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Solitary Wave Solutions of the Phi-Four Equation and the Breaking Soliton System by Means of Jacobi Elliptic Sine-Cosine Expansion Method

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Abstract: The goal of this study is twofold. The Jacobi elliptic expansion method is used to extract new solutions for the phi-four equation and the breaking soliton system. Special values of the Jacobi elliptic module and other involved parameters are chosen to produce solutions of soliton type and singular periodic solutions. The obtained solutions are verified and presented graphically.

Keywords: *Jacobi elliptic sine-cosine expansion method; phi-four equation; breaking soliton system.*

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1 Introduction

Solitary waves occur due to nonlinear phenomena appearing in different fields of science and engineering. These nonlinear phenomena are interpreted as $(n + 1)$-dimensional nonlinear partial differential equations. Seeking the exact solutions to these equations provides essential information about the physical structure of such phenomena. Since there is no specific method that produces such solutions, researchers made all the efforts to construct and modify methods to retrieve different kind of solutions for the same nonlinear model. We may mention some of these well-known techniques such as: the simplified bilinear method [11, 18, 31], sine-cosine method [4, 5], rational trigonometric function method [6], tanh method [7], extended tanh method [12,27], Yan transformation

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method [33–35], sech-tanh method [8–10,32], exponential-function method [25], the first integral method [2,29], the (G'/G)-expansion method [3,23,24], etc.

In this work, we use the Jacobi elliptic expansion method to explore further new solutions for two physical models: the phi-four equation that reads [17]

$$u_{tt} - \alpha u_{xx} - \lambda u + \beta u^2 = 0, \tag{1}$$

and the breaking soliton system

$$\begin{aligned} u_t &= -\alpha u_{xxy} - 4\alpha(uv)_x, \\ u_y &= v_x. \end{aligned} \tag{2}$$

The phi-four equation is a mathematical model that is used in nuclear and particle physics. Many methods have been used to study the solutions of this model. In [13], the modified simple equation method is used and tanh-coth solutions are derived. The modified (G'/G)-expansion method is adopted in [26] and produced the same solutions as in [13]. In [28], $\tan^2$ and $\cot^2$ solutions are obtained by using the extended direct algebraic method. Finally, the exponential-function method is used and rational trigonometric solutions of the phi-four equation are obtained in [14].

Different versions of the breaking soliton model are also studied by many researchers. In [30], the mapping method is used to obtain propagating solutions. The tanh-coth method is implemented [15] to construct solitary and soliton solutions of the breaking soliton equations. Finally, the exponential-function method is used [16] to obtain multiple soliton solutions of $(2+1)$ and $(3+1)$-dimensional breaking soliton equations.

2 Jacobi Elliptic Sine-Cosine Expansion Method

Partial differential equations can be written as a polynomial of the unknown function and its partial derivatives, i.e.

$$f(u, u_t, u_x, u_{xt}, u_{xx}...) = 0, \quad u = u(x,t). \tag{3}$$

By using the variable of the form $\xi = \mu(x - ct)$ and the chain rule, equation (3) is transformed into

$$g(u, -c\mu u', \mu u', -c\mu^2 u'', \mu^2 u'', ...) = 0, \quad u = u(\xi). \tag{4}$$

For the Jacobi elliptic sine-cosine technique [1,19–22], we write the solution as a power series of order n in terms of either the Jacobi elliptic sine $sn(\xi, m)$ or cosine $cn(\xi, m)$. The index m is regarded as the Jacobi module and $0 \le m \le 1$, i.e.

$$u(\xi) = \sum_{i=0}^{n} a_i Y^i, \tag{5}$$

where

$$Y = sn(\xi, m), \tag{6}$$

or

$$Y = cn(\xi, m). \tag{7}$$

Then, we determine the value of n by matching the order of Y in the highest derivative term with its order in the other nonlinear terms of the equation. Once n is obtained, we substitute (5) in (4) and collect the coefficients of $Y^i : \ i = 0, 1, 2, ..., n, ...$ Setting these coefficients to zero and solving the resulting non algebraic system lead to identifying the required $a_0,\ a_1,\ ...,\ a_n,\ \mu$ and c.

3 The Phi-Four Equation

Consider the phi-four equation that reads

$$u_{tt} - \alpha u_{xx} - \lambda u + \beta u^2 = 0. \tag{8}$$

By the wave variable $\xi = k(x - ct)$, equation (8) is turned into the differential equation:

$$k^2(c^2 - \alpha)u'' - \lambda u + \beta u^2 = 0. \tag{9}$$

Balancing u'' with u^2 produces the algebraic equation $n+2 = 2n$ whose solution is $n = 2$. Thus, the solution of (8) in terms of the elliptic sine function will have the form

$$u(\xi) = a_0 + a_1\ sn(\xi, m) + a_2\ sn^2(\xi, m). \tag{10}$$

Substituting (10) into (9) and collecting the coefficients of the same power of sn lead to the nonlinear algebraic system

$$\begin{aligned}
0 &= 2a_2k^2(c^2 - \alpha) + a_0(a_0\beta - \lambda),\\
0 &= -a_1(c^2k^2(1 + m^2) - k^2(1 + m^2)\alpha - 2a_0\beta + \lambda),\\
0 &= a_1^2\beta - a_2(4c^2k^2(1 + m^2) - 4k^2(1 + m^2)\alpha - 2a_0\beta + \lambda),\\
0 &= 2a_1(c^2k^2m^2 - k^2m^2\alpha + a_2\beta),\\
0 &= a_2(6c^2k^2m^2 - 6k^2m^2\alpha + a_2\beta).
\end{aligned} \tag{11}$$

By solving the above system for the parameters a_0, a_1, a_2, c and k, we get

$$\begin{aligned}
a_0 &= \frac{\lambda}{2\beta}\left(1 - \frac{1 + m^2}{\sqrt{1 - m^2 + m^4}}\right),\\
a_1 &= 0,\\
a_2 &= \frac{3m^2\lambda}{2\beta\sqrt{1 - m^2 + m^4}},\\
c &= \frac{1}{2}\sqrt{4\alpha - \frac{\lambda}{k^2\sqrt{1 - m^2 + m^4}}}\ ,
\end{aligned} \tag{12}$$

where k is a free parameter. Thus, our first solution to the phi-four model is

$$\begin{aligned}
u(x,t) &= \frac{3m^2\lambda}{2\beta\sqrt{1 - m^2 + m^4}}\ sn^2(k(x - \frac{1}{2}\sqrt{4\alpha - \frac{\lambda}{k^2\sqrt{1 - m^2 + m^4}}}t), m)\\
&+ \frac{\lambda}{2\beta}\left(1 - \frac{1 + m^2}{\sqrt{1 - m^2 + m^4}}\right).
\end{aligned} \tag{13}$$

Substituting $m = 1$ in (13) produces the soliton solution

$$u(x,t) = -\frac{\lambda}{2\beta} + \frac{3\lambda}{2\beta}\ \tanh^2(k(x - \frac{1}{2}t\sqrt{4\alpha - \frac{\lambda}{k^2}})). \tag{14}$$

Now, replacing sn in (10) by cn will lead to a second solution, which is

$$\begin{aligned}
u(x,t) &= \frac{-3m^2\lambda}{2\beta\sqrt{1 - m^2 + m^4}}\ cn^2(k(x - \frac{1}{2}\sqrt{4\alpha - \frac{\lambda}{k^2\sqrt{1 - m^2 + m^4}}}t), m)\\
&+ \frac{\lambda}{2\beta}\left(1 + \frac{2m^2 - 1}{\sqrt{1 - m^2 + m^4}}\right).
\end{aligned} \tag{15}$$

Let $m = 1$ in (15), this produces the soliton solution

$$u(x,t) = \frac{\lambda}{\beta} - \frac{3\lambda}{2\beta}\ \mathrm{sech}^2(k(x - \frac{1}{2}t\sqrt{4\alpha - \frac{\lambda}{k^2}})). \tag{16}$$

Remark 1 The obtained solution given in (16) can be obtained directly from (14) by using the identity $\mathrm{sech}^2(x) = 1 - \tanh^2(x)$.

Remark 2 If we replace the free parameter k in (14) by $i\gamma$ with $i = \sqrt{-1}$, we obtain the singular periodic solution

$$u(x,t) = -\frac{\lambda}{2\beta} - \frac{3\lambda}{2\beta}\ \tan^2(\gamma(x - \frac{1}{2}t\sqrt{4\alpha + \frac{\lambda}{\gamma^2}})). \tag{17}$$

Also, in (16), we obtain the singular periodic solution

$$u(x,t) = \frac{\lambda}{\beta} - \frac{3\lambda}{2\beta}\ \sec^2(\gamma(x - \frac{1}{2}t\sqrt{4\alpha + \frac{\lambda}{\gamma^2}})), \tag{18}$$

where the singularities occur on the line characteristics $\gamma(x - \frac{1}{2}t\sqrt{4\alpha + \frac{\lambda}{\gamma^2}}) = \frac{\pi}{2} + n\pi$.

Proof: Use the fact that $\tanh(ix) = i\tan(x)$ and $\mathrm{sech}\,(ix) = \sec(x)$.

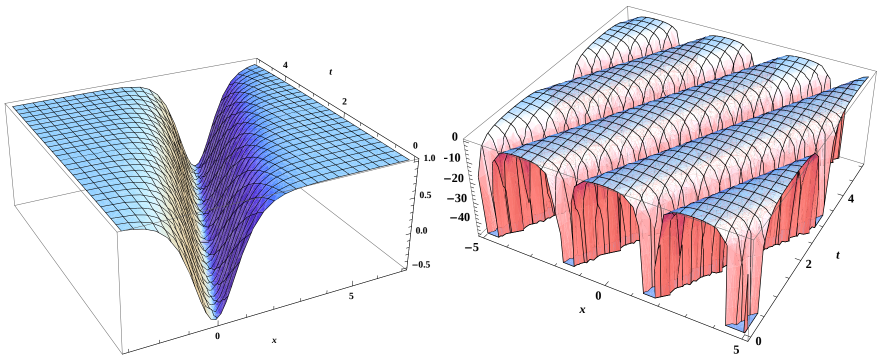

Figure 1: The obtained solutions given in (14) and (18) respectively, when $\lambda = \alpha = \beta = k = 1$.

4 $(2+1)$-Dimensional Breaking Soliton Equations

We recall the following (2+1)-dimensional breaking soliton equations

$$\begin{aligned} u_t &= -\alpha u_{xxy} - 4\alpha(uv)_x, \\ u_y &= v_x. \end{aligned} \tag{19}$$

Substituting $\xi = \mu x + \lambda y - ct$ into (19) yields

$$-cu + \alpha\mu^2\lambda u'' + 4\alpha\mu(uv) = 0 \tag{20}$$

and

$$\lambda u' = \mu v'. \tag{21}$$

From (21) we get

$$v = \frac{\lambda}{\mu} u. \tag{22}$$

Substituting (22) in (20) yields

$$-cu + 4\alpha\lambda u^2 + \alpha\lambda\mu^2 u'' = 0. \tag{23}$$

Balancing u'' with u^2 in (23) produces the algebraic equation $n + 2 = 2n$ whose solution is $n = 2$. Thus, by the Jacobi elliptic sine expansion, the solution has the form

$$u(\xi) = a_0 + a_1\ sn(\xi, m) + a_2\ sn^2(\xi, m). \tag{24}$$

Substitute (24) into (23) to get the following algebraic system

$$\begin{aligned}
0 &= -a_0 c + 4a_0^2\alpha\lambda + 2a_2\alpha\lambda\mu^2, \\
0 &= -a_1(c + \alpha\lambda(-8a_0 + (1 + m^2)\mu^2)), \\
0 &= 4a_1^2\alpha\lambda - a_2(c + 4\alpha\lambda(-2a_0 + (1 + m^2)\mu^2)), \\
0 &= 2a_1\alpha\lambda(4a_2 + m^2\mu^2), \\
0 &= 2a_2\alpha\lambda(2a_2 + 3m^2\mu^2).
\end{aligned} \tag{25}$$

Solving the above system with respect to a_0, a_1, a_2, μ, λ and c, we get

$$\begin{aligned}
a_0 &= \frac{1}{2}\left(1 + m^2 \pm \sqrt{1 - m^2 + m^4}\right)\mu^2, \\
a_1 &= 0, \\
a_2 &= \frac{-3}{2} m^2\mu^2, \\
c &= \pm 4\alpha\lambda\mu^2\sqrt{1 - m^2 + m^4}.
\end{aligned} \tag{26}$$

Thus, the solution is

$$\begin{aligned}
u(x, y, t) &= \frac{1}{2}\mu^2\{1 + m^2 + A - 3m^2 sn^2(\mu x + \lambda y - 4A\alpha\lambda\mu^2 t, m)\}, \\
v(x, y, t) &= \frac{1}{2}\lambda\mu\{1 + m^2 + A - 3m^2 sn^2(\mu x + \lambda y - 4A\alpha\lambda\mu^2 t, m)\},
\end{aligned} \tag{27}$$

where $A = \sqrt{1 - m^2 + m^4}$. When $m = 1$ in (27), we obtain

$$\begin{aligned}
u(x, y, t) &= \frac{3}{2}\mu^2\left(1 - \tanh^2(\mu x + \lambda y - 4\alpha\lambda\mu^2 t)\right), \\
v(x, y, t) &= \frac{3}{2}\lambda\mu\left(1 - \tanh^2(\mu x + \lambda y - 4\alpha\lambda\mu^2 t)\right).
\end{aligned} \tag{28}$$

Now, by the Jacobi elliptic cosine expansion, the solution has the form

$$u(\xi) = a_0 + a_1\ cn(\xi, m) + a_2\ cn^2(\xi, m) \tag{29}$$

Substituting (29) into (23) and solving the resulting algebraic system, we arrive at

$$\begin{aligned} u(x,y,t) &= \frac{1}{4}\mu^2\{1-2m^2-m^4+B+2(m^2+2m^4)\ cn^2(\mu x+\lambda y-2B\alpha\lambda\mu^2 t,m)\},\\ v(x,y,t) &= \frac{1}{4}\lambda\mu\{1-2m^2-m^4+B+2(m^2+2m^4)\ cn^2(\mu x+\lambda y-2B\alpha\lambda\mu^2 t,m)\}, \end{aligned} \tag{30}$$

where $B=\sqrt{1+6m^4-4m^6+m^8}$ and $\lambda,\ \mu$ are free variables. When $m=1$, the solution is

$$\begin{aligned} u(x,y,t) &= \frac{3}{2}\mu^2\ \mathrm{sech}^2(\mu x+\lambda y-4\alpha\lambda\mu^2 t),\\ v(x,y,t) &= \frac{3}{2}\lambda\mu\ \mathrm{sech}^2(\mu x+\lambda y-4\alpha\lambda\mu^2 t). \end{aligned} \tag{31}$$

Remark 3 If we replace λ by $\theta\lambda_1$ and μ by $\theta\mu_1$ and θ by $i\theta_1$ in both (28) and (31), where $i=\sqrt{-1}$, two singular periodic solutions are obtained

$$\begin{aligned} u(x,y,t) &= -\frac{3}{2}\theta_1^2\mu_1^2\left(1+\tan^2(\theta_1(\mu_1 x+\lambda_1 y+\theta_1^2 4\alpha\lambda_1\mu_1^2 t))\right),\\ v(x,y,t) &= -\frac{3}{2}\theta_1^2\lambda_1\mu_1\left(1+\tan^2(\theta_1(\mu_1 x+\lambda_1 y+\theta_1^2 4\alpha\lambda_1\mu_1^2 t))\right) \end{aligned} \tag{32}$$

and

$$\begin{aligned} u(x,y,t) &= -\frac{3}{2}\theta_1^2\mu_1^2\sec^2(\theta_1(\mu_1 x+\lambda_1 y+\theta_1^2 4\alpha\lambda_1\mu_1^2 t)),\\ v(x,y,t) &= -\frac{3}{2}\theta_1^2\lambda_1\mu_1\sec^2(\theta_1(\mu_1 x+\lambda_1 y+\theta_1^2 4\alpha\lambda_1\mu_1^2 t)). \end{aligned} \tag{33}$$

The singularities of the last two solutions occur on the plane characteristics $\theta_1(\mu_1 x+\lambda_1 y+\theta_1^2 4\alpha\lambda_1\mu_1^2 t)=\frac{\pi}{2}+n\pi$.

5 Conclusion

In this paper, we used the Jacobi elliptic sine-cosine expansion method to study the solutions of two physical models, the phi-four equation and the $(2+1)$-dimensional breaking soliton system. Special values of the Jacobi elliptic module and the free parameters lead us to different types of solutions to these models such as soliton, singular-soliton and periodic solution. This work reveals that the proposed method is a reliable technique that provides different types of solutions and is relatively easy when applied to nonlinear equations.

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