta_1(t) x_1 - \theta_2(t) x_2 - \theta_3(t) x_3,
\end{aligned}
\tag{31}
$$

where $\alpha_1(t) = \sin t$, $\alpha_2(t) = 1$, $\alpha_3(t) = \frac{1}{t+1}$, $\alpha_4(t) = 2$, $\beta_1(t) = 3$, $\beta_2(t) = \cos t$, $\beta_3(t) = 4$, $\gamma_1(t) = e - t$, $\gamma_2(t) = 2$, $\gamma_3(t) = 0$, $\gamma_4(t) = \frac{1}{t^2+1}$, $\theta_1(t) = \frac{t}{t+1}$, $\theta_2(t) = 0$, $\theta_3(t) = \sin 3t$. So,

$$
\alpha_3(t) \gamma_1(t) - \alpha_1(t) \gamma_3(t) = \frac{1}{e^t(t+1)} \neq 0.
\tag{32}
$$

The coexistence of IFSHFPS and FSHFPS, in this example, is achieved when the control matrix C is selected as

$$
C = \begin{pmatrix} 0 & 18 & 0 & 1 \\ 40 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 1 \end{pmatrix},
\tag{33}
$$

and the controllers u_i, $1 \leq i \leq 4$, are constructed according to (8) as follows:

$$
\begin{aligned}
u_1 &= -2e^t(-e_2 - R_2) + e^t(-3e_3 - R_3) - \frac{e^t}{t^2+1}(-e_4 - R_4), &(34)\\
u_2 &= -e_2 + 5y_1y_3 - 40y_1 - y_4 - R_2, \\
u_3 &= -(t+1)(-18e_1 - R_1) + e^t(t+1) &(35)\\
&\quad \left[-(2 + 2e_2 + 2R_2 + 3e_3 + R_3)\sin t + \left(\frac{\sin t}{t^2+1} - e^{-t} \right)(-e_4 - R_4) \right], \\
u_4 &= -e_4 + y_1 + y_2 - R_4,
\end{aligned}
$$

where

$$
R_1 = 2.9(x_2 - x_1) - y_1 \cos t + \frac{1}{(t+1)^2} y_3 - \sin t \left(18(y_2 - y_1) + y_2 y_3 \right)
\tag{36}
$$

$$
+ \frac{1}{t+1}(4y_1y_2 - 3y_3) - y_1 - y_2,
\tag{37}
$$

$$
R_2 = -5y_1y_3 + 40y_1 + y_4 + x_2 \sin t - 8.7(x_2 - x_1) - x_1 x_3 \cos t,
$$

$$
R_3 = 50 - 0.7x_1^2 - 0.6x_3 + e^{-t} y_1 + \frac{2t}{(t^2+1)^2} y_4 - e^{-t}\left(18(y_2 - y_1) + y_2 y_3 \right)
\tag{38}
$$

$$
+ 10y_1y_3 + 80y_1 - 2y_4 + \frac{1}{t^2+1}(y_1 + y_2),
$$

$$
R_4 = -y_1 - y_2 - \frac{t+1-t^2}{(t+1)^2} x_1 - 3x_3 \cos 3t - \frac{2.9t}{t+1}(x_2 - x_1)
\tag{39}
$$

$$
- \left(50 - 0.7x_1^2 - 0.6x_3 \right) \sin 3t.
$$
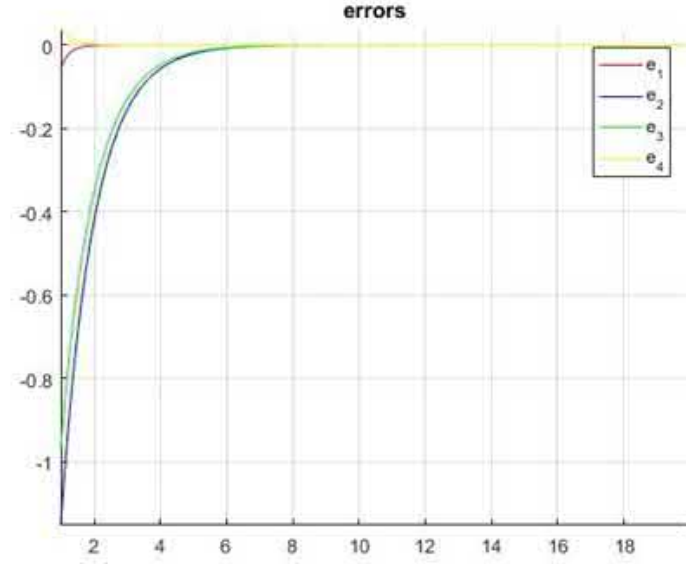
Figure 3: Time evolution of the errors e1, e2, e3 and e4.

We can show that all eigenvalues of $B - C$ have negative real parts. It can be seen that all conditions of Theorem 1 are satisfied. Consequently, the error functions between systems (29) and (30) are described by

$$
\begin{aligned}
\dot{e}_1 &= -18e_1, &\text{(40)}\\
\dot{e}_2 &= -e_2,\\
\dot{e}_3 &= -3e_3,\\
\dot{e}_4 &= -e_4.
\end{aligned}
$$

Numerical results plotted in Figure 3 are obtained, indicating that the coexistence of IFSHFPS and FSHFPS is effectively achieved in 4D.

5.2   Example 2

Herein, the master system is selected as a 3D chaotic system proposed in [22] by the following ODE system

$$
\begin{aligned}
\dot{x}_1 &= x_2, &\text{(41)}\\
\dot{x}_2 &= x_3,\\
\dot{x}_3 &= -c_1 x_1 (1 - x_1) - x_2 + c_2 x_2^2.
\end{aligned}
$$

System (41), when $(c_1, c_2) = (0.2, 0.01)$ and $(x_1(0), x_2(0), x_3(0)) = (0.0.1, -0.0.1, 0.0.1)$, possesses chaotic attractors plotted in Figures 4.

Using the notations presented in Section 4, the linear part A and the nonlinear part

Figure 4: Phase portraits of the master system (33) in 2D.

f of the master system (41) are given as follows

$$A = (a_{ij})_{3\times3} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -0.2 & -1 & 0 \end{pmatrix} \text{ and } f = \begin{pmatrix} 0 \\ 2 \\ 0.2x_1^2 + 0.01x_1^2 \end{pmatrix}.$$
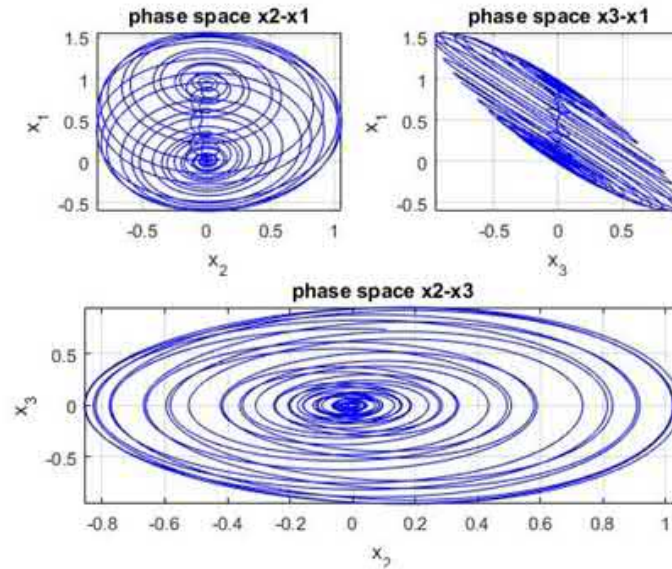
As the slave master system, we consider a novel 4D hyperchaotic system introduced in [23] by the following ODE system

$$\begin{aligned}
\dot{y}_1 &= d_1(y_2 - y_1) + y_2y_3 - y_4 + u_1, &(42)\\
\dot{y}_2 &= d_2y_2 - y_1y_3 + y_4 + u_2, \\
\dot{y}_3 &= y_1y_2 - d_3y_3 + u_3, \\
\dot{y}_4 &= -d_4(y_1 + y_2) + u_4.
\end{aligned}$$

System (42), when $u_1 = u_2 = u_3 = u_4 = 0$, $(d_1, d_2, d_3, d_4) = (40, 20.5, 5, 2.5)$ and $(y_1(0), y_2(0), y_3(0), y_4(0)) = (0.5, 0.8, 0.6, 0.2)$, displays hyperchaotic attractors shown in Figure 5.

In this example, according to the control scheme presented in Section 4, the synchronization errors are given as

$$\begin{aligned}
e_1 &= y_1 - \lambda_1(t)x_1 - \lambda_1(t)x_1 - \lambda_1(t)x_1, &(43)\\
e_2 &= \mu_1(t)y_1 + \mu_2(t)y_2 + \mu_3(t)y_3 + \mu_4(t)y_4 - x_2, \\
e_3 &= y_3 - \sigma_1(t)x_1 - \sigma_2(t)x_2 - \sigma_3(t)x_3,
\end{aligned}$$

where $\lambda_1(t) = e^{-t}$, $\lambda_2(t) = \sin 2t$, $\lambda_3(t) = 0$, $\mu_1(t) = 0$, $\mu_2(t) = \frac{1}{\sqrt{t+1}}$, $\mu_3(t) = \frac{1}{1+\cos^2 t}$, $\mu_4(t) = 4$, $\sigma_1(t) = \frac{1}{\lin(t+1)}$, $\sigma_2(t) = \frac{1}{1+\sin^2 t}$ and $\sigma_3(t) = 0$.
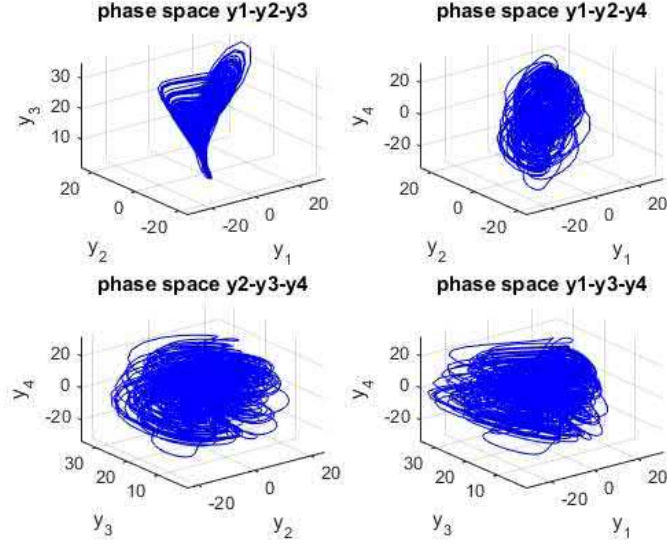
Figure 5: Phase portraits of the slave system (34) without control in 3D.

We selecte the control matrix L as

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \tag{44}$$

and by using (19), the controllers u_1, u_2, u_3 and u_4 are designed as follows

$$u_1 = -e_1 - R_1, \tag{45}$$

$$u_2 = -\left(\sqrt{t}+1\right)(-2e_2 - R_2) - \frac{\sqrt{t}+1}{1+\cos^2 t}(-3e_3 - R_3),$$

$$u_3 = -3e_3 - R_3,$$

$$u_4 = 0,$$

where

$$R_1 = 40(y_2 - y_1) + y_2 y_3 - y_4 + e_1 + e^{-t}x_1 - 2x_2\cos 2t - x_2 e^{-t} - x_3\sin 2t, \tag{46}$$

$$R_2 = x_3 + 2e_2 - \frac{y_2}{2\sqrt{t}\left(\sqrt{t}+1\right)^2} - y_3\frac{2\sin t\cos t}{(1+\cos^2 t)} - \frac{1}{\sqrt{t}+1} \tag{47}$$

$$(20.5y_2 - y_1 y_3 + y_4) - \frac{1}{1+\cos^2 t}(y_1 y_2 - 5.5y_3) + 10(y_1 + y_2),$$

$$R_3 = y_1 y_2 - 5.5y_3 + 3e_3 + \frac{1}{(t+1)\ln^2(t+1)}x_1 + \frac{2\sin t\cos t}{(1+\sin^2 t)}x_2 \tag{48}$$

$$-\frac{x_2}{\ln(t+1)} - \frac{x_3}{1+\sin^2 t}.$$

It is easy to see that $(A-L)^T + (A-L)$ is a negative definite matrix. It can be readily shown that all conditions of Theorem 2 are satisfied. Consequently, the error functions
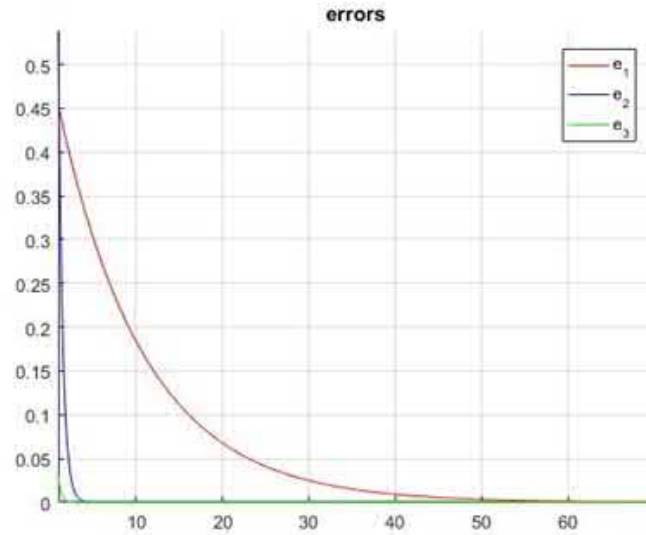
Figure 6: Time evolution of the errors e1, e2 and e3.

between systems (41) and (42) are described by

$$
\begin{aligned}
\dot{e}_1 &= -0.1e_1, \\
\dot{e}_2 &= -2e_2, \\
\dot{e}_3 &= -3e_2.
\end{aligned}
\tag{49}
$$

According to numerical results obtained in Figure 6, it can be concluded that the coexistence of FSHFPS and IFSHFPS synchronization is effectively achieved in 3D.

6 Conclusion

When analyzing the synchronization of chaotic systems, an interesting phenomenon that may occur is the co-existence of some synchronization types. Based on these considerations, this paper has presented new results related to the co-existence of FSHFPS and IFSHFPS between non-identical and different dimensions chaotic systems characterized. Specifically, the manuscript has proposed new schemes, which assures the co-existence of FSHFPS and IFSHFPS between a three-dimensional master system and a four-dimensional slave system. Note that the approach developed herein enables to prove the co-existence of FSHFPS and IFSHFPS in several cases. Specifically, the approach can be applied to: i) wide classes of chaotic (hyperchaotic) master-slave systems; ii) non-identical systems with different dimensions; iii) schemes wherein the scaling factor of the linear combination can be any arbitrary differentiable function. Numerical examples, describing the co-existence of FSHFPS and IFSHFPS between chaotic and hyperchaotic systems, have clearly highlighted the effectiveness of the approach proposed herein.

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On Multi-Switching Synchronization of Non-Identical Chaotic Systems via Active Backstepping Technique

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Abstract: An active backstepping scheme is proposed to attain three different types of synchronization between the chaotic Cai system and the Chen system. Complete synchronization, anti-synchronization and hybrid synchronization are accomplished by using the active backstepping method between different switches of the Cai and Chen systems, where the Cai system is considered as a master system and the Chen system is considered as a slave system. The goal is to design appropriate controllers by using the Lyapunov stability criteria and active backstepping method so that asymptotically stable synchronized state for different switches of the master and slave systems can be obtained. The results obtained by theoretical and graphical analysis are in agreement.

Keywords: *active backstepping method; multi-switching synchronization; chaotic systems; Lyapunov stability theory.*

Mathematics Subject Classification (2010): 34D06, 34H10, 93C10.

1 Introduction

In the area of applied sciences "chaos" is an important field as one of its beautiful features is its applications in several areas such as ecology, secure communication, medicine, biology etc. So many integer order chaotic and hyperchaotic systems have been obtained after the invention of the classical "Lorenz system" in 1963, and so many chaotic and hyperchaotic systems have also been developed in the field of fractional calculus. In the field of chaos, synchronization has been a fascinating branch for the last three decades and researchers have shown their interest to this branch.

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Since 1990, when the important phenomenon of synchronization was discovered by Pecora and Carroll [10], the field of synchronization has been growing day by day. Numerous new researches have been done theoretically and experimentally in the field of synchronization. Several researches have been done to extend the phenomenon of synchronization from complete synchronization [10] to a new range of synchronizations [2,9,17] and to develop the new techniques [6,15,16] to achieve synchronization. The active backstepping technique has been applied widely to achieve synchronization in different cases. In the last few years, outstanding work has been done on synchronization via active backstepping such as complete synchronization between identical systems [1], combination synchronization [11], reduced order synchronization [8], multi-switching synchronization for three chaotic systems [13] etc. The active backstepping method is found very effective for the cases given above. Some of these works have been done on multi-switching synchronization. Since 2008, when a new type of synchronization was achieved for two identical chaotic systems by Ucar [12], multi-switching synchronization has been a hot topic among researchers. Later, multi-switching synchronization between the Lorenz system and the Chen system with fully unknown parameters [14] has also been achieved. Inspite of all the work that has been done, there is a large scope of work in the field of multi-switching synchronization.

In this paper, for different switches of the chaotic Cai system and the Chen system three types of synchronizations are achieved by the active backstepping method. It is clear from numerical simulations that the active backstepping method is very fast, by which synchronization can be achieved very quickly. The proposed scheme has significant applications in the field of secure communications as the synchronization attained by any arbitrary pair increases the grade of security. Secure communication [5] is a field where synchronization is being used very widely. It was found by some researchers that because of arbitrary multiplying factor projective synchronization is an important tool to make communication more reliable [7]. Multi-switching synchronization is defined in such a manner that any pair of state variables may achieve synchronization, which increases the level of security. The advantage of the presented scheme is that by choosing different values of scaling factors different synchronizations can be achieved by a single approach.

This manuscript has been arranged in the following manner. Problem formulation is given in Section 2. In Section 3 dynamics of the Cai system and the Chen system is given. Section 4 contains the scheme for multi-switching synchronization achieved by the active backstepping method and Section 5 contains simulation results for three types of synchronization between the Cai system and the Chen system. In Section 6, main features of this work are highlighted.

2 Problem Formulation

Suppose an n-dimensional system is considered as the master system

$$\dot{v}_1 = h_{11}(v_1, v_2, \ldots, v_n), \quad \dot{v}_2 = h_{21}(v_1, v_2, \ldots, v_n), \ldots, \dot{v}_n = h_{n1}(v_1, v_2, \ldots, v_n), \quad (1)$$

and the n-dimensional slave system is

$$\dot{w}_1 = h_{12}(w_1, w_2, \ldots, w_n) + u_1, \quad \dot{w}_2 = h_{22}(w_1, w_2, \\ \ldots, w_n) + u_2, \ldots, \dot{w}_n = h_{n2}(w_1, w_2, \ldots, w_n) + u_n, \quad (2)$$

where $u_1, u_2, \ldots, u_n \in R^n \rightarrow R$ are the controllers and $h_{i1}, h_{i2} \in R^n \rightarrow R$ for $i = 1, 2, \ldots, n$ are continuous functions. Suppose the errors are defined as

$$\dot{e}_1 = p_1 w_1 + q_1 v_1, \quad \dot{e}_2 = p_2 w_2 + q_2 v_2, \ldots, \dot{e}_n = p_n w_n + q_n v_n, \tag{3}$$

where, $p_i, q_i, i = 1, 2, \ldots, n$ are arbitrary scaling factors. By using equations (1) and (2) the error dynamical system can be expressed as

$$\dot{e}_1 = g_1 + f_1 + p_1 u_1, \quad \dot{e}_2 = g_2 + f_2 + p_2 u_2, \ldots, \dot{e}_n = g_n + f_n + p_n u_n, \tag{4}$$

where $e = (e_1, e_2, \ldots. e_n)'$ is the error vector, $g_1, g_2, \ldots, g_n$ are the functions which contain only error components and $f_1, f_2, \ldots, f_n$ are the nonlinear functions which contain the terms of master and slave systems. First, put $l_1 = e_1$ and consider the l_1 subsystem so that $\dot{l}_1 = G_1(l_1, f_1, p_1 u_1)$, where a virtual controller $e_2 = \vartheta(l_1)$ is assumed. The aim is to design the virtual controller $\vartheta(l_1)$ and the controller $p_1 u_1$ by using the Lyapunov stability criteria so that the l_1 subsystem will be stabilized. The same procedure will be repeated in the next step to stabilize the (l_1, l_2) subsystem, where $l_2 = e_2 - \vartheta(l_1)$ and the virtual controller $e_3 = \vartheta(l_1, l_2)$. Thus, eventually an asymptotically stable $(l_1, l_2, \ldots, l_n)$ system will be achieved so that the master and slave systems will attain asymptotically stable synchronization state.

3 The Cai System and the Chen System

The Cai system [3] is considered as the master system which is given below

$$\begin{aligned}
\dot{v}_1 &= \zeta_1(v_2 - v_1), \\
\dot{v}_2 &= \eta_1 v_1 + \theta_1 v_2 - v_1 v_3, \\
\dot{v}_3 &= v_1{}^2 - \delta_1 v_3,
\end{aligned} \tag{5}$$

which shows chaotic behavior for the parameter values $\zeta_1 = 20, \eta_1 = 14, \theta_1 = 10.6, \delta_1 = 2.8$ and the well known Chen system [4] is considered as the slave system which is given below

$$\begin{aligned}
\dot{w}_1 &= \zeta_2(w_2 - w_1), \\
\dot{w}_2 &= (\theta_2 - \zeta_2)w_1 + \theta_2 w_2 - w_1 w_3, \\
\dot{w}_3 &= w_1 w_2 - \eta_2 w_3,
\end{aligned} \tag{6}$$

which exhibits chaotic behavior for the parameter values $\zeta_2 = 35, \eta_2 = 3, \theta_2 = 28$.

4 Multi-Switching Synchronization Methodology

The slave system with controller is

$$\begin{aligned}
\dot{w}_1 &= \zeta_2(w_2 - w_1) + u_{1j}, \\
\dot{w}_2 &= (\theta_2 - \zeta_2)w_1 + \theta_1 w_2 - w_1 w_3 + u_{2j}, \\
\dot{w}_3 &= w_1 w_2 - \eta_2 w_3 + u_{3j},
\end{aligned} \tag{7}$$

where u_{1j}, u_{2j}, u_{3j}, represent different controllers and $j = \overline{1,6}$ represent different switching states.

First, we will define a general synchronization methodology using the active back-stepping technique. In order to explain the method, for $j = 1$, the errors are defined as follows:

$$
\begin{aligned}
e_{11} &= p_1 w_1 + q_1 v_1, \\
e_{21} &= p_2 w_2 + q_2 v_2, \\
e_{31} &= p_3 w_3 + q_3 v_3,
\end{aligned}
\tag{8}
$$

where $p_i, q_i, i = 1, 2, 3$ are arbitrary scaling factors. If $p_1 = p_2 = p_3 = 1$ and $q_1 = q_2 = q_3 = 1$, then anti-synchronization will be achieved for the pairs of state variables $(w_1, v_1), (w_2, v_2), (w_3, v_3)$. If $p_1 = p_2 = p_3 = 1$ and $q_1 = q_2 = q_3 = -1$, then complete synchronization will be achieved and if $p_1 = p_2 = p_3 = 1$ and $q_1 = 1, q_2 = -1, q_3 = 1$, then hybrid synchronization will be achieved.

Hybrid synchronization has been defined as the synchronization for which some state variables attain completely synchronized state and some state variables attain anti-synchronized state. But in this paper, since we have chosen the master-slave combination in multi-switching manner, we assume in the case of hybrid synchronization that any state variable which is taken with w_1 and w_3 will be completely synchronized with these state variables and the state variable which is taken with w_2 will be anti-synchronized. From (8) the error dynamics can be written as

$$
\begin{cases}
\dot{e}_{11} = p_1 \dot{w}_1 + q_1 \dot{v}_1, \\
\dot{e}_{21} = p_2 \dot{w}_2 + q_2 \dot{v}_2, \\
\dot{e}_{31} = p_3 \dot{w}_3 + q_3 \dot{v}_3.
\end{cases}
\tag{9}
$$

By using (5) and (7) in (9), we get

$$
\begin{aligned}
\dot{e}_{11} &= p_1 \left\{ \zeta_2(w_2 - w_1) + u_{11} \right\} + q_1 \left\{ \zeta_1(v_2 - v_1) \right\}, \\
\dot{e}_{21} &= p_2 \left\{ (\theta_2 - \zeta_2)w_1 + \theta_2 w_2 - w_1 w_3 + u_{21} \right\} + q_2 \left(\eta_1 v_1 + \theta_1 v_2 - v_1 v_3 \right), \\
\dot{e}_{31} &= p_3 \left(w_1 w_2 - \eta_2 w_3 + u_{31} \right) + q_3 \left({v_1}^2 - \delta_1 v_3 \right).
\end{aligned}
\tag{10}
$$

Hence the error dynamical system can be written as

$$
\begin{aligned}
\dot{e}_{11} &= \zeta_2(p_1 w_2 - p_1 w_1) + \zeta_1(q_1 v_2 - q_1 v_1) + p_1 u_{11} \\
&= \frac{p_1 \zeta_2}{p_2}(e_{21} - q_2 v_2) - \zeta_2(e_{11} - q_1 v_1) + \zeta_1(q_1 v_2 - q_1 v_1) + p_1 u_{11}, \\
&= \frac{p_1 \zeta_2}{p_2} e_{21} - \zeta_2 e_{11} + f_1 + p_1 u_{11}.
\end{aligned}
\tag{11}
$$

Similarly

$$
\begin{aligned}
\dot{e}_{21} &= p_2 \left\{ (\theta_2 - \zeta_2)w_1 + \theta_2 w_2 - w_1 w_3 \right\} + q_2(\eta_1 v_1 + \theta_1 v_2 - v_1 v_3) + p_2 u_{21} \\
&= \frac{p_2(\theta_2 - \zeta_2)}{p_1}(e_{11} - q_1 v_1) + \theta_2(e_{21} - q_2 v_2) - \frac{p_2}{p_1 p_3}(e_{11} - q_1 v_1)(e_{31} - q_3 v_3) \\
&\quad + q_2(\eta_1 v_1 - \theta_1 v_2 - v_1 v_3) + p_2 u_{21} = \frac{p_2(\theta_2 - \zeta_2)}{p_1} e_{11} - \frac{p_2}{p_1 p_3} e_{11} e_{31} + \frac{p_2 q_3}{p_1 p_3} e_{11} v_3 \\
&\quad + \frac{p_2 q_1}{p_1 p_3} e_{31} v_1 + \theta_2 e_{21} + f_2 + p_2 u_{21}
\end{aligned}
\tag{12}
$$

and

$$\begin{aligned}
\dot{e}_{31} &= p_3(w_1 w_2 - \eta_2 w_3) + q_3(v_1{}^2 - \delta_1 v_3) + p_3 u_{31} \\
&= \frac{p_3}{p_1 p_2}(e_{11} - q_1 v_1)(e_{21} - q_2 v_2) - \eta_2(e_{31} - q_3 v_3) + q_3(v_1{}^2 - \delta_2 v_3) + p_3 u_{33} \\
&= \frac{p_3}{p_1 p_2}e_{11}e_{21} - \frac{p_3 q_2}{p_1 p_2}e_{11}v_2 - \frac{p_3 q_1 v_1}{p_1 p_2}e_{21} - \eta_2 e_{31} + f_3 + p_3 u_{31},
\end{aligned} \tag{13}$$

where

$$\begin{aligned}
f_1 &= \frac{-p_1 \zeta_2}{p_2}q_2 v_2 + \zeta_2 q_1 v_1 + \zeta_1 q_1 v_2 - \zeta_1 q_1 v_1, \\
f_2 &= -\frac{q_1 p_2}{p_1}(\theta_2 - \zeta_2)v_1 - \frac{p_2 q_3 q_1}{p_1 p_3}v_1 v_3 - \theta_2 q_2 v_2 + q_2(\eta_1 v_1 - \theta_1 v_2 - v_1 v_3), \\
f_3 &= \frac{p_3}{p_1 p_2}q_1 q_2 v_1 v_2 + \eta_2 q_3 v_3 + q_3(v_1{}^2 - \delta_2 v_3).
\end{aligned} \tag{14}$$

Let $l_1 = e_{11}$. Then its derivative will be

$$\dot{l}_1 = \dot{e}_{11} = \frac{p_1 \zeta_2}{p_2}e_{21} - \zeta_2 l_1 + f_1 + p_1 u_{11}, \tag{15}$$

where $e_{21} = \vartheta_1(l_1)$ is considered as a virtual controller. Our aim is to design $\vartheta_1(l_1)$ so that the l_1 subsystem (15) could be stabilized. Consider the following Lyapunov function

$$K_1 = 0.5 l_1{}^2. \tag{16}$$

Then the derivative of K_1 will be

$$\dot{K}_1 = l_1 \dot{l}_1 = l_1\left(\frac{p_1 \zeta_2}{p_2}\vartheta_1(l_1) - \zeta_2 e_{11} + f_1 + p_1 u_{11}\right). \tag{17}$$

If $\vartheta_1(l_1) = 0$ and $u_{11} = -\frac{1}{p_1}(f_1)$, then $\dot{K}_1 = -\zeta_2 e_{11}^2$ which is negative definite. Hence by the Lyapunov stability criteria the l_1 subsystem is asymptotically stable. Suppose the error between e_{21} and $\vartheta_1(l_1)$ is denoted by $l_2 = e_{21} - \vartheta_1(l_1)$. Then we have the (l_1, l_2) subsystem given below

$$\begin{aligned}
\dot{l}_1 &= \frac{p_1 \zeta_2}{p_2}l_2 - \zeta_2 l_1, \\
\dot{l}_2 &= \left(\frac{p_2(\theta_2 - \zeta_2)}{p_1} - \frac{p_2}{p_1 p_3}e_{31} + \frac{p_2 q_3}{p_1 p_3}v_3\right)l_1 + \theta_2 l_2 + \frac{p_2 q_1}{p_1 p_3}e_{31}v_1 + f_2 + p_2 u_{21}.
\end{aligned} \tag{18}$$

In order to make the (l_1, l_2) subsystem stable, $e_{31} = \vartheta_2(l_1, l_2)$ is taken as a virtual controller. Now, we take the Lyapunov function and its derivatives as

$$\begin{aligned}
K_2 &= K_1 + (0.5)l_2{}^2, \\
\dot{K}_2 &= -\zeta_2 l_1{}^2 - \theta_2 l_2{}^2 + l_2\left[\left(\frac{p_2(\theta_2 - \zeta_2)}{p_1} - \frac{p_2}{p_1 p_3}\vartheta_2(l_1, l_2) + \frac{p_2 q_3}{p_1 p_3}v_3\right)l_1 + 2\theta_2 l_2 \right. \\
&\quad \left. + \frac{p_2 q_1}{p_1 p_3}v_1 \vartheta_2(l_1, l_2) + \frac{p_1 \zeta_2}{p_2}l_1 + f_2 + p_2 u_{21}\right].
\end{aligned} \tag{19}$$

Hence by choosing the controller u_{21} in the following way

$$u_{21} = -\frac{1}{p_2}\left(\frac{p_2}{p_1}(\theta_2 - \zeta_2)l_1 + \frac{p_2q_3}{p_1p_3}l_1v_3 + 2\theta_2l_2 + \frac{p_1\zeta_2}{p_2}l_1 + f_2\right) \tag{20}$$

and the virtual controller $\vartheta_2(l_1, l_2) = 0$, we get $\dot{K}_2 = -\zeta_2{l_1}^2 - \theta_2{l_2}^2$ which is negative definite. Hence the (l_1, l_2) subsystem is asymptotically stable. Now, suppose the error between e_{31} and $\vartheta_2(l_1, l_2)$ is $l_3 = e_{31} - \vartheta_2(l_1, l_2)$. Then

$$\dot{l_3} = \frac{p_3}{p_1p_2}l_1l_2 - \frac{p_3q_2}{p_1p_2}l_1v_2 - \frac{p_3}{p_1p_2}l_2q_1v_1 - \eta_2l_3 + f_3 + p_3u_{31}. \tag{21}$$

Now to stabilize the (l_1, l_2, l_3) system, the controller u_{31} is defined as

$$u_{31} = -\frac{1}{p_3}\left[\left\{\left(\frac{p_3}{p_1p_2} - \frac{p_2}{p_1p_3}\right)l_2 - \frac{p_3q_2}{p_1p_2}v_2\right\}l_1 - \left(\frac{p_3}{p_1p_2} - \frac{p_2}{p_1p_3}\right)q_1l_2v_1 + f_3\right] \tag{22}$$

and the Lyapunov function K_3 as

$$K_3 = K_2 + 0.5{l_3}^2. \tag{23}$$

Its derivative will be

$$\dot{K}_3 = -\zeta_2{l_1}^2 - \theta_2{l_2}^2 - \eta_2{l_3}^2 \tag{24}$$

which is negative definite. Hence according to the Lyapunov stability theory $(0, 0, 0)$ equilibrium point of (l_1, l_2, l_3) system is now asymptotically stable. The (l_1, l_2, l_3) system is given by

$$\begin{cases} \dot{l_1} = \dfrac{p_1\zeta_2}{p_2}l_2 - \zeta_2l_1, \\[2mm] \dot{l_2} = -\dfrac{p_2}{p_1p_3}l_1l_3 + \dfrac{p_2}{p_1p_3}q_1v_1l_3 - \theta_2l_2 - \dfrac{p_1\zeta_2}{p_2}l_1, \\[2mm] \dot{l_3} = \dfrac{p_2}{p_1p_3}l_2l_1 - \eta_2l_3 - \dfrac{p_2q_1}{p_1p_3}l_2v_1. \end{cases} \tag{25}$$

Now, for the second switch the errors are defined as follows

$$e_{12} = p_1w_1 + q_1v_2, e_{22} = p_2w_2 + q_2v_3, e_{32} = p_3w_3 + q_3v_1. \tag{26}$$

Then, the controllers are

$$\begin{cases} u_{12} = -\dfrac{1}{p_1}(f_1), \\[2mm] u_{22} = -\dfrac{1}{p_2}\left(\dfrac{p_2}{p_1}(\theta_2 - \zeta_2)l_1 + \dfrac{p_2q_3}{p_1p_3}l_1v_1 + 2\theta_2l_2 + \dfrac{p_1\zeta_2}{p_2}l_1 + f_2\right), \\[2mm] u_{32} = -\dfrac{1}{p_3}\left[\left(\left(\dfrac{p_3}{p_1p_2} - \dfrac{p_2}{p_1p_3}\right)l_2 - \dfrac{p_3q_2}{p_1p_2}v_3\right)l_1 - \left(\dfrac{p_3}{p_1p_2} - \dfrac{p_2}{p_1p_3}\right)q_1l_2v_2 + f_3\right], \end{cases} \tag{27}$$

where $l_1 = e_{12}, l_2 = e_{22}, l_3 = e_{32}$. For the third switch the errors are

$$e_{13} = p_1w_1 + q_1v_3, e_{22} = p_2w_2 + q_2v_1, e_{32} = p_3w_3 + q_3v_2. \tag{28}$$

The controllers defined by using the above procedure are

$$
\begin{cases}
u_{13} = -\dfrac{1}{p_1}(f_1), \\[2mm]
u_{23} = -\dfrac{1}{p_2}\left(\dfrac{p_2}{p_1}(\theta_2 - \zeta_2)l_1 + \dfrac{p_2 q_3}{p_1 p_3}l_1 v_2 + 2\theta_2 l_2 + \dfrac{p_1 \zeta_2}{p_2}l_1 + f_2\right), \\[2mm]
u_{33} = -\dfrac{1}{p_3}\left[\left(\left(\dfrac{p_3}{p_1 p_2} - \dfrac{p_2}{p_1 p_3}\right)l_2 - \dfrac{p_3 q_2}{p_1 p_2}v_1\right)l_1 - \left(\dfrac{p_3}{p_1 p_2} - \dfrac{p_2}{p_1 p_3}\right)q_1 l_2 v_3 + f_3\right],
\end{cases}
\tag{29}
$$

where $l_1 = e_{13}, l_2 = e_{23}, l_3 = e_{33}$. In case of switch four the errors are defined as

$$
e_{14} = p_1 w_1 + q_1 v_1,\ e_{24} = p_2 w_2 + q_2 v_3,\ e_{34} = p_3 w_3 + q_3 v_2
\tag{30}
$$

and the controllers are

$$
\begin{cases}
u_{14} = -\dfrac{1}{p_1}(f_1), \\[2mm]
u_{24} = -\dfrac{1}{p_2}\left(\dfrac{p_2}{p_1}(\theta_2 - \zeta_2)l_1 + \dfrac{p_2 q_3}{p_1 p_3}l_1 v_2 + 2\theta_2 l_2 + \dfrac{p_1 \zeta_2}{p_2}l_1 + f_2\right), \\[2mm]
u_{34} = -\dfrac{1}{p_3}\left[\left(\left(\dfrac{p_3}{p_1 p_2} - \dfrac{p_2}{p_1 p_3}\right)l_2 - \dfrac{p_3 q_2}{p_1 p_2}v_3\right)l_1 - \left(\dfrac{p_3}{p_1 p_2} - \dfrac{p_2}{p_1 p_3}\right)q_1 l_2 v_1 + f_3\right),
\end{cases}
\tag{31}
$$

where $l_1 = e_{14}, l_2 = e_{24}, l_3 = e_{34}$. For switch five the errors are taken as

$$
e_{15} = p_1 w_1 + q_1 v_3,\ e_{25} = p_2 w_2 + q_2 v_2,\ e_{35} = p_3 w_3 + q_3 v_1.
\tag{32}
$$

For switch five the controllers are

$$
\begin{cases}
u_{15} = -\dfrac{1}{p_1}(f_1), \\[2mm]
u_{25} = -\dfrac{1}{p_2}\left(\dfrac{p_2}{p_1}(\theta_2 - \zeta_2)l_1 + \dfrac{p_2 q_3}{p_1 p_3}l_1 v_1 + 2\theta_2 l_2 + \dfrac{p_1 \zeta_2}{p_2}l_1 + f_2\right), \\[2mm]
u_{35} = -\dfrac{1}{p_3}\left[\left(\left(\dfrac{p_3}{p_1 p_2} - \dfrac{p_2}{p_1 p_3}\right)l_2 - \dfrac{p_3 q_2}{p_1 p_2}v_2\right)l_1 - \left(\dfrac{p_3}{p_1 p_2} - \dfrac{p_2}{p_1 p_3}\right)q_1 l_2 v_3 + f_3\right],
\end{cases}
\tag{33}
$$

where $l_1 = e_{15}, l_2 = e_{25}, l_3 = e_{35}$. For switch six the errors are

$$
e_{16} = p_1 w_1 + q_1 v_2,\ e_{26} = p_2 w_2 + q_2 v_1,\ e_{36} = p_3 w_3 + q_3 v_3
\tag{34}
$$

and the controllers are

$$
\begin{cases}
u_{16} = -\dfrac{1}{p_1}(f_1), \\[2mm]
u_{26} = -\dfrac{1}{p_2}\left(\dfrac{p_2}{p_1}(\theta_2 - \zeta_2)l_1 + \dfrac{p_2 q_3}{p_1 p_3}l_1 v_3 + 2\theta_2 l_2 + \dfrac{p_1 \zeta_2}{p_2}l_1 + f_2\right), \\[2mm]
u_{36} = -\dfrac{1}{p_3}\left[\left(\left(\dfrac{p_3}{p_1 p_2} - \dfrac{p_2}{p_1 p_3}\right)l_2 - \dfrac{p_3 q_2}{p_1 p_2}v_1\right)l_1 - \left(\dfrac{p_3}{p_1 p_2} - \dfrac{p_2}{p_1 p_3}\right)q_1 l_2 v_2 + f_3\right],
\end{cases}
\tag{35}
$$

where $l_1 = e_{16}, l_2 = e_{26}, l_3 = e_{36}$. It is obvious that the values of f_1, f_2, f_3 will be different in all the switches, since the values of f_1, f_2, f_3 will be changed according to
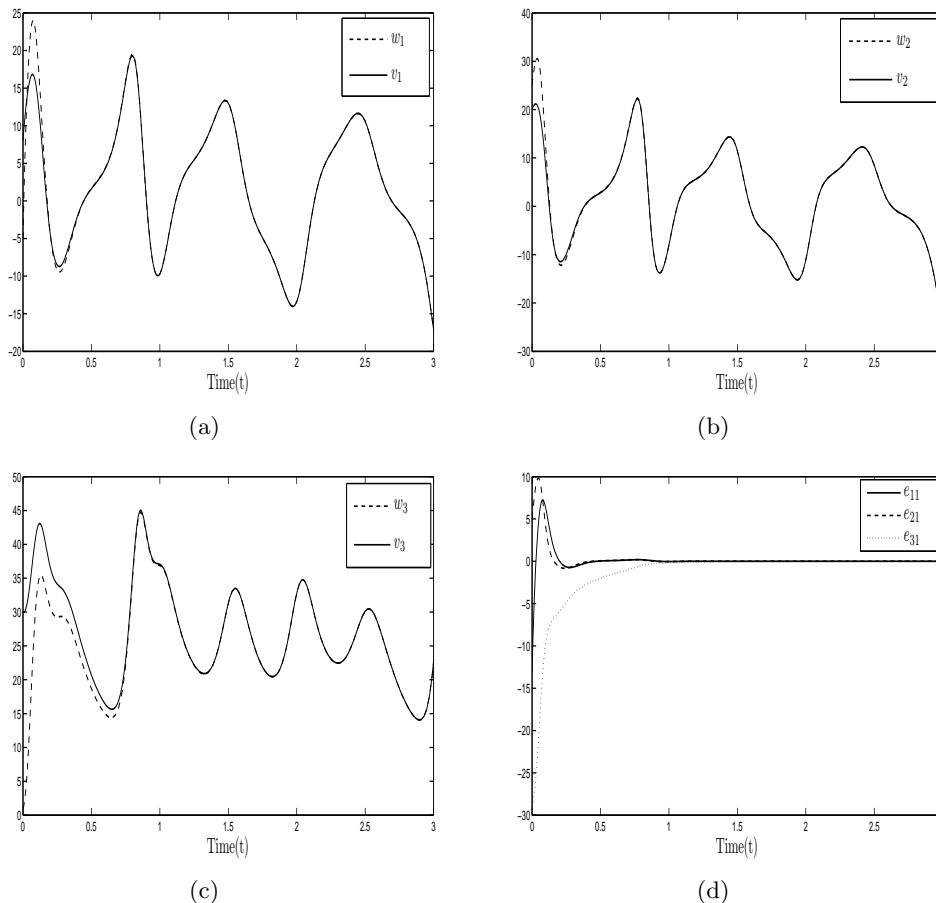
Figure 1: (a) Synchronization between the state variables w_1, v_1 in switch one; (b) Synchronization between the state variables w_2, v_2 in switch one; (c) Synchronization between the state variables w_3, v_3 in switch one; (d) Convergence of e_{11}, e_{21}, e_{31} to zero for switch one.

the error defined. If q_1, q_2, q_3 are chosen as any arbitrary scalars but not equal and all $p_1 = p_2 = p_3 = 1$, then this will become a case of modified projective synchronization. If $q_1 = q_2 = q_3$ are chosen as any arbitrary scalars and all $p_1 = p_2 = p_3 = 1$, then the problem will be reduced to projective synchronization which is a particular case of modified projective synchronization. The method described above is easy to apply for the dynamical systems having dimension greater than three also.

5   Numerical Simulations

5.1   Complete synchronization

Numerical simulations are shown only for three switches as the remaining ones can be achieved in a similar manner. The values of $p_i's$ and $q_i's$ are chosen in such a manner which lead to complete synchronization, anti-synchronization and hybrid synchronization
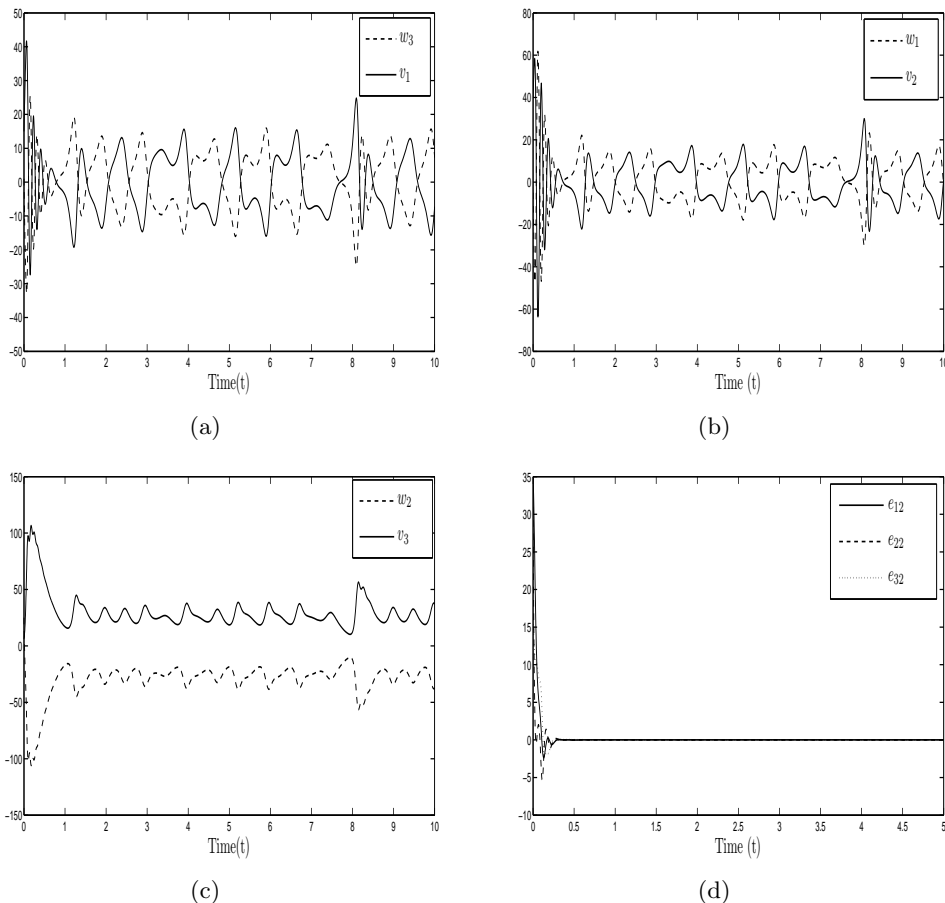
Figure 2: (a) Anti-synchronization between the state variables w_1, v_2 in switch two; (b) Anti-synchronization between the state variables w_2, v_3 for switch two; (c) Anti-synchronization between the state variables w_3, v_1 in switch two; (d) Convergence of the errors e_{12}, e_{22}, e_{32} to zero for switch two.

between different state variables of the drive and response systems.

The case of complete synchronization is considered for the first switch and the values of scaling factors are $p_1 = p_2 = p_3 = 1$ and $q_1 = q_2 = q_3 = -1$.

The initial conditions are kept fixed for the slave system throughout the paper, which are $(-5, 25, 1)$, but in each type of synchronization the initial conditions for the master system are different. In the case of complete synchronization the initial conditions for the master system are $(8, 20, 30)$. Hence for the first switch the initial conditions for the errors are $(-13, 5, -29)$. Complete synchronization between w_1, v_1 and w_2, v_2 is shown in Figure 1a-b. Figure 1c-d show synchronization between w_3, v_3 and the errors e_{11}, e_{21}, e_{31} converging to zero.
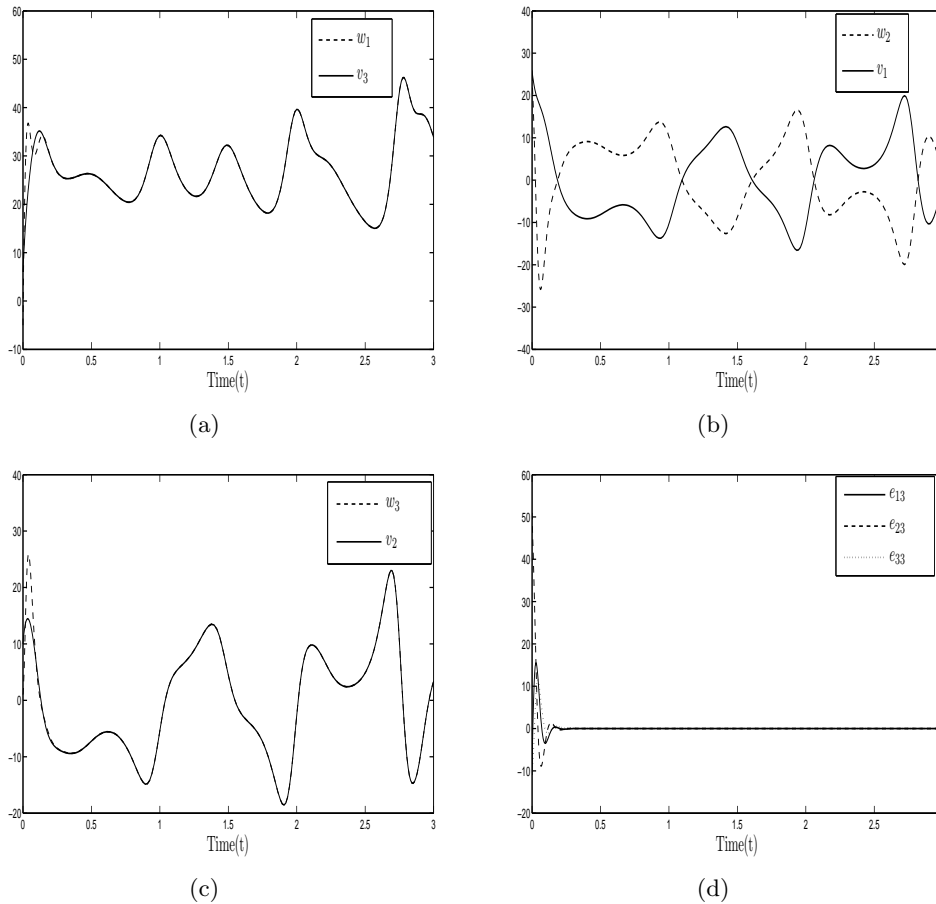
Figure 3: (a) Complete synchronization between the state variables w_1, v_3; (b) Anti-synchronization between w_2, v_1 for switch three; (c) Complete synchronization between the state variables w_3, v_2; (d) Convergence of e_{13}, e_{23}, e_{33} to zero for switch three.

5.2 Anti-synchronization

Anti-synchronization is shown for the second switch. In order to achieve anti-synchronization, the values of $p_1 = p_2 = p_3 = 1$ and $q_1 = q_2 = q_3 = 1$ are chosen. Since the initial conditions for the master and slave systems are $(15, 40, 6)$ and $(-5, 25, 1)$, the initial conditions for the errors are $(35, 31, 16)$. Figure 2 a-b show anti-synchronization between w_1, v_2 and w_2, v_3, and Figure 2 c-d show anti-synchronization between the state variables w_3, v_1 and the errors e_{12}, e_{22}, e_{32} converging to zero.

5.3 Hybrid synchronization

In this subsection the case of hybrid synchronization is considered for the third switch. In order to attain hybrid synchronization, the values of $p_1 = p_2 = p_3 = 1$ and $q_1 = 1, q_2 = -1, q_3 = -1$ are chosen. In the case of hybrid synchronization the initial conditions for the master system are $(26, 10, 6)$. According to the initial conditions for the master and slave

systems $(26, 10, 6)$ and $(-5, 25, 1)$ respectively, for the third switch the initial conditions for the errors are $(-11, 51, -9)$. Figure 3 a-b show complete synchronization between w_1, v_3 and anti-synchronization for w_2, v_1. Figure 3 c-d exhibit the state variables w_3, v_2 in complete synchronized states and the errors e_{13}, e_{23}, e_{33} converging to zero.

The numerical results presented in this paper are obtained by using Matlab software. In numerical simulations, complete synchronization, anti-synchronization and hybrid synchronization are shown and other types of synchronization can be achieved by choosing different scaling factors.

6 Conclusion

In this manuscript, we have investigated multi-switching synchronization between the Cai system and the Chen system by using the active backstepping method. An efficient and easy method is proposed to design suitable controllers and fruitful results are obtained. Both theoretical and graphical analysis lead to the same conclusion. The controllers designed by this approach are very effective as synchronizations are achieved very rapidly by this method. Since the chaos synchronization has its applications in secure communications and multi-switching increases the grade of security as it is very difficult to guess which pair of state variables will attain synchronization, the proposed method has significant applications in the field of secure communication. The approach is also significant in the sense that by simply taking different scaling factors, various types of synchronization can be achieved. This work can be extended to fractional order systems and various higher dimensional systems.

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A Novel Method for Solving Caputo-Time-Fractional Dispersive Long Wave Wu-Zhang System

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Abstract: In this paper we presented a reliable efficient numerical scheme to find analytical supportive solution of Caputo-time-fractional Wu-Zhang system. A modified version of generalized Taylor power series method is used in this work. Graphical justifications of the reliability of the proposed method are provided. Finally, the effects of the fractional order on the solution of Wu-Zhang system is also discussed.

Keywords: *Caputo-time-fractional Wu-Zhang system; approximate solutions; generalized Taylor series.*

Mathematics Subject Classification (2010): 26A33, 35F25, 35C10.

1 Introduction

Wu-Zhang system is known also as (1+1)-dimensional dispersive long wave equations [25]. It is very helpful for coastal and civil engineers to apply the nonlinear water wave model in harbor and coastal design. Abundant soliton solutions are obtained to this model using the extended hyperbolic tangent expansion method. In [20], the Wu-Zhang system is considered to study dispersive long waves. The extended trial equation method is used and solitary wave solutions are obtained. Also, they used the mapping method to extract more solitonic solutions.

   Finding analytical solution to fractional nonlinear differential equations is a difficult task. In the literature, different computational schemes were developed for either finding numerical solutions over a specific range or considering a few terms of an iterative computational series solution as an approximate. Such available methods are the variational

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iteration method [21], the iterative Laplace transform method [17], Adomian's decomposition method [23], homotopy analysis (perturbation) methods [12, 15, 16, 22], generalized iterative-monotone methods [11, 13, 24] and the fractional power series method [1–10, 14, 18, 19].

The motivation of this work is to study for the first time the fractional Wu-Zhang system

$$
\begin{aligned}
D_t^\alpha v(x,t) &= -v(x,t)v_x(x,t) - w_x(x,t), \\
D_t^\alpha w(x,t) &= -(v(x,t)w(x,t))_x - \frac{1}{3}v_{xxx}(x,t),
\end{aligned}
\tag{1}
$$

where $0 < \alpha \le 1$ in Caputo sense and $0 < t < R < 1$. Also, we desire to study the effect of the fractional derivative α on the solution of (1).

The generalized Taylor fractional series will be used as an alternative method to extract a reliable analytical supportive solution of the time-fractional Wu-Zhang system. The accuracy of the method will be provided and graphical analysis is conducted to study the effect of the fractional order α on the behavior of the obtained solution.

2 Analysis of the Proposed Method

In this section, we present in details the construction of the generalized Taylor fractional series. The suggested solutions of the problem are sought to have the form

$$
v(x,t) = \sum_{j=0}^{\infty} c_j(x)\frac{t^{j\alpha}}{\Gamma(j\alpha+1)},
\tag{2}
$$

$$
w(x,t) = \sum_{j=0}^{\infty} d_j(x)\frac{t^{j\alpha}}{\Gamma(j\alpha+1)}.
\tag{3}
$$

The target of this study is obtaining a supportive approximate solution to the proposed model. Thus, we may write the suggested solution as

$$
v(x,t) = \sum_{j=0}^{m} c_j(x)\frac{t^{j\alpha}}{\Gamma(j\alpha+1)} = c_0(x) + \sum_{j=1}^{m} c_j(x)\frac{t^{j\alpha}}{\Gamma(j\alpha+1)},
\tag{4}
$$

$$
w(x,t) = \sum_{j=0}^{m} d_j(x)\frac{t^{j\alpha}}{\Gamma(j\alpha+1)} = d_0(x) + \sum_{j=1}^{m} d_j(x)\frac{t^{j\alpha}}{\Gamma(j\alpha+1)}.
\tag{5}
$$

In Caputo sense, we recall the fact that

$$
D_t^\alpha t^\beta =
\begin{cases}
\frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}t^{\beta-\alpha}, & \beta \ge \alpha, \\[2mm]
0, & \beta < \alpha.
\end{cases}
$$

Therefore, applying the operator D_t^α on equations (4) and (5), will produce the formulas

$$
D_t^\alpha v_m(x,t) = \sum_{j=0}^{m-1} c_{j+1}(x)\frac{t^{j\alpha}}{\Gamma(j\alpha+1)},
\tag{6}
$$

$$
D_t^\alpha w_m(x,t) = \sum_{j=0}^{m-1} d_{j+1}(x)\frac{t^{j\alpha}}{\Gamma(j\alpha+1)}.
\tag{7}
$$

Next, we substitute both (4)-(7) in the fractional equation (1). Therefore, we arrive at the following recurrence relations

$$
\begin{aligned}
0 \;=\; & \sum_{j=0}^{m-1} c_{j+1}(x)\frac{t^{j\alpha}}{\Gamma(j\alpha+1)} + \sum_{j=0}^{m} d_j'(x)\frac{t^{j\alpha}}{\Gamma(j\alpha+1)} \\
& + \left(\sum_{j=0}^{m} c_j(x)\frac{t^{j\alpha}}{\Gamma(j\alpha+1)}\right)\left(\sum_{j=0}^{m} c_j'(x)\frac{t^{j\alpha}}{\Gamma(j\alpha+1)}\right)
\end{aligned}
\tag{8}
$$

and

$$
\begin{aligned}
0 \;=\; & \sum_{j=0}^{m-1} d_{j+1}(x)\frac{t^{j\alpha}}{\Gamma(j\alpha+1)} + \frac{1}{3}\sum_{j=0}^{m} c_j'''(x)\frac{t^{j\alpha}}{\Gamma(j\alpha+1)} \\
& + \left(\sum_{j=0}^{m} c_j(x)\frac{t^{j\alpha}}{\Gamma(j\alpha+1)}\right)\left(\sum_{j=0}^{m} d_j'(x)\frac{t^{j\alpha}}{\Gamma(j\alpha+1)}\right) \\
& + \left(\sum_{j=0}^{m} c_j'(x)\frac{t^{j\alpha}}{\Gamma(j\alpha+1)}\right)\left(\sum_{j=0}^{m} d_j(x)\frac{t^{j\alpha}}{\Gamma(j\alpha+1)}\right).
\end{aligned}
\tag{9}
$$

We follow the same analogue used in obtaining the Taylor series coefficients. In particular, to determine the functions $c_n(x)$, $d_n(x)$, $n = 1, 2, 3, ...$, we have to solve the following two systems simultaneously

$$
\begin{aligned}
D_t^{(m-1)\alpha}\{L_1(x,t,\alpha,m)\} \downarrow_{t=0} = 0, \\
D_t^{(m-1)\alpha}\{L_2(x,t,\alpha,m)\} \downarrow_{t=0} = 0,
\end{aligned}
\tag{10}
$$

where

$$
\begin{aligned}
L_1(x,t,\alpha,m) \;=\; & \sum_{j=0}^{m-1} c_{j+1}(x)\frac{t^{j\alpha}}{\Gamma(j\alpha+1)} + \sum_{j=0}^{m} d_j'(x)\frac{t^{j\alpha}}{\Gamma(j\alpha+1)} \\
& + \left(\sum_{j=0}^{m} c_j(x)\frac{t^{j\alpha}}{\Gamma(j\alpha+1)}\right)\left(\sum_{j=0}^{m} c_j'(x)\frac{t^{j\alpha}}{\Gamma(j\alpha+1)}\right)
\end{aligned}
\tag{11}
$$

and

$$
\begin{aligned}
L_2(x,t,\alpha,m) \;=\; & \sum_{j=0}^{m-1} d_{j+1}(x)\frac{t^{j\alpha}}{\Gamma(j\alpha+1)} + \frac{1}{3}\sum_{j=0}^{m} c_j'''(x)\frac{t^{j\alpha}}{\Gamma(j\alpha+1)} \\
& + \left(\sum_{j=0}^{m} c_j(x)\frac{t^{j\alpha}}{\Gamma(j\alpha+1)}\right)\left(\sum_{j=0}^{m} d_j'(x)\frac{t^{j\alpha}}{\Gamma(j\alpha+1)}\right) \\
& + \left(\sum_{j=0}^{m} c_j'(x)\frac{t^{j\alpha}}{\Gamma(j\alpha+1)}\right)\left(\sum_{j=0}^{m} d_j(x)\frac{t^{j\alpha}}{\Gamma(j\alpha+1)}\right).
\end{aligned}
\tag{12}
$$

Now, we explain the derivations of the first few terms of the sequence $\{c_m(x)\}_1^N$ and $\{d_m(x)\}_1^N$. We start with the index $m = 1$;

$$
\begin{aligned}
L_1(x,t,\alpha,1) &= c_1(x) + d_0'(x) + d_1'(x)\frac{t^\alpha}{\Gamma(\alpha+1)} \\
&+ \left(c_0(x) + f_1(x)\frac{t^\alpha}{\Gamma(\alpha+1)}\right)\left(c_0'(x) + c_1'(x)\frac{t^\alpha}{\Gamma(\alpha+1)}\right), \\
L_2(x,t,\alpha,1) &= d_1(x) + \frac{1}{3}\left(c_0'''(x) + c_1'''(x)\frac{t^\alpha}{\Gamma(\alpha+1)}\right) \\
&+ \left(c_0(x) + c_1(x)\frac{t^\alpha}{\Gamma(\alpha+1)}\right)\left(d_0'(x) + d_1'(x)\frac{t^\alpha}{\Gamma(\alpha+1)}\right) \\
&+ \left(c_0'(x) + c_1'(x)\frac{t^\alpha}{\Gamma(\alpha+1)}\right)\left(d_0(x) + d_1(x)\frac{t^\alpha}{\Gamma(\alpha+1)}\right). \quad (13)
\end{aligned}
$$

Solving $L_1(x,0,\alpha,1) = 0$ and $L_2(x,0,\alpha,1) = 0$, yields

$$
\begin{aligned}
c_1(x) &= -c_0(x)c_0'(x) - d_0'(x), \\
d_1(x) &= -\frac{1}{3}c_0'''(x) - (c_0(x)d_0'(x) + c_0'(x)d_0(x)). \quad (14)
\end{aligned}
$$

To determine $c_2(x)$ and $d_2(x)$, we consider $L_1(x,t,\alpha,2)$ & $L_2(x,t,\alpha,2)$ and we solve $D_t^\alpha\{L_1(x,t,\alpha,2)\}\downarrow_{t=0}= 0$ and $D_t^\alpha\{L_2(x,t,\alpha,2)\}\downarrow_{t=0}= 0$. Therefore

$$
\begin{aligned}
c_2(x) &= -(c_1(x)c_0'(x) + c_1'(x)c_0(x)) - d_1'(x), \\
d_2(x) &= -\frac{1}{3}c_1'''(x) - (c_0(x)d_1'(x) + c_0'(x)d_1(x)) \\
&\quad - (c_1(x)d_0'(x) + c_1'(x)d_0(x)). \quad (15)
\end{aligned}
$$

We should point here that chain rule differentiation is not applicable when using Caputo sense. Thus, in the preceding step "as well as the forthcoming steps" we had to expand all the terms involved in both $L_1(x,t,\alpha,2)$, $L_2(x,t,\alpha,2)$ "in general $L_1(x,t,\alpha,n)$, $L_2(x,t,\alpha,n)$" and use the following fact

$$
D_t^\alpha t^\beta \downarrow_{t=0} = \begin{cases} 0, & \beta < \alpha, \\ \Gamma(\alpha+1), & \beta = \alpha, \\ 0, & \beta > \alpha. \end{cases}
$$

To determine $c_3(x)$ and $d_3(x)$, we consider $L_1(x,t,\alpha,3)$ and $L_2(x,t,\alpha,3)$ and we solve $D_t^{2\alpha}\{L_1(x,t,\alpha,3)\}\downarrow_{t=0}= 0$ and $D_t^{2\alpha}\{L_2(x,t,\alpha,3)\}\downarrow_{t=0}= 0$. Therefore

$$
\begin{aligned}
c_3(x) &= -(c_2(x)c_0'(x) + c_2'(x)c_0(x)) - \frac{\Gamma(1+2\alpha)}{\Gamma^2(1+\alpha)}c_1(x)c_1'(x) - d_2'(x), \\
d_3(x) &= -\frac{1}{3}c_2'''(x) - (c_0(x)d_2'(x) + c_0'(x)d_2(x)) - (c_2(x)d_0'(x) + c_2'(x)d_0(x)) \\
&\quad - \frac{\Gamma(1+2\alpha)}{\Gamma^2(1+\alpha)}(c_1(x)d_1'(x) + c_1'(x)d_1(x)). \quad (16)
\end{aligned}
$$

Finally, we proceed as above to obtain the other coefficient functions $c_k(x)$ and $d_k(x)$ by solving $D_t^{(k-1)\alpha}\{L_1(x,t,\alpha,k)\}\downarrow_{t=0}= 0$ and $D_t^{(k-1)\alpha}\{L_2(x,t,\alpha,k)\}\downarrow_{t=0}= 0$.

3   Discussion and Concluding Remarks

The purpose of this section is to test the validity of the proposed scheme and to study the effect of the fractional order α on the solution of the time-fractional Wu-Zhang system. To achieve these goals, we solve (1) subject to the initial conditions

$$
\begin{aligned}
v(x,0) &= \frac{2}{3}\left(1 - \tanh\left(\sqrt{\frac{1}{3}}x\right)\right), \\
w(x,0) &= \frac{2}{9}\left(1 - \tanh^2\left(\sqrt{\frac{1}{3}}x\right)\right).
\end{aligned}
\tag{17}
$$

Provided that the exact solution of (1) when $\alpha = 1$ is [25]

$$
\begin{aligned}
v(x,t) &= \frac{2}{3}\left(1 - \tanh\left(\sqrt{\frac{1}{3}}\left(x - \frac{2}{3}t\right)\right)\right), \\
w(x,t) &= \frac{2}{9}\left(1 - \tanh^2\left(\sqrt{\frac{1}{3}}\left(x - \frac{2}{3}t\right)\right)\right).
\end{aligned}
\tag{18}
$$

For a reliability verification, we consider the 4-th order approximation

$$
\begin{aligned}
v_4(x,t) &= v(x,0) + \sum_{k=1}^{4} c_k(x)\frac{t^{k\alpha}}{\Gamma(k\alpha + 1)}, \\
w_4(x,t) &= w(x,0) + \sum_{k=1}^{4} d_k(x)\frac{t^{k\alpha}}{\Gamma(k\alpha + 1)},
\end{aligned}
\tag{19}
$$

as the supportive solution of the time-fractional Wu-Zhang system. We present here the plots of this obtained approximate solution against the exact solution (18) when $\alpha = 1$, see Figure 1 (i) and (ii) and Figure 4 (a) and (b). For the accuracy of the used method, we provide Figures 2 and 5 which represent respectively $|v(x,t) - v_4(x,t)|$ and $|w(x,t) - w_4(x,t)|$.

Figure 3 provides profile solutions of the function $v(x,t)$ for different values of the fractional order α, the plot on the left when t is fixed, $t = 0.2$, and the plot on the right when x is fixed, $x = 0.5$. Figure 6 provides profile solutions of the function $w(x,t)$ for different values of the fractional order α, the plot on the left when t is fixed, $t = 0.2$, and the plot on the right when x is fixed, $x = 0.5$.

We point here that the proposed method is effective for all nonlinear equations. If the order of the nonlinear terms involved in the equation is small, then a few terms of the fractional power series provide a high accuracy approximation. But, if the order of the nonlinear term is big, it is required to add more terms to reach the desired reasonable approximation.
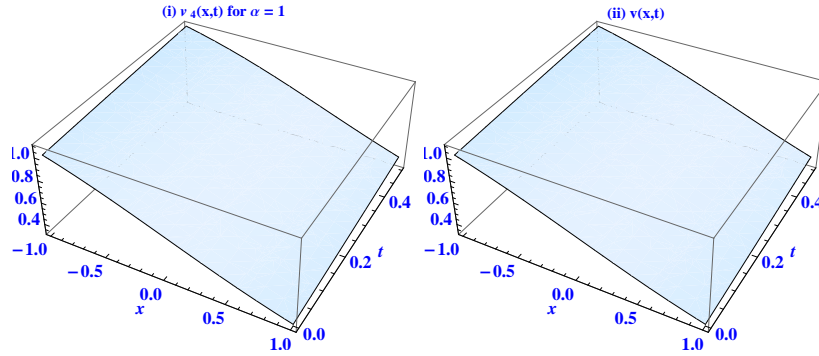
Figure 1: The approximate $v_4(x, t)$ and exact $v(x, t)$ solutions, respectively, when $-1 < x < 15$ and $0 < t < 0.5$ and $\alpha = 1$.

Figure 2: Absolute error $|v(x, t) - v_4(x, t)|$

Figure 3: Profile solutions of $v_4(x, 0.2)$ on the left and $v_4(0.5, t)$ on the right for different values of the fractional order α.

Figure 4: The approximate $w_4(x, t)$ and exact $w(x, t)$ solutions, respectively, when $-1 < x < 15$ and $0 < t < 0.5$ and $\alpha = 1$.
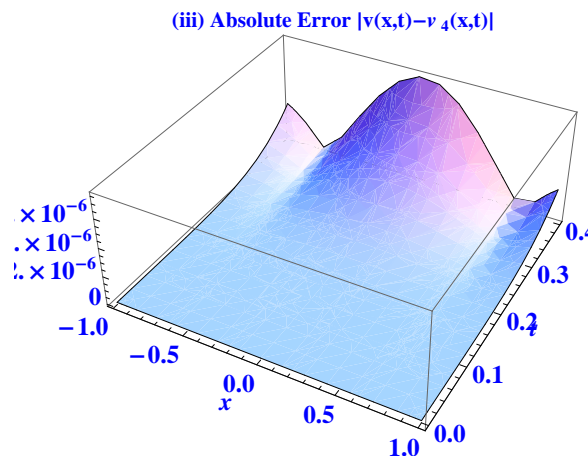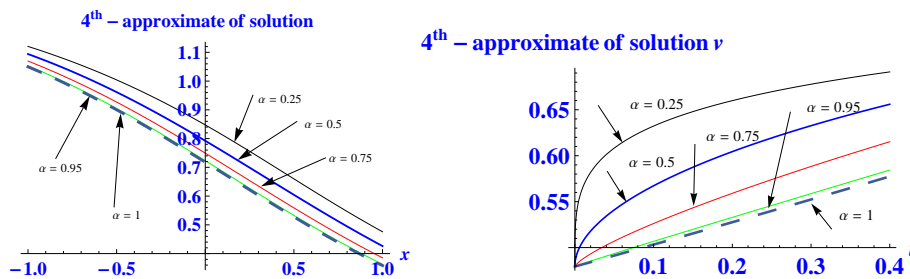

Figure 5: Absolute error $|w(x, t) - w_4(x, t)|$.

Figure 6: Profile solutions of $w_4(x, 0.2)$ on the left and $w_4(0.5, t)$ on the right for different values of the fractional order α.
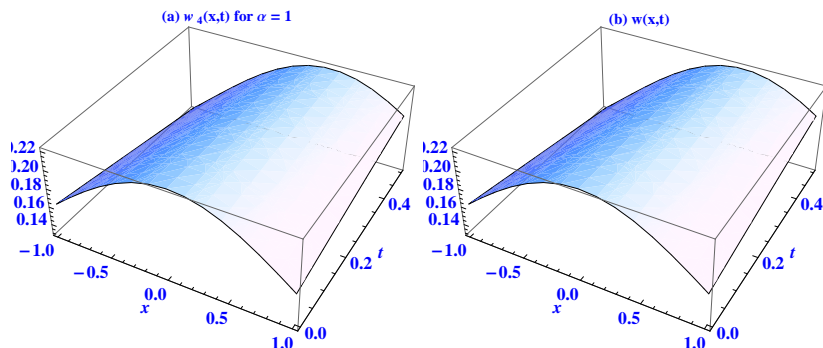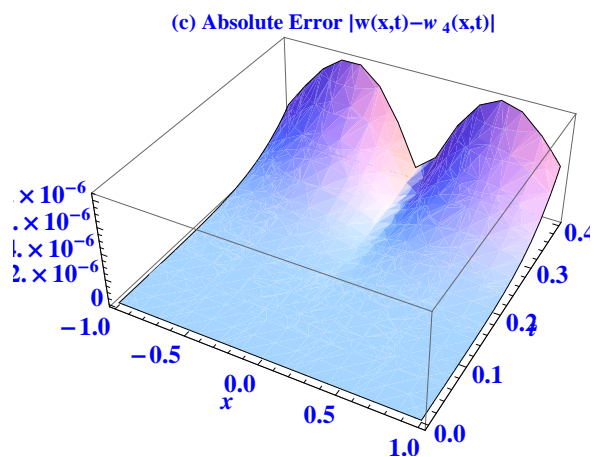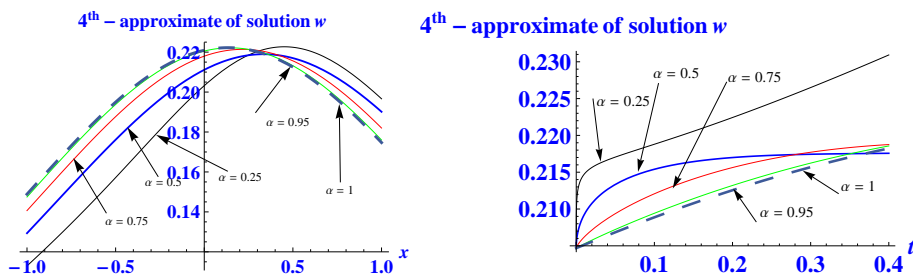
Acknowledgment

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Closed-Form Solution of European Option under Fractional Heston Model

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Abstract: In this paper, we give a closed-form solution of a European option generated by the fractional Heston stochastic volatility model based on the Adomian decomposition method.

Keywords: *pricing European option; stochastic volatility; fractional Heston model; Adomian decomposition.*

Mathematics Subject Classification (2010): 91Gxx; 26A33; 34A08.

1 Introduction

The valuation of options is one of the most popular problems in financial mathematical literature. This problem is of interest for both academics and traders. As compared to the case of the Black and Scholes model, where the volatility is constant, the Heston model is more common since the volatility is stochastic, inasmuch as the dynamics of the volatility is fundamental to elaborate strategies for hedging and for arbitrage and a model based on a constant volatility cannot explain the reality of the financial markets. So, the pricing of option under stochastic volatility model is then very important and required.

During the last few decades, several papers studied the existence of closed-form solution of the European option using many methods and generated by different models, for example, the Black and Scholes case [3–5], the Hull and White model [14], the Heston model [6, 12] and recently, Jerbi has given a new closed-form solution for the European option [15] based on a new stochastic process.

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The fractional calculus is used in several research axes. Recently, it has been introduced in the mathematical finance field [9–11, 16], and especially to generate the underlying asset price in order to give a closed form solution for the evaluation of a European option problem [10, 11, 16, 18, 20]. Many methods are proposed in order to resolve linear and nonlinear fractional differential equations, see, for example, [2, 19]. In this work we use the Adomian decomposition method [1, 7, 8].

In the following, we shall need to introduce the dynamic of the Heston model. Let S_t and V_t represent two stochastic processes so that X_t is generated by the following process :

$$dS_t = rS_tdt + S_t\sqrt{V_t}dW_t^S \tag{1}$$

and V_t follows a mean reversion and a square-root diffusion process given by:

$$dV_t = k_V(\theta_V - V_t)dt + \sigma_V\sqrt{V_t}dW_t^V, \tag{2}$$

where r is supposed to be constant, W_t^S and W_t^V are two correlated standard Brownian motions, i.e. $W_t^S = \sqrt{1-\rho^2}B_t^1 + \rho B_t^2$ and $W_t^V = B_t^2$, where B is a standard 2-dimensional Brownian motion and $\rho \in]-1,1[$. The parameters θ_V, k_V and σ_V are respectively, the long-term mean, the rate of mean reversion, and the volatility of the stochastic process V_t. We assume that the volatility process V_t is strictly positive.
So, based on the Heston stochastic volatility model a two dimensional parabolic partial differential equation can be derived for the value of the European option, see, for instance, [13].

2   Preliminaries

In what follows, we give some definitions related to the fractional calculus which constitute the basis of our work. For an organic presentation of the fractional theory, we can refer the readers to Podlubny's book [17].

Definition 2.1 The Riemann-Liouville fractional integral of order $\alpha > 0$ is defined as

$$I_{t_0}^\alpha x(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^t (t-\tau)^{\alpha-1}x(\tau)d\tau,$$

where $\Gamma(\alpha) = \int_0^{+\infty} e^{-t}t^{\alpha-1}dt$.

Definition 2.2 The Caputo fractional derivative is defined as

$$D_{t_0,t}^\alpha x(t) = \frac{1}{\Gamma(m-\alpha)} \int_{t_0}^t (t-\tau)^{m-\alpha-1}\frac{d^m}{d\tau^m}x(\tau)d\tau, \ (m-1 < \alpha < m).$$

When $0 < \alpha < 1$, then the Caputo fractional derivative of order α of f reduces to

$$D_{t_0,t}^\alpha x(t) = \frac{1}{\Gamma(1-\alpha)} \int_{t_0}^t (t-\tau)^{-\alpha}\frac{d}{d\tau}x(\tau)d\tau. \tag{3}$$

Note that the relation between the Riemann-Liouville operator and the Caputo fractional differential operator is given by the following equality:

$$I_{t_0}^\alpha D_{t_0,t}^\alpha f(t) = D_{t_0,t}^{-\alpha}D_{t_0,t}^\alpha f(t) = f(t) - \sum_{k=0}^{m-1} \frac{t^k}{k!}f^k(0), \quad m-1 < \alpha \leq m. \tag{4}$$

Similarly to the exponential function used in the solutions of integer-order differential systems, the Mittag-Leffler function is frequently used in the solutions of fractional-order differential systems.

Definition 2.3 The Mittag-Leffler function with two parameters is defined as

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{+\infty} \frac{z^k}{\Gamma(k\alpha + \beta)},$$

where $\alpha > 0$, $\beta > 0$, $z \in C$.

When $\beta = 1$, we have $E_\alpha(z) = E_{\alpha,1}(z)$, furthermore, $E_{1,1}(z) = e^z$.

3   Main Results

When the volatility is stochastic, the value $P(S_t, V_t)$ of a European option is given by the following nonlinear fractional differential equation

$$D_t^\alpha P(S_t, V_t) + A[P](S_t, V_t) = 0, \qquad 0 < \alpha \leq 1, \tag{5}$$

in the unbounded domain $\{(S_t, V_t)|S_t \geq 0, V_t \geq 0 \ and \ t \in [0, T]\}$ with the initial value

$$P(S_0, V_0). \tag{6}$$

For the boundary conditions, in the case of a call option, at maturity T with an exercise price K, the payoff function is

$$\max(S_T - K, 0) \tag{7}$$

and for the put option the payoff function is equal to

$$\max(K - S_T, 0), \tag{8}$$

where $D_t^\alpha = \frac{\partial^\alpha}{\partial t}$ and

$$A[P] = rS\frac{\partial P}{\partial S} + k(\theta - V)\frac{\partial P}{\partial V} + \frac{1}{2}VS^2\frac{\partial^2 P}{\partial S^2} + \rho\sigma VS\frac{\partial^2 P}{\partial S\partial V} - \frac{1}{2}\sigma V\frac{\partial^2 P}{\partial V^2} - rP.$$

Theorem 3.1 *Let $(P_t)_{t\geq 0}$ be the European option price, a function of the underlying asset price and the volatility. Under the same hypotheses of the Heston model, the price of the European option is given by the following formula:*

$$P(S_t, V_t) = E_\alpha(-t^\alpha A[P(S_0, V_0)]),$$

where $0 < \alpha \leq 1$, E_α is the Mittag-Leffler function and $A[P] = rS\frac{\partial P}{\partial S} + k(\theta - V)\frac{\partial P}{\partial V} + \frac{1}{2}VS^2\frac{\partial^2 P}{\partial S^2} + \rho\sigma VS\frac{\partial^2 P}{\partial S\partial V} - \frac{1}{2}\sigma V\frac{\partial^2 P}{\partial V^2} - rP$.

Proof. Multiplying equation (5) by the operator $D_t^{-\alpha}$ and on taking into account (4), we get

$$P(S_t, V_t) = P(S_0, V_0) + D_t^{-\alpha}(-A[P](S_t, V_t)), \tag{9}$$

so, using the Adomian decomposition method we get the solution in the following form

$$P(S_t, V_t) = P_0(S_t, V_t) + \sum_{k=1}^{\infty} P_k(S_t, V_t), \tag{10}$$

by substituting (10) into (5), we have:

$$P_{n+1}(S_t, V_t) = D_t^{-\alpha}(-A[P_n](S_t, V_t))$$

$$= -A[P(S_0, V_0)]^n D_t^{-\alpha}\Big(\frac{t^{n\alpha}}{\Gamma(1 + n\alpha)}\Big) \tag{11}$$

with $P_0(S_t, V_t) = P(S_0, V_0)$, we get:

$$P(S_t, V_t) = \sum_{k=0}^{\infty}(-1)^k \frac{t^{k\alpha}}{\Gamma(1 + k\alpha)} A[P(S_0, V_0)]^k$$

$$= E_\alpha(-t^\alpha A[P(S_0, V_0)]). \tag{12}$$

The convergence of the power series of the fractional Heston model is guaranteed for a real and positive α.

4 Conclusion

In this paper, we have elaborated a new closed-form solution of a European option generated by the fractional Heston stochastic volatility model. In this work, we have performed two extensions: when we take $\alpha = 1$, we return to the standard Heston model and for a constant volatility, we have the fractional Black-Scholes model.

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Sufficient Conditions for the Existence of Optimal Controls for Some Classes of Functional-Differential Equations

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Abstract: Sufficient conditions for the existence of optimal controls for system of functional-differential equations which is nonlinear by phase variables and linear by control function are given. These conditions are obtained in terms of right-hand sides of the system and the quality criterion function, which makes them convenient for verification. The main differences from the previously obtained results are that the control is in the system as a functional, and the optimal control problem is considered until the exit of the solution from the area of a certain functional space.

Keywords: *optimal control, functional, weakly convergence, convexity, minimizing sequence.*

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1 Introduction

Let $h > 0$ be a value of delay, $|\cdot|$ denote the norm of the vector in the space $\mathbb{R}^d$, $\|\cdot\|$ be the norm of $d \times m$ -dimensional matrix which is consistent with the norm of the vector.

Let us denote by $C = C\left([-h,0];\mathbb{R}^d\right)$ the Banach space of continuous maps of $[-h,0]$ into $\mathbb{R}^d$ with the uniform norm $\|\varphi\|_C = \max\limits_{\theta \in [-h;0]} |\varphi(\theta)|$. Also denote by $L_p = L_p\left([-h,0];\mathbb{R}^m\right)$, $p > 1$, the Banach space of p-integrable m-dimensional vector-functions with standard norm $\|\varphi\|_{L_p} = \left(\int_{-h}^0 |\varphi(\tau)|^p d\tau\right)^{\frac{1}{p}}$.

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Let $x \in C\left([0,T]; R^d\right)$, $\varphi \in C$. If $x(0) = \varphi(0)$, then the function

$$x(t,\varphi) = \begin{cases} \varphi(t), t \in [-h,0], \\ x(t), t \geq 0, \end{cases}$$

is continuous on $[-h,T]$.

For each $t \in [0,T]$ in the standart way by $\theta \in [-h,0]$ we put an element $x_t(\varphi) \in C$ as $x_t(\varphi) = x(t+\theta,\varphi)$. In what follows we shall write x_t instead of $x_t(\varphi)$.

Let $t \in [0,T]$, D be some domain in $[0,T] \times C$, ∂D be its boundary and $\overline{D} = D \cup \partial D$.

In this paper we consider the optimal control problems for systems of functional differential equations

$$\dot{x} = f_1(t,x_t) + \int_{-h}^{0} f_2(t,x_t,y) u(t,y) \, dy, \; t \in [0,\tau], \tag{1.1}$$

$$x(t) = \varphi_0(t), \; t \in [-h,0],$$

with the quality criterion

$$J[u] = \int_0^\tau L(t,x_t,u(t,\cdot)) \, dt \to \inf \tag{1.2}$$

on $[0,T]$, where $\varphi_0 \in C$ is a fixed element such that $(0,\varphi_0) \in D$, $x(t)$ is the phase vector in $\mathbb{R}^d$, x_t is the phase vector in C, τ is the moment of the first exit (t,x_t) on the boundary ∂D, $f_1 : D \to \mathbb{R}^d$, $f_2 : D \times [-h,0] \to M^{d \times m}$ are $d \times m$ -dimensional matrices, and for each $(t,\varphi) \in D$, $f_2(t,\varphi,\cdot) \in L_q\left([-h,0]; M^{d \times m}\right)$ with the norm $\|f_2(t,\varphi,\cdot)\|_{L_q} = \left(\int_{-h}^0 \|f_2(t,\varphi,y)\|^q dy\right)^{\frac{1}{q}}$, $\frac{1}{q} + \frac{1}{p} = 1$, $L : D \times L_p \to \mathbb{R}^1$.

The control parameter $u \in L_p\left([0,T] \times [-h,0]\right)$ is such that $u(t,y) \in U$, and U is a convex and closed set in $\mathbb{R}^m$ for almost all t,y.

Many works are devoted to the optimal control problems for functional-differential equations systems. We note the monograph [1] devoted to the application of the method of dynamic programming and the principle of maximum to such problems. There is also a wide bibliography. Although these methods, as a rule, give the necessary conditions of optimality, it would be desirable to have suitable sufficient conditions for checking to apply them.

In this regard, we cite the work [2] in which in the case of compactness of the set of admissible controls an analogue of the Filippov theorem on optimal control existence for ordinary differential equations was obtained.

For noncompact set of admissible controls an analogue of the Cessari theorem is obtained in [4]. In the mentioned work the condition of compactness is imposed on a set of constraints and a certain condition of growth is established which connects the right-hand sides of the system and the quality criterion.

In [5] under the condition of compactness of the set of admissible controls values sufficient conditions for optimality on a fixed interval $[t_0,t_1]$ for neutral-type equations are obtained.

In [6] the problem of optimal control of a delayed linear system is rewritten in a form that does not depend on the delay and which is studied by the methods of ordinary differential equations. In the works [7]- [9] the optimal control problem of the system

$$\dot{x}(t) = rx(t) + f_0\left(x(t), \int_{-T}^{0} a(\varsigma) x(t+\varsigma) \, d\varsigma\right) - u(t)$$

is considered.

In [7] certain Hamilton-Jacobi-Bellman equations are obtained for certain quality functionals and, in terms of their solutions, sufficient conditions for optimality in the form of a reverse link are obtained.

In [8] similar questions are considered for problems with phase restriction.

In [9] for such problems the authors obtained sufficient conditions for optimality under the condition of nondecreasing function $rx + f_0(x, y)$ in both variables and for the quality criterion

$$J(u) = \int_0^\infty e^{-\varphi t} a^{t\sigma}(t) dt, \quad \sigma \in (0, 1).$$

The main goal of this work is to obtain the theorem on the existence of optimal controls for a wider class of problems under weaker conditions as compared with the above mentioned works [2]- [9].

This paper is organized as follows. In Section 2 we give rigorous formulations of the considered problems and state main results. Section 3 is devoted to the proof of the main results.

In Subection 3.1 we prove the existence theorem, the uniqueness and extension of the solution of the initial problem (1.1) to the boundary ∂D of the domain D.

In Subsection 3.2 the theorem on the existence of optimal control for problem (1.1)-(1.2) is proved.

Examples of the application of the results obtained for ordinary differential equations, equations with delaying argument and equations with maxima are given in Section 4.

2   Statement of the Problems and Main Results

Now we give exact statement of the problem and formulate the main results of this paper. The main conditions for the problem (1.1)-(1.2) are assumed as follows.

Assumption 2.1 Admissible controls are m-dimensional vector functions $u \in L_p([0,T][-h,0], \mathbb{R}^m)$, such that $u(t,y) \in U$ for almost all $t \in [0,T]$ and $y \in [-h,0]$.

The set of admissible controls is denoted by $\mathcal{U}$.

Assumption 2.2 The maps $f_1(t,\varphi) : D \to \mathbb{R}^d$ and $f_2(t,\varphi,y) : D \times [-h,0] \to M^{d\times m}$ are defined and measurable with respect to all their arguments in the domains D and $D_1 = \{(t,\varphi) \in D, y \in [-h,0]\}$ respectively, and satisfy the linear growth condition and the Lipschitz condition with respect to φ, i.e. there exists a constant $K > 0$ such that

$$|f_1(t,\varphi)| + \|f_2(t,\varphi,y)\| \leq K(1 + \|\varphi\|_C) \tag{2.1}$$

for any $(t,\varphi) \in D, y \in [-h,0]$,

$$|f_1(t,\varphi_1) - f_1(t,\varphi_2)| + \|f_2(t,\varphi_1,y) - f_2(t,\varphi_2,y)\| \leq K\|\varphi_1 - \varphi_2\|_C \tag{2.2}$$

for all $(t,\varphi_1), (t,\varphi_2) \in D, \ y \in [-h,0]$.

Assumption 2.3 Conditions for the criterion function are:

1) the map $L(t,\varphi,z) : D \times L_p \to \mathbb{R}^1$ is defined and continuous with respect to all its arguments in the domain $D_2 = \{(t,\varphi) \in D, z \in L_p\}$;

2) there exists $a > 0$ such that

$$|L(t, \varphi_1, z) - L(t, \varphi_2, z)\| \leq a\|\varphi_1 - \varphi_2\|_C$$

for all $(t, \varphi_1, z), (t, \varphi_2, z) \in D_2$;

3) the Frechet derivative L_u of the map L is continuous with respect to all its arguments in the domain D_2, and there exist constants $C_1 > 0$, $\alpha > 0$ such that for all $(t, \varphi, z) \in D_2$ the following inequality holds:

$$\|L_u(t, \varphi, z)\|_{L_q} \leq C_1(1 + \|\varphi\|_C^\alpha + \|z\|_{L_p}^{p-1});$$

4) there exists a constant $C > 0$ such that $L(t, \varphi, z) \geq C\|z\|_{L_p}^p$ for all $(t, \varphi, z) \in D_2$;

5) $L(t, \varphi, z)$ is convex with respect to z for any fixed t, φ;

Our first result concerns the existence, uniqueness and extension of the solution of the original problem (1.1) to the boundary ∂D of the domain D. It is some analogue of the Carathéodory theorem for ordinary differential equations.

Definition 2.1 The solution of the initial problem (1.1) on the segment $[-h, A]$, $A > 0$, is called a continuous on the segment $[-h, A]$ function $x(t)$ such that

1) $x(t) = \varphi_0(t)$, $t \in [-h, 0]$;

2) $(t, x_t) \in D$ on $t \in [0, A]$;

3) for $t \in [0, A]$ the function $x(t)$ satisfies the integral equation

$$x(t) = \varphi_0(0) + \int_0^t \left[f_1(s, x_s) + \int_{-h}^0 f_2(s, x_s, y)u(s, y)dy \right] ds. \tag{2.3}$$

.

Remark 2.1 It is obvious that for $t \in [0, A]$ the solution $x(t)$ is an absolutely continuous function and satisfies the equation (1.1) for almost all t on $[0, A]$.

Theorem 2.1 *Suppose that Assumptions 2.1 and 2.2 are satisfied. Then there exists a solution of the initial problem (2.3) on the maximal segment $[-h, \tau]$, $\tau > 0$ and $(\tau, x_\tau) \in \partial D$.*

The following theorem gives for the problem (1.1)-(1.2) the existence conditions of the optimal pair $x^*(t), u^*(t, \theta))$, which provides the minimum of the quality criterion (1.2).

In this case $u^* \in \mathcal{U}$ is called the optimal control and the corresponding trajectory $x^*(t)$ (1.1) is called the optimal trajectory.

Theorem 2.2 *Suppose that Assumptions 2.1-2.3 are satisfied. Then there exists a solution of the optimal control problem (1.1)-(1.2).*

3　Proofs of the Theorems

3.1　Proof of Theorem 2.1

Let us fix an admissible control $u^* \in \mathcal{U}$. First, we shall prove the local existence and uniqueness of the solution of the problem (1.1), on some segment $[-h, \alpha]$, $\alpha > 0$.

To do this, we use the standard principle of contraction mappings.

Obviously, there exist $\alpha_0 > 0$ and $\beta_0 > 0$ such that all (t, φ) for which $0 \leq t \leq \alpha_0$, and $\|\varphi - \varphi_0\|_C \leq \beta_0$ belong to D are equivalent to φ_0 on $[-h, 0]$.

Now we consider the class $B(\alpha, \beta_0)$ of all continuous on $[-h, \alpha]$ functions $x(t)$ that are equivalent to φ_0 on $[-h, 0]$ and $|x(t) - \varphi_0(0)| \leq \beta_0$ for $t \in [0, \alpha]$.

Obviously, the set $B(\alpha, \beta_0)$ is closed relatively uniformly metric on $[-h, \alpha]$.

In this case there exists $\alpha_0 \geq \alpha_1 > 0$ such that if $x(t) \in B(\alpha, \beta_0)$ at $0 < \alpha \leq \alpha_1$, then the following inequality holds:

$$\|x_t - \varphi_0(0)\|_C \leq \beta_0, \quad t \in [0, \alpha]. \tag{3.1}$$

Indeed, under the condition of uniform continuity of φ_0 on $[-h, 0]$ there exists $\alpha_1 > 0$ such that if $|\theta_1 - \theta_2| \leq \alpha_1$, then

$$|\varphi_0(\theta_1) - \varphi_0(\theta_2)| \leq \frac{\beta_0}{3} \tag{3.2}$$

for all $\theta_1, \theta_2 \in [-h, 0]$.

Hence, for each $t \in [0, \alpha_1]$ at $\alpha \leq \alpha_1$ from (3.2) and the properties of the set $B(\alpha, \beta_0)$ we have

$$\|x_t - \varphi_0\|_C \leq \sup_{\theta \in [-h, -t]} |x(t + \theta) - \varphi_0(\theta)| + \sup_{\theta \in [-t, 0]} |x(t + \theta) - \varphi_0(\theta)| \leq$$

$$\leq \sup_{\theta \in [-h, -t]} |\varphi_0(t + \theta) - \varphi_0(\theta)| + \sup_{\theta \in [-t, 0]} |x(t + \theta) - \varphi_0(\theta)| +$$

$$+ \sup_{\theta \in [-t, 0]} |\varphi_0(\theta) - \varphi_0(0)| \leq \frac{\beta_0}{3} + \frac{\beta_0}{3} + \frac{\beta_0}{3} = \beta_0.$$

Next we shall prove that $\alpha > 0$ can be choosen so that the operator

$$(Ax)(t) = \begin{cases} \varphi_0(t), \ t \in [-h, 0], \\ \varphi_0(0) + \int_0^t f_1(s, x_s) ds + \int_0^t \int_{-h}^0 f_2(s, x_s, y) u(s, y) dy ds, \ t \in [0, \alpha], \end{cases} \tag{3.3}$$

maps the set $B(\alpha, \beta_0)$ into itself and this operator is a contraction.

Indeed, by Lemma 2.2.1 [10] it follows that x_t is a continuous function with respect to $t \in [0, \alpha]$. Therefore, from Assumption 2.1 $f_1(s, x_s)$ is continuous with respect to $s \in [0, \alpha]$ and the function

$$\int_{-h}^0 f_2(s, x_s, y) u(s, y) dy \tag{3.4}$$

is measurable with respect to s and satisfies the estimate

$$\left| \int_{-h}^0 f_2(s, x_s, y) u(s, y) dy \right| \leq C_2 \left(\int_{-h}^0 |u(s, y)|^p dy \right)^{\frac{1}{p}}$$

for some constant $C_2 > 0$. From this we have the integrability of (3.4) with respect to s and hence the absolute continuity of the second integral in (3.4).

Now we evaluate the difference $|(Ax)(t) - \varphi_0(0)|$ at $t \in [0, \alpha]$, $\alpha \leq \alpha_0$.

With (2.1) and (2.2) using Holder's inequality and Fubini's theorem we get

$$|(Ax)(t) - \varphi_0(0)| \leq \int_0^t |f_1(s, x_s)|\, ds + \int_0^t \left(\int_{-h}^0 \|f_2(s, x_s, y)\| \, |u(s, y)|\, dy \right) ds \leq$$

$$\leq \int_0^t K(1 + \|x_s\|_C)ds + \int_0^t \left(\int_{-h}^0 K^q \left(1 + \|x_s\|_C\right)^q dy \right)^{\frac{1}{q}} \cdot \left(\int_{-h}^0 |u(s, y)|^p \, dy \right)^{\frac{1}{p}} ds \leq$$

$$\leq K(1 + \beta_0 + \|\varphi_0\|_C)\alpha + Kh^{\frac{1}{q}}(1 + \beta_0 + \|\varphi_0\|_C)\left(\int_0^\alpha \int_{-h}^0 |u(s, y)|^p \, ds \right)^{\frac{1}{p}} \alpha^{\frac{1}{q}}.$$

Here $q = \frac{p}{p-1}$.

Let us choose now $\alpha_2 \leq \alpha_1$ from the condition

$$K(1 + \beta_0 + \|\varphi_0\|_C \left(\alpha + \alpha^{\frac{1}{q}} h^{\frac{1}{q}} \left(\int_0^\alpha \int_{-h}^0 |u(s, y)|^p \, dyds \right)^{\frac{1}{p}} \leq \frac{\beta_0}{3}. \tag{3.5}$$

Thus, for all $\alpha \leq \alpha_1$ the operator A maps $B(\alpha, \beta_0)$ into itself.

Let us show that there exists $\alpha_3 \in [0, \alpha_2]$ such that the operator A will be a contraction on $B(\alpha_3, \beta_0)$.

Let x and $z \in B(\alpha, \beta_0)$. By (2.2) we have

$$|(Ax)(t) - (Az)(t)| \leq \int_0^t K \|x_s - z_s\|_C \, ds + \int_0^t K \|x_s - z_s\|_C \int_{-h}^0 |u(s, y)| \, dyds \leq$$

$$\leq \left(K\alpha + K\alpha^{\frac{1}{q}} h^{\frac{1}{q}} \left(\int_0^\alpha \int_{-h}^0 |u(s, y)|^p \, dyds \right)^{\frac{1}{p}} \right) \sup_{t \in [-h, \alpha]} |x(t) - z(t)|.$$

And now from this we have

$$\sup_{t \in [-h, \alpha]} |(Ax)(t) - (Az)(t)| \leq$$

$$\leq \left(K\alpha + K\alpha^{\frac{1}{q}} h^{\frac{1}{q}} \left(\int_0^\alpha \int_{-h}^0 |u(s, y)|^p \, dyds \right)^{\frac{1}{p}} \right) \sup_{t \in [-h, \alpha]} |x(t) - z(t)|. \tag{3.6}$$

Now choosing $0 < \alpha_3 \leq \alpha_2$ from the condition

$$K\alpha + K\alpha^{\frac{1}{q}} h^{\frac{1}{q}} \left(\int_0^\alpha \int_{-h}^0 |u(s, y)|^p \, dyds \right)^{\frac{1}{p}} < 1$$

we get that the operator $A : B(\alpha_3, \beta_0) \to B(\alpha_3, \beta_0)$ is a contraction. Thus, on the segment $[-h, \alpha_3)$ there exists a unique solution to the initial problem (1.1).

To prove the extension of this solution to the boundary ∂D, we use the approach of Theorem 2.3.2 [10]. Note that by (2.1) for $(t, \varphi) \in D$ the following estimates hold:

$$|f_1(t, \varphi)| \leq K(1 + \|\varphi\|_C) \tag{3.7}$$

and

$$\left| \int_{-h}^{0} f_2(t, \varphi, y)u(t, y)dy \right| \leq K(1 + \|\varphi\|_C)h^{\frac{1}{q}} \int_{-h}^{0} |u(s, y)| \, dy. \tag{3.8}$$

Let $[-h, \tau]$ be the maximum interval of existence of the solution $x(t)$. For its extension to the boundary ∂D it is necessary to show that for any closed set $G \in Dt_G$ there exists t_G such that $(t, x_t) \notin G$ for $t \in [t_G, \tau]$. The last statement can be proved by contradiction. Indeed, if this is not the case, then, similar to Theorem 2.3.2 [10], the set $\bar{Q} = \{(t, x_t) : t \in [-h, \tau]\}$ is closed and bounded in D.

Therefore, the estimates (3.7) and (3.8) imply the existence of a constant M such that for $(t, \varphi) \in \bar{Q}$ we have $|f_1(t, \varphi)| \leq M$ and $\left| \int_{-h}^{0} f_2(t, \varphi, y)u(t, y)dy \right| \leq M \int_{-h}^{0} |u(s, y)| \, dy$.

From (2.3) for each $t_1, t \in [0, \tau]$ we have

$$|x(t_2) - x(t_1)| \leq M(t_2 - t_1) + Mh(t_2 - t_1)^{\frac{1}{q}} \left(\int_{Q}^{T} \int_{-h}^{0} |u(s, y)|^p \, dyds \right)^{\frac{1}{p}}. \tag{3.9}$$

This implies that $\{(t, x) : t \in [-h, \tau]\}$ belongs to a compact set in D. The last statement contradicts Corollary 2.3.1 from [10]. The theorem is proved. □

3.2   Proof of Theorem 2.2

First note that controls $u(t, \theta) = u(\theta)$ are admissible.

Let $x(t)$ be a solution that coresponds to $u(t)$ and τ be a moment of the first exit (t, x_t) on the boundary ∂D.

Now we shall prove that $x(t)$ is bounded on $[0, \tau]$.

From (2.3) for $\in [0, \tau]$ we have

$$|x(t)| \leq |\varphi_0(0)| + \int_0^t K(1 + \|x_s\|_C)ds + Kh^{\frac{1}{q}} \int_0^t (1 + \|x_s\|_C)ds \, \|u\|_{L_p} \leq$$

$$\leq |\varphi_0(0)| + KT + Kh^{\frac{1}{q}} \|u\|_{L_p} T + (K + h^{\frac{1}{q}} u \, \|u\|_{L_p}) \int_0^t \|x_s\|_C \, ds =$$

$$= C_3 + C_4 \int_0^t \|x_s\| \, ds \leq C_3 + C_4 \int_0^t \max_{s_1 \in [-h,s]} |x(s_1)| \, ds \tag{3.10}$$

Since

$$\max_{s \in [-h,t]} |x(s)| \leq \max_{s \in [-h,0]} |\varphi_0(s)| + \max_{s \in [0,t]} |x_1(s_1)|,$$

and from (3.10) we have

$$\max_{s \in [-h,t]} |x(s)| \leq C_5 + C_4 \int_0^t \max_{s \in [-h,s]} |x(s_1)| \, ds$$

for some constant $C_5 > 0$. Using Gronwall's inequality we have $\max_{s \in [-h,t]} |x(s)| \leq C_5$, $t \in [0, \tau]$ for some constant $C_6 > 0$ which does not depend on t. From this it follows that

$$\max_{s \in [-h,\tau]} |x(s)| \leq C_6$$

and $\max_{s \in [-h,\tau]} \|x_t\| \leq C_6$.

Since from Lemma 2.2.1 [10] x_t is continuous with respect to $t \in [0, \tau]$, under the first condition of Assumption 2.3 we have $L(t, x_t, u(\theta))$ (where $(u(t, \theta)) = u(\theta)$ is continuous with respect to t and hence

$$\int_0^\tau L(t, x_t, u(\theta))dt \qquad (3.11)$$

is bounded. Therefore $\inf_{u \in U} J(u) < \infty$. Since $J(u) \geq 0$, there exists a nonnegative lower limit m of the values $J(u)$. Let $u^{(n)}(t, \theta)$ be a minimizing sequence such that $J(u^n) \to m$, $n \to \infty$ monotonously.

Let $x^{(n)}$ be a sequence of coresponding to $u^{(n)}$ solutions of equation (2.3), $[-h, \tau_n]$ be a maximal interval of its existense. From Theorem 2.1 it follows that $[\tau_n, x_{\tau_n}^{(n)}] \in \partial D$.

We have

$$m + 1 \geq \int_0^{\tau_n} L(t, x_t^{(n)}, u^{(n)})dt \geq C \int_0^T \int_{-h}^0 \left| u^{(n)}(t, y) \right|^p dy dt \qquad (3.12)$$

for sufficiently large n. Consequently $u^{(n)}(t, y)$ is weakly compact in $L_p([0, T] \times [-h, 0])$.

Therefore one can choose a sequence (also denoted by $u^{(n)}(t, y)$) which is weakly converging to $u^k(f) \in L_p([0, T] \times [-h, 0])$ in $L_p([0, T] \times [-h, 0])$.

By Mazur's lemma ([11], Ch. 5) there exists a convex combination $b_k(t, y) = \sum_{\tau=1}^{n(k)} \alpha_i \cdot (K) u^{(i)} \cdot (t, u)$ of elements $u^{(i)}(t, y)$ such that $b_k \to u^{(*)}$ strongly converges in $L_p([0, T] \times [-h, 0])$.

Therefore there exists a subsequence $b_{k_j}(t, y)$ of sequence $b_k(t, y)$ such that for almost all (t, y) on $[0, T] \times [-h, 0]$ it converges to $u^x(t, y)$.

Since U is convex, we have $b_{k_j}(t, y) \in U$, and from the closedness of U it follows that $u^*(t, y) \in U$ for almost all (t, y). So, the control function $u^*(t, y)$ is admissible.

Let us prove uniform boundedness of solutions $x^{(n)}$ on $[-h, \tau_n]$. From (2.3) under Assumptions 2.1 and 2.2 we have for $t \in [0, \tau_n]$

$$\left| x^{(n)}(t) \right|^q \leq 3^{q-1} |\varphi(0)|^q + K^q T^{\frac{q}{p}} 2^{q-1} \int_0^t (1 + \left\| x_s^{(n)} \right\|_C^q) ds +$$

$$+ h \left(\int_0^T \int_{-h}^0 \left| u^{(n)}(t, y) \right|^p \right)^{\frac{1}{p}} K^q h \int_0^t 2^{q-1} (1 + \left\| x_s^{(n)} \right\|_C^q) ds.$$

With (3.12), from the last inequality for some positive constants C_7 and C_8 which do not depend on t, y and n, we have

$$\left| x^{(n)}(t) \right|^q \leq C_7 + C_8 \int_0^t \|x_s\|^q ds$$

for $t \in [0, \tau_n]$.

Thus we have the estimate

$$\max_{s \in [-h, t]} |x(s)| \leq C_9 + C_8 \int_0^t \max_{s_1 \in [-h, s]} |x(s_1)| ds$$

for some constant C_9.

Using Gronwall's inequality we have

$$\max_{t \in [-h, \tau_n]} \left| x^{(a)}(t) \right| \leq C_{10}, \tag{3.13}$$

where C_{10} is a positive constant which does not depend on n.

So $x^{(n)}(t)$ are uniform bounded. Let us extend the functions $x^{(n)}(t)$ to the whole segment $[0, T]$ as follows

$$y^{(n)}(t) = \begin{cases} x^{(n)}(t), t \in [0, \tau_n], \\ x^{(n)}(\tau_n), t \in [\tau_n, T]. \end{cases} \tag{3.14}$$

If $s_1 \leq s_2 \leq \tau_n$, then from (3.9) it follows the estimate

$$\left| y^{(n)}(s_1) - y^n(s_2) \right| \leq C_{11}(s_2 - s_1) + C_{12}(s_2 - s_1)^{\frac{1}{q}}. \tag{3.15}$$

If $s_1 \leq \tau_n \leq s_2$, then similarly to (3.15) we have

$$\left| y^{(n)}(s_1) - y^n(s_2) \right| = \left| x^{(n)}(s_1) - x^{(n)}(\tau_n) \right| \leq C_{11}|\tau_n - s_1| + C_{12}|\tau_n - s_1| \leq$$

$$\leq C_{11}(s_2 - s_1) + C_{12}(s_2 - s_1)^{\frac{1}{q}}.$$

This implies the equicontinuity of the function set $\{y^{(n)}(t)\}$ on $[0, T]$ and from (3.14) and (3.13) it follows uniform boundedness of this set. Hence the set $\{y^{(n)}(t)\}$ includes a subsequence which converges uniformly on $[0, T]$ and which we denote as $\{y^{(n)}(t)\}$. Let $y^{(x)}(t)$ be its uniform limit on $[0, T]$.

Function $y^*(t)$ is defined and continuous on $[0, T]$. Therefore, we also have $y_t^* = y^*(t + \theta)$ for all $t \in [0, T]$. Let τ^* be a moment of the first exit (t, y_t^*) on the boundary ∂D, i.e.

$$\tau^* = \begin{cases} \inf\{t \in [O, T] : (t, y_t^*) \in \partial D\}, \\ T, \quad if \quad (t, y_t^*) \in D, \forall t \in [O, T]. \end{cases}$$

Note that if

$$y_{\tau_n}^{(n)} = y^{(n)}(\tau_n + \theta) = x^{(n)}(\tau_n + \theta) = x_{\tau_n}^{(n)},$$

then τ_n is the moment of the first exit (t, y_t^n) on ∂D.

Let us prove that

$$\tau^* \leq \lim_{n \to \infty} \inf \tau_n. \tag{3.16}$$

Suppose that it is not true. Then

$$\tau^* > \lim_{n \to \infty} \inf \tau_n = \tau. \tag{3.17}$$

Obviously, there exists a subsequence τ_{n_k} such that $\tau_{n_k} \to \tau$ for $n_k \to \infty$. Therefore for sufficiently large n_k we have $\tau < \tau^*$ and

$$(\tau, y_\tau^*) \in D. \tag{3.18}$$

But $(\tau_{n_k}, y_{\tau_{n_k}}^{(n_k)}) \in \partial D$.

On the other hand, taking into account the uniform convergence of the sequence $y^n(t)$ to $y^*(t)$ on $[-h, T]$ and uniform on $[-h, T]$ continuity of $y^*(t)$ it is not difficult to see that $y_{\tau_{n_k}}^{(n_k)} \to y_\tau^*$ in C for $n_k \to \infty$.

Since the set ∂D is closed, we have $(\tau, y_\tau^*) \in D$. The latter contradicts (3.18). Therefore

$$\tau^* \leq \tau = \liminf_{n \to \infty} \tau_n.$$

Let $x^*(t) = y^*(t)$ for $t \in [0, \tau^*]$. Show that $x^*(t)$ is a solution of the equation (1.1) which corresponds to the equation $u^*(t)$.

We consider two cases.

1. Let $\tau^* < \tau$. Then by the theorem of the characterization of the lower bound, the set $\{n \in N : \tau_n \leq \tau^*\}$ is finite. Consequently, there exists a subsequence $\{\tau_{n_k}\}$ of the sequence τ_n, such that $\tau_{n_k} > \tau^*$. Then $y^{(n_k)}(t) = x^{(n_k)}(t)$ for $t \in [0, \tau^*]$ and $x^{(n_k)}(t)$ converges uniformly to $x^*(t)$ for $n_k \to \infty$. We have

$$x^{(n_k)}(t) = \varphi_0(0) + \int_0^t f_1(s, x_s^{n_k}) ds + \int_0^t \int_{-h}^0 f_2(s, x_s^{n_k} y) u^{n_k}(s, y) dy ds \qquad (3.19)$$

for $t \in [0, \tau^*]$.

Then we get

$$x^{(u_k)}(t) = \varphi_0(0) + \int_0^t f_1(s, x_s^{n_k}) ds + \int_0^t \int_{-h}^0 f_2(s, x_s^{n_k} y) u^*(s, y) dy ds +$$

$$+ \int_0^t \int_{-h}^0 (f_2(s, x_s^{(n_k)}, y) - f_2(s, x_s^*, y))(u^{(u_k)}(s, y) - u^*(s, y) dy ds +$$

$$+ \int_0^t \int_{-h}^0 f_2(s, x_s^*, y)(u^{(u_k)}(s, y) - u^*(s, y) dy ds. \qquad (3.20)$$

It is obvious that $x_t^{(n_k)} \to x_t^*$ on C for all $t \in [0, \tau^*]$.

From (2.2) we have

$$\int_0^t f_1(s, x_s^{(n_k)}) ds \to \int_0^t f_1(s, x_s^*) ds \qquad (3.21)$$

and in view of the Lebesgue theorem on dominanted convergence, we also obtain that

$$\int_0^t \int_{-h}^0 f_2(s, x_s^{(n_k)}, y) u^*(s, y) dy ds \to \int_0^t \int_{-h}^0 f_2(s, x_s^*, y) u^*(s, y) dy ds. \qquad (3.22)$$

Similarly, we establish that the third integral in (3.20) tends to zero for $n_k \to \infty$.

Taking into account Assumption 2.2 with respect to f_2, it is easy to see that the expression $\int_0^t \int_{-h}^0 f_2(s, x_s^*, y) u(s, y) dy ds$ defines a linear continuous functional on $L_2([0, t] \times [-h, 0])$.

Therefore the last integral in (3.20) tends to zero because of the weak convergence of $u^{(n_k)}(s, y)$ to $u^*(s, y)$. Using the limiting transition in (3.20) we obtain that $x^*(t)$ is the solution of the initial problem (1.1) on $[0, \tau^*]$ which corresponds to the control $u^*(t, y)$.

2. Let $\tau^* = \tau$. Take an arbitrary $t_1 \in [0, \tau]$ such that $t_1 < \tau^*$. Then the set $\{n \in N : \tau_n \leq t_1\}$ is finite.

In the case of the finiteness of the set $Z = \{n \in N : t_1 < \tau_n \leq t^*\}$ the proof reduces to the preceding case. Let Z be infinite and δ_{n_k} be a subsequence of the sequence τ_n such that $\tau_{n_k} \in Z$. Then for each $t \in [0, t]$ we have $y^{(n_k)}(t) = x^{(n_k)}(t)$ $y^*(t) = x^*(t)$.

Similarly to the previous case then $x^*(t)$ is the solution of the initial problem (1.1) on $[0, t_1]$ corresponding to the control $u^*(t, y)$, that is

$$x^*(t) = \varphi_0(0) + \int_0^t f_1(s, x_s^*)ds + \int_0^t \int_{-h}^0 f_2(s, x_s^*, y)u^*(s, y)dyds \qquad (3.23)$$

for $t \in [0, t_1]$. Since $t_1 < \tau^*$ is arbitrary, the equality (3.23) holds on the interval $[0, \tau^*]$.

Let us show it holds also for $t = \tau^*$. Let $t_n \in [0, \tau^*]$ and $t_n \to \tau^*$, then $x^*(t_n) \to x^*(\tau^*)$.

Similarly to the inequality (3.9) we get for $n \to \infty$

$$\left| \int_0^{\tau^*} [f_1(s, x_s^*) + \int_{-h}^0 f_2(s, x_s^*, y)u^*(s, y)dy]ds - \int_0^{t_n} [f_1(s, x_s^*) + \int_{-h}^0 f_2(s, x_s^*, y)u^*(s, y)dy]ds \right| \to 0.$$

Therefore $x^*(t)$ satisfies (3.23) for $t = \tau^*$ too.

It remains to show that the control $u^*(s, y)$ is optimal. We have two cases.

1. Let $\tau^* < T$.

a) Let $\tau^* < \varliminf_{n \to \infty} \tau_n = \tau$. Then, similarly to the above, there exists a subsequence τ_{n_k} of the sequence τ_n such that $\tau_{n_k} > \tau^*$ and for $t \in [0, \tau^*]$ $y^{(n_k)}(t) = x^{(n_k)}(t)$ and $y^*(t) = x^*(t)$.

We show the integrability of the function $L(t, x_t^*, u^{(n_k)}(t, \cdot))$ on $[0, \tau^*]$

Using inequality

$$\left| L(t, x_t^*, u^{(n_k)}(t, \cdot)) - L(t, x_t^*, u_0) \right| \leq$$

$$\leq \sup_{\lambda \in [0,1]} \left\| L_u(t, x_t^*, u_0 + \lambda(u^{(n_k)}(t, \cdot) - u_0)) \right\|_{L_q} \left\| u^{(n_k)}(t, \cdot) - u_0 \right\|_{L_p},$$

where $u_0 = const$, $u_0 \in \mathcal{U}$, we have

$$L(t, x_t^* u^{(n_k)}(t, \cdot)) \leq L(t, x_t^*, u_0) +$$

$$+ \sup_{\lambda \in [0,1]} \left\| L_u(t, x_t^*, u_0 + \lambda(u^{(n_k)}(t, \cdot) - u_0)) \right\|_{\lambda_q} \left\| u^{(n_k)}(t, \cdot) - u_0 \right\|_{L_p}.$$

Using condition 3) of Assumption 2.3 we have

$$L(t, x_t^*, u^{(n_k)}(t, \cdot)) \leq L(t, x_t^*, u_0) + C_1 \left\| u^{(n_k)}(t, \cdot) - u_0 \right\|_{L_p} +$$

$$+ C_1 \left\| x_t^* \right\|_C^\alpha \left\| u^{(n_k)}(t, \cdot) - u_0 \right\|_{L_p} +$$

$$+ C_1 \sup_{\lambda \in [0,1]} \left\| u_0 + \lambda(u^{(n_k)}(t, \cdot) - u_0) \right\|_{L_p}^{p-1} \left\| u^{(n_k)}(t, \cdot) - u_0 \right\|_{L_p}. \qquad (3.24)$$

The first term in (3.24) is integrable in accordance with (3.11). The second and third terms are also integrable on $[0, \tau^*]$ due to (3.12), (3.13) and the uniform convergence of $x^{(n_k)}(t)$ to $x^*(t)$ on $[0, \tau^*]$.

The integrability of the last term in (3.24) follows from the estimate

$$\int_0^{\tau^*} (\|u_0\|_{L_p} + \|u^{n_k}(t, \cdot) - u_0\|)^{p-1} \left\| u^{(n_k)}(t, \cdot) - u_0 \right\|_{L_p} dt \leq$$

$$\leq 2^{\frac{(p-1)^2}{p}} \left(\int_0^{\tau^*} (\|u_0\|_{L_p}^p + \|u^{n_k}(t,\cdot) - u_0\|_{L_p}^p) dt \right)^{\frac{p-1}{p}} \left(\int_0^{\tau^*} (\|u^{n_k}(t,\cdot) - u_0\|_{L_p}^p) dt \right)^{\frac{1}{p}}.$$

Therefore the function $\alpha(t, x_t^*, u^{(n_k)}(t,\cdot)$ is integrable on $[0, \tau^*]$.

Let $\chi_R(t)$ be a characteristic function of the set $\left\{ t \in [0,t] : \|u^*(t,\cdot)\|_{L_p} < R \right\}$ for some $R > 0$.

Since $L(t, y, z)$ is convex on z (condition 5) from Assumption 2.3), the following inequality holds

$$L(t, x_t^*, \upsilon(t,\cdot)\chi_R(t) \geq L(t, x_t^*, u^*(t,\cdot))\chi_R(t)+$$

$$\left\langle L_u'(t, x_t^*, u^*(t,\cdot), \upsilon(t,\cdot) - u^*(t,\cdot) \right\rangle \chi_R(t) \tag{3.25}$$

for any admissible control $\upsilon(t, y) \in U_p$ $t \in [0, \tau^*]$. Here $\left\langle L_u', \upsilon - u^* \right\rangle$ is the action of the linear continuous functional L_u on the element $\upsilon(t,\cdot) - u^*(t,\cdot) \in L_p$. Putting in (3.25) $\upsilon(t,\cdot) = u^{(n_k)}(t,\cdot)$ we have

$$\int_0^{\tau^*} L(t, x_t^*, u^{(n_k)}(t,\cdot))\chi_R(t)dt \geq \int_0^{\tau^*} L(t, x_t^*, u^*(t,\cdot))\chi_R(t)dt+$$

$$+ \int_0^{\tau^*} \left\langle L_u'(t, x_t^*, u^*(t,\cdot), u^{(n_k)}(t,\cdot) - u^*(t,\cdot) \right\rangle \chi_R(t)dt. \tag{3.26}$$

Under condition 3) of Assumption 2.3 we have

$$\|L_u(t, x_t^*, u^*(t,\cdot)\|_{L_q} \chi_R(t) \leq K(1 + \|x_t^*\|_C^\alpha + R)^{p-1},$$

therefore, the second term defines a linear continuous functional in $L_p([0, \tau^*] \times [-h, 0])$. So, the second integral in (3.26) tends to zero, because of the weak convergence of $u^{n_k}(t, s)$ to $u^*(t, s)$.

Therefore

$$\lim_{n \to \infty} \inf \int_0^{\tau^*} L(t, x_t^*, u^{(u_k)}(t,\cdot))\chi_R(t)dt \geq \int_0^{\tau^*} L(t, x_t^*, u^*(t,\cdot))\chi_R(t)dt.$$

Since $L(t, y, z) \geq 0$, $\chi_R(t) \leq 1$ and $\chi_R(t) \to 1$ for $R \to \infty$, we get from the last inequality that

$$\int_0^{\tau^*} L(t, x_t^*, u^*(t,\cdot)dt \leq \lim_{n \to \infty} \inf \int_0^{\tau^*} \alpha(t, x_t^*, u^{(u_k)}(t,\cdot)dt. \tag{3.27}$$

The integrability of $L(t, x_t^*, u^{(n_k)}(t,\cdot)$ on $[0, \tau^*]$ is taken into account.

Let us also consider the difference

$$\int_0^{\tau^*} \left| L(t, x_t^{(n_k)}, u^{(u_k)}(t,\cdot)) - L(t, x_t^*, u^{(u_k)}(t,\cdot)) \right| dt. \tag{3.28}$$

Using condition 2) of Assumption 2.3 we have

$$\int_0^{\tau^*} \left| L(t, x_t^{(n_k)}, u^{(u_k)}(t,\cdot)) - L(t, x_t^*, u^{(u_k)}(t,\cdot)) \right| dt \leq$$

$$\leq \alpha \int_0^{\tau^*} \left\| x_t^{(n_k)} - x_t^* \right\| dt \to 0, n_k \to \infty. \tag{3.29}$$

The limit transition in (3.29) is possible by the Lebesgue theorem on the majorization of convergence (3.13) and the uniform convergence of $x^{(n_k)}(t)$ to $x^*(t)$ on $[0, \tau^*]$. From (3.29) we find that the expression (3.28) tends to zero for $n_k \to \infty$.

Further, we have

$$\liminf_{n \to \infty} \int_0^{\tau^*} L(t, x_t^{(n_k)}, u^{(n_k)}(t, \cdot)) dt \geq$$

$$\geq \liminf_{n \to \infty} \int_0^{\tau^*} [L(t, x_t^{(n_k)}, u^{(n_k)}(t, \cdot) - L(t, x^*, u^{(n_k)})] dt +$$

$$+ \liminf_{n \to \infty} \int_0^{\tau^*} \left| L(t, x_t^*, u^{(u_k)}(t, \cdot)) - L(t, x_t^*, u^*(t, \cdot)) \right| dt + \int_0^{\tau^*} L(t, x_t^*, u^*(t, \cdot)) dt. \tag{3.30}$$

As is shown above, the first limit on the right-hand side (3.30) is zero, and the second limit is non-negative with (3.27).

Then

$$m = \liminf_{n_k \to \infty} \int_0^{\tau_{n_k}} L(t, x_t^{(n_k)}, u^{(n_k)}(t, \cdot)) dt \geq \liminf_{n_k \to \infty} \int_0^{\tau^*} L(t, x_t^{(n_k)}, u^{(n_k)}(t, \cdot)) dt \geq$$

$$\geq \int_0^{\tau^*} L(t, x_t^*, u^*(t, \cdot)) dt.$$

Thus $J(u^*) = m$, so the pair $(x^*(t), u^*(t, s))$ is optimal.

b) Let $\tau^* = \tau = \liminf_{n \to \infty} \tau_n$.

Let us consider the set $Z = \{n \in N : t_1 < \tau_n \leq \tau^*\}$, where we again take an arbitrary $t_1 \in [0, T]$ such that $t_1 < \tau^*$. It is enough to consider the case when this set is infinite. Then, in the same way as in a), we can show that

$$\int_0^{t_1} L(t, x_t^*, u^*(t, \cdot)) dt \leq m.$$

Thus, by the limit transition for $t_1 \to \tau^*$ we establish that

$$\int_0^{\tau^*} L(t, x_t^*, u^*(t, \cdot)) dt \leq m.$$

Hence $J(u^*) = m$.

2. Let $\tau^* = T$. Then from (3.16) we have $\tau = \liminf_{n \to \infty} \tau_n = \tau^*$, and the proof reduces to case 1, b). The theorem is proved. $\square$

Remark 3.1 The method of proving the existence of optimal control and optimal trajectory is constructive if we take into account the fact that the approach of works [16, Chapter 7], or [17, Chapter 4] can be used to construct a minimizing control sequence.

4   Applications

As an application of the obtained results, we consider some particular cases of problem (1.1)-(1.2).

Example 4.1 If $u = u(t)$ and does not depend on the value y, then the problem (1.1)-(1.2) reduces to the "ordinary" optimal control problem for functional-differential equations

$$\dot{x}(t) = f_1(t, x_t) + g(t, x_t)u(t), \ t \in [0, \tau],$$

$$x(t) = \varphi_0(t), \ t \in [-h, 0],$$

$$J[u] = \int_0^\tau L(t, x_t, u(t))dt \ \to \inf,$$

where $g(t, x_t) \in M^{d \times m}$ and $g(t, x_t) = \int_{-h}^0 f_2(t, x_t, y)dy$, $u(t) \in L_p([0, T])$, $u(t) \in \mathcal{U}$.

Example 4.2 Equations with maximum.

A particular case of the problem (1.1)-(1.2) is the optimal control problem with maximum on the interval $[-h, T], h > 0$.

$$\dot{x}(t) = f_1(t, x_t, \max_{s \in I(t)} x(s)) + f_2(t, x_t, \max_{s \in I(t)} x(s))u(t), \qquad (4.1)$$

$$x(t) = \varphi(t), \ t \in [-h, 0],$$

$$J[u] = \int_0^\tau L(t, x(t), u(t))dt \ \to \inf, \qquad (4.2)$$

where $I(t) = [\beta(t), \alpha(t)]$, $\max x(s) = (\max x_1(s), \ldots, \max x_d(s))$, $\beta(t), \alpha(t)$ are continuous on $[0, T]$ functions such that $\beta(t) \le \alpha(t) \le t$ and $\min_{t \in [0,T]} (\beta(t) - t) = -h$, G is a domain in $\mathbb{R}^d$, $f(t, x, y) : [0, T] \times G \times G \to M^{d \times m}$, $u \in U \subset \mathbb{R}^m$, $L(t, x, u) : [0, T] \times G \times U \to \mathbb{R}^1$.

The general theory of equations with maxima is presented in the monograph [12].

The problem (4.1)-(4.2) reduces to problem (1.1)-(1.2) if we put

$$u(t, y) = u(t) \in L_p([0, T]),$$

$$\tilde{f}_1(t, \varphi) = f_1(t, \varphi(0), \max_{\theta \in [\beta(t)-t, \alpha(t)-t]} \varphi(\theta)),$$

$$\tilde{f}_2(t, \varphi) = \int_{-h}^0 f_2(t, \varphi(0), \max_{\theta \in [\beta(t)-t, \alpha(t)-t]} \varphi(\theta), s)ds.$$

Let the following conditions be satisfied:

4.A. Functions $f_1(t, x, y)$ and $f_2(t, x, y, s)$ are defined and measurable with respect to all its arguments in domains $Q = \{t \in [0, T], x \in G, y \in G\}$, and $Q_1 = \{t \in [0, T], x \in G, y \in G, s \in [-h, 0]\}$ and satisfies with respect to x, y the linear growth and the Lipschitz condition with constant $K > 0$ in these domains, i. e.

$$|f_1(t, x, y)| + \|f_2(t, x, y, s)\| \le K(1 + |x| + |y|) \qquad (4.3)$$

$$|f_1(t, x, y) - f_1(t, x_1, y_1)| + \|f_2(t, x, y, s) - f_2(t, x_1, y_1, s)\| \le$$

$$\le K(|x_0 - x_1| + |y - y_1|) \qquad (4.4)$$

for all $t \in [0, T]$, $x, y, x_1, y_1 \in Q$ $s \in [-h, 0]$.

4.B. 1) The function $L(t, x, y) : [0, T] \times G \times U \to \mathbb{R}^1$ is defined and continuous with respect to all its arguments and satisfies the Lipschitz condition with respect to x;

2) the partial derivative L_u is continuous in the domain of definition and satisfies for some $C_0 > 0$, $\alpha > 0$ the following estimate:

$$\|L_u(t, x, u)\| \leq C_0(1 + |x|^\alpha + |u|^{p-1});$$

3) there exists a constant $C_1 > 0$ such that

$$L(t, x, u) \geq C_1 |u|^p, p > 1;$$

4) the function $L(t, x, u)$ is convex with respect to u for each fixed $t \in [0, T]$, $x \in G$. The optimal control problem (4.1)-(4.2) can be written as follows:

$$\dot{x}(\theta) = f_1(t, x_t), \max_{\theta \in I(t)} x(t)) + \int_{-h}^0 f_2(t, x(t)) \max_{\theta \in I(t)} x_t, s) ds\, u(t) \qquad (4.5)$$

$$x(t) = \varphi(t), t \in [-h, 0]$$

$$I(t) = [\beta(t) - t, \alpha(t) - t]$$

$$I(u) = \int_0^\tau L(t, x(t), u(t)) dt \to \inf. \qquad (4.6)$$

Then all of the conditions of Assumptions 2.1-2.3 hold.

Moreover, the domain $D \subset [-h, T] \times C$ is a set $\{(t, \varphi) : t \in [-h, T], \varphi \in \Omega$, where Ω is a set of functions $\varphi \in C$ such that $\varphi(\theta) \in G$ for $\theta \in [-h, 0]$, $\partial\Omega$ is a set of functions $\varphi \in C$ such that $\varphi(\theta) \in \bar{G}$, and for each of these functions there exists a point $\theta \in [-h, 0]$ such that $\varphi(\theta) \in \partial G$. It is obvious that the set $[0, T] \times \Omega = D$ is open, and $\partial D = ([0, T] \times \partial\Omega) \cup (\{T\} \times \bar{\Omega})$ is closed.

Let us check the conditions of Assumptions 2.1-2.3. Indeed, by 4.A we have

$$\left| \tilde{f}_1(t, \varphi) \right| = \left| f_1(t, \varphi(0), \max_{\theta \in [\beta(t)-t, \alpha(t)-t]} \varphi(\theta)) \right| \leq$$

$$\leq K \left(1 + |\varphi(0)| + \left| \max_{\theta \in [\beta(t)-t, \alpha(t)-t]} \varphi(\theta) \right| \right) \leq K \left(1 + \|\varphi\|_C + \|\varphi\|_C \right)$$

and for $\tilde{f}_1(t, \varphi)$ the condition (2.1) holds. For $\tilde{f}_2(t, \varphi)$ the situation is similar. Further

$$\left| \tilde{f}_1(t, \varphi) - \tilde{f}_1(t, \varphi_1) \right| \leq$$

$$\leq K \left(|\varphi(0) - \varphi_1(0)| + \left| \max_{\theta \in [\beta(t)]-t, \alpha(t)-t]} \varphi(\theta) - \max_{\theta \in [\beta(t)-t, \alpha(t)-t]} \varphi_1(\theta) \right| \right) \leq$$

$$\leq K \left(\|(\varphi - \varphi_1\|_C + \left| \max_{\theta \in [\beta(t)-t, \alpha(t)-t]} |\varphi(\theta) - \varphi_1(\theta)| \right| \right) \leq 2K \|\varphi - \varphi_1\|_C,$$

that is the condition (2.2) holds.

Further, since the mapping L is finite-dimensional with respect to u, the Frechet derivative L_u is the Jacobi matrix $\frac{\partial L}{\partial u}$, and the norm $\|L_u\|_{L_q} = \|L_u\|$. Therefore, condition 3) from Assumption 2.3 is trivially satisfied. It is obvious that other conditions from Assumption 2.3 are also satisfied. Therefore, for problems (4.1) and (4.2), when conditions 4.1 and 4.2 are satisfied, Theorems 2.1 and 2.2 hold.

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