Nonlinear Dynamics and Systems Theory, **18** (3) (2018) 241–252

Average Edge Betweenness of a Graph

A. Aytaç * and H.A. Öztürk

Department of Mathematics, Faculty of Science, Ege University, Izmir/TURKEY

Received: February 11, 2016; Revised: June 19, 2018

Abstract: Vulnerability is an important concept in network analysis. When a failure occurs in some of the components of the network, vulnerability measures the ability of the network to disruption in order to avoid the external or internal effects. Graph theory is an important concept in network vulnerability analysis. If a network is modeled as an undirected and unweighted graph composed of processing vertices and communication links, there have been several proposals for measuring graph vulnerability under link or vertex failures. In this paper, we consider the concept of average edge betweenness of a graph in order to measure the network stability. The average edge betweenness is related to the edge betweenness of an edge. The edge betweenness of a given edge is the fraction of shortest paths, counted over all pairs of vertices that pass through that edge. The average edge betweenness considers both the local and the global structure of the graph. In this paper, we obtain exact values for average edge betweenness and normalized average edge betweenness for some special graphs and E_p^t graph.

Keywords: *network vulnerability; network design and communication; stability; average edge betweenness.*

Mathematics Subject Classification (2010): 05C40, 68M10, 68R10.

1 Introduction

Many complex systems in the real world can be conceptually described as networks, where vertices represent the system constituents and edges depict the interaction between them, such as social networks (collaboration network), technological networks (communication networks, the Internet), information networks (the World Wide Web), biological networks (protein-protein interaction networks, neural networks) and etc. [10, 11] . A central issue in the analysis of complex networks is the assessment of their stability and vulnerability. Vulnerability is an important concept in network analysis related with the ability of

* Corresponding author: mailto:aysun.aytac@ege.edu.tr

the network to avoid intentional attacks or disruption when a failure is produced in some of its components. Often enough, the network is modeled as an undirected and unweighted graph. Different measures for graph vulnerability have been introduced so far to study different aspects of the graph behavior after removal of vertices or links such as connectivity, toughness, scattering number, integrity, residual closeness and exponential domination number [1, 4, 9, 12–15].

As an important parameter in the study of networks associated with complex systems in both modeling and measuring the reliability, the graph-theoretical concept of vertex betweenness was first proposed by Freeman [7] in 1977. Then, Girvan and Newman in [8] generalize this definition to edges and introduce the edge betweenness of an edge as the fraction of shortest paths between pairs of vertices that run along it. The edge betweenness of a given edge is the fraction of shortest paths, counted over all pairs of vertices that pass through that edge. This measure considers both the local and the global structure of the graph. Since average edge betweenness gives information on which edge carries the most of the network vulnerability, it is important to determine the average edge betweenness of several graph classes.

In this paper, we consider simple finite undirected graphs without loops and multiple edges. Let $G = (V, E)$ be a graph with a vertex set $V = V(G)$ and an edge set $E = E(G)$. The *complement* $\overline{G}$ of a graph G is the graph with a vertex set $V(G)$ such that two vertices are adjacent in $\overline{G}$ if and only if these vertices are not adjacent in G. A *vertex dominating set* for a graph G is a set S of vertices such that every vertex G belongs to S or is adjacent to a vertex of S. The minimum cardinality of a vertex dominating set in a graph G is called the *vertex dominating number* of G and is denoted by $\gamma(G)$. The *distance* $d(u, v)$ between two vertices u and v in G is the length of the shortest path between them. If u and v are not connected, then $d(u, v) = \infty$, and for $u = v$, $d(u, v) = 0$. In addition, the distance between the vertices u and v in G can be denoted by $d(u, v\,|G)$. The *diameter* of G, denoted by $diam\,(G)$, is the largest distance between two vertices in $V(G)$ [2].

The paper proceeds as follows. In Section 2, definitions and known results for average edge betweenness and normalized average edge betweenness are given. In Sections 3 and 4, average edge betweenness and normalized average edge betweenness of some special graphs are respectively determined and exact values are given. Conclusions are addressed in Section 5.

2 Average Edge Betweenness and Normalized Average Edge Betweenness

In this paper, we consider a simple finite undirected graph that has no self-loops and possesses no more than one edge between any two different vertices. Let $G = (V, E)$ be a graph with a vertex set $V = V(G)$ and an edge set $E = E(G)$.

Average edge betweenness of the graph G is defined as $b(G) = \frac{1}{|E|}\sum\limits_{e\in E} b_e$, where $|E|$ is the number of the edges, and b_e is the edge betweenness of the edge e, defined as $b_e = \sum\limits_{i\neq j} b_e(i,j)$, where $b_e(i,j) = n_{ij}(e)/n_{ij}$, $n_{ij}(e)$ is the number of geodesics (shortest paths) from vertex i to vertex j that contain the edge e, and n_{ij} is the total number of shortest paths [3, 5].

Let us compare two graphs G_1 and G_2. If $b(G_1) < b(G_2)$, then $G_1$is more stable than G_2. Since for a graph with a fixed number of vertices $b(G)$ decreases as the number of edges in the graph increases, it can be said that they represent how "well connected"the graph is. The higher the values of $b(G)$, the more vulnerable G is to the loss of edges. We consider the concept of average edge betweenness of a graph because when computing $b(G)$, we can gather information on which edge carries the most of the graph vulnerability.

A complete graph is a simple graph in which every pair of distinct vertices is connected by an edge. The complete graph with n vertices has $n(n-1)/2$ edges. For a complete graph, we have $b\left(G_{complete}\right) = 1$. A path graph is a particularly simple example of a tree, which is not branched at all, that is, it contains only vertices of degree two and one. In particular, two of its vertices have degree 1 and all others (if any) have degree 2. For a path graph with n vertices, $|E| = n-1$, and therefore: $b\left(G_{path}\right) = n\left(n+1\right)/6$.
It is easy to see that $b\left(G_{complete}\right) \leq b\left(G\right) \leq b\left(G_{path}\right)$. As a consequence, we can define the normalized average edge betweenness of a graph G

$$b_{nor}(G) = \frac{b(G) - b(G_{complete})}{b(G_{path}) - b(G_{complete})} = (b(G) - 1)/(n(n+1)/6 - 1).$$

Clearly $0 \leq b_{nor}\left(G\right) \leq 1$; if the normalized average edge betweenness is close to 0, it means that the network is more robust, when it is close to 1, then the graph is more vulnerable.

The following lemma provides some basic properties for the betweenness related parameters. Let us recall that for a graph G, b_e is the betweenness of edge e, $b(G)$ is the average edge betweenness of G.

Example 2.1 Let us find the edge betweenness value of each edge of the graph G with sixvertices and sevenedges given in Fig. 1. Let us find the average edge betweenness value of the graph G.

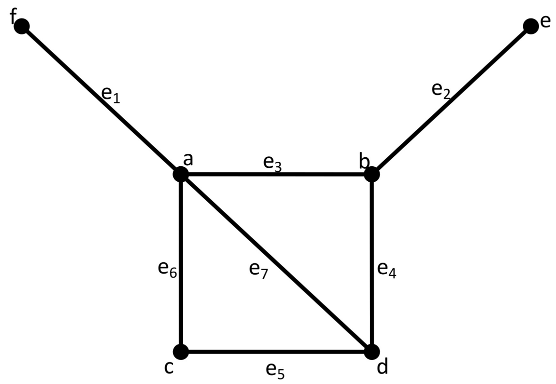

Figure 1: The graph G with sixvertices and sevenedges.

In Table 1, the shortest paths between all pairs of vertices of the graph G are found. According to these shortest paths, the edge betweenness values of each edge are calculated. Next, the normalization process is performed by finding the average edge betweenness value of the graph G.

As can be seen in line SUM of Table 1, the edge betweenness values of the edges $e_1, e_2, e_3, e_4, e_5, e_6$ *and* e_7 are found to be 5, 5, 5, 3, 2, 3 and 2, respectively. Here, the highest edge betweenness value is 5. This shows that the edges e_1, e_2 *and* e_3 are the most important positions in the graph. According to Table 1, the lowest value is 2. The edges

pairs of vertices	the shortest paths	e_1	e_2	e_3	e_4	e_5	e_6	e_7
a,b	e_3	0	0	1	0	0	0	0
a,c	e_6	0	0	0	0	0	1	0
a,d	e_7	0	0	0	0	0	0	1
a,e	e_3e_2	0	1	1	0	0	0	0
a,f	e_1	1	0	0	0	0	0	0
b,c	e_3e_6, e_4e_5	0	0	1/2	1/2	1/2	1/2	0
b,d	e_4	0	0	0	1	0	0	0
b,e	e_2	0	1	0	0	0	0	0
b,f	e_3e_1	1	0	1	0	0	0	0
c,d	e_5	0	0	0	0	1	0	0
c,e	$e_6e_3e_2$, $e_5e_4e_2$	0	2/2	1/2	1/2	1/2	1/2	0
c,f	e_6e_1	1	0	0	0	0	1	0
d,e	e_4e_2	0	1	0	1	0	0	0
d,f	e_7e_1	1	0	0	0	0	0	1
e,f	$e_2e_3e_1$	1	1	1	0	0	0	0
	SUM	5	5	5	3	2	3	2

Table 1: The edge betweenness values of the edges and the average edge betweenness value of the graph G.

e_5 and e_7 have this value. This fact shows that these edges play a more passive role than other edges of the graph. By using these values, the average edge betweenness value of G is obtained as

$$b(G) = \frac{1}{7}\sum_{i=1}^{7} b_{e_i} = \frac{25}{7} = 3,57.$$

For $n = 6$, the normalized average edge betweenness value of the G graph is as follows:

$$b_{nor}(G) = \frac{b(G)-1}{\frac{n(n+1)}{6}-1} = \frac{\frac{25}{7}-1}{\frac{42}{6}-1} = \frac{18}{42} = 0,4.$$

Lemma 2.1 *[5] Let G be a connected graph and let $e \in E$ be an edge with end vertices i, $j \in V$, then*

1. $b_e(i,j) = 1 = b_e(j,i)$.
2. $2 \leq b_e \leq n^2/2$ *if n is even and* $2 \leq b_e \leq (n-1)^2/2$ *if n is odd.*
3. $b_e = 2(n-1)$ *if one of the end vertices of e has degree* 1.

Lemma 2.2 *[5] Let G be a graph of order n, then*

1. *If e is an edge-bridge of the graph G connecting G_1 with $G \backslash G_1$, where $|V(G_1)| = n_1$, then $b_e = 2n_1(n-n_1)$.*
2. *If C is a cut-set of edges of the graph G connecting two sets of vertices X and $V(G) \backslash X$ and*

$$|X| = n_x \ , \quad \textit{then} \ \sum\nolimits_{e\in C} b_e = 2n_x(n-n_x) \ .$$

Theorem 2.1 *Let $\overline{G}$ be the complement graph of G. Then, if G has n vertices and m edges with domination number $\gamma(G) > 2$, then the average edge betweenness of $\overline{G}$ is*

$$b(\overline{G}) = (n(n-1) + 2m)/(n(n-1) - 2m).$$

Proof. Let i and j be the vertices of G and e be any edge of G. We have two cases according to $d(i,j)$:

Case 1. If $d(i,j) > 1$ is in G graph, then $d(i,j)=1$ is in $\overline{G}$. Therefore, there are $(n(n-1)/2\ -m)$ paths with length 1 in $\overline{G}$. Thus, for all vertex pairs i and j, the summation of the values of edge betweenness of e is

$$\sum_{i \neq j} b_e(i,j) = (n(n-1)/2\ -m)\,.$$

Case 2. If $d(i,j) = 1$ is in G graph, then $d(i,j) > 1$ is in $\overline{G}$. Let t be the number of vertices which are not adjacent to vertices i and j. Since $\gamma(G) > 2$, it is clear that $t \geq 1$. Thus, there are t paths with length 2 in $\overline{G}$. Hence, for all vertex pairs i and j , the summation of the values of edge betweenness of e is

$$\sum_{i \neq j} b_e(i,j) = t\,(1/t\ )\,2m = 2m.$$

By summing up *Cases 1* and *2*, we obtain

$$\sum_{e \in E} b_e = (n(n-1)/2\ -m) + 2m = n(n-1)/2\ +m.$$

As a consequence, the average edge betweenness of $\overline{G}$ is

$$b(\overline{G}) = 1/(n(n-1)/2\ -m)\,(n(n-1)/2\ +m) = (n(n-1) + 2m)/(n(n-1) - 2m).$$

The proof is completed. □

3 The Average Edge Betweenness of Some Special Graphs

In this section, we give some results on average edge betweennesses of some special graphs. These graphs are: C_n is a cycle graph, $S_{1,n}$ is a star graph, $W_{1,n}$ is a wheel graph, and $K_{m,n}$ is a complete bipartite graph. Finally we give average edge betweenness of E_p^t graph.

Lemma 3.1 *Label the vertices of C_n as $1, 2, 3, \ldots, n$ and the edges of C_n as e_1, e_2, e_3, $\ldots$, e_n, respectively. Let $d_{ij}(e_k)$ be the distance between i and j including the edge e_k. $n_{ij}(e_k)$ is the number of paths which include the edge e_k with length $d_{ij}(e_k)$ $(1 \leq i, j, k \leq n$ and $i \neq j)$. The relation between $d_{ij}(e_k)$ and $n_{ij}(e_k)$in graph C_n is the following*

$$If\, d_{ij}(e_k) = 1, \quad then \quad n_{ij}(e_k) = 1 \tag{1}$$
$$If\, d_{ij}(e_k) = 2, \quad then \quad n_{ij}(e_k) = 2 \tag{2}$$
$$If\, d_{ij}(e_k) = 3, \quad then \quad n_{ij}(e_k) = 3 \tag{3}$$
$$\vdots \tag{4}$$
$$If\, d_{ij}(e_k) = (n-1)/2, \quad then \quad n_{ij}(e_k) = (n-1)/2\,. \tag{5}$$
$$\tag{6}$$

Theorem 3.1 *If C_n is a cycle graph, then the average edge betweenness for the cycle graph C_n with n vertices is*

$$b(C_n) = \begin{cases} (n^2-1)/8\,, & n \text{ is odd} \\ n^2/8\,, & n \text{ is even.} \end{cases}$$

Proof. There exist two cases according to n:

Case 1. If n is odd, then $n_{ij} = 1$ for $\forall i,j \quad (1 \le i,j,k \le n \text{ and } i \neq j)$, we get $b_{e_k} = \sum\limits_{i\neq j} \frac{n_{ij}(e_k)}{n_{ij}} = \sum\limits_{i\neq j} n_{ij}(e_k)$. From *Lemma* 3.1 and $d_{ij}(e_k) \le diam(C_n) = (n-1)/2$, we obtain $b_{e_k} = \sum\limits_{i\neq j} n_{ij}(e_k) = 1+2+3+\ldots+((n-1)/2\ ) = (n^2-1)/8$. By the definition of the average edge betweenness of a graph,

$$b(C_n) = \frac{1}{|E|}\left(\sum_{i=1}^{n} b_{e_i}\right) = (n^2-1)/8.$$

Case 2. If n is even, then we have two subcases for $d_{ij}(e_k)$.

Subcase 1. If $d_{ij}(e_k) < diam(C_n) = n/2$, then $n_{ij} = 1$ for $\forall i,j \quad (1 \le i,j,k \le n \text{ and } i \neq j)$. In this case we proceed in a similar way as in Case 1 and

$$b_{e_k}(i,j) = 1+2+3+\ldots+[(n/2)-1] = \left(n^2-2n\right)/8$$

is obtained.

Subcase 2. If $d_{ij}(e_k) = diam(C_n) = n/2$, then $n_{ij}(e_k) = n/2$ and $n_{ij} = 2$ for $\forall i,j \quad (1 \le i,j,k \le n \text{ and } i \neq j)$, we get

$$b_{e_k}(i,j) = \sum_{i\neq j} n_{ij}(e)/n_{ij} = (n/2)(1/2) = n/4.$$

By Subcase 1 and Subcase 2, it is clear that

$$b_{e_k} = (n^2-2n)/8\ + n/4\ = n^2/8 \quad (\forall k = \overline{1,n}).$$

Consequently, we obtain the average edge betweenness of C_n

$$b(C_n) = \frac{1}{|E|}\sum_{i=1}^{n} b_{e_i} = n^2/8.$$

Thus, the proof is completed. □

Theorem 3.2 *If $S_{1,n}$ is a star graph, then the average edge betweenness for the star graph $S_{1,n}$ with $n+1$ vertices is $b(S_{1,n}) = n$.*

Proof. The vertices of $S_{1,n}$ are of two kinds: one vertex of degree n and n vertices of degree one. The vertices of degree one will be referred to as the minor vertices and the vertex of degree n as the center vertex. Label the minor vertices as $1,2,3,\ldots,n$, the center vertex as c, and the edges of $S_{1,n}$ as $e_i \ (i = \overline{1,n})$. We have two cases in order to find the shortest paths.

Case 1. The shortest path between central vertex c and any minor vertex i:
There is only one path e_i in this case. By the definition of the edge betweenness, we obtain the value of the edge e_i $(i = \overline{1,n})$

$$b_{e_i}(c,i) = 1.$$

Case 2. The shortest path between any two different minor vertices:
There is only one path e_ie_j between the minor vertices i and j $(1 \leq i,j \leq n)$. By using *Lemma* 2.1, for $\forall i,j$, we get $n_{ij} = n_{ji} = 1$ and $n_{ij}(e_k) = 1$ $(k = i \vee j)$. Thus, we have $b_{e_k}(i,j) = 1/1 = 1$. There are $n-1$ different pairs of vertices that include e_k. Hence, the value of the edge betweenness of e_k

$$b_{e_k}(i,j) = (n-1)\ 1 = n-1.$$

By summing up *Cases 1* and *2*, we clearly see that

$$b_{e_i} = 1 + n - 1 = n.$$

Consequently, the average edge betweenness of $S_{1,n}$ is

$$b(S_{1,n}) = \frac{1}{|E|}\left(\sum_{i=1}^{n} b_{e_i}\right) = n.$$

Thus, the proof is completed. □

Theorem 3.3 *If $W_{1,n}$ is a wheel graph, then the average edge betweenness for the wheel graph $W_{1,n}$ $(n \geq 5)$ with $n+1$ vertices is $b(W_{1,n}) = (n-1)/2$.*

Proof. The vertices of $W_{1,n}$ are of two kinds: n vertices which are of degree 3 will be referred to as the minor vertices and the vertex of degree n will be referred to as the central vertex. Label the minor vertices as $1,2,3,\dots,n$, the central vertex as c, the edges between the central vertex and the minor vertices as $e_{c\,i}$ $(i = \overline{1,n})$ and the other remaining edges as e_i $(i = \overline{1,n})$. There exist two cases for the shortest paths between the pairs of vertices.

Case 1. If the pair of vertices includes the central vertex and the minor vertices:
There exists only one path e_{ci} between those vertices that has the length $d(c,i)$=1. It is clear that for the path e_{ci}, we have $n_{ci} = 1$ and $n_{ci}(e_{ci}) = 1$. Hence, the value of the edge betweenness of e_{ci}

$$b_{e_{ci}}(c,i) = 1.$$

Case 2. If the pair of vertices includes any two different minor vertices i and j:
We have three subcases for these minor vertices according to the length of the shortest path between the vertices:

Subcase 1. If $d(i,j) = 1$, then there is only one path e_k $(k = i \vee j)$ between i and j. It is clear that for this path e_k, we have $n_{ij} = 1$ and $n_{ij}(e_k) = 1$. Hence, the value of the edge betweenness of e_k

$$b_{e_k}(i,j) = 1.$$

Subcase 2. If $d(i,j) = 2$, then there are two paths: the paths $e_i e_j$ and $e_{ci} e_{cj}$ between the vertices i and j. The lengths of the paths between the vertices i and j including the edges e_k and e_{ck} $(k = i \vee j)$ are $d_{ij}(e_k) = 2$ and $d_{ij}(e_{ck}) = 2$, respectively. By Lemma 3.1, $n_{ij}(e_k) = 2$, $n_{ij}(e_{ck}) = 2$ and $n_{ij} = 2$. Thus we have

$$b_{e_k}(i,j) = 2/2 = 1, \quad b_{e_{ck}}(i,j) = 2/2 = 1.$$

Subcase 3. If $d(i,j) > 2$, then there is only one path between the vertices i and j with length 2, that is $e_{ci} e_{cj}$. It is clear that for this path $e_{ci} e_{cj}$, we have $n_{ij} = 1$ and $n_{ij}(e_{ck}) = 1$ $(k = i \vee j)$. Hence,

$$b_{e_{ck}}(i,j) = 1.$$

In this way, since there are $n - 5$ different pairs of vertices that include the edge e_{ck}, the value of the edge betweenness of e_{ck} is

$$b_{e_{ck}}(i,j) = 1(n-5) = n-5.$$

By summing up Subcases 1 and 2, we get the value of the edge betweenness of e_k as

$$b_{e_k} = 1 + 1 = 2.$$

By summing up Case 1 and Subcases 2 and 3, we get the value of the edge betweenness of e_{ck} as

$$b_{e_{ck}} = 1 + 1 + n - 5 = n - 3.$$

Consequently, the average edge betweenness of $W_{1,n}$ is

$$b(W_{1,n}) = \frac{1}{|E|}\left(\sum_{i=1}^{n} b_{e_i} + \sum_{i=1}^{n} b_{e_{ci}}\right) = (n-1)/2.$$

Thus, the proof is completed. □

Theorem 3.4 *If $K_{m,n}$ is a complete bipartite graph, then the average edge betweenness for the complete bipartite graph $K_{m,n}$ with $m+n$ vertices is $b(K_{m,n}) = (m^2 + n^2 - (m+n))/mn + 1$.*

Proof. Let $G = K_{m,n}$, where S_1 and S_2 are the partite sets of G with cardinality m and n respectively. The set of edges of $K_{m,n}$ is $E = \{e_{pk} \mid p \in S_1 \ \ and \ \ k \in S_2\}$ and $|E| = mn$. We have 3 cases in order to find the shortest paths according to the vertices being either in S_1 or in S_2 . Let i and j be the vertices of $K_{m,n}$.

Case 1. If $i \in S_1$ and $j \in S_2$, then there is only one path e_{ij} between the vertices i and j. Therefore, it is straightforward that $n_{ij} = 1$ and $n_{ij}(e_{ij}) = 1$. Thus

$$b_{e_{ij}}(i,j) = 1.$$

Case 2. If $i, j \in S_1$, then there are n paths $e_{ik} e_{jk}$ with length 2 between the vertices i and j $(k \in S_2)$. Clearly, $n_{ij} = n$, $n_{ij}(e_{pk}) = 1$ $(p = i \vee j)$. Hence,

$$b_{e_{pk}}(i,j) = 1/n.$$

There are $m - 1$ different pairs of vertices that include e_{pk}, the value of the edge betweenness of e_{pk} is

$$b_{e_{pk}}(i,j) = (m-1)/n.$$

Case 3. If $i, j \in S_2$, then there are m paths $e_{ik}e_{jk}$ with length 2 between the vertices i and j ($k \in S_1$). This case is similar to *Case 2*, and for $n-1$ different pairs of vertices that include e_{pk}, the value of the edge betweenness of e_{pk} is

$$b_{e_{pk}}(i, j) = (n-1)/m.$$

By summing up Cases 1, 2 and 3, we get the value of the edge betweenness of e_{pk} as

$$b_{e_{pk}} = (m^2 + n^2 - (m+n))/mn + 1.$$

Consequently, the average edge betweenness of $K_{m,n}$ is

$$b(K_{m,n}) = \frac{1}{|E|}\sum_{p=1}^{m}\sum_{k=1}^{n} b_{e_{pk}} = (m^2 + n^2 - (m+n))/mn + 1.$$

Hence the desired result holds. □

Definition 3.1 [6] The graph E_p^t has t legs and each leg has p vertices (Figure 2). Thus E_p^t has $pt+2$ vertices and $pt+1$ edges.

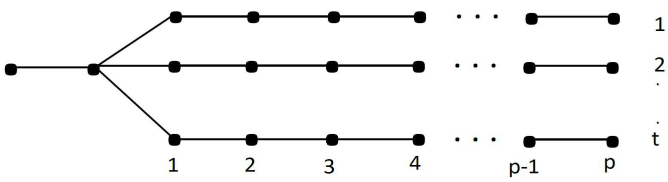

Figure 2: E_p^t graph with $pt+2$ vertices.

Theorem 3.5 *Let t and p be integers $(t \geq 2, p \geq 2)$. The average edge betweenness of graph E_p^t is*

$$b(E_p^t) = [(pt\,(p+1))\,/6\,(pt+1)]\;[3pt - 2p + 5] + 1.$$

Proof. Label the vertex with degree $t+1$ as v, the neighbor of v with degree 1 as u, the vertices of jth leg as (i, j) ($i = \overline{1,p}$ *and* $j = \overline{1,t}$), the edge between the vertices u and v as e, the edge between the vertices v and (i, j) as bridge e_{ij}, where $i = 1$, and the edges of jth leg as e_{ij} respectively ($i = \overline{2,p}$ and $j = \overline{1,t}$).

This labeling is shown in Figure 3. Since E_p^t is a tree, there is only one path between any pairs of vertices. Clearly, $n_{ij} = 1$ and $b_e = \sum\limits_{i \neq j}(n_{ij}(e)/n_{ij}) = \sum\limits_{i \neq j} n_{ij}(e)$ ($i = \overline{1,p},\ j = \overline{1,t}$). We have four cases for the vertex pairs of E_p^t.

Case 1. Consider the shortest paths between the vertex u and the other vertices. There exist $(pt+1)$ paths. Each of these paths includes the edge e. The value of the edge betweenness of this edge e is

$$b_e = b_e(u,\,(i,j)) \,+\, b_e(u,v) = pt + 1.$$

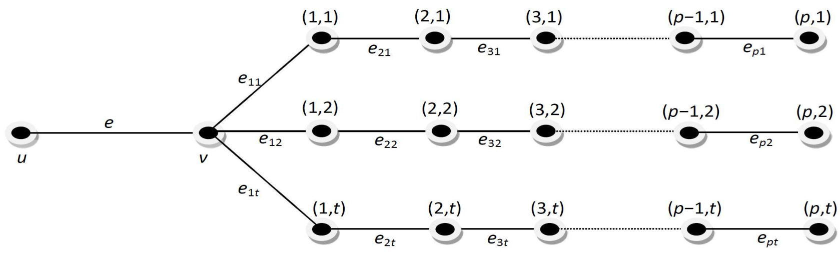

Figure 3: The labeling of vertices and edges of E_p^t graph.

Each of these paths also includes the edge e_{ij}. The edge e_{ij}, that is at distance i to the vertex v, is on $p+1-i$ different paths. The value of the edge betweenness of this edge e_{ij} that is between the vertices u and (k,m) ($k=\overline{1,p}$ and $m=\overline{1,t}$) is

$$b_{e_{ij}}\left(u,(k,m)\right)=p+1-i.$$

Case 2. Consider the shortest paths between the vertex v and the other vertices on the legs. Each of these paths includes only the edge e_{ij}. The value of the edge betweenness of this edge e_{ij} that is between the vertices v and (k,m) ($k=\overline{1,p}$ and $m=\overline{1,t}$) is

$$b_{e_{ij}}\left(v,(k,m)\right)=p+1-i.$$

Case 3. Consider the shortest paths between the vertices of any leg. The initial vertex is $(1,j)$ and the last vertex is (p,j) ($j=\overline{1,\,t}$) on a leg. Thus we have t paths with p vertices, that is P_p. Those paths include the edge e_{ij}. The value of the edge betweenness of this edge e_{ij} equals the number of the left-hand side vertices of e_{ij} multiplied by the number of the right-hand side vertices of e_{ij}. If the edge e_{ij} is between the vertices (k,m) and (k',m) ($k,k'=\overline{1,p}$ and $m=\overline{1,t}$), then we have

$$b_{e_{ij}}\left((k,m),(k',m)\right)=(i-1)\left(p+1-i\right).$$

Case 4. Consider the shortest paths between the vertices of any leg and the vertices of the other legs. This case is similar to Case 3. If the edge e_{ij} is between the vertices (k,m) and (i,j) ($k=\overline{1,p}$ and $m=\overline{1,t}$), then we get

$$b_{e_{ij}}\left((k,m),(i,j)\right)=\left[p\left(t-1\right)\right]\left(p+1-i\right).$$

By summing up Cases 1, 2, 3, and 4, we obtain

$$b_{e_{ij}}=(p+1)\left(p\left(t-1\right)+1\right)+i\left(p\left(2-t\right)\right)-i^2.$$

The summation for all the edges e_{ij} of the graph is

$$\sum_{i=1}^{p}\sum_{j=1}^{t}b_{e_{ij}}=(pt/6)\left[3p^2t+3pt+3p-2p^2+5\right].$$

Consequently, the average edge betweenness of E_p^t graph is

$$b(E_p^t) = [\,1/\,(1+pt)\,]\ \left[(pt+1) + (pt/6)\ \left(3p^2t + 3pt + 3p - 2p^2 + 5\right)\,\right]$$

$$b(E_p^t) = [(pt\,(p+1))\,/6\,(pt+1)]\ [3pt - 2p + 5] + 1.$$

Thus the proof is completed. □

4 The Normalized Average Edge Betweenness of Some Special Graphs

In this section, we give the normalized average edge betweennesses of some special graphs whose average edge betweennesses values are calculated in Section 3.

1. $b_{nor}(C_n) = \begin{cases} [3(n-3)]/[4(n-2)], & n \text{ is odd} \\ \left[3(n^2-8)\right]/\left[4(n^2+n-6)\right], & n \text{ is even.} \end{cases}$
2. $b_{nor}(W_{1,n}) = (3n-9)/(n^2+3n-4)$.
3. $b_{nor}(S_{1,n}) = 6/(n+4)$.
4. $K_{m,n}$ *and* $p = m+n$, $b_{nor}(K_{m,n}) = \left[6(m^2+n^2-p)\right]/\left[mn(p^2+p-6)\right]$.
5. $b_{nor}(E_p^t) = \left[3p^2t - 2p^2 + p + 6pt + 13\right]/[pt+5] - [2p-5]/[pt(pt+5)]$.
6. $\overline{G}$ is a complement graph of G with $\gamma(G) > 2$,

$$b_{nor}(\overline{G}) = 24m/\left[(n^2-n-2m)(n^2+n-6)\right].$$

5 Conclusion

In this paper, we evaluate the average edge betweenness and the normalized average edge betweenness of some special graphs and E_p^t graph. The average edge betweenness is a new characteristic for graph vulnerability introduced in [8]. Calculation of average edge betweenness for simple graph types is important because we can gather information on which edge is the most vulnerable. The average edge betweenness of a given edge is the fraction of shortest paths, counted over all pairs of vertices that pass through that edge. This measure considers both the local and the global structure of the graph.

References

[1] Barefoot, C. A., Entringer, R. and Swart, H. Vulnerability in graphs — a comparative survey. Proceedings of the first Carbondale combinatorics conference (Carbondale, III., 1986) *J. Combin. Math. Combin. Comput.* **1** (1987) 13–22.

[2] Bondy, J. A. and Murty, U. S. R. *Graph Theory with Applications.* American Elsevier Publishing Co., Inc., New York, 1976.

[3] Boccaletti, S., Buldu, J. M., Criado, R., Flores, J., Latora, V., Pello, J. and Romance, M. Multiscale vulnerability of complex networks. *Chaos* **17** (2007) 043110.

[4] Chvàtal, V. Tough graphs and Hamiltonian circuits. *Discrete Math.* **5** (1973) 215–228.

[5] Comellas, F. and Gago, S. Spectral bounds for the betweenness of a graph. *Linear Algebra Appl.* **423** (1) (2007) 74–80.

[6] Cormen, T. H., Leiserson, C. E. and Rivest, R. L. *Introduction to Algorithms.* MIT Press, Cambridge, MA; McGraw-Hill Book Co., New York, 1990.

[7] Freeman, L. C. A Set of Measures of Centrality Based on Betweenness. *Sociometry, American Sociological Association* **40** (1) (1977) 35–41.

[8] Girvan, M. and Newman, M. E. J. Community structure in social and biological networks. *Proc. Natl. Acad. Sci.* **99** (12) (2002) 7821–7826.

[9] Jung, H. A. On a class of posets and the corresponding comparability graphs. *J. Combinatorial Theory Ser. B* **24**(2) 1978 125–133.

[10] Feng, Z. and Michel, A. N. Robustness analysis of a class of discrete-time systems with applications to neural networks. *Nonlinear Dyn. Syst. Theory* **3** (1) (2003) 75–86.

[11] Martynyuk, A. A. Stability in the models of real world phenomena. *Nonlinear Dyn. Syst. Theory* **11** (1) (2011) 7–52.

[12] Aytaç, A. and Atay, B. On exponential domination of some graphs. *Nonlinear Dyn. Syst. Theory* **16** (1) (2016) 12–19.

[13] Turacı, T. and Ökten, M. Vulnerability of Mycielski graphs via residual closeness. *Ars Combin.* **118** (2015) 419–427.

[14] Turacı, T. and Aytaç, V. Residual Closeness of Splitting Networks. *Ars Combin.* **130** (2017) 17–27.

[15] Aytaç, V. and Turacı, T. Exponential domination and bondage numbers in some graceful cyclic structure. *Nonlinear Dyn. Syst. Theory* **17** (2) (2017) 139–149.