Nonlinear Dynamics and Systems Theory, **19** (1-SI) (2019) 117–123

On Stability of a Second Order Integro-Differential Equation

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Received: June 16, 2018; Revised: January 27, 2019

Abstract: There exists a well-developed stability theory for integro-differential equations of the first order and only a few results on integro-differential equations of the second order. The aim of this paper is to fill up this gap. Explicit tests for uniform exponential stability of linear scalar delay integro-differential equations of the second order

$$\ddot{x}(t) + \int_{g(t)}^{t} G(t,s)\dot{x}(s)ds + \int_{h(t)}^{t} H(t,s)x(s)ds = 0$$

are obtained.

Keywords: *exponential stability; second order delay integro-differential equations; a priory estimation; Bohl-Perron theorem.*

Mathematics Subject Classification (2010): 34K40, 34K20, 34K06.

1 Introduction

Beginning with the classical book of Volterra [1] integro-differential equations and, more generally, functional differential equations have many applications in biology, physics, mechanics (see, for example, [2, 4–7, 22, 26]). In particular, second order integro-differential equations appear in stability problems of viscoelastic shells [3]. There are many papers devoted to stability of the first order integro-differential equations [8–11, 18] and only few papers on stability for the second order equations [12–14]. Oscillation conditions for the first and the second order functional differential equations can be found in papers [15–17].

The aim of the present paper is to fill up this gap and obtain new explicit exponential stability conditions for the equation

$$\ddot{x}(t) + \int_{g(t)}^{t} G(t,s)\dot{x}(s)ds + \int_{h(t)}^{t} H(t,s)x(s)ds = 0. \tag{1}$$

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Papers [12–14] are devoted to some asymptotic properties of partial cases of (1). In [12] an asymptotic behavior of solutions is studied using analysis of a generalized characteristic equation. In [14] the authors obtain stability results by an application of the Lyapunov functional method. In [13] the authors use a connection between asymptotic properties of (1) (for some special kernels $G(t,s), H(t,s)$) and a system of infinite number of ordinary differential equations.

To obtain new stability tests, we apply the method based on the Bohl-Perron theorem together with a priori estimations of solutions, integral inequalities for fundamental functions of linear delay equations and various transformations of a given equation. We consider equation (1) in more general assumptions than in the above mentioned papers: all kernels and delays are measurable functions, derivative of a solution is an absolutely continuous function.

2 Preliminaries

Denote

$$a(t)=\int_{g(t)}^{t} G(t,s)ds,\ b(t)=\int_{h(t)}^{t} H(t,s)ds,$$

$$a_1(t)=\int_{g(t)}^{t} G(t,s)(t-s)ds,\ b_1(t)=\int_{h(t)}^{t} H(t,s)(t-s)ds.$$

We consider scalar delay differential equation (1) under the following conditions:
(a1) $G(t,s)\geq 0, H(t,s)\geq 0$ are Lebesgue measurable on $t\geq s\geq 0$, h,g are measurable on $[0,\infty)$ functions, a,b,a_1,b_1 are essentially bounded on $[0,\infty)$ functions;
(a2) $0<a_0\leq a(t)\leq A_0, 0<b_0\leq b(t)\leq B_0$ for all $t\geq t_0\geq 0$ and some fixed $t_0\geq 0$;
(a3) $0\leq t-g(t)\leq \sigma, 0\leq t-h(t)\leq \tau$ for $t\geq t_0$ and some $\sigma>0,\tau>0$ and $t_0\geq 0$.

Along with (1), we consider for each $t_0\geq 0$ an initial value problem

$$\ddot{x}(t)+\int_{g(t)}^{t} G(t,s)\dot{x}(s)ds+\int_{h(t)}^{t} H(t,s)x(s)ds=f(t), \tag{2}$$

$$x(t)=\varphi(t),\ \dot{x}(t)=\psi(t),\ t\leq t_0, \tag{3}$$

where $f:[t_0,\infty)\to R$ is a Lebesgue measurable locally essentially bounded function, $\varphi:(-\infty,t_0]\to R$, $\psi:(-\infty,t_0)\to R$ are Borel measurable bounded functions.

Further, we assume that the above conditions hold without mentioning it.

A function x with a locally absolutely continuous on $[t_0,\infty)$ derivative $x':R\to R$ is called **a solution of problem** (2) if it satisfies the equation (2) for almost all $t\in[t_0,\infty)$ and the equalities in (3) for $t\leq t_0$.

There exists a unique solution of problem (2)-(3), see [6,21].

Equation (1) is **(uniformly) exponentially stable** if there exist positive numbers M and γ such that the solution of problem (3)with $f\equiv 0$ satisfies the estimate

$$\max\{|x(t)|,|\dot{x}(t)|\}\leq Me^{-\gamma(t-t_0)}\sup_{t\in(-\infty,t_0]}\max\{|\psi(t)|,|\varphi(t)|\},\quad t\geq t_0, \tag{4}$$

where M and γ do not depend on $t_0\geq 0$ and functions ψ,φ.

Next, we present the Bohl-Perron theorem [6,19].

Lemma 2.1 *Assume that the solution x of the problem* (2) *with the initial conditions $x(t) = \dot{x}(t) = 0, t \leq t_0$, and its derivative $\dot{x}$ are bounded on $[t_0, +\infty)$ for any essentially bounded function f on $[t_0, +\infty)$. Then equation (1) is exponentially stable.*

Consider now an ordinary differential equation

$$\ddot{x}(t) + a(t)\dot{x}(t) + b(t)x(t) = 0 \tag{5}$$

and denote by $X(t,s)$ the fundamental function of (5).

Lemma 2.2 *[20] If $A_0 \geq a(t) \geq a_0 > 0, B_0 \geq b(t) \geq b_0 > 0, t \geq t_0$ and $a_0^2 \geq 4B_0$, then $X(t,s) \geq 0$, equation* (5) *is exponentially stable and*

$$\int_{t_0}^{t} X(t,s)b(s)ds < 1.$$

For a fixed bounded interval $I = [t_0, t_1]$, consider the space $L_\infty[t_0, t_1]$ of all essentially bounded on I functions with the norm $\|y\|_{[t_0,t_1]} = \mathrm{esssup}_{t\in I}|y(t)|$, denote

$$\|f\|_{[t_0,+\infty)} = \mathrm{esssup}_{t\geq t_0}|f(t)|$$

for an unbounded interval, E is the identity operator.

In the sequel, we use the concept of a non-singular M-matrix. For convenience, we recall this notion.

Definition 2.1 [[24]] An $m \times m$ matrix $A = (a_{ij})_{i,j=1}^{m}$ is called a non-singular M-matrix if $a_{ij} \leq 0,\ i,j = 1,\ldots,m,\ i \neq j$ and one of the following equivalent conditions holds:

1. There exists a positive inverse matrix A^{-1}.
2. All the principal minors of matrix A are positive.

3 Explicit Stability Conditions

Theorem 3.1 *Assume that for some $t_0 \geq 0$ and $t \geq t_0$ $a_0^2 \geq 4B_0$ and the following condition holds*

$$\begin{aligned} &\|a\|_{[t_0,\infty)} \left\|\frac{a_1}{a}\right\|_{[t_0,\infty)} + \left\|\frac{b_1}{b}\right\|_{[t_0,\infty)} \left(\left\|\frac{b}{a}\right\|_{[t_0,\infty)} + \|b\|_{[t_0,\infty)} \left\|\frac{a_1}{a}\right\|_{[t_0,\infty)} \right) \\ &\qquad + \left\|\frac{a_1}{b}\right\|_{[t_0,\infty)} \left(\|b\|_{[t_0,\infty)} + \|a\|_{[t_0,\infty)} \left\|\frac{b}{a}\right\|_{[t_0,\infty)} \right) < 1. \end{aligned} \tag{6}$$

Then equation (1) *is exponentially stable.*

Proof. For simplicity we omit the index in the norm $\|\cdot\|_{[t_0,+\infty)}$ of functions.

Consider problem (2) with $\|f\| < \infty$, where $x(t) = \dot{x}(t) = 0,\ t \leq t_0$. We will prove that the solution x and its derivative are bounded functions on $[t_0, +\infty)$. First we have to obtain estimates for $x, \dot{x}, \ddot{x}, t \in I = [t_0, t_1], t_1 > t_0$. Rewrite equation (2)

$$\ddot{x}(t) + a(t)\dot{x}(t) + b(t)x(t) = \int_{g(t)}^{t} G(t,s)(\dot{x}(t) - \dot{x}(s))ds + \int_{h(t)}^{t} H(t,s)(x(t) - x(s))ds + f(t)$$

$$= \int_{g(t)}^{t} G(t,s) \int_{s}^{t} \ddot{x}(\tau) d\tau ds + \int_{h(t)}^{t} H(t,s) \int_{s}^{t} \dot{x}(\tau) d\tau ds + f(t).$$

Hence

$$x(t) = \int_{t_0}^{t} X(t,s) b(s) \left[\frac{1}{b(s)} \int_{g(s)}^{s} G(s,\xi) \int_{\xi}^{s} \ddot{x}(\tau) d\tau d\xi \right.$$

$$\left. + \frac{1}{b(s)} \int_{h(s)}^{s} H(s,\xi) \int_{\xi}^{s} \dot{x}(\tau) d\tau d\xi \right] ds + f_1(t),$$

where $X(t,s)$ is the fundamental function of equation (5) and $f_1(t) = \int_{t_0}^{t} X(t,s) f(s) ds$. Since $X(t,s)$ has an exponential estimate, f_1 is essentially bounded on $[t_0, \infty)$.

By Lemma 2.2 we have

$$\|x\|_{[t_0,t_1]} \leq \left\| \frac{a_1}{b} \right\| \|\ddot{x}\|_{[t_0,t_1]} + \left\| \frac{b_1}{b} \right\| \|\dot{x}\|_{[t_0,t_1]} + \|f_1\|. \quad (7)$$

Rewrite now (2) in another form:

$$\ddot{x}(t) + a(t)\dot{x}(t) = \int_{g(t)}^{t} G(t,s) \int_{s}^{t} \ddot{x}(\tau) d\tau ds - \int_{h(t)}^{t} H(t,s) x(s) ds + f(t).$$

Hence

$$\dot{x}(t) = \int_{t_0}^{t} e^{-\int_s^t a(\xi) d\xi} a(s) \left[\frac{1}{a(s)} \int_{g(s)}^{s} G(s,\xi) \int_{\xi}^{s} \ddot{x}(\tau) d\tau d\xi \right.$$

$$\left. - \frac{1}{a(s)} \int_{h(s)}^{s} H(s,\xi) x(\xi) d\xi \right] ds + f_2(t),$$

where $f_2(t) = \int_{t_0}^{t} e^{-\int_s^t a(\xi) d\xi} f(s) ds$ is an essential bounded on $[t_0, \infty)$ function.

Hence

$$\|\dot{x}\|_{[t_0,t_1]} \leq \left\| \frac{a_1}{a} \right\| \|\ddot{x}\|_{[t_0,t_1]} + \left\| \frac{b}{a} \right\| \|x\|_{[t_0,t_1]} + \|f_2\|. \quad (8)$$

From equation (2) we have

$$\|\ddot{x}\|_{[t_0,t_1]} \leq \|a\| \|\dot{x}\|_{[t_0,t_1]} + \|b\| \|x\|_{[t_0,t_1]} + \|f\|. \quad (9)$$

Denote $Y = \{\|x\|_{[t_0,t_1]}, \|\dot{x}\|_{[t_0,t_1]}, \|\ddot{x}\|_{[t_0,t_1]}\}^T, F = \{\|f_1\|, \|f_2\|, \|f\|, \}^T$. Inequalities (7)-(9) imply $Y \leq BY + F$, where

$$B = \begin{pmatrix} 0 & \left\| \frac{b_1}{b} \right\| & \left\| \frac{a_1}{b} \right\| \\ \left\| \frac{b_1}{b} \right\| & 0 & \left\| \frac{a_1}{a} \right\| \\ \|b\| & \|a\| & 0 \end{pmatrix}.$$

Hence $AY \leq F$, where $A = E - B$. Theorem conditions imply that A is an M-matrix then $Y \leq A^{-1}F$, where $A^{-1}F$ is a constant vector which does not depend on the interval I. Hence the solution of (2) with its derivative are bounded functions on $[t_0, \infty)$, therefore by Lemma 2.1 equation (1) is exponentially stable.

Corollary 3.1 *Assume that for some $t_0 \geq 0$ and $t \geq t_0$, $a_0^2 \geq 4B_0$ and the following condition holds*

$$\sigma\|a\|_{[t_0,\infty)} + \tau\Big(\left\| \frac{b}{a} \right\|_{[t_0,\infty)} + \sigma\|b\|_{[t_0,\infty)} \Big) + \sigma \left\| \frac{a}{b} \right\|_{[t_0,\infty)} \Big(\|a\|_{[t_0,\infty)} \left\| \frac{b}{a} \right\|_{[t_0,\infty)} + \|b\|_{[t_0,\infty)} \Big) < 1. \tag{10}$$

Then equation (1) *is exponentially stable.*

Proof. For simplicity we omit the index in the norm on functions. We have $t - s \leq t - g(t) \leq \sigma$ for $g(t) \leq s \leq t$. Similarly, $t - s \leq t - h(t) \leq \tau$ for $h(t) \leq s \leq t$. Hence

$$a_1(t) = \int_{g(t)}^{t} G(t,s)(t-s)ds \leq \int_{g(t)}^{t} G(t,s)\sigma ds = \sigma a(t),$$

$$b_1(t) = \int_{h(t)}^{t} H(t,s)(t-s)ds \leq \int_{h(t)}^{t} H(t,s)\tau ds = \tau b(t).$$

Then

$$\|a\| \left\| \frac{a_1}{a} \right\| + \left\| \frac{b_1}{b} \right\| \left(\left\| \frac{b}{a} \right\| + \|b\| \left\| \frac{a_1}{a} \right\| \right) + \left\| \frac{a_1}{b} \right\| \left(\|b\| + \|a\| \left\| \frac{b}{a} \right\| \right)$$

$$\leq \sigma\|a\| + \tau \left(\left\| \frac{b}{a} \right\| + \sigma\|b\| \right) + \sigma \left\| \frac{a}{b} \right\| \left(\|a\| \left\| \frac{b}{a} \right\| + \|b\| \right) < 1.$$

By Theorem 3.1 equation (1) is exponentially stable.

Corollary 3.2 *Assume there exist*

$$\lim_{t\to\infty} a(t) = a > 0, \lim_{t\to\infty} b(t) = b > 0, \lim_{t\to\infty} a_1(t) = a_1 > 0, \lim_{t\to\infty} b_1(t) = b_1 > 0.$$

If

$$a^2 \geq 4b, \;\; 3a_1 + \frac{b_1(1+a_1)}{a} < 1,$$

then the equation (1) is exponentially stable.

Limits in the corollary 3.2 exist, for example, for kernels of the form $M(t-s)^n e^{-\gamma(t-s)}$ where $n \geq 0$ is a natural number.

Example 3.1 Consider the following equation

$$\ddot{x}(t) + M_1 \int_{t-\sigma}^{t} e^{-\alpha_1(t-s)}\dot{x}(s)ds + M_2 \int_{t-\tau}^{t} e^{-\alpha_2(t-s)}x(s)ds = 0, \tag{11}$$

where $\alpha > 0, \beta > 0, \sigma > 0, \tau > 0$.

We have

$$a(t) = a = M_1 \int_{t-\sigma}^{t} e^{-\alpha_1(t-s)}ds = \frac{M_1}{\alpha_1}\left(1 - e^{-\alpha_1\sigma}\right),$$

$$b(t) = b = M_2 \int_{t-\tau}^{t} e^{-\alpha_2(t-s)}ds = \frac{M_2}{\alpha_2}\left(1 - e^{-\alpha_2\tau}\right),$$

$$a_1(t) = a_1 = M_1 \int_{t-\sigma}^{t} (t-s)e^{-\alpha(t-s)}ds = \frac{M_1}{\alpha}\left(\frac{1}{\alpha} - e^{-\alpha\sigma}(\sigma + \frac{1}{\alpha})\right),$$

$$b_1(t) = b_1 = M_2 \int_{t-\tau}^{t} (t-s)e^{-\alpha_2(t-s)}ds = \frac{M_2}{\alpha_2}\left(\frac{1}{\alpha_2} - e^{-\alpha_2\tau}(\tau + \frac{1}{\beta})\right).$$

Hence, if $a^2 \geq 4b, 3a_1 + \frac{b_1(1+a_1)}{a} < 1$, then equation (11) is exponentially stable.

Corollary 3.3 *Assume for $t \geq t_0$*

$$0 < a_0 \leq a(t) \leq A_0, 0 < b_0 \leq b(t) \leq B_0, a_0^2 \geq 4B_0,$$

$$0 < \sigma_0 \leq t - g(t) \leq \sigma, 0 < \tau_0 \leq t - h(t) \leq \tau$$

and

$$\frac{A_0\sigma^3}{2a_0\sigma_0} + \frac{B_0^2\tau^3}{2a_0b_0\tau_0\sigma_0}(1 + \frac{A_0\sigma^2}{2}) + \frac{A_0B_0\tau\sigma_2}{2b_0\tau_0}\left(1 + \frac{A_0\sigma}{a_0\sigma_0}\right) < 1.$$

Then the equation (1) is exponentially stable.

Proof. The proof follows from the inequalities

$$a_0\sigma_0 \leq a(t) \leq A_0\sigma, b_0\tau_0 \leq b(t) \leq B_0\tau, a_1(t) \leq A_0\frac{\sigma^2}{2}, b_1(t) \leq B_0\frac{\tau^2}{2}$$

and Theorem 3.1.

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