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Approximate Analytical Solutions for Transient Heat Transfer in Two-Dimensional Straight Fins

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Abstract: In this paper we analyse the heat transfer in two-dimensional straight fins. Both heat transfer coefficient and thermal conductivity are temperature dependent. The resulting 2+1 dimension partial differential equation (PDE) is rendered nonlinear and difficult to solve exactly, particularly with prescribed initial and boundary conditions. The three-dimensional differential transform method (3D DTM) is used to construct the approximate analytical solutions. The effects of parameters, appearing in the boundary value problem (BVP), on temperature profile of the fin are studied.

Keywords: *3D DTM; approximate solutions; 2D staright fins, heat transfer.*

Mathematics Subject Classification (2010): 35K57, 35G30, 35K05, 74A15, 41A58.

1 Introduction

Fins are surfaces that extend from a primary body to a surrounding fluid. They are predominantly used to increase the heat transfer rate between the body and its surroundings. Fins are designed in such a way that they increase the surface area of an object and hence its contact with the environment. They come in various shapes, geometries and profiles that cater for a diverse range of problems and applications (the reader is referred to [1] for a detailed theory). Fins are widely used in devices that exchange heat, common examples would include vehicle engine radiators, refrigerators, air conditioning devices and compressors. Consequently, the study of heat transfer in fins continues to be of interest.

Two-dimensional fin problems have received much attention, however, it is assumed in most works that the thermal conductivity and the heat transfer coefficient are constants, and the internal heat generation is omitted. In [2], the authors provided the approximate

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solutions using homotopy analysis for the transient problem with constant thermal properties. Moitsheki and Rowjee [3] constructed exact solutions for a two-dimensional steady state problem with the temperature-dependent thermal conductivity, heat transfer coefficient and internal heat generation. Analysis of transient heat transfer in straight fins of various shapes and with constant heat flux was carried out in [4]. A two-dimensional rectangular fin with variable heat transfer coefficient was analysed using the Fourier series approach [5]. In [6], two-dimensional trapezoidal fins were analysed wherein heat loss through fins at various slopes were compared. Exact solutions for heat transfer in rectangular fins were constructed in [7].

In this paper, the two-dimension heat flow in straight fins is analysed using the 3D DTM. The DTM was introduced in [8] and an account for the higher dimension DTM may be found in [9]. In Section 2, a mathematical description of the problem in question is provided. A brief account of the DTM is provided in Section 3. In Section 4, approximate analytical solutions are constructed. Some discussions and conclusion are given in Section 5.

2 Mathematical Description

The fin is attached to a primary surface of temperature T_b. The coordinate system has the origin at the intersection of the fin surface and the fin tip, with the X-axis extending towards the fin base and the Y-axis extending towards the centre of the fin. The fin height is L and the length from the X-axis to the center of the fin is δ. The temperature of the surrounding fluid into which the fin extends is designated by T_s. The thermal conductivity and the heat transfer coefficient are dependent on temperature and are denoted by $K(T)$ and $H(T)$, respectively. For our problem under consideration we assume no internal heat generation. Therefore, in the dimensionless variables and parameters, the governing BVP is given by (see also [1])

$$\frac{\partial\theta}{\partial\tau} = \frac{\partial}{\partial x}\left[k(\theta)\frac{\partial\theta}{\partial x}\right] + E^2\frac{\partial}{\partial y}\left[k(\theta)\frac{\partial\theta}{\partial y}\right], \tag{1}$$

subject to the initial condition

$$\theta(0,x,y) = 0, \quad 0 \le x \le 1, \quad 0 \le y \le 1, \tag{2}$$

and the boundary conditions

$$\theta(\tau,1,y) = 1, \quad 0 \le y \le 1, \quad \tau > 0, \tag{3}$$

$$\frac{\partial\theta}{\partial x} = 0, \quad x = 0, \quad 0 \le y \le 1, \quad \tau > 0, \tag{4}$$

$$k(\theta)\frac{\partial\theta}{\partial y} = -Bih(\theta)\theta, \quad y = 0, \quad 0 \le x \le 1, \quad \tau > 0, \tag{5}$$

$$\frac{\partial\theta}{\partial y} = 0, \quad y = 1, \quad 0 \le x \le 1, \quad \tau > 0, \tag{6}$$

where the dimensionless quantities are given by

$$t = \frac{L^2\rho c_p}{K_a}\tau, \quad X = Lx, \quad Y = \delta y, K = K_a k, \quad H = H_b h, \quad T = (T_b - T_s)\theta + T_s,$$

with τ, x, y, k, h and θ being the dimensionless variables. K_a and H_b are the ambient thermal conductivity and the fin base heat transfer coefficient, respectively, and $E = \frac{L}{\delta}$ and $Bi = \frac{\delta H_b}{K_a}$ are the fin extension factor and the Biot number, respectively. An account of studies of diffusion equations in higher dimensions may be found, for example, in [10].

For practicality purposes, two cases will be considered for the relation of the thermal conductivity and temperature [11], namely, the linear function relation and the power law. We also consider the heat transfer coefficient given by the power law. The two cases for the thermal conductivity (see e.g. [12,13]) are given by
Case (i) the power law

$$k(\theta) = \theta^n, \tag{7}$$

where n is a dimensionless constant and
Case (ii) the linear function

$$k(\theta) = 1 + \beta\theta, \tag{8}$$

where $\beta = \epsilon(T_b - T_s)$ is the thermal conductivity parameter and ϵ is the thermal conductivity gradient. For most engineering applications the heat transfer coefficient has a power law relation with temperature [1], that is,

$$h(\theta) = \theta^m. \tag{9}$$

Here m is a dimensionless constant, which in engineering applications takes values from -3 to 3.

3 A Brief Account of the p-Dimensional DTM

For an analytic multivariable function $f(x_1, x_2, ..., x_p)$, we have the p-dimensional transform given by

$$F(k_1, k_2, ..., k_p) = \frac{1}{k_1!k_2!...k_p!}\left[\frac{\partial^{k_1+k_2+...+k_p} f(x_1, x_2, ..., x_p)}{\partial x_1^{k_1}\partial x_2^{k_2}...\partial x_p^{k_p}}\right]\Bigg|_{(x_1,x_2,...,x_p)=(0,0,...,0)}. \tag{10}$$

The upper and lower case letters stand for the transformed and the original functions, respectively. The transformed function is also referred to as the T-function, the differential inverse transform is given by

$$f(x_1, x_2, ..., x_p) = \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}...\sum_{k_p=0}^{\infty} F(k_1, k_2, ..., k_p)\prod_{l=1}^{p} x_l^{k_l}. \tag{11}$$

It can be easily deduced that the substitution of equation (10) into equation (11) gives the Taylor series expansion of the function $f(x_1, x_2, ..., x_p)$ about the point $(x_1, x_2, ..., x_p) = (0, 0, ..., 0)$. This is given by

$$f(x_1, x_2, ..., x_p) = \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}...\sum_{k_p=0}^{\infty} \frac{\prod_{l=1}^{p} x_l^{k_l}}{k_1!k_2!...k_p!}\left[\frac{\partial^{k_1+k_2+...+k_p} f(x_1, x_2, ..., x_p)}{\partial x_1^{k_1}\partial x_2^{k_2}...\partial x_p^{k_p}}\right]\Bigg|_{x_1=0,...,x_p=0}. \tag{12}$$

For real world applications the function $f(x_1, x_2, ..., x_p)$ is given in terms of a finite series for some $q, r, s \in \mathbb{Z}$. Then equation (11) becomes

$$f(x_1, x_2, ..., x_p) = \sum_{k_1=0}^{q}\sum_{k_2=0}^{r}...\sum_{k_p=0}^{s} F(k_1, k_2, ..., k_p)\prod_{l=1}^{p} x_l^{k_l}. \tag{13}$$

Original function $f(x_1, x_2, ..., x_p)$	T-function $F(k_1, k_2, ..., k_p)$
$f(x_1, x_2, ..., x_p) = \lambda g(x_1, x_2, ..., x_p)$	$F(k_1, k_2, ..., k_p) = \lambda G(k_1, k_2, ..., k_p)$
$f(x_1, x_2, ..., x_p) = g(x_1, x_2, ..., x_p) \pm p(x_1, x_2, ..., x_p)$	$F(k_1, k_2, ..., k_p) = G(k_1, k_2, ..., k_p)$ $\pm P(k_1, k_2, ..., k_p)$
$f(x_1, x_2, ..., x_p) = \frac{\partial^{r_1+r_2+...+r_p} g(x_1,x_2,...,x_p)}{\partial x_1^{r_1} \partial x_2^{r_2} ... \partial x_p^{r_p}}$	$F(k_1, k_2, ..., k_p) = \frac{(k_1+r_1)!...(k_p+r_p)!}{k_1!...k_p!}$ $(k_1 + r_1, ..., k_p + r_p)$
$f(x_1, x_2, ..., x_p) = \prod_{l=1}^{p} x_l^{e_l}$	$F(k_1, k_2, ..., k_p) = \delta(k_1 - e_1,$ $k_2 - e_2, ..., k_p - e_p)$

Table 1: Theorems and operations performed in the p-dimensional DTM.

We now give some important operations and theorems performed in the p-dimensional DTM in Table 1. Those have been derived using the definition in (10) together with previously obtained results [14].

In the table

$$\delta(k_1 - e_1, k_2 - e_2, ..., k_p - e_p) = \begin{cases} 1, & \text{if } k_i = e_i \text{ for } i = 1, 2, .., p. \\ 0, & \text{otherwise.} \end{cases}$$

4 Approximate Analytical Solutions

4.1 Constant and linear function thermal conductivity

The work presented in this section will cover two cases, namely, the linear model case with $\beta = 0$, and the nonlinear case with $\beta \neq 0$. Equation (1) may be given by

$$\frac{\partial \theta}{\partial \tau} = \frac{\partial}{\partial x}\left[(1 + \beta\theta)\frac{\partial \theta}{\partial x}\right] + E^2 \frac{\partial}{\partial y}\left[(1 + \beta\theta)\frac{\partial \theta}{\partial y}\right], \tag{14}$$

subject to the conditions (2) - (6). We now apply the three-dimensional DTM to the governing equation (14) and the above mentioned conditions to obtain the approximate analytical solution

$$\begin{aligned} \theta(\tau, x, y) = \quad & c\tau + c\tau y + c\tau y^2 + c\tau y^3 + c\tau y^4 + c\tau y^5 + c\tau y^6 + c\tau y^7 + ... \\ & + c\tau x^2 - \frac{Bic^{m+1}}{(1+\beta c)}\tau y x^2 - \frac{5c}{E^2}\tau y^2 x^2 + \frac{5Bic^{m+1}}{3E^2(1+\beta c)}\tau y^3 x^2 + ... \\ & + c\tau x^3 - \frac{Bic^{m+1}}{(1+\beta c)}\tau y x^3 - \frac{9c}{E^2}\tau y^2 x^3 + \frac{3Bic^{m+1}}{E^2(1+\beta c)}\tau y^3 x^3 + ... \\ & \vdots \end{aligned} \tag{15}$$

For this problem we will choose the boundary $x = 1$. Along this boundary c must satisfy the equation

$$c\tau + c\tau y + c\tau y^2 + ... + c\tau - \frac{Bic^{m+1}}{(1+\beta c)}\tau y - \frac{5c}{2E^2}\tau y^2 ... + c\tau - \frac{Bic^{m+1}}{(1+\beta c)}\tau y - \frac{9c}{E^2}\tau y^2 + ... = 1. \tag{16}$$

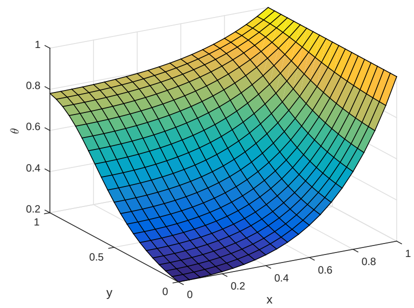

Figure 1: Approximate analytical solutions for a two-dimensional rectangular fin with a constant thermal conductivity ($\beta = 0$) for $\tau = 0.4$. The parameters are set such that $E = 2.8$, $Bi = 0.2$, and $m = 3$.

(y,x)	0	0.2	0.4	0.6	0.8	1
0	0.2000	0.2100	0.2520	0.3567	0.5779	1
0.2	0.2493	0.2590	0.2993	0.3986	0.6064	1
0.4	0.3465	0.3558	0.3933	0.4829	0.6646	1
0.6	0.5075	0.5157	0.5475	0.6194	0.7574	1
0.8	0.6883	0.6943	0.7168	0.7650	0.8528	1
1	0.7801	0.7837	0.7975	0.8286	0.8894	1

Table 2: Approximate analytical solutions for a two-dimensional rectangular fin with a constant thermal conductivity for $\tau = 0.4$.

Upon solution of (16) one obtains an expression for $\theta(\tau, x, y)$.The solution $\theta(\tau, x, y)$ will be discontinuous in the y direction. Taking the first six terms in every direction, that is, taking the first 216 terms of the series, we give the profile and plot for the case $\beta = 0$ over the (x, y) plane. The solution is depicted in Figures 1 and 2, and the numerical account is provided in Table 2.

4.2 Power law thermal conductivity

In this section we focus on the rectangular fin with a power law thermal conductivity. The problem is given by the equation

$$\frac{\partial \theta}{\partial \tau} = \frac{\partial}{\partial x}\left[\theta^n \frac{\partial \theta}{\partial x}\right] + E^2 \frac{\partial}{\partial y}\left[\theta^n \frac{\partial \theta}{\partial y}\right], \tag{17}$$

which is subject to the conditions presented in (2)- (6). Applying the three-dimensional DTM to the governing equation (17) and the above mentioned conditions one obtains

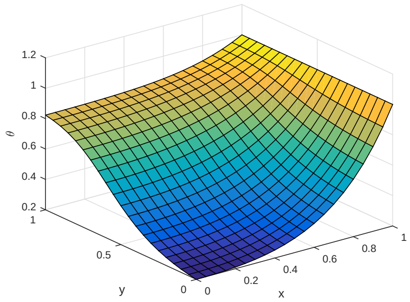

Figure 2: Approximate analytical solutions for a two-dimensional rectangular fin with a linear function thermal conductivity ($\beta = 2$) for $\tau = 0.4$. The parameters are set such that $E = 3.2$, $Bi = 0.2$, and $m = 3$.

the series solution

$$\begin{aligned}\theta(\tau,x,y) = \quad & c\tau - Bic^m\tau y + c\tau y^2 + c\tau y^3 + c\tau y^4 + c\tau y^5 + c\tau y^6 + c\tau y^7 + ... \\ & c\tau x^2 - Bic^m\tau y x^2 - \frac{18c + 2E^2c - 2BiE^2c^m - 3}{2E^2}\tau y^2 x^2 + ... \\ & c\tau x^3 - Bic^m\tau y x^3 - \frac{40c + 2E^2c - 2BiE^2c^m - 3}{2E^2}\tau y^2 x^3 + ... \\ & \vdots\end{aligned} \tag{18}$$

In order to find a value for c, we choose the boundary $x = 1$. This results in an equation in terms of τ and y given by

$$\begin{aligned} & c\tau - Bic^m\tau y + ... + c\tau - Bic^m\tau y + ... + c\tau - Bic^m\tau y + ... = 1. \\ & \vdots \end{aligned} \tag{19}$$

The obtained value of c can then be substituted back into (18) to get an expression for $\theta(\tau, x, y)$. The solution is depicted in Figure 3. It turns out that the 3D DTM works well only when $n = 1$, which is equivalent to rescaling of the linear thermal conductivity in equation (15). A question arises of whether this observation is the only case in this problem for which DTM is efficient. Figures 4 and 5 depict the temperature profiles for transient heat transfer.

5 Conclusion

As far as we know, the 3D DTM has never been applied to transient problems of heat transfer in 2D straight fins with temperature-dependent thermal properties. We have demonstrated that these methods are effective in providing approximate analytical solutions. Figures 1 to 3 provide the temperature profile of heat transfer in the 2D rectangular straight fins. One may notice that the transient solutions approach the steady state solution in Figures 4 and 5. Numerical results are provided in Table 2. The dependency of the thermal properties on temperature rendered the considered equation nonlinear. The effects of the Biot number and aspect ratio were studied in [3]. Similar results are

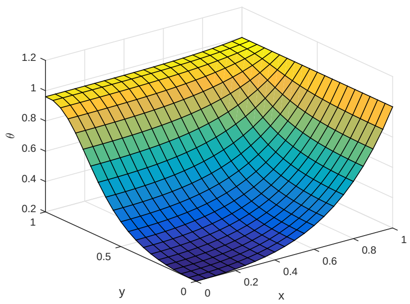

Figure 3: Approximate analytical solutions for a two-dimensional rectangular fin with a power law thermal conductivity for $\tau = 0.4$. The parameters are set such that $E = 25$, $Bi = 0.2$, and $m = 3$.

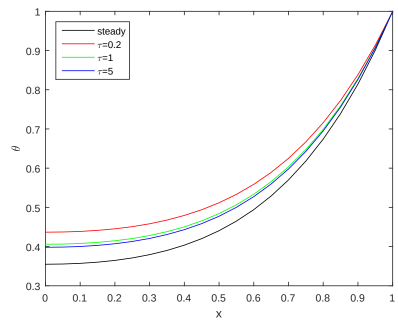

Figure 4: Plots of the transient profile for varying τ, against the steady state profile along $y = 0.5$. The parameters are set such that $E = 2.9$, $Bi = 0.2$, and $m = 3$.

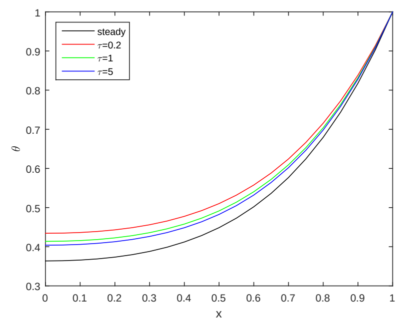

Figure 5: Plots of the transient profile for varying τ, against the steady state profile along $y = 0.5$. The parameters are set such that $E = 2.9$, $Bi = 0.1$, and $m = 2$.

obtained in this study, namely, that the fin performance decreases with the increased aspect ratio and the large Biot number yields a decreased fin efficiency.

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