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Dual Phase Synchronization of Chaotic Systems Using Nonlinear Observer Based Technique

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Abstract: The present paper reports an investigation on dual phase synchronization results among chaotic systems with nonlinear observer controller. The dual phase synchronization is achieved using the nonlinear state observer technique and the stability theory. The Qi system and the Newton-Leipnik system are considered during the demonstration of dual phase synchronization. The nonlinear state observer technique is found to be very effective and convenient to achieve dual phase synchronization of various types of chaotic systems. Numerical simulation and graphical results demonstrate the effectiveness of the control technique during dual phase synchronization among chaotic systems.

Keywords: *dual synchronization, phase synchronization, chaotic systems, nonlinear state observer technique.*

Mathematics Subject Classification (2010): 34D06, 74H65, 34C28.

1 Introduction

Chaos theory is a developing field since 1970 and still the theory has not yet been understood very well. If a dynamical system is bounded and has infinite recurrences with dependency on initial conditions, then it is known as chaotic [1]. Several researchers have studied chaotic dynamical systems in various fields and effect of chaos in nonlinear dynamics is studied during the last few years. This effect is most common and has been detected in a number of dynamical systems of various types of physical nature. Chaos theory is also used to analyze the problems of dynamical and non-linear dynamical systems related with complex networks which are generally used in biological and social systems in ecology, medicine and in the field of business strategy. The most important achievement in the research of chaos is that chaotic systems can be made to synchronize with each other. The first idea of synchronization of two identical chaotic systems was

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analyzed by Pecora and Carrols [2]. In 2011, Runzi et al. [3] discussed combination synchronization using two master and one slave systems, before that synchronization was confined to one master and one slave systems. Yadav et al. [4] obtained dual function projective synchronization of fractional order complex chaotic systems.

In recent years, a lot of methods have been used to analyse synchronizations of the chaotic systems theoretically and experimentally, viz., the active control method, observer based method, backstepping method, nonlinear control method etc. Also, these methods are applied to study some new types of synchronizations, viz., combination synchronization, combination-combination synchronization, compound synchronization, multi-switching synchronization, compound-combination synchronization etc. ([5]- [9]). Juan and Xing-yuan [10] discussed nonlinear observer based phase synchronization of chaotic systems. Singh et. al. [11] explained dual combination synchronization of the fractional order complex chaotic systems.

The purpose of this paper is the investigation of dual phase synchronization of chaotic systems with nonlinear observer controllers. Dual synchronization is a special circumstance in synchronization in which two identical/non-identical pairs of chaotic systems are synchronized. The dual synchronization of systems plays an important role in many fields including chaotic secure communication. But it has received less attention of the researchers. There are only a few results available in the literature on dual synchronization between chaotic systems ([12]– [13]). In phase synchronization, the coupled chaotic systems keep their phase difference bounded by a constant while their amplitudes remain uncorrelated. The phase synchronization is usually applied upon two waveforms of the same frequency with identical phase angles with each cycle. However it can be applied if there is an integer relationship of frequency such that the cyclic signals share a repeating sequence of phase angles over consecutive cycles. There are few results about the phase synchronizations for the chaotic systems ([14]– [17]). Observer design, having vital importance in the area of systems and control theory, arises whenever some components of the state are not directly measured. After the solution of multivariate problems in the linear time invariant case by Luenberger [18], many researchers were motivated to extend the basic ideas of his work to the nonlinear context. Though the applications of linear observer theory to nonlinear problems had been a success, still the researchers were reduced to construct nonlinear observers using tools from nonlinear systems theory. A brief introduction to some of these nonlinear approaches to the problem of observer design can be found in the paper of Primbs [19]. In 2012, Beikzadeh and Taghirad [20] presented a novel nonlinear continuous-time observer based on differential state-dependent Riccati equation filter with guaranteed exponential stability of the estimation error dynamics utilising Lyapunov stability analysis which is used to obtain the required conditions for exponential stability of the estimation error dynamics.

These results have motivated the authors to study the dual phase synchronization between two identical pairs of different chaotic systems with nonlinear state observer algorithm using stability theory.The numerical example is provided to illustrate the obtained results. Dual phase synchronization between the systems with time delays ([21]– [25]) using the similar method will be considered for future study.

2 Problem Formulation

Let us consider the following two chaotic systems:

$$\dot{x} = Ax + Bf(x), \tag{1}$$

$$\dot{y} = Cy + Dg(y), \tag{2}$$

where $x, y \in R^n$ are the state vectors of the systems (1) and (2). $A, B \in R^{n\times n}$,$C, D \in R^{n\times m}$ are the constant matrices and $f, g : R^n \to R^m$ are the nonlinear functions. Suppose the dynamical systems (1) and (2) with the output are represented as

$$s(x) = f(x) + K_j x, \tag{3}$$

$$S(y) = g(y) + K'_j y, \tag{4}$$

where $K_j, K'_j \in R^{m\times n}$ denote the feedback gain matrices. Let us define the observer as

$$\dot{\hat{x}} = A\hat{x} + Bf(\hat{x}) + B[s(x) - s(\hat{x})], \tag{5}$$

$$\dot{\hat{y}} = C\hat{y} + Dg(\hat{y}) + D[S(y) - S(\hat{y})]. \tag{6}$$

The synchronization errors among the systems (1), (2) and (5), (6) are defined as

$$e_{x\hat{x}} = x - \hat{x}, \tag{7}$$

$$e_{y\hat{y}} = y - \hat{y}. \tag{8}$$

Then the error systems can be obtained as

$$\dot{e}_{x\hat{x}} = \dot{x} - \dot{\hat{x}} = Ae_{x\hat{x}} + Bf(x) - Bf(\hat{x}) - B[s(x) - s(\hat{x})],$$

$$\dot{e}_{y\hat{y}} = \dot{y} - \dot{\hat{y}} = Ce_{y\hat{y}} + Dg(y) - Dg(\hat{y}) - D[S(y) - S(\hat{y})].$$

From equations (3) and (4), the error systems reduce in the following form

$$\dot{e}_{x\hat{x}} = [A - BK_j]e_{x\hat{x}}, \tag{9}$$

$$\dot{e}_{y\hat{y}} = [C - DK'_j]e_{y\hat{y}}. \tag{10}$$

In order to make systems (9) and (10) controllable with the controllable matrices $[B, AB, ...A^{n-1}B]$ and $[D, CD, ...C^{n-1}D]$ of full ranks, the choices of the feedback gain matrices,K_j, K'_j will be in such a way that the characteristic polynomials of the matrices $[A - BK_j]$ and $[C - DK'_j]$ must have all the eigenvalues with negative real parts. Then the error systems will be stabilized and the dual synchronization among the systems under consideration is achieved. If there is any eigenvalue of the error system equal to zero, then another type of synchronization phenomenon called the phase synchronization occurs, in which the difference between various states of synchronized systems may not necessarily converge to zero, but is less than or equal to a constant.

3 Systems' Descriptions

3.1 Qi chaotic system

Consider the following Qi system [26]:

$$\dot{x}_1 = a_1(x_2 - x_1) + x_2x_3;\ \ \dot{x}_2 = a_3x_1 - x_2 - x_1x_3;\ \ \dot{x}_3 = -a_2x_3 + x_1x_2, \tag{11}$$

where x_1, x_2, x_3 are the state variables. The phase portrait of the system (11) for the parameter values $a_1 = 35, a_2 = 8/3, a_3 = 80$ and the initial condition $(3, 2, 1)$ is depicted in Fig. 1(a).

3.2 Newton-Leipnik system

The Newton-Leipnik system [27] is defined as

$$\dot{y}_1 = -b_1 y_1 + y_2 + 10 y_2 y_3;\ \ \dot{y}_2 = -y_1 - 0.4 y_2 + 5 y_1 y_3;\ \ \dot{y}_3 = b_2 y_3 - 5 y_1 y_2. \tag{12}$$

The phase portrait of the Newton-Leipnik system (12) is depicted in Fig. 1(b) for the values of the parameters $b_1 = 0.4, b_2 = 0.175$ and the initial condition $(0.394, 0, -0.16)$.

4 Dual Phase Synchronization of Chaotic Systems

In this section we are taking two systems, viz., Qi and Newton-Leipnik, to perform dual phase synchronization. The systems (11) and (12) can be rewritten as

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \begin{bmatrix}-a_1 & a_1 & 0\\ a_3 & -1 & 0\\ 0 & 0 & -a_2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \begin{bmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}x_2x_3\\ x_1x_3\\ x_1x_2\end{bmatrix} \tag{13}$$

and

$$\begin{bmatrix}\dot{y}_1\\ \dot{y}_2\\ \dot{y}_3\end{bmatrix} = \begin{bmatrix}-b_1 & 1 & 0\\ -1 & -0.4 & 0\\ 0 & 0 & b_2\end{bmatrix}\begin{bmatrix}y_1\\ y_2\\ y_3\end{bmatrix} + \begin{bmatrix}10 & 0 & 0\\ 0 & 5 & 0\\ 0 & 0 & -5\end{bmatrix}\begin{bmatrix}y_2y_3\\ y_1y_3\\ y_1y_2\end{bmatrix}. \tag{14}$$

Comparing equations (13) and (14) with equations (1) and (2), we get

$$A = \begin{bmatrix}-a_1 & a_1 & 0\\ a_3 & -1 & 0\\ 0 & 0 & -a_2\end{bmatrix}, B = \begin{bmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 1\end{bmatrix}, C = \begin{bmatrix}-b_1 & 1 & 0\\ -1 & -0.4 & 0\\ 0 & 0 & b_2\end{bmatrix}, D = \begin{bmatrix}10 & 0 & 0\\ 0 & 5 & 0\\ 0 & 0 & -5\end{bmatrix}.$$

The observers of the systems (11) and (12) are designed as

$$\begin{bmatrix}\dot{\hat{x}}_1\\ \dot{\hat{x}}_2\\ \dot{\hat{x}}_3\end{bmatrix} = \begin{bmatrix}-a_1 & a_1 & 0\\ a_3 & -1 & 0\\ 0 & 0 & -a_2\end{bmatrix}\begin{bmatrix}\hat{x}_1\\ \hat{x}_2\\ \hat{x}_3\end{bmatrix} + \begin{bmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\hat{x}_2\hat{x}_3\\ \hat{x}_1\hat{x}_3\\ \hat{x}_1\hat{x}_2\end{bmatrix} + B[s(x) - s(\hat{x})], \tag{15}$$

$$\begin{bmatrix}\dot{\hat{y}}_1\\ \dot{\hat{y}}_2\\ \dot{\hat{y}}_3\end{bmatrix} = \begin{bmatrix}-b_1 & 1 & 0\\ -1 & -0.4 & 0\\ 0 & 0 & b_2\end{bmatrix}\begin{bmatrix}\hat{y}_1\\ \hat{y}_2\\ \hat{y}_3\end{bmatrix} + \begin{bmatrix}10 & 0 & 0\\ 0 & 5 & 0\\ 0 & 0 & -5\end{bmatrix}\begin{bmatrix}\hat{y}_2\hat{y}_3\\ \hat{y}_1\hat{y}_3\\ \hat{y}_1\hat{y}_2\end{bmatrix} + D[S(y) - S(\hat{y})], \tag{16}$$

where $B[s(x)-s(\hat{x})], D[S(y)-S(\hat{y})]$ are outputs of the systems. Now by defining the error function towards dual synchronization as $e_{x_1\hat{x}_1} = x_1 - \hat{x}_1, e_{x_2\hat{x}_2} = x_2 - \hat{x}_2, e_{x_3\hat{x}_3} = x_3 - \hat{x}_3$, $e_{y_1\hat{y}_1} = y_1 - \hat{y}_1, e_{y_2\hat{y}_2} = y_2 - \hat{y}_2, e_{y_3\hat{y}_3} = y_3 - \hat{y}_3$, the error systems can be obtained as

$$\begin{bmatrix}\dot{e}_{x_1\hat{x}_1}\\ \dot{e}_{x_2\hat{x}_2}\\ \dot{e}_{x_3\hat{x}_3}\end{bmatrix} = \left\{\begin{bmatrix}-a_1 & a_1 & 0\\ a_3 & -1 & 0\\ 0 & 0 & -a_2\end{bmatrix} - \begin{bmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 1\end{bmatrix} K_1\right\}\begin{bmatrix}e_{x_1\hat{x}_1}\\ e_{x_2\hat{x}_2}\\ e_{x_3\hat{x}_3}\end{bmatrix}, \tag{17}$$

$$\begin{bmatrix}\dot{e}_{y_1\hat{y}_1}\\ \dot{e}_{y_2\hat{y}_2}\\ \dot{e}_{y_3\hat{y}_3}\end{bmatrix} = \left\{\begin{bmatrix}-b_1 & 1 & 0\\ -1 & -0.4 & 0\\ 0 & 0 & b_2\end{bmatrix} - \begin{bmatrix}10 & 0 & 0\\ 0 & 5 & 0\\ 0 & 0 & -5\end{bmatrix} K_1'\right\}\begin{bmatrix}e_{y_1\hat{y}_1}\\ e_{y_2\hat{y}_2}\\ e_{y_3\hat{y}_3}\end{bmatrix}. \tag{18}$$

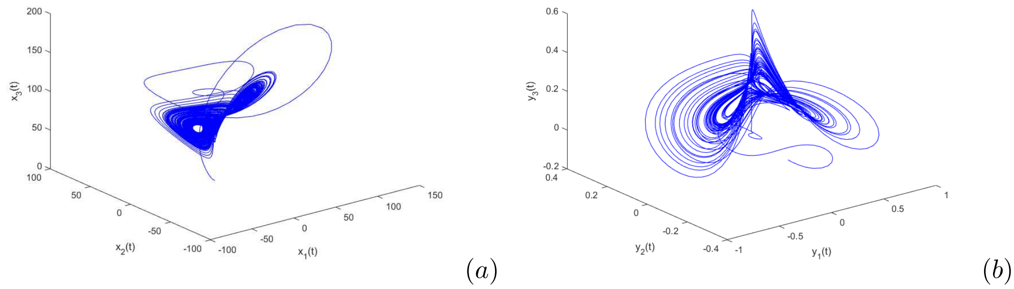


Figure 1: Phase portraits of chaotic systems: (a) the Qi system; (b) the Newton-Leipnik system.

The matrices $[B, AB, A^2B]$ and $[D, CD, C^2D]$ are in full ranks, so the systems (15) and (16) are the global observers of systems (13) and (14) through proper choices of the feedback gain matrices towards the synchronization

$$K_1 = \begin{bmatrix} -34 & 35 & 0 \\ -80 & 0 & 0 \\ 0 & 0 & -5/3 \end{bmatrix}, \quad K_1' = \begin{bmatrix} -3/50 & 1/10 & 0 \\ -1/5 & 3/25 & 0 \\ 0 & 0 & -0.235 \end{bmatrix}.$$

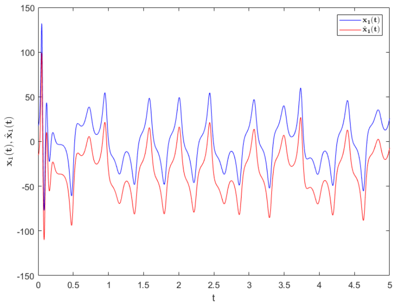
For phase synchronization of the above-mentioned systems, the feedback gain matrices are taken as

$$K_1 = \begin{bmatrix} -35 & 35 & 0 \\ -80 & 1 & 0 \\ 0 & 0 & -8/3 \end{bmatrix}, \quad K_1' = \begin{bmatrix} -2/50 & 1/10 & 0 \\ -1/5 & -2/25 & 0 \\ 0 & 0 & -0.035 \end{bmatrix}.$$

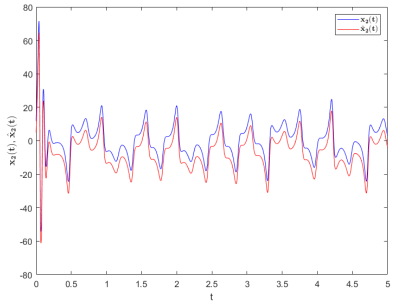
5 Numerical Simulation and Results

During numerical simulation the earlier considered parameters of the chaotic systems are taken. For the dual phase synchronization the initial conditions of the master systems I, II and slave systems I, II are taken as $(x_1(0), x_2(0), x_3(0)) = (18, 12, 10), (y_1(0), y_2(0), y_3(0)) = (0.349, 1.5, -0.16)$ and $(\hat{x}_1(0), \hat{x}_2(0), \hat{x}_3(0)) = (-15, 5, 1), (\hat{y}_1(0), \hat{y}_2(0), \hat{y}_3(0)) = (0.5, 2.5, 0.5)$, respectively. Hence the initial conditions of error system for dual phase synchronization will be $(33, 7, 9, -0.151, -1, -0.66)$. During dual synchronization of the systems, the time step size is taken as 0.005. Now, by choosing $\lambda_1 = 0, \lambda_2 = -1, \lambda_3 = -1, \lambda_4 = -1, \lambda_5 = -1, \lambda_6 = -1$, the phase synchronization between signals $x_1(t)$ and $\hat{x}_1(t)$ is achieved. It should be noted that, when $\lambda_1 = 0, \lambda_2 = -1, \lambda_3 = -1, \lambda_4 = -1, \lambda_5 = -1, \lambda_6 = -1$, the signals $x_2(t)$ and $\hat{x}_2(t)$ and $x_3(t)$ and $\hat{x}_3(t)$ and $y_1(t)$ and $\hat{y}_1(t)$ and $y_2(t)$ and $\hat{y}_2(t)$ and $y_3(t)$ and $\hat{y}_3(t)$ become synchronized. Similarly, if $\lambda_1 = -1, \lambda_2 = 0, \lambda_3 = -1, \lambda_4 = -1, \lambda_5 = -1, \lambda_6 = -1; \lambda_1 = -1, \lambda_2 = -1, \lambda_3 = 0, \lambda_4 = -1, \lambda_5 = -1, \lambda_6 = -1; \lambda_1 = -1, \lambda_2 = -1, \lambda_3 = -1, \lambda_4 = 0, \lambda_5 = -1, \lambda_6 = -1; \lambda_1 = -1, \lambda_2 = -1, \lambda_3 = -1, \lambda_4 = -1, \lambda_5 = 0, \lambda_6 = -1$ and $\lambda_1 = -1, \lambda_2 = -1, \lambda_3 = -1, \lambda_4 = -1, \lambda_5 = -1, \lambda_6 = 0$ are taken, phase synchronizations between signals $x_2(t)$ and $\hat{x}_2(t)$ and $x_3(t)$ and $\hat{x}_3(t)$ and $y_1(t)$ and $\hat{y}_1(t)$ and

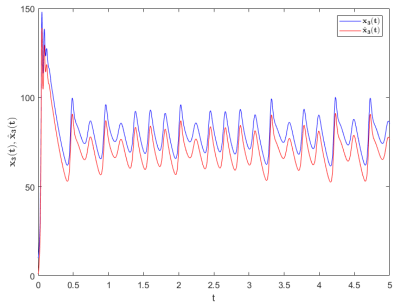
$y_2(t)$ and $\hat{y}_2(t)$ and $y_3(t)$ and $\hat{y}_3(t)$ are obtained, respectively. State trajectories of the dual phase synchronization of chaotic systems are depicted in Fig. 2(a)-(f). The plot of the error function for dual synchronization is depicted in Fig. 2(g), which shows that error states converge to zero when time becomes large. This implies that the dual phase synchronization between identical pairs of different chaotic systems consisting of the Qi and Newton-Leipnik systems occurs with the help of nonlinear observers.



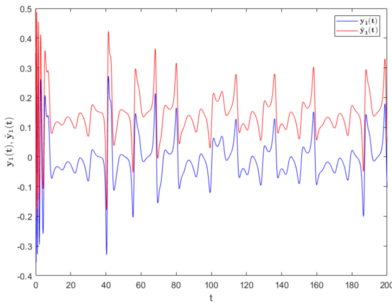
(a)



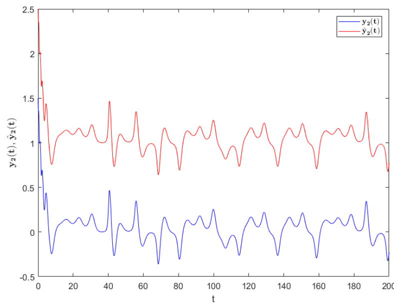
(b)



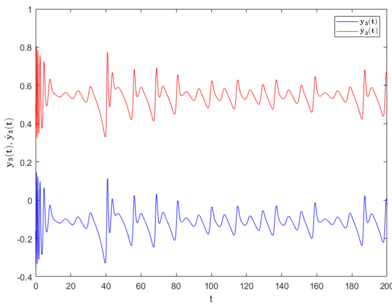
(c)



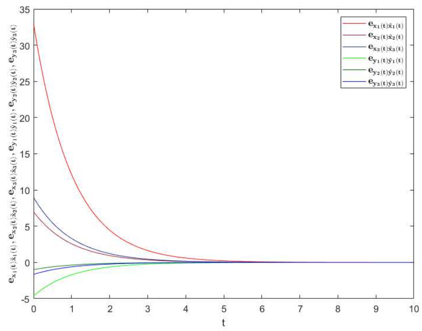
(d)



(e)



(f)



(g)

Figure 2: Phase synchronization for signals (a) between $x_1(t)$ and $\hat{x}_1(t)$, (b) between $x_2(t)$ and $\hat{x}_2(t)$, (c) between $x_3(t)$ and $\hat{x}_3(t)$, (d) between $y_1(t)$ and $\hat{y}_1(t)$, (e) between $y_2(t)$ and $\hat{y}_2(t)$, (f) between $y_3(t)$ and $\hat{y}_3(t)$, (g) The evolution of the error functions of chaotic systems during synchronization.

6 Conclusion

The present paper has successfully demonstrated the dual phase synchronization between the Qi and Newton-Leipnik systems using the nonlinear observer based technique. Based on the stability analysis, the dual phase synchronization of chaotic systems through nonlinear controller input parameters on the respective systems has been achieved and the components of the error system tend to zero as time becomes large, which helps to find the time required for dual phase synchronization between different chaotic systems. Numerical simulations are given to exhibit the reliability and effectiveness of the proposed dual combination synchronization scheme towards predicting the accuracy of the theory. The authors are optimistic that the outcome of this chapter will be utilized by the researchers involved in the field of chaotic systems.

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