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Generalized Monotone Method for Riemann-Liouville Fractional Reaction Diffusion Equation with Applications

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Abstract: Initially, we have obtained the integral representation for the solution of the linear Riemann-Liouville fractional reaction diffusion equation of order q, where $0 < q < 1$, in terms of Green's function. We have developed a generalized monotone method for the non-linear Riemann-Liouville reaction diffusion equation when the forcing term is the sum of an increasing and decreasing functions. The generalized monotone method yields monotone sequences which converge uniformly and monotonically to coupled minimal and maximal solutions. Under uniqueness assumption, we prove the existence of a unique solution for the non-linear Riemann-Liouville fractional reaction diffusion equation.

Keywords: *Riemann-Liouville fractional derivative; representation form; eigenfunction expansion; Mittag-Leffler function; coupled upper and lower solutions; generalized monotone method.*

Mathematics Subject Classification (2010): 26A33, 26A48.

1 Introduction

Computation of explicit solutions of non-linear dynamic equation is rarely possible. It is more so with non-linear fractional dynamic equations with initial and boundary conditions. In general, the existence and uniqueness of solution of the fractional dynamic equation has been established mostly, using some kind of fixed point approach. See [1, 3, 7–9, 15–17, 28, 29, 31, 32] and the references therein for the existence, uniqueness and applications of fractional dynamic equations. The drawback of fixed point theorem results for the initial and/or boundary value problem is that they do not guarantee the

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interval of existence. The method of upper and lower solutions combined with the monotone iterative technique not only guarantees the interval of existence but also the method is both theoretical and computational. See [4,5,12–14,24–27] for monotone methods and generalized monotone methods for nonlinear dynamic equations. The monotone method is feasible only when the non-linear function is increasing or could be made increasing. In this case, we obtain a sequences of approximate solutions which are either monotonically increasing or monotonically decreasing if the approximation is the lower solution or the upper solution respectively. If the non-linear function is decreasing, the monotone method will yield alternating sequences. However, from practical application problems, the non-linear forcing term will be a sum of increasing and decreasing functions as in the population models and chemical combustion models, see [19]. In order to handle such problems, a generalized monotone method has been developed in [20,22,23,30].

In this work, we consider the non-linear Riemann Liouville fractional reaction diffusion equation where the forcing function is the sum of increasing and decreasing functions. We develop a generalized monotone method for the non-linear Riemann-Liouville fractional reaction diffusion equation using coupled lower and upper solutions. Initially, we have obtained a representation form for the solution of the linear Riemann-Liouville fractional reaction diffusion equation using the eigen function expansion method and Green's identity. We have also developed the maximum principle and comparision results relative to one dimentional time fractional parabolic equations. These results are useful in proving that the sequences developed in the generalized monotone method converge to the coupled minimal and maximal solutions of the non-linear fractional reaction diffusion equation. The convergence of the sequences is monotonic and uniform in the weighted norm. Finally, under the uniqueness assumption, we can prove that there exists a unique solution to the non-linear Riemann-Liouville fractional reaction diffusion equation.

2 Preliminary Results

In this section, we recall some known definitions and known results which are useful to develop our main results. Here and throughout, the notation $\Gamma(q)$ denotes the gamma function of order q.

Definition 2.1 The Riemann-Liouville fractional integral of $u(t)$ of order q is defined by

$$D_t^{-q}u = \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}u(s)ds, \tag{1}$$

where $0 < q \leq 1$.

Definition 2.2 The Riemann-Liouville (left-sided) fractional derivative of $u(t)$ of order q, when $0 < q < 1$, is defined as:

$$D^q u(t) = \frac{1}{\Gamma(1-q)}\frac{d}{dt}\int_0^t (t-s)^{q-1}u(s)ds,\ t > 0. \tag{2}$$

Next we define the Mittag-Leffler function which is useful in computing the solution of linear fractional differential equation explicitly.

Definition 2.3 The two parameter Mittag-Leffler function is defined as

$$E_{q,r}(\lambda t^q) = \sum_{k=0}^{\infty}\frac{(\lambda t^q)^k}{\Gamma(qk+r)}. \tag{3}$$

If $r = q$, (3) reduces to

$$E_{q,q}(\lambda t^q) = \sum_{k=0}^{\infty} \frac{(\lambda t^q)^k}{\Gamma q(k+1)}. \tag{4}$$

If $r = 1$, the Mittag-leffler function is defined as

$$E_{q,1}(\lambda t^q) = \sum_{k=0}^{\infty} \frac{(\lambda t^q)^k}{\Gamma q(k+1)}. \tag{5}$$

Further, if $q = r = 1$, $E_{1,1} = e^{\lambda t}$ is the exponential function.

For more details, see [6, 11, 18, 19, 21]. In our next definition we assume $p = 1 - q$, when $0 < q < 1$, $J = (0, T]$ and $J_0 = [0, T]$.

Definition 2.4 A function $\phi(t) \in C(J, R)$ is a C_p continuous function, if $t^{1-q}\phi(t) \in C(J_0, R)$. The set of C_p continuous functions is denoted by $C_p(J, R)$. Further, given a function $\phi(t) \in C_p(J, R)$, we call the function $t^{1-q}\phi(t)$ the continuous extension of $\phi(t)$.

Note that any continuous function in J_0 is also a C_p continuous function.

Consider the initial value problem for the linear Riemann-Liouville fractional reaction differential equation of order q as

$$D^q u = \lambda u + f(t),\ \Gamma(q)u(t)\ t^{1-q}|_{t=0} = u^0, \tag{6}$$

where λ is a real number and $f \in C[[0, T], \mathbb{R}]$. The integral representation of the solution of equation(6) is:

$$u(t) = u^0 t^{q-1} E_{q,q}(\lambda t^q) + \int_0^t (t-s)^{q-1} E_{q,q}[\lambda(t-s)^q] f(s) ds. \tag{7}$$

For details, see [10, 11, 21]. The next result is a basic comparison result involving the q^{th} order fractional Riemann-Liouville derivative with respect to time.

Lemma 2.1 *Let $m(t) \in C_p[[0, T], R]$ be such that for some $t_1 \in (0, T]$, $m(t_1) = 0$, and $t^{1-q}m(t) \leq 0$ on $[0, t_1]$, then $D^q m(t_1) \geq 0$. See more in [4, 5].*

Remark: In the above theorem, if m is a function of (x, t), then the conclusion is true with the partial fractional derivative of m with respect to t of order q. This is what we need in our work.

3 Auxiliary Results

In this section, we obtain a representation form for the solution of the linear Riemann-Liouville fractional reaction diffusion equation with the fractional time derivative. We achieve this by using the eigen function expansion method. Then we will develop comparison results for the non-linear Riemann-Liouville fractional reaction diffusion equation with initial and boundary conditions. The first comparison theorem is with respect to the natural lower and upper solutions when the non-linear term is of the form $F(x, t, u)$, where $F(x, t, u)$ satisfies the one sided Lipschitz condition. The second comparison theorem is relative to coupled lower and upper solutions. In this case, we assume the non-linear term as the sum of two functions $f(x,t,u)$ and $g(x,t,u)$, where $f(x,t,u)$ is a non-decreasing function in u and $g(x,t,u)$ is a non-increasing function in u for (x,t) in $[0,L]\times[0,T]$. In order to present our result, consider the linear Riemann-Liouville fractional reaction diffusion equation with initial and boundary conditions of the form

$$\begin{gathered}\partial_t^q u - k u_{xx} = Q(x,t) \; on \; Q_T,\\ u(0,t) = A(t),\; u(L,t) = B(t) \; in \; \Gamma_T,\\ \Gamma(q)t^{1-q}u(x,t)|_{t=0} = f^0(x),\; x \in \overline{\Omega},\end{gathered} \tag{8}$$

where $\Omega=[0,L]$, $J=(0,T]$, $Q_T=J\times\Omega$, $k>0$ and $\Gamma_T=(0,T)\times\partial\Omega$. Here, ∂_t^q is the partial Riemann-Liuoville fractional derivative with respect to time 't' of order q, $0<q<1$.

In order for the initial boundary value problem to be compatible, we assume that $f^0(0)=A(0)=f^0(L)=B(0)=0$, $\Gamma(q)\; t^{1-q}u(x,t)|_{t=0}=f^0(x)$. Here and throughout this work, we assume the initial and boundary conditions satisfy the compatibility conditions. Using the method of eigenfunction expansion on equation (8), we have the solution of the form:

$$u(x,t)=\sum_{n=1}^{\infty} b_n(t)\phi_n(x), \tag{9}$$

where the eigenfunctions of the related homogeneous problem are known to be $\phi_n(x) = sin\frac{n\pi x}{L}$ and its corresponding eigenvalues are $\lambda_n=(\frac{n\pi}{L})^2$. Using the same approach as in [22], we can compute $b_n(t)$, where $b_n(t)$ will be the solution of the ordinary linear Riemann-Liuoville differential equation.

Here, our aim is to find $b_n(t)$. Using the standard arguments, one can compute $b_n(t)$. The explicit form of $b_n(t)$ is

$$\begin{aligned} b_n(t) &= b_n^0 t^{q-1}E_{q,q}(-k\lambda_n t^q) \qquad (10)\\ &+\int_0^t (t-s)^{q-1}E_{q,q}(-k\lambda_n t^q)q_n(s) + k\frac{2n\pi}{L^2}(A(s)-(-1)^n B(s)ds,\end{aligned}$$

where

$$b_n^0 = \frac{2}{L}\int_0^L f^0(y)\phi_n(y)dy \; and \tag{11}$$

$$q_n(t) = \frac{2}{L}\int_0^L Q(y,t)\phi_n(y)dy. \tag{12}$$

Therefore,

$$\begin{aligned} b_n(t) &= \frac{2}{L}\int_0^L f^0(y)\phi_n(y)dy t^{q-1}E_{q,q}(-k\lambda_n t^q) \qquad (13)\\ &+\int_0^t (t-s)^{q-1}E_{q,q}(-k\lambda_n t^q)\frac{2}{L}\int_0^L Q(y,s)\phi_n(y)dyds\\ &+k\frac{2n\pi}{L^2}\int_0^t (t-s)^{q-1}E_{q,q}(-k\lambda_n t^q)(A(s)-(-1)^n B(s))ds.\end{aligned}$$

So, using $b_n(t)$ in (9), we can get the solution $u(x,t)$ of the form

$$\begin{aligned} u(x,t) &= \int_0^L t^{q-1}[\sum_{n=1}^{\infty}\frac{2}{L}E_{q,q}(-k\lambda_n(t^q))\phi_n(x)\phi_n(y)]f^0(y)dy \\ &+\int_0^t\int_0^L[\sum_{n=1}^{\infty}\frac{2}{L}(t-s)^{q-1}E_{q,q}(-k\lambda_n(t-s)^q)\phi_n(x)\phi_n(y)]Q(y,s)dyds \\ &+k\int_0^t[\frac{2n\pi}{L^2}(t-s)^{q-1}E_{q,q}(-k\lambda_n(t-s)^q)\phi_n(x)]A(s)ds \\ &-k\int_0^t[\frac{2n\pi}{L^2}(t-s)^{q-1}E_{q,q}(-k\lambda_n(t-s)^q)\phi_n(x)]B(s)ds. \end{aligned} \tag{14}$$

Finally, we can write

$$\begin{aligned} u(x,t) = \int_0^L t^{q-1}G(x,y,t)f^0(y)dy + \int_0^t\int_0^L G(x,y,t-s)Q(y,s)dyds \\ +k\int_0^t G_y(x,0,t-s)A(s)ds - k\int_0^t G_y(x,L,t-s)B(s)ds, \end{aligned} \tag{15}$$

where

$$G(x,y,t) = \sum_{n=1}^{\infty}\frac{2}{L}E_{q,q}(-k\lambda_n t^q)\phi_n(x)\phi_n(y).$$

This result will be useful in our main result when we are computing the linear approximations of the generalized monotone iterates.

Here, we can find the steady state condition with homogeneous boundary conditions in which the source term $Q(x,t) = Q(x)$ is independent of time:

$$ku_{xx} + Q(x) = 0.$$

Now the form $u_{xx} = g(x)$, in which $g(x) = -\frac{Q(x)}{k}$.
Therefore,

$$u(x,t) = \int_0^L f^0(y)t^{q-1}G(x,t;y,0)dy + \int_0^L -kg(y)\big[\int_0^t G(x,t;y,s)ds\big]dy, \tag{16}$$

where
$t^{q-1}G(x,t;y,s) = t^{q-1}\sum_{n=1}^{\infty}\frac{2}{L}E_{q,q}(-k\lambda_n(t-s)^q)\phi_n(x)\phi_n(y)$.
As $t \to \infty$, $G(x,t;y,0) \to 0$ such that the effect of the initial condition $t^{1-q}u(x,t)|_{t=0} = f^0(x)$ vanishes as $t \to \infty$. But, as $t^{q-1}G(x,t;y,s) \to 0$ as $t \to \infty$, the steady source is still important as $t \to \infty$ since $\int_0^t E_{q,q}(-k\lambda_n(t-s)^q)ds = \frac{1-E_{q,q}(-k\lambda_n t^q)}{k(\frac{n\pi}{L})^2}$.
Thus, as $t \to \infty$,

$$u(x,t) \to u(x) = \int_0^L g(y)G(x,y)dy,$$

where

$$G(x,y) = -\sum_{n=1}^{\infty}\frac{2}{L}\phi_n(x)\phi_n(y).$$

Hence, we obtained the steady-state temperature distribution $u(x)$ by taking the limit as $t \to \infty$ of the time-dependent problem with a steady source $Q(x) = -kg(x)$.
We recall two known lemmas regarding the Mittag-Leffler functions series from [2].

Lemma 3.1 *Let $E_{q,1}(-\lambda t^q)$ be the Mittag-Leffler function of order q, where $0 < q \leq 1$. Then, $\frac{E_{q,1}(-\lambda_1 t^q)}{E_{q,1}(-\lambda_2 t^q)} < 1$, where $\lambda_1, \lambda_2 > 0$ such that $\lambda_1 = \lambda_2 + k$ for $k > 0$.*

Lemma 3.2 *Let $E_{q,q}(-\lambda t^q)$ be the Mittag-Leffler function of order q, where $0 < q \leq 1$. Then $\frac{E_{q,q}(-\lambda_1 t^q)}{E_{q,q}(-\lambda_2 t^q)} < 1$, where $\lambda_1, \lambda_2 > 0$ such that $\lambda_1 = \lambda_2 + k$ for $k > 0$.*

Now, we can show the convergence of the above solution using the two lemmas above, i.e Lemma 3.1 and Lemma 3.2. We can split the solution of (8) as $u_1(x,t)$, $u_2(x,t)$ and $u_3(x,t)$ respectively as follows:
(a) $u_1(x,t)$ is the solution of (8), when $Q(x,t) = 0, A(t) = 0 = B(t)$,
(b) $u_2(x,t)$ is the solution of (8), when $A(t) = 0 = B(t), f^0(x) = 0$,
(c) $u_3(x,t)$ is the solution of (8), when $Q(x,t) = 0, f^0(x) = 0$.

Theorem 3.1 *$u_1(x,t)$, $u_2(x,t)$ and $u_3(x,t)$ converge when $|f^0(x)| < N_1$, $N_1 > 0$, $|Q(x,t)| < N_2$, $N_2 > 0$, $|A(t)| < M_1$ and $|B(t)| < M_2$, $M_1, M_2 > 0$ respectively.*

Proof of the above theorem follows as an application of Lemma 3.1 and Lemma 3.2. The details of the proof can be found in [2]. Next we will consider the non-linear Riemann-Louiville fractional reaction diffusion equation of the type:

$$\begin{aligned} \partial_t^q u - k\frac{\partial^2 u}{\partial x^2} &= f(x,t,u) + g(x,t,u),\ (x,t) \in Q_T, \\ & \Gamma(q)(t)^{1-q} u(x,t)|_{t=0} = f^0(x),\ x \in \overline{\Omega}, \\ u(0,t) &= A(t), u(L,t) = B(t) \text{ on } \Gamma_T, \\ & J = (0,T],\ Q_T = J \times \Omega,\ k > 0 \text{ and } \Gamma_T = (0 \times T) \times \partial\Omega, \\ & f, g \in\ C^{2,q}[[0,L] \times J \times \mathbb{R}, \mathbb{R}]. \end{aligned} \tag{17}$$

In this work, we seek the classical solution such that $u(x,t) \in C_p^{2,q}$ *on* Q_T, *and* $u(x,t) \in C_p$ *on* $\overline{Q}_T$. In order to develop the generalized monotone method for (17), we need the following definitions.

Definition 3.1 $v(x,t)$, $w(x,t) \in C^{2,q}[Q_T, \mathbb{R}]$. Then
(a) $v(x,t)$ and $w(x,t)$ are called the natural lower and upper solutions of (17) if the following inequalities are satisfied:

$$\begin{aligned} \partial_t^q v(x,t) - k\frac{\partial^2 v(x,t)}{\partial x^2} \leq f(x,t,v(x,t)) + g(x,t,v(x,t)) \text{ on } Q_T, \\ \Gamma(q)(t-t_0)^{1-q} v(x,t)|_{t=0} \leq f^0(x),\ x \in\ \overline{\Omega}, \\ v(x,0) \leq A(t), v(L,t) \leq B(t) \text{ in } \Gamma_T, \end{aligned} \tag{18}$$

$$\begin{aligned} \partial_t^q w(x,t) - k\frac{\partial^2 w(x,t)}{\partial x^2} \geq f(x,t,w(x,t)) + g(x,t,w(x,t)) \text{ on } Q_T, \\ \Gamma(q)(t-t_0)^{1-q} w(x,t)|_{t=0} \geq f^0(x),\ x \in\ \overline{\Omega}, \\ w(x,0) \geq A(t), w(L,t) \geq B(t) \text{ in } \Gamma_T. \end{aligned} \tag{19}$$

(b) $v(x,t)$ and $w(x,t)$ are called coupled lower and upper solutions of type I if the following inequalities are satisfied:

$$\begin{aligned} \partial_t^q v(x,t) - k\frac{\partial^2 v(x,t)}{\partial x^2} &\leq f(x,t,v(x,t)) + g(x,t,w(x,t)) \text{ on } Q_T, \\ \Gamma(q)(t-t_0)^{1-q} v(x,t)|_{t=0} &\leq f^0(x), \; x \in \overline{\Omega}, \\ v(x,0) &\leq A(t), v(L,t) \leq B(t) \text{ in } \Gamma_T, \end{aligned} \tag{20}$$

$$\begin{aligned} \partial_t^q w(x,t) - k\frac{\partial^2 w(x,t)}{\partial x^2} &\geq f(x,t,w(x,t)) + g(x,t,v(x,t)) \text{ on } Q_T, \\ \Gamma(q)(t-t_0)^{1-q} w(x,t)|_{t=0} &\geq f^0(x), \; x \in \overline{\Omega}, \\ w(x,0) &\geq A(t), w(L,t) \geq B(t) \text{ in } \Gamma_T. \end{aligned} \tag{21}$$

The next result is a comparison result relative to lower and upper solutions of (17) of natural type. For that purpose, we write $F(x,t,u) = f(x,t,u) + g(x,t,u)$.

Theorem 3.2 *Assume that*
(i) $v(x,t)$, $w(x,t) \in C^{2,q}[Q_T, \mathbb{R}]$ are natural lower and upper solutions of (17), respectively. Furthermore, $\Gamma(q)t^{1-q}v(x,t)|_{t=0} \leq \Gamma(q)t^{1-q}w(x,t)|_{t=0}$, $v(0,t) \leq w(0,t)$ and $v(L,t) \leq w(L,t)$;
(ii) $F(x,t,u)$ satisfies the one sided Lipschitz condition of the form

$$F(x,t,u_1) - F(x,t,u_2) \leq L(u_1 - u_2),$$

whenever $u_1 \geq u_2$ and $L > 0$. Then $v(x,t) \leq w(x,t)$ on $J \times \Omega$.

Proof. Initially, we will prove the theorem when one of the inequalities in (i) is strict. For that purpose, let $m(x,t) = v(x,t) - w(x,t)$. We claim that $m(x,t) < 0$, $(x,t) \in [0,L] \times (0,T]$. Suppose that the conclusion is not true, then there exists a $t_1 \in J$ and $x_1 \in \Omega$ such that $t^{1-q}m(x_1,t) < 0$ on $[0,t_1)$, $m(x_1,t_1) = 0$. It is easy to check $m_x(x_1,t_1) = 0$ and $m_{xx}(x_1,t_1) \leq 0$.
Then, using Lemma 3.2, we get $\partial_t^q m(x_1,t_1) \geq 0$.
From the hypothesis, we also have

$$\begin{aligned} &\partial_t^q m(x_1,t_1) \\ &\quad = \partial_t^q v(x_1,t_1) - \partial_t^q w(x_1,t_1) \\ &\quad < k\frac{\partial^2 v(x_1,t_1)}{\partial x^2} + F(x_1,t_1,v(x_1,t_1)) - k\frac{\partial^2 w(x_1,t_1)}{\partial x^2} - F(x_1,t_1,w(x_1,t_1)) \\ &\quad < F(x_1,t_1,v(x_1,t_1)) - F(x_1,t_1,w(x_1,t_1)) = 0, \end{aligned} \tag{22}$$

which is a contradiction. Therefore, $v(x,t) < w(x,t)$ on $\overline{Q}_T$.

In order to prove the theorem for the non strict inequalities, let

$$\overline{w}(x,t) = w(x,t) + \epsilon t^{q-1} E_{q,q}[2Lt^q],$$

$$\overline{v}(x,t) = v(x,t) - \epsilon t^{q-1} E_{q,q}[2Lt^q].$$

From this it follows

$$\overline{w}(0,t) > \overline{v}(0,t),$$

$$\overline{w}(L,t) > \overline{v}(L,t),$$

$$\Gamma(q)t^{1-q}\overline{w}(x,t)|_{t=0} > \Gamma(q)t^{1-q}w(x,t)|_{t=0} > \Gamma(q)t^{1-q}v(x,t)|_{t=0} > \Gamma(q)t^{1-q}\overline{v}(x,t)|_{t=0}.$$

Then,

$$\begin{aligned}
&\partial_t^q \overline{w}(x,t) - k\frac{\partial^2 \overline{w}(x,t)}{\partial x^2} \\
&= \partial_t^q w(x,t) - k\frac{\partial^2 w(x,t)}{\partial x^2} + \partial_t^q \epsilon t^{q-1}E_{q,q}[2Lt^q] \\
&\geq F(x,t,w(x,t)) + \epsilon t^{q-1}E_{q,q}2LEq,q[2Lt^q] \\
&= F(x,t,w(x,t)) + 2\epsilon Lt^{q-1}E_{q,q}[2Lt^q] - F(x,t,\overline{w}(x,t)) + F(x,t,\overline{w}(x,t)) \\
&\geq -L(\overline{w}-w) + F(x,t,\overline{w}(x,t)) + \epsilon 2LE_{q,q}(2Lt^q) \\
&= -L\epsilon t^{q-1}E_{q,q}[2Lt^q] + F(x,t,\overline{w}(x,t)) + 2L\epsilon E_{q,q}(2Lt^q) \\
&= F(x,t,\overline{w}(x,t)) + \epsilon Lt^{q-1}E_{q,q}[2Lt^q] \\
&> F(x,t,\overline{w}(x,t)) \ on \ \overline{Q}_T.
\end{aligned} \tag{23}$$

Similarly,

$$\partial_t^q \overline{v}(x,t) - k\frac{\partial^2 \overline{v}(x,t)}{\partial x^2} > F(x,t,\overline{v}(x,t)) \ on \ \overline{Q}_T. \tag{24}$$

By the strict inequality result, $\overline{v} < \overline{w}$ on $\overline{Q}_T$. Letting $\epsilon \to 0$, we have $v \leq w$ on $\overline{Q}_T$.

The next result is related to coupled lower and upper solutions of type I related to (17).

Theorem 3.3 *Assume that*
(i) $v(x,t)$, $w(x,t) \in C^{2,q}[Q_T,\mathbb{R}]$ are coupled lower and upper solutions of type I of (17), respectively.
(ii) Assume $F(x,t,u) = f(x,t,u) + g(x,t,u)$, where f is a nondecreasing function and g is a nonincreasing function respectively for $(x,t) \in \overline{Q}_T$ in u.
(iii) Let $f(x,t,u)$ and $g(x,t,u)$ satisfy the one sided Lipschitz condition of the form

$$f(x,t,u_1) - f(x,t,u_2) \leq L(u_1 - u_2),$$

$$g(x,t,u_1) - g(x,t,u_2) \geq -M(u_1 - u_2),$$

whenever $u_1 \geq u_2$ and $L, M > 0$. Then $v(x,t) \leq w(x,t)$ on $J \times \Omega$.

Proof. Initially, we will prove the theorem when one of the inequalities in (i) is strict. For that purpose, let $m(x,t) = v(x,t) - w(x,t)$. It is easy to see that $m(x,0) < 0$ on $[0,L]$. Also, $m(0,t) < 0$ and $m(L,t) < 0$, $t \in (0,T]$. Suppose the conclusion is not true, then there exists a $t_1 \in J$ and $x_1 \in \Omega$ such that $t^{1-q}m(x,t) < 0$ on $(0,t_1]$, $m(x_1,t_1) = 0$. This implies $v(x_1,t_1) = w(x_1,t_1)$ and $\frac{\partial^2 m(x_1,t_1)}{\partial x^2} \leq 0$, where $t_1 > 0$ and $x_1 \in (0,L)$. Using Lemma 3.2, $\partial_t^q m(x_1,t_1) \geq 0$.

From the hypothesis, we also have

$$
\begin{aligned}
&\partial_t^q m(x_1,t_1)\\
&\qquad = \partial_t^q v(x_1,t_1) - \partial_t^q w(x_1,t_1)\\
&\qquad < k\frac{\partial^2 v(x_1,t_1)}{\partial x^2} + f(x_1,t_1,v(x_1,t_1)) + g(x_1,t_1,w(x_1,t_1))\\
&\qquad - k\frac{\partial^2 w(x_1,t_1)}{\partial x^2} - f(x_1,t_1,w(x_1,t_1)) - g(x_1,t_1,v(x_1,t_1))\\
&\qquad \le 0,
\end{aligned}
\tag{25}
$$

which leads to a contradiction. Therefore, $v(x,t) < w(x,t)$ on $\overline{Q}_T$.
In order to prove the theorem for the non strict inequalities, let

$$\overline{w}(x,t) = w(x,t) + \epsilon(t-t_0)^{q-1}E_{q,q}[2(L+M)(t-t_0)^q],$$

$$\overline{v}(x,t) = v(x,t) - \epsilon(t-t_0)^{q-1}E_{q,q}[2(L+M)(t-t_0)^q].$$

One can show $\overline{v}(x,t)$ *and* $\overline{w}(x,t)$ satisfy the hypothesis with strict inequalities. Using the strict inequality result, $\overline{v} < \overline{w}$ on $\overline{Q}_T$. Letting $\epsilon \to 0$, we have $v \le w$ on $\overline{Q}_T$.
The next result is the maximum principle for the Riemann-Liouville parabolic equation in one dimensional space which will be useful in proving the uniqueness of the solution.

Corollary 3.1 *Let*

$$\partial_t^q m(x,t) - k\frac{\partial^2 m(x,t)}{\partial x^2} \le 0 \ \text{on}\ Q_T,$$

$$m(0,t) \le 0, m(L,t) \le 0 \ \text{on}\ \Gamma_T,$$

$$\Gamma(q)t^{1-q}m(x,t)|_{t=0} \le 0 \ \text{on}\ \overline{\Omega}.$$

Then $m(x,t) \le 0$ *on* Q_T.

Proof. Suppose $m(x,t)$ has a positive maximum at (x_1,t_1). Let $m(x_1,t_1) = K$. Let $\overline{m}(x,t) = m(x,t) - K$. Then, $t^{1-q}\overline{m}(x,t) \le 0$ *on* $(0,t_1]$ and $\overline{m}(x_1,t_1) = 0$. Using Lemma 2.1, we get $\partial_t^q \overline{m}(x_1,t_1) \ge 0$. Also, $\overline{m}_{xx}(x_1,t_1) \le 0$. Combining these two, we get $\partial_t^q \overline{m}(x_1,t_1) - K\overline{m}_{xx}(x_1,t_1) \ge 0$.
We can also observe

$$\partial_t^q \overline{m}(x,t) - K\overline{m}_{xx} = \partial_t^q m - Km_{xx} - K\frac{t^{q-1}}{\Gamma q} < \partial_t^q m - Km_{xx} < 0, \tag{26}$$

which gives a contradiction. Hence, $m(x,t) \le 0$.

We can also prove this corolary by other method. We can show it is true first for the strict inequality and then for the instrict inequality by using the strict inequality. The solution of the linear problem is unique which follows from this maximum principle. This maximum principle is used to show the uniqueness of iterates and the monotonicity of the iterates. In next section, we will develop a generalized monotone method for the nonlinear Riemann-Liuoville fractional reaction diffusion equation (17) using coupled lower and upper solutions of type I. The generalized monotone method yields monotone sequences which converge uniformly and monotonically to coupled minimal and maximal solutions of (17). Further using the uniqueness condition, we prove the uniqueness of the solution of (17). The next result is a generalized monotone method for (17).

4 Main Results

Theorem 4.1 *(i) Let (v_0, w_0) be the coupled lower and upper solutions of (17) such that $t^{1-q}v_0 \leq t^{1-q}w_0$ on $\overline{Q}_T$.*
(ii) Suppose $f(x,t,u)$ is nondecreasing and $g(x,t,u)$ is nonincreasing in u on $\overline{Q}_T$, respectively. Then there exist monotone sequences $\{t^{1-q}v_n(x,t)\}$ and $\{t^{1-q}w_n(x,t)\}$ such that $t^{1-q}v_n(x,t) \to t^{1-q}\rho(x,t)$ and $t^{1-q}w_n(x,t) \to t^{1-q}r(x,t)$ uniformly and monotonically on $\overline{Q}_T$, where $\rho(x,t)$ and $r(x,t)$ are coupled minimal and maximal solutions of (17) respectively. That is, $\rho(x,t)$ and $r(x,t)$ satisfy

$$\partial_t^q \rho(x,t) - k\frac{\partial^2 \rho(x,t)}{\partial x^2} = f(x,t,\rho) + g(x,t,r) \text{ on } Q_T,$$
$$\rho(0,t) = A(t), \rho(L,t) = B(t) \text{ on } \Gamma_T,$$
$$\Gamma(q)t^{1-q}\rho(x,t)|_{t=0} = f^0(x) \text{ on } \Omega$$

and

$$\partial_t^q r(x,t) - k\frac{\partial^2 r(x,t)}{\partial x^2} = f(x,t,r) + g(x,t,\rho) \text{ on } Q_T,$$
$$r(0,t) = A(t), r(L,t) = B(t) \text{ on } \Gamma_T,$$
$$\Gamma(q)t^{1-q}r(x,t)|_{t=0} = f^0(x) \text{ on } \Omega$$

such that $t^{1-q}v_0(x,t) < t^{1-q}\rho(x,t) < t^{1-q}u(x,t) < t^{1-q}r(x,t) < t^{1-q}w_0(x,t)$.

Proof. We construct the sequences $\{v_n(x,t)\}$ and $\{w_n(x,t)\}$ as follows:

$$\begin{aligned} \partial_t^q v_n(x,t) - k\frac{\partial^2 v_n(x,t)}{\partial x^2} &= f(x,t,v_{n-1}) + g(x,t,w_{n-1}) \text{ on } Q_T, \\ v_n(0,t) &= A(t), v_n(L,t) = B(t), \\ \Gamma(q)t^{1-q}v_n(x,t)|_{t=0} &= f^0(x) \end{aligned} \tag{27}$$

and

$$\begin{aligned} \partial_t^q w_n(x,t) - k\frac{\partial^2 w_n(x,t)}{\partial x^2} &= f(x,t,w_{n-1}) + g(x,t,v_{n-1}) \text{ on } Q_T, \\ w_n(0,t) &= A(t), w_n(L,t) = B(t), \\ \Gamma(q)t^{1-q}w_n(x,t)|_{t=0} &= f^0(x). \end{aligned} \tag{28}$$

It is easy to observe that $v_1(x,t)$ and $w_1(x,t)$ exist and are unique by the representation form of linear equation and Corollary 3.1. By induction and the assumptions on f and g, we can prove that the solutions $v_n(x,t)$ and $w_n(x,t)$ exist and are unique by Corollary 3.1, for any n.

Let us prove first that $v_0(x,t) \leq v_1(x,t)$ and that $w_1(x,t) \leq w_0(x,t)$ on $\overline{Q}_T$. Let $p(x,t) = v_0(x,t) - v_1(x,t)$. Then

$$\partial_t^q p(x,t) - k\frac{\partial^2 p(x,t)}{\partial x^2}$$
$$= \partial_t^q v_0(x,t) - k\frac{\partial^2 v_0(x,t)}{\partial x^2} - \big(\partial_t^q v_1(x,t) - k\frac{\partial^2 v_1(x,t)}{\partial x^2}\big)$$

$$\leq f(x,t,v_0(x,t)) + g(x,t,w_0(x,t)) - (f(x,t,v_0(x,t)) + g(x,t,w_0(x,t))) = 0,$$

$p(0,t) = 0,\ p(L,t) = 0$ on $\overline{\Omega}$ and $\Gamma(q)t^{1-q}p(x,t)|_{t=0} = 0$ on Γ_T. Therefore, by Corollary 3.1, it follows that $p(x,t) \leq 0$ on $\overline{Q}_T$ and $t^{1-q}v_0(x,t) \leq t^{1-q}v_1(x,t)$ on $\overline{Q}_T$.
Similarly, we can show that $w_1(x,t) \leq w_0(x,t)$ on $\overline{Q}_T$.
Then, we prove that $v_1(x,t) \leq w_1(x,t)$. Let $p(x,t) = v_1(x,t) - w_1(x,t)$. Then from our hypothesis, we get

$$\partial_t^q p(x,t) - k\frac{\partial^2 p(x,t)}{\partial x^2}$$

$$= \partial_t^q v_1(x,t) - k\frac{\partial^2 v_1(x,t)}{\partial x^2} - (\partial_t^q w_1(x,t) - k\frac{\partial^2 w_1(x,t)}{\partial x^2})$$

$$\leq f(x,t,v_0(x,t)) + g(x,t,w_0(x,t)) - (f(x,t,v_0(x,t)) + g(x,t,w_0(x,t))) = 0,$$

$p(0,t) = 0,\ p(L,t) = 0$ on $\overline{\Omega}$ and $\Gamma(q)t^{1-q}p(x,t)|_{t=0} = 0$ on Γ_T. Therefore, by Corollary 3.1, it follows that $p(x,t) \leq 0$ on $\overline{Q}_T$ and $t^{1-q}v_1(x,t) \leq t^{1-q}w_1(x,t)$ *on* $\overline{Q}_T$. Hence,

$$t^{1-q}v_0(x,t) \leq t^{1-q}v_1(x,t) \leq t^{1-q}w_1(x,t) \leq t^{1-q}w_0(x,t) \ on\ \overline{Q}_T.$$

By mathematical induction, we have

$$t^{1-q}v_0(x,t) \leq ... \leq t^{1-q}v_n(x,t) \leq t^{1-q}w_n(x,t) \leq ... \leq t^{1-q}w_0(x,t) \ on\ \ \overline{Q}_T \ for\ all\ n.$$

Furthermore, if $t^{1-q}v_0(x,t) \leq t^{1-q}u(x,t) \leq t^{1-q}w_0(x,t)$ *on* $\overline{Q}_T$, then for any $u(x,t)$ of (17), we establish the following inequality by the method of induction.

$$t^{1-q}v_0(x,t) \leq ... \leq t^{1-q}v_n(x,t) \leq t^{1-q}u(x,t) \leq t^{1-q}w_n(x,t) \leq ... \leq t^{1-q}w_0(x,t) \quad (29)$$

$$on\ \overline{Q}_T\ for\ all\ n.$$

It is certainly true for $n = 0$, by hypothesis. Assume the inequality (29) to be true for $n = k$, that is

$$t^{1-q}v_0(x,t) \leq ... \leq t^{1-q}v_k(x,t) \leq t^{1-q}u(x,t) \leq t^{1-q}w-k(x,t) \leq ... \leq t^{1-q}w_0(x,t) \quad (30)$$

$$on\ \overline{Q}_T\ for\ all\ n.$$

Let $p(x,t) = v_{k+1}(x,t) - u(x,t)$. Then from our hypothesis, we get

$$\partial_t^q p(x,t) - k\frac{\partial^2 p(x,t)}{\partial x^2}$$

$$= \partial_t^q v_{k+1}(x,t) - k\frac{\partial^2 v_{k+1}(x,t)}{\partial x^2} - (\partial_t^q u(x,t) - k\frac{\partial^2 u(x,t)}{\partial x^2})$$

$$\leq f(x,t,v_k) + g(x,t,w_k) - (f(x,t,u) + g(x,t,u)) \leq 0,$$

$p(0,t) = 0, p(L,t) = 0$ on $\overline{\Omega}$ and $\Gamma(q)t^{1-q}p(x,0)|_{t=0} = 0$ on Γ_T. Therefore, by Corollary 3.1, it follows that $p(x,t) \leq 0$ on $\overline{Q}_T$. Therefore, $t^{1-q}v_{k+1}(x,t) \leq t^{1-q}u(x,t)$ on $\overline{Q}_T$. In a similar way, we can show that $t^{1-q}u(x,t) \leq t^{1-q}w_{k+1}(x,t)$ on $\overline{Q}_T$.

Hence we constructed the monotonic sequences. Using the integral representation of the linear problem and an appropriate computation process, we can show that the sequences $\{t^{1-q}v_n(x,t)\}$ and $\{t^{1-q}w_n(x,t)\}$ are uniformly bounded and equicontinuous. Using the Ascoli-Arzela theorem, we obtain subsequences of $\{t^{1-q}v_n(x,t)\}$ and $\{t^{1-q}w_n(x,t)\}$ which converge uniformly and monotonically on $\overline{Q}_T$. Since the sequences

$\{t^{1-q}v_n(x,t)\}$ and $\{t^{1-q}w_n(x,t)\}$ are monotone, the entire sequences $\{t^{1-q}v_n(x,t)\}$ and $\{t^{1-q}w_n(x,t)\}$ converge to $t^{1-q}\rho(x,t)$ and $t^{1-q}r(x,t)$, respectively. From this it follows that

$$\begin{aligned} t^{1-q}v_0(x,t) \le t^{1-q}v_1(x,t) \le ... \le t^{1-q}v_n(x,t) \le ... \le t^{1-q}\rho(x,t) \le t^{1-q}u(x,t) \\ \le t^{1-q}r(x,t) \le ...t^{1-q}w_n(x,t) \le ... \le t^{1-q}w_0(x,t) \textit{ on } \overline{Q}_T. \end{aligned} \tag{31}$$

Consequently, $\rho(x,t)$ and $r(x,t)$ are coupled minimal and maximal solutions of (17) since

$$t^{1-q}v_0(x,t) \le t^{1-q}\rho(x,t) \le t^{1-q}u(x,t) \le t^{1-q}r(x,t) \le t^{1-q}w_0(x,t) \textit{ on } \overline{Q}_T. \tag{32}$$

Since $f(x,t,u)$ and $g(x,t,u)$ satisfy the one sided Lipschitz condition, we prove the uniqueness of the solution of (17). The next result is precisely this.

Theorem 4.2 *Let all the assumptions of Theorem 4.1 hold. Further, let $f(x,t,u)$ and $g(x,t,u)$ satisfy the one sided Lipschitz condition of the form*

$$f(x,t,u_1) - f(x,t,u_2) \le L_1(u_1 - u_2),$$

$$g(x,t,u_1) - g(x,t,u_2) \ge -L_2(u_1 - u_2),$$

whenever $u_1 \ge u_2$ and $L_1, L_2 > 0$. Then the solution $u(x,t)$ of (17) exists and is unique.

Proof. We have already proved (ρ, r) are coupled minimal and maximal solutions of (17) on $\overline{Q}_T$. Hence it is enough to show that $r(x,t) \le \rho(x,t)$ on $\overline{Q}_T$.
It is known from Theorem 4.1 that $\rho(x,t) \le r(x,t)$ on $\overline{Q}_T$.
Let $p(x,t) = r(x,t) - \rho(x,t)$. By the hypothesis, we get

$$\begin{aligned} &\partial_t^q p(x,t) - k\frac{\partial^2 p(x,t)}{\partial x^2} \\ &= \partial_t^q r(x,t) - k\frac{\partial^2 r(x,t)}{\partial x^2} - \big(\partial_t^q \rho(x,t) - k\frac{\partial^2 \rho(x,t)}{\partial x^2}\big) \\ &\le f(x,t,r) + p(x,t,\rho) - (f(x,t,\rho) + g(x,t,r)) \\ &\le t^{1-q}L_1|r-\rho| + t^{1-q}L_2|r-\rho| \\ &\le (L_1 + L_2)|p|, \end{aligned}$$

$p(0,t) = 0,\ p(L,t) = 0$ on $\overline{\Omega}$ and $\Gamma(q)t^{1-q}p(x,t)|_{t=0} = 0$ on Γ_T. It follows from Corollary 3.1 that $p(x,t) \le 0$. This proves that $r(x,t) = \rho(x,t) = u(x,t)$ on $\overline{Q}_T$ and the proof is complete.

5 Conclusion

In this work, initially we have obtained an integral representation for the solution of the Riemann-Liouville reaction diffusion equation with $Q(x,t)$, $f^0(x)$, $A(t)$, $B(t)$ being the non-homogeneous term, the initial function and the boundary functions respectively. In addition, we assume that the boundary conditions and the initial function satisfy the compatibility condition. We also establish, when $Q(x,t)$, $f^0(x)$, $A(t)$ and $B(t)$ are bounded, the solution $u(x,t)$ converges, by using the convergence of the series involving the Mittag-Leffler function. In addition, when $Q(x,t) = Q(x)$ is independent of t and

$A(t) = B(t) = 0$, we have proved that the solution of the Riemann-Liouville fractional reaction diffusion equation converges to the steady state solution. We have proved the maximum principle and comparison theorem relative to the non-linear Riemann-Liouville fractional reaction diffusion equation of (17) on $\overline{Q}_T$. Using the comparison result as a tool, we have developed a generalized monotone method for the Riemann-Liouville fractional reaction diffusion equation of (17). The generalized monotone method yields monotone sequences which converge uniformly and monotonically to coupled minimal and maximal solutions of (17). Under the uniqueness assumption, we have proved that the unique solution of $u(x,t)$ of (17) exists and is unique. In our future work, we plan to use our method relative to the physical application problem.

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