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Oscillation of Second Order Nonlinear Differential Equations with Several Sub-Linear Neutral Terms

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Abstract: Some new sufficient conditions for oscillation of all solutions of a class of second order differential equations with several sub-linear neutral terms are given. Our results generalize and extend those reported in the literature. Examples are included to illustrate the importance of the results obtained.

Keywords: *second order neutral differential equation; sub-linear neutral term; oscillation.*

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1 Introduction

In this paper, we study the oscillatory behavior of second order differential equations with several sub-linear neutral terms of the form

$$(a(t)z'(t))' + q(t)x^{\beta}(\sigma(t)) = 0, \quad t \geq t_0 > 0, \tag{1}$$

where $m > 0$ is an integer, $z(t) = x(t) + \sum_{i=1}^{m} p_i(t)x^{\alpha_i}(\tau_i(t))$ and we assume that

(H_1) $0 \leq \alpha_i \leq 1$ for $i = 1, 2, ..., m$ and β are the ratios of odd positive integers;

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(H_2) $a, p_i, q : [t_0, \infty) \to \mathbb{R}^+$ are continuous functions for $i = 1, 2, ..., m$ with

$$\int_{t_0}^{\infty} \frac{1}{a(t)} dt = \infty; \tag{2}$$

(H_3) $\tau_i, \sigma : [t_0, \infty) \to \mathbb{R}$ are continuous functions with $\tau_i(t) < t$, $\sigma(t) \le t$, $\sigma'(t) > 0$ and $\tau_i(t), \sigma(t) \to \infty$ as $t \to \infty$ for $i = 1, 2, ..., m$.

By a solution of equation (1), we mean a function $x \in C([T_x, \infty), \mathbb{R})$, $T_x \ge t_0$, which has the property $az' \in C^1([T_x, \infty), \mathbb{R})$ and satisfies equation (1) on $[T_x, \infty)$. We consider only those solutions x of equation (1) which satisfy $\sup\{|x(t)| : t \ge T\} > 0$ for all $T \ge T_x$, and assume that the equation (1) possesses such solutions. As usual, a solution of equation (1) is called oscillatory if it has a zero on $[T, \infty)$ for all $T \ge T_x$; otherwise it is called nonoscillatory. If all solutions of a differential equation are oscillatory, then the equation itself is called oscillatory.

The problem of investigating the oscillatory behavior of solutions of particular functional differential equations received a great attention in the past decades, see, for example, [1] – [20] for recent references. However, there are few results dealing with the oscillation of second order differential equations with a sub-linear neutral term, see [3, 8, 19], even though, such equations arise in many applications, see [9]. In establishing some new criteria for the oscillation of solutions of such equations, we reduce the equation to an equation with linear neutral term, using some inequalities.

Thus, by using some elementary inequalities, we obtained in this paper some new oscillation results, which are new, extend and complement those established in [2–5, 14–17, 19, 20].

2 Oscillation Results

In what follows, all functional inequalities considered here are assumed to hold eventually, that is, they are satisfied for all t large enough. Due to the assumptions and the form of the equation (1), we can deal only with eventually positive solutions of equation (1).

We begin with the following lemma.

Lemma 2.1 *If a and b are nonnegative, then*

$$a^{\alpha} b^{1-\alpha} \le \alpha a + (1-\alpha) b \quad \textit{for} \;\; 0 < \alpha \le 1, \tag{3}$$

where equality holds if and only if $a = b$.

Proof. The proof of the lemma can be found in [9]. □

To simplify our notation, for any function $\rho : [t_0, \infty) \to \mathbb{R}^+$ which is positive, continuous decreasing to zero, we set

$$\begin{aligned} P(t) &= \left(1 - \sum_{i=1}^{m} \alpha_i p_i(t) - \frac{1}{\rho(t)} \sum_{i=1}^{m} (1-\alpha_i) p_i(t)\right), \\ Q(t) &= q(t) P^{\beta}(\sigma(t)) \end{aligned}$$

and

$$R(t) = \int_{t_1}^{t} \frac{1}{a(s)} ds.$$

Remark 2.1 It follows from condition (2), that the lower bound t_1 is an absolutely unimportant constant in the intended oscillatory criteria.

Lemma 2.2 *Assume condition* (2) *and let x be a positive solution of equation* (1). *Then the corresponding function z satisfies*

$$z(t) > 0,\ z'(t) > 0,\ \textit{and}\ (a(t)z'(t))' < 0,\ t \geq t_1 \geq t_0, \tag{4}$$

$$z(t) \geq R(t)a(t)z'(t),\ t \geq t_1 \tag{5}$$

and

$$\frac{z(t)}{R(t)}\ \textit{is decreasing for}\ t \geq t_1. \tag{6}$$

Proof. Assume that x is a positive solution of (1). Then $(a(t)z'(t))' < 0$ for $t \geq t_1 \geq t_0$ which in view of (2) implies $z'(t) > 0$ for $t \geq t_1 \geq t_0$. Since $a(t)z'(t)$ is decreasing, we have

$$z(t) \geq \int_{t_1}^{t} a(s)z'(s)\frac{1}{a(s)}ds \geq a(t)z'(t)R(t).$$

Moreover, using the previous inequality, we have

$$\left(\frac{z(t)}{R(t)}\right)' = \frac{a(t)z'(t)R(t) - z(t)}{a(t)R^2(t)} \leq 0.$$

We can conclude that $\frac{z(t)}{R(t)}$ is decreasing for $t \geq t_1$. $\square$

Theorem 2.1 *Let $\beta > 1$ and conditions $(H_1) - (H_3)$ and* (2) *hold. Let*

$$\int_{t_1}^{\infty} \frac{1}{a(u)} \int_{u}^{\infty} q(s)P^{\beta}(\sigma(s))ds\ du = \infty. \tag{7}$$

Assume that there is a positive continuous decreasing function $\rho : [t_0, \infty) \to (0, \infty)$ tending to zero, such that $P(t)$ is positive for $t \geq t_0$. If there exists a positive function $\mu(t) \in C^1([t_0, \infty), \mathbb{R})$ such that

$$\lim_{t\to\infty} \sup \int_{t_0}^{t} \left[\mu(s)Q(s) - \frac{a(\sigma(s))(\mu'(s))^2}{4\mu(s)\sigma'(s)}\right] ds = \infty, \tag{8}$$

then every solution of equation (1) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1), say $x(t) > 0,\ x(\tau_i(t)) > 0$ and $x(\sigma(t)) > 0$ for $t \geq t_1$, some $t_1 \geq t_0$ and for $i = 1, 2, ..., m$. It is easy to see that $z(t) > 0$ for $t \geq t_1$, and from Lemma 2.2 (4) holds.

Now from the definition of z, we have

$$\begin{aligned} x(t) &= z(t) - \sum_{i=1}^{m} p_i(t)x^{\alpha_i}(\tau_i(t)) \\ &\geq z(t) - \sum_{i=1}^{m} p_i(t)z^{\alpha_i}(t) \\ &\geq z(t) - \sum_{i=1}^{m} p_i(t)(\alpha_i z(t) + (1-\alpha_i)) \\ &= \left(1 - \sum_{i=1}^{m} \alpha_i p_i(t)\right) z(t) - \sum_{i=1}^{m}(1-\alpha_i)p_i(t), \end{aligned} \tag{9}$$

where we have used inequality (3) with $b = 1$. Since $z(t)$ is positive and increasing and $\rho(t)$ is positive and decreasing to zero, there is a $t_2 \geq t_1$ such that

$$z(t) \geq \rho(t) \text{ for } t \geq t_2. \tag{10}$$

Using (10) in (9), we obtain

$$x(t) \geq \left(1 - \sum_{i=1}^{m} \alpha_i p_i(t) - \frac{1}{\rho(t)} \sum_{i=1}^{m} (1-\alpha_i) p_i(t)\right) z(t) = P(t) z(t)$$

and substituting this in equation (1) yields

$$(a(t)z'(t))' + q(t)P^{\beta}(\sigma(t))z^{\beta}(\sigma(t)) \leq 0, \ t \geq t_2. \tag{11}$$

From condition (7) it follows that $z(t) \to \infty$ as for $t \to \infty$ and for $\beta > 1$, inequality

$$z^{\beta}(\sigma(t)) > z(\sigma(t))$$

holds. Using this inequality in (11), we obtain

$$(a(t)z'(t))' + Q(t)z(\sigma(t)) \leq 0, t \geq t_2. \tag{12}$$

Define the function

$$w(t) = \mu(t) \frac{a(t)z'(t)}{z(\sigma(t))}, \ t \geq t_2.$$

Then $w(t) > 0$ for $t \geq t_2$ and

$$w'(t) = \mu'(t) \frac{a(t)z'(t)}{z(\sigma(t))} + \mu(t) \frac{(a(t)z'(t))'}{z(\sigma(t))} - \frac{\mu(t)a(t)z'(t)}{z^2(\sigma(t))} z'(\sigma(t)).\sigma'(t). \tag{13}$$

Since $a(t)z'(t)$ is positive and nonincreasing, we obtain

$$a(t)z'(t) \leq a(\sigma(t))z'(\sigma(t)). \tag{14}$$

Using (14) and (12) in (13), and completing the square, we see that

$$w'(t) \leq -\mu(t)Q(t) + \frac{a(\sigma(t))(\mu'(t))^2}{4\mu(t)\sigma'(t)}.$$

An integration of the last inequality from t_2 to t yields

$$\int_{t_2}^{t} \left[\mu(s)Q(s) - \frac{a(\sigma(s))(\mu'(s))^2}{4\mu(s)\sigma'(s)}\right] ds \leq w(t_2),$$

and on taking lim sup as $t \to \infty$, we obtain a contradiction with (8). This completes the proof. □

Next, we present new oscillation results for equation (1) with $\beta > 1$.

Theorem 2.2 *Let $\beta > 1$ and conditions $(H_1) - (H_3)$ and (2) hold. Assume that there is a positive continuous and decreasing function $\rho : [t_0, \infty) \to \mathbb{R}^+$ tending to zero as $t \to \infty$ such that $P(t)$ is positive for all $t \geq t_0$. If there exists a positive function $\mu(t) \in C^1([t_0, \infty), \mathbb{R})$ such that*

$$\lim_{t\to\infty} \sup \int_{t_0}^{t} \left[\mu(s)q(s)P^{\beta}(\sigma(s))\rho^{\beta-1}(\sigma(s)) - \frac{a(\sigma(s))(\mu'(s))^2}{4\mu(s)\sigma'(s)}\right] = \infty, \tag{15}$$

then every solution of equation (1) is oscillatory.

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1), say $x(t) > 0,\ x(\tau_i(t)) > 0$ and $x(\sigma(t)) > 0$ for $t \geq t_1$, some $t_1 \geq t_0$ and $i = 1, 2, ..., m$. Proceeding as in the proof of Theorem 2.1, we see that (11) holds. Now using (10) in (11), we obtain

$$(a(t)z'(t))' + q(t)P^{\beta}(\sigma(t))\rho^{\beta-1}(\sigma(t))z(\sigma(t)) \leq 0,\ t \geq t_2.$$

The rest of the proof is similar to that of Theorem 2.1 and hence it is omitted. □

If $\beta = 1$, then from Theorem 2.2 one can immediately obtain the following oscillation results for the equation (1).

Theorem 2.3 *Let $\beta = 1$ and conditions $(H_1) - (H_3)$ and (2) hold. Assume that there is a positive continuous and decreasing function $\rho : [t_0, \infty) \to \mathbb{R}^+$ tending to zero as $t \to \infty$, such that $P(t)$ is positive for all $t \geq t_0$. If there exists a positive function $\mu(t) \in C^1([t_0, \infty), \mathbb{R})$ such that*

$$\lim_{t\to\infty} \sup \int_{t_0}^{t} \left[\mu(s)q(s)P(\sigma(s)) - \frac{a(\sigma(s))(\mu'(s))^2}{4\mu(s)\sigma'(s)}\right] ds = \infty, \tag{16}$$

then every solution of equation (1) is oscillatory.

Next, we obtain an oscillation result for the equation (1) in the case $0 < \beta < 1$.

Theorem 2.4 *Let $0 < \beta < 1$ and conditions $(H_1) - (H_3)$ and (2) hold. Assume that there is a positive continuous and decreasing function $\rho(t) : [t_0, \infty) \to \mathbb{R}^+$ tending to zero as $t \to \infty$, such that $P(t)$ is positive for all $t \geq t_0$. If there exists a positive function $\mu(t) \in C^1([t_0, \infty), \mathbb{R})$ such that*

$$\lim_{t\to\infty} \sup \int_{t_0}^{t} \left[\frac{\mu(s)q(s)P^{\beta}(\sigma(s))R^{\beta-1}(\sigma(s))}{K^{1-\beta}} - \frac{a(\sigma(s))(\mu'(s))^2}{4\mu(s)\sigma'(s)}\right] ds = \infty \tag{17}$$

for every constant $K > 0$, then every solution of equation (1) is oscillatory.

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1), say $x(t) > 0,\ x(\tau_i(t)) > 0$ and $x(\sigma(t)) > 0$ for $t \geq t_1$, for some $t_1 \geq t_0$ and $i = 1, 2, ..., m$. Proceeding as in the proof of Theorem 2.1, we obtain (11). Now (11) can be written as

$$(a(t)z'(t))' + q(t)P^{\beta}(\sigma(t))R^{\beta-1}(\sigma(t))\frac{z^{\beta-1}(\sigma(t))}{R^{\beta-1}(\sigma(t))}z(\sigma(t)) \leq 0 \tag{18}$$

for all $t \geq t_2 \geq t_1$. Since $\frac{z(t)}{R(t)}$ is decreasing, there is a constant $K > 0$ such that

$$\frac{z(t)}{R(t)} \leq K \text{ for } t \geq t_2. \tag{19}$$

Using (19) and $\beta < 1$, in (18), we have

$$(a(t)z'(t))' + q(t)\frac{P^{\beta}(\sigma(t))R^{\beta-1}(\sigma(t))}{K^{1-\beta}}z(\sigma(t)) \leq 0,\ t \geq t_2.$$

We define function $w(t)$ as in proof of Theorem 2.1. Proceeding exactly as in the proof of Theorem 2.1, we get

$$w'(t) \leq -\mu(t)q(t)\frac{P^{\beta}(\sigma(t))R^{\beta-1}(\sigma(t))}{K^{1-\beta}} + \frac{a(\sigma(t))(\mu'(t))^2}{4\mu(t)\sigma'(t)}.$$

Integrating the last inequality from t_2 to t, we obtain

$$\int_{t_0}^{t} \left[\frac{\mu(s)q(s)P^{\beta}(\sigma(s))R^{\beta-1}(\sigma(s))}{K^{1-\beta}} - \frac{a(\sigma(s))(\mu'(s))^2}{4\mu(s)\sigma'(s)} \right] ds \le w(t_2),$$

and on taking limsup as $t \to \infty$, we have a contradiction with (17). □

Next, we use a comparison method to prove our results for the case $\beta \in (0, \infty)$.

Theorem 2.5 *Let conditions $(H_1) - (H_3)$ and (2) hold. Assume that there is a positive, continuous and decreasing function $\rho(t) : [t_0, \infty) \to \mathbb{R}^+$ tending to zero such that $P(t)$ is positive for all $t \ge t_0$. If the first order delay differential equation*

$$w'(t) + q(t)P^{\beta}(\sigma(t))R^{\beta}(\sigma(t))w^{\beta}(\sigma(t)) = 0, \ t \ge t_1 \tag{20}$$

is oscillatory, then every solution of equation (1) is oscillatory.

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1), say $x(t) > 0, \ x(\tau_i(t)) > 0$ and $x(\sigma(t)) > 0$ for $t \ge t_1$, for some $t_1 \ge t_0$ and $i = 1, 2, ..., m$. Proceeding as in the proof of Theorem 2.1, we see that (11) holds. Using (5) in (11), we obtain

$$(a(t)z'(t))' + q(t)P^{\beta}(\sigma(t))R^{\beta}(\sigma(t))(a(\sigma(t))z'(\sigma(t)))^{\beta} \le 0, \ t \ge t_1. \tag{21}$$

Set $w(t) = a(t)z'(t)$. Thus $w(t) > 0$, and

$$w'(t) + q(t)P^{\beta}(\sigma(t))R^{\beta}(\sigma(t))w^{\beta}(\sigma(t)) \le 0.$$

By Lemma 2.2 of [17], the equation (20) has a positive solution which is a contradiction. This completes the proof. □

Using the results of [8] and [18], one can easily obtain the following corollaries from Theorem 2.5.

Corollary 2.1 *Let all conditions of Theorem 2.5 hold with $\beta = 1$ for all $t \ge t_0$. If*

$$\lim_{t\to\infty} \inf \int_{\sigma(t)}^{t} q(s)P(\sigma(s))R(\sigma(s))ds > \frac{1}{e},$$

then every solution of equation (1) is oscillatory.

Corollary 2.2 *Let all conditions of Theorem 2.5 hold with $0 < \beta < 1$ for all $t \ge t_0$. If*

$$\int_{t_0}^{\infty} q(t)P^{\beta}(\sigma(t))R^{\beta}(\sigma(t))dt = \infty,$$

then every solution of equation (1) is oscillatory.

Corollary 2.3 *Let all conditions of Theorem 2.5 hold with $\beta > 1$ for all $t \ge t_0$. If $\sigma(t) = t - \delta, \ \delta > 0$, and*

$$\lim_{t\to\infty} \inf \beta^{-\frac{t}{\delta}} \log(q(t)P^{\beta}(t-\delta)R^{\beta}(t-\delta)) > 0,$$

then every solution of equation (1) is oscillatory.

3 Examples

In this section, we provide some examples to illustrate the main results.

Example 3.1 Consider the differential equation with sub-linear neutral terms

$$\left(t\left(x(t)+\frac{1}{t}x^{\frac{1}{3}}\left(\frac{t}{2}\right)+\frac{1}{t^2}x^{\frac{1}{5}}\left(\frac{t}{3}\right)\right)'\right)'+t^{\gamma}x^3\left(\frac{t}{2}\right)=0,\ t\geq 8. \tag{22}$$

Here $a(t)=t,\ p_1(t)=\frac{1}{t},\ p_2(t)=\frac{1}{t^2},\ \tau_1(t)=\frac{t}{2},\ \tau_2(t)=\frac{t}{3},\ \sigma(t)=\frac{t}{2},\ q(t)=t^{\gamma}$, $\alpha_1=\frac{1}{3},\ \alpha_2=\frac{1}{5}$ and $\beta=3$. Let $\rho(t)=\frac{1}{t}$ then $\rho(t)\to 0$ as $t\to\infty$ and $\eta(t)=\frac{1}{t}$ and

$$\begin{aligned} P(t) &= \left(1-\frac{1}{3t}-\frac{1}{5t^2}-t\left(\frac{2}{3t}+\frac{4}{5t^2}\right)\right) \\ &= \left(\frac{1}{3}-\frac{1}{3t}-\frac{1}{5t^2}-\frac{4}{5t}\right)=\frac{5t^2-17t-3}{15t^2}>0 \ \text{ for } \ t\geq 8. \end{aligned}$$

By taking $\mu(t)=t$, we see that

$$\limsup_{t\to\infty}\int_8^t\left(\frac{3}{2}s^{\gamma-1}\left(\frac{5s^2-34s-12}{15s^2}\right)^3-\frac{1}{4}\right)ds=\infty$$

provides $\gamma>1$. So by Theorem 2.2, every solution of equation (22) is oscillatory.

Example 3.2 Consider the differential equation with sub-linear neutral terms

$$\left(t\left(x(t)+\frac{1}{t}x^{\frac{3}{5}}\left(\frac{t}{2}\right)+\frac{1}{t^2}x^{\frac{1}{3}}\left(\frac{t}{3}\right)\right)'\right)'+t^{\gamma}x\left(\frac{t}{2}\right)=0. \tag{23}$$

Here $a(t)=t,\ p_1(t)=\frac{1}{t},\ p_2(t)=\frac{1}{t^2},\ \tau_1(t)=\frac{t}{2},\ \tau_2(t)=\frac{t}{3},\ \sigma(t)=\frac{t}{2},\ q(t)=t^{\gamma}$, $\alpha_1=\frac{3}{5},\ \alpha_2=\frac{1}{3}$ and $\beta=1$. Let $\rho(t)=\frac{1}{t}$ then $\rho(t)\to 0$ as $t\to\infty$ and

$$\begin{aligned} P(t) &= 1-\frac{3}{5t}-\frac{1}{3t^2}-t\left(\frac{2}{5t}+\frac{2}{3t^2}\right) \\ &= \left(1-\frac{3}{5t}-\frac{1}{3t^2}-\frac{2}{5}-\frac{2}{3t}\right)=\frac{3}{5}-\frac{19}{15t}-\frac{1}{3t^2} \\ &= \frac{1}{15t^2}(9t^2-19t-5), \\ P\left(\frac{t}{2}\right) &= \left(\frac{9t^2-38t-20}{15t^2}\right)>0 \text{ for } t\geq 8. \end{aligned}$$

By taking $\mu(t)=t$, we see that

$$\limsup_{t\to\infty}\int_8^t\left(s^{\gamma+1}\left(\frac{9s^2-38s-20}{15s^2}\right)-\frac{1}{4}\right)ds=\infty$$

provides $\gamma\geq -1$. By Theorem 2.3, every solution of equation (23) is oscillatory.

Example 3.3 Consider the differential equation with sub-linear neutral terms

$$\left(t^{\frac{1}{2}}\left(x(t)+\frac{1}{t}x^{\frac{1}{3}}\left(\frac{t}{2}\right)+\frac{1}{t^2}x^{\frac{5}{7}}\left(\frac{t}{3}\right)\right)'\right)'+t^{\gamma}x^{\frac{1}{3}}\left(\frac{t}{2}\right)=0. \tag{24}$$

Here $a(t)=t^{\frac{1}{2}}$, $p_1(t)=\frac{1}{t}$, $p_2(t)=\frac{1}{t^2}$, $\alpha_1=\frac{1}{3}$, $\alpha_2=\frac{5}{7}$, $\beta=\frac{1}{3}$, $q(t)=t^{\gamma}$, $\tau_1(t)=\frac{t}{2}$, $\tau_2(t)=\frac{t}{3}$ and $\sigma(t)=\frac{t}{2}$. Let $\rho(t)=\frac{1}{t}$, then $\rho(t)\to 0$ as $t\to\infty$ and

$$\begin{aligned} P(t) &= 1-\frac{1}{3t}-\frac{5}{7t^2}-t\left(\frac{2}{3t}+\frac{2}{7t^2}\right) \\ &= 1-\frac{1}{3t}-\frac{5}{7t^2}-\frac{2}{3}-\frac{2}{7t}=\left(\frac{1}{3}-\frac{13}{21t}-\frac{5}{7t^2}\right), \\ P(\sigma(t)) &= \left(\frac{1}{3}-\frac{26}{21t}-\frac{20}{7t^2}\right)=\frac{(7t^2-26t-60)}{21t^2}>0,\ t\geq 8, \\ R(t) &= \int_8^t \frac{1}{s^{1/2}}ds=2\sqrt{t}-4\sqrt{2}. \end{aligned}$$

By taking $\mu(t)=1$, we see that

$$\lim_{t\to\infty}\sup\int_8^t K^{1/3-1}s^{\gamma}\left(\frac{7s^2-26s-60}{21s^2}\right)^{\frac{1}{3}}\left(2s^{\frac{1}{2}}-4\sqrt{2}\right)^{-\frac{2}{3}}ds=\infty$$

provides $\gamma\geq\frac{1}{3}$. By Theorem 2.4, every solution of equation (22) is oscillatory.

4 Conclusion

The results presented in this paper are new and complement to those of [3, 17, 19, 20]. Further it would be of interest to use this method to study equation (1) with $\alpha_i>1$ for $i=1,2,...,m$, that is, equation (1) with several superlinear neutral terms. Also, the results established in [2–5, 14–17, 19, 20] cannot be applied to equations (22) to (24), since the neutral term contains more than one sub-linear neutral term. Thus the results obtained in this paper are applicable to several classes of neutral type differential equations.

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