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Solution of 2D Fractional Order Integral Equations by Bernstein Polynomials Operational Matrices

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Abstract: In this paper, we construct a new two-dimensional Bernstein polynomials operational matrix for solving 2-dimensional fractional order Volterra integral equations (2DFOVIE). By using this operational matrix, we reduce the original problem to a linear or nonlinear system of algebraic equations. We present some numerical examples to show the efficiency of the proposed method.

Keywords: *two-dimensional fractional integral equations; two-dimensional Bernstein polynomials; block pulse operational matrix; operational matrix of integration.*

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1 Introduction

In the last few decades, various engineering and scientific problems involving fractional calculus were discussed. For example, electrochemical process [1, 2], earthquakes [3], economics [4], bioengineering [5], orthogonal splin collocation [6] and fractional optimal control problems [7, 8]. There are several analytical and numerical methods for solving one-dimensional and two-dimensional differential and integral equations of fractional order such as the Adomian decomposition [9], Variational iteration method [10, 11], Transform method [12], Homotopy perturbation method [13], and the methods of Harr and Chebyshev wavelet [14, 15] and Bernstein polynomials [16, 17].

The Bernstein polynomials play a conspicuous role in several areas of mathematics. These polynomials have been commonly used in the solution of differential equations,

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integral equations, fractional optimal control problems and approximation theory [7, 8, 17–23]. In this work, we consider the following type of 2DVIEFO

$$u(x,y) - I_0^q u^p(x,y) = g(x,y), \quad q = (\alpha,\beta) \in (0,\infty)\times(0,\infty), \tag{1}$$

where $g(x,y)$ is a known function and $I_0^q u(x,y)$ is the left-sided mixed Riemann-Liouville integral of order q which is defined as [24]

$$(I_0^q u)(x,y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^x \int_0^y (x-\xi)^{\alpha-1}(y-\tau)^{\beta-1} u(\xi,\tau)\, d\tau\, d\xi. \tag{2}$$

Note: For $\alpha > 0$, the Riemann-Liouville integral (I^α) on the Lebesgue space $L^1[a,b]$ is defined as

$$(I_0^\alpha u)(t) = (I^\alpha u)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} u(\tau)\, d\tau. \tag{3}$$

In particular, for (2), we have

1. $(I_0^0 u)(x,y) = u(x,y)$,
2. $(I_0^\sigma u)(x,y) = \int_0^x \int_0^y u(\xi,\tau)\, d\tau\, d\xi, \quad (x,y) \in J,\ \sigma = (1,1)$,
3. $(I_0^r u)(x,0) = (I_0^r u)(0,y) = 0, \quad x \in [0,a], y \in [0,b]$,
4. $I_0^r x^\lambda y^\omega = \frac{\Gamma(1+\lambda)\Gamma(1+\omega)}{\Gamma(1+\lambda+\alpha)\Gamma(1+\omega+\beta)} x^{\lambda+\alpha} y^{\omega+\beta}, \quad (x,y) \in J, \quad \lambda,\omega \in (-1,\infty)$.

We are looking for $u \in L^1(J)$, $J := [0,a]\times[0,b]$. The existence and uniqueness of (1) is investigated in [25].

We want to obtain the numerical solution of (1) by using two-dimensional Bernstein polynomials and block pulse functions. The rest of this paper is organized as follows. First, we briefly review some general concepts concerning one-dimensional and two-dimensional Bernstein polynomials, block pulse functions and derive the Bernstein polynomials operational matrix of two-dimensional integration of fractional order. In Section 3, the method is applied to solve linear or nonlinear 2DVIEFO. Section 4 exhibits an error estimation for the presented method. Section 5 illustrates several numerical examples to show the convergence and accuracy of the proposed method.

2 Bernstein Polynomials and Block Pulse Functions

2.1 One dimensional Bernstein polynomials (1D-BPs)

The nth degree Bernstein polynomials (BPs) on the interval $[0,1]$ are defined as

$$B_{i,n}(\tau) = \binom{n}{i} \tau^i (1-\tau)^{n-i}, \quad 0 \le i \le n. \tag{4}$$

The BPs on $[0,1]$ have the following properties [7]:

1. $B_{i,n}(\tau) \ge 0, i = 0,1,\ldots,n, \quad \tau \in [0,1]$,
2. $\sum_{i=0}^{n} B_{i,n}(t) = 1$,

3. $B_{i,n}(\tau) = (1-\tau)B_{i,n-1}(\tau) + \tau B_{i-1,n-1}(\tau), \quad i = 0, 1, \ldots, n,$
4. $B_{i,n}(\tau) = \sum_{k=0}^{n-i}(-1)^k \begin{pmatrix} n \\ i \end{pmatrix} \begin{pmatrix} n-i \\ k \end{pmatrix} \tau^{i+k}, \quad i = 0, 1, \ldots, n.$

Theorem 2.1 *[26] Suppose that $H = L^2[0,1]$ is a Hilbert space with the inner product and $X = Span\{B_{0,n}(t), B_{1,n}(t), \ldots, B_{n,n}(t)\}$ is a closed subspace with finite dimensions, therefore X is a complete subspace of H. So, if $u \in H$ is an arbitrary element, it has a unique best approximation out of X such as x_0, that is*

$$\exists \ x_0 \in Y \quad s.t. \ \forall x \in X, \ \ \| u - x_0 \|_2 \leq \| u - x \|_2, \tag{5}$$

where $\| u \|_2 = \sqrt{< u, u >}$, $< u, v > = \int_0^1 u(\tau)v(\tau)\, d\tau$.

Thus, there exist unique coefficients $c_0, c_1, \ldots, c_n$ such that

$$u(t) \simeq x_0 = \sum_{i=0}^{n} c_i B_{i,n}(t) = c^T \varphi(t), \tag{6}$$

where $c^T = [c_0, c_1, \ldots, c_n]$, $\varphi(\tau) = [B_{0,n}(\tau), B_{1,n}(\tau), \ldots, B_{n,n}(\tau)]^T$.

Lemma 2.1 *If $\varphi_n(\tau) = [B_{0,n}(\tau), B_{1,n}(\tau), \ldots, B_{n,n}(\tau)]^T$ is a complete basis, then $\varphi_n(t) = AT_n(t)$, where A is an $(n+1) \times (n+1)$ upper triangular matrix with*

$$a_{i+1,j+1} = \begin{cases} (-1)^{j-i} \begin{pmatrix} n \\ i \end{pmatrix} \begin{pmatrix} n-i \\ j-i \end{pmatrix}, & i \leqslant j, \\ 0, & i > j, \end{cases} \tag{7}$$

for $i, j = 0, 1, \ldots, n$ and $T_n(\tau) = [1, \tau, \tau^2, \ldots, \tau^n]^T$.

2.2 BPF and operational matrix

A set of BPF on $[0, 1)$ is defined as follows:

$$b_i(t) = \begin{cases} 1, & \frac{i}{m} \leq t < \frac{i+1}{m}, \quad i, j = 0, 1, \ldots, m-1, \\ 0, & otherwise. \end{cases} \tag{8}$$

The above functions are orthogonal and disjoint, i.e.
$b_i(t)b_j(t) = \begin{cases} b_i(t) & i = j, \\ 0 & i \neq j, \end{cases}$ and $\int_0^1 b_i(t)b_j(t)\, dt = \dfrac{1}{m}\delta_{ij}$, where δ_{ij} is the Kronecker delta.

If $B_m(\tau) = [b_0(\tau), b_1(\tau), \ldots, b_{m-1}(\tau)]^T$, the block pulse operational matrix of the fractional order integration F^α is [27]

$$I^\alpha B_m(\tau) = F^\alpha B_m(\tau),$$

where

$$F^\alpha = \frac{1}{m^\alpha} \frac{1}{\Gamma(\alpha+2)} \begin{bmatrix} 1 & \xi_1 & \xi_2 & \xi_3 & \ldots & \xi_{m-1} \\ 0 & 1 & \xi_1 & \xi_2 & \ldots & \xi_{m-2} \\ 0 & 0 & 1 & \xi_1 & \ldots & \xi_{m-3} \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 & 1 & \xi_1 \\ 0 & 0 & \ldots & 0 & 0 & 1 \end{bmatrix}, \tag{9}$$

with $\xi_s = (s+1)^{\alpha+1} - 2s^{\alpha+1} + (s-1)^{\alpha+1}$.

2.3 Operational matrix for fractional integral equation(1D)

If

$$\varphi(\tau) = \varphi_n(\tau) = [B_{0,n}(\tau), B_{1,n}(\tau), \ldots, B_{n,n}(\tau)]^T,$$

then for the fractional integral equation (3), we have

$$I^\alpha \varphi_n(\tau) = P^\alpha \varphi_n(\tau), \tag{10}$$

with $n = m-1$, the Bernstein polynomial might be expanded into an m-term BPF as

$$\varphi_m(\tau) = \phi_{m\times m} B_m(\tau), \tag{11}$$

now

$$I^\alpha \varphi_m(\tau) = I^\alpha \phi_{m\times m} B_m(\tau) = \phi_{m\times m} I^\alpha B_m(\tau) = \phi_{m\times m} F^\alpha B_m(\tau). \tag{12}$$

From equations (11) and (12), we have

$$I^\alpha \varphi_m(\tau) = \phi_{m\times m} F^\alpha B_m(\tau) = \phi_{m\times m} F^\alpha \phi^{-1}_{m\times m} \varphi_m(\tau). \tag{13}$$

Therefore,

$$P^\alpha_{m\times m} = \phi_{m\times m} F^\alpha \phi^{-1}_{m\times m}. \tag{14}$$

P^α is called an operational matrix for fractional integration based on the Bernstein polynomials [28].

2.4 Two-dimensional Bernstein polynomials (2D-BPs)

The Bernstein polynomials of degree mn on the interval $[0,1]\times[0,1]$ are defined by

$$B_{(i,m)(j,n)}(\mu,\nu) = \binom{m}{i}\binom{n}{j} \mu^i (1-\mu)^{m-i} \nu^j (1-\nu)^{n-j} \tag{15}$$

for $i = 0, 1, \ldots, m,\ \ j = 0, 1, \ldots, n.$

Similar to the 1D case, we have [19]:

1. $B_{(i,m)(j,n)}(\mu,\nu) \geq 0,$
2. $B_{(i,m)(j,n)}(\mu,\nu) = B_{(i,m)}(\mu) B_{(j,n)}(\nu),$
3. $B_{(i,m)(j,n)}(\mu,\nu) = \sum_{k=0}^{m-i}\sum_{t=0}^{n-j} (-1)^{r+t} \binom{m}{i}\binom{n}{j}\binom{m-i}{k}\binom{n-j}{t} \mu^{i+k}\nu^{j+t},$
4. $Q = < B_{(i,m)(j,n)}(\mu,\nu), B_{(k,m)(t,n)}(\mu,\nu) >$

$$= \int_0^1\int_0^1 B_{(i,m)(j,n)}(\mu,\nu) B_{(k,m)(t,n)}(\mu,\nu)\, d\mu d\nu = \frac{\binom{m}{i}\binom{n}{j}\binom{m}{k}\binom{n}{t}}{(2m+1)(2n+1)\binom{2m}{i+k}\binom{2n}{j+t}},$$

for $i, k = 0, 1, \ldots, m,\ \ j, t = 0, 1, \ldots, n.$

Now, if we define $(m+1)\times(n+1)$-vector

$$\begin{aligned}\varphi_{mn}(\mu,\nu) = [&B_{(0,m)(0,n)}(\mu,\nu), \ldots, B_{(0,m)(n,n)}(\mu,\nu),\\ &\ldots, B_{(m,m)(0,n)}(\mu,\nu), \ldots, B_{(m,m)(n,n)}(\mu,\nu)]^T,\end{aligned} \tag{16}$$

where $(\mu,\nu) \in [0,1]\times[0,1]$, then $\varphi_{mn}(\mu,\nu)$ is a complete basis.

2.5 Function expansion with 2D-BPs

We expand $u(\mu,\nu) \in L^2([0,1]\times[0,1])$ by 2D-BPs as

$$u(\mu,\nu)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}u_{ij}\varphi_{ij}(\mu,\nu)\simeq\sum_{i=0}^{m}\sum_{j=0}^{n}u_{ij}\varphi_{ij}(\mu,\nu)=U^T\varphi(\mu,\nu)=\varphi^T(\mu,\nu)U, \quad (17)$$

where $\varphi(\mu,\nu)$ and U are $(m+1)(n+1)$ vectors. Components u_{ij} of U are obtained as

$$u_{ij}=<u(\mu,\nu),\varphi(\mu,\nu)>=\int_0^1\int_0^1 u(\mu,\nu)B_{(i,m)(j,n)}(\mu,\nu)\,d\mu\,d\nu. \quad (18)$$

Similarly, let $k(\mu,\nu,s,t)$ be defined on $[0,1]\times[0,1]\times[0,1]\times[0,1]$. It can be expanded with respect to 2D-BPs as

$$k(\mu,\nu,s,t)\simeq\varphi^T(\mu,\nu)K\psi(s,t), \quad (19)$$

where $\varphi(\mu,\nu)$ and $\psi(s,t)$ are 2D-BPs vectors of dimension $(m_1+1)(n_1+1)$ and $(m_2+1)(n_2+1)$, respectively, and K is the $(m_1+1)(n_1+1)\times(m_2+1)(n_2+1)$ two-dimensional Bernstein polynomials coefficient matrix.

2.6 Operational matrix for fractional integral equation(2D)

Suppose $B_{(i,m)}(\mu)=A_1T_m(\mu)$ and $B_{(j,n)}(\nu)=A_2T_n(\nu)$. Then

$$\varphi_{mn}(\mu,\nu)=MT_{mn}(\mu,\nu),$$

where

$$T_{mn}(\mu,\nu)=[1,\nu,\nu^2,\ldots,\nu^n,\mu,\mu\nu,\ldots,\mu\nu^n,\ldots,\mu^m,\mu^m\nu,\ldots,\mu^m\nu^n]^T,$$

and $M=A_1\otimes A_2$ and $\otimes$ denotes the Kronecker product.
Now, we present two-dimensional Bernstein polynomials operational matrices of fractional mode. Let $\varphi_{mn}(\mu,\nu)$ be defined as in (16). The fractional integration of the $\varphi_{mn}(\mu,\nu)$ can be approximately obtained as

$$\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^x\int_0^y(x-\xi)^{\alpha-1}(y-\tau)^{\beta-1}\varphi_{mn}(\xi,\tau)\,d\xi\,d\tau\simeq P^r\varphi_{mn}(\mu,\nu), \quad (20)$$

where P^r is a $(m+1)(n+1)\times(m+1)(n+1)$ matrix and is called an operational matrix.
Let operational matrices P^α and P^β satisfy (14), i.e.

$$\begin{aligned}I^\alpha\varphi_m(\mu)&=P^\alpha\varphi_m(\mu)=\phi_{m\times m}F^\alpha\phi^{-1}_{m\times m}\varphi_m(\mu),\\ I^\beta\varphi_n(\nu)&=P^\beta\varphi_n(\nu)=\phi_{n\times n}F^\beta\phi^{-1}_{n\times n}\varphi_n(\nu).\end{aligned} \quad (21)$$

From the disjointness property of two-dimensional Bernstein polynomials, we get

$$\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^x\int_0^y(x-\xi)^{\alpha-1}(y-\tau)^{\beta-1}\varphi_{mn}(\xi,\tau)\,d\xi\,d\tau=\frac{1}{\Gamma(\alpha)}\int_0^x(x-\xi)^{\alpha-1}\varphi_m(\xi)\,d\xi$$
$$\times\frac{1}{\Gamma(\beta)}\int_0^y(y-\tau)^{\beta-1}\varphi_n(\tau)\,d\tau.$$

By using (21), we have

$$P^r=P^\alpha\otimes P^\beta. \quad (22)$$

2.7 Product operational matrix

In view of (1), we have $u^p(x, y)$. So, we need to evaluate the product of $\varphi_(x, y)$ and $\varphi^T(x, y)$, which is called the product matrix.

Lemma 2.2 *Suppose that $C_{(m+1)(n+1)}$ is an arbitrary vector. The operational matrix of product $\hat{C}_{(m+1)(n+1)\times(m+1)(n+1)}$ using BPs can be given as follows [29] :*

$$\varphi(x, y)\varphi^T(x, y)C \simeq \hat{C}^T \varphi(x, y). \tag{23}$$

Corollary 2.1 *Suppose $u(x, y) = U^T\varphi(x, y) = \varphi^T(x, y)U$ and $\hat{U}$ is the operational matrix of product. Then*

$$(u(x, y))^k = \varphi^T(x, y)\overline{U}_k, \tag{24}$$

where $k \in N$ and $\overline{U}_k = \hat{U}^{k-1}U$.

Proof. By using Lemma 2.2, for $k = 2$, we get

$$(u(x, y))^2 = U^T\varphi(x, y)\varphi^T(x, y)U = \varphi^T(x, y)\hat{U}U = \varphi^T(x, y)\overline{U}_2.$$

Also, if $k = 3$,

$$(u(x, y))^3 = U^T\varphi(x, y)\varphi^T(x, y)\hat{U}U = \varphi^T(x, y)\hat{U}^2U = \varphi^T(x, y)\overline{U}_3.$$

So, by induction we have

$$(u(x, y))^k = U^T\varphi(x, y)\varphi^T(x, y)\hat{U}^{k-2}U = \varphi^T(x, y)\hat{U}^{k-1}U = \varphi^T(x, y)\overline{U}_k.$$

3 Solving 2DFOVIE

In this section, two-dimensional Bernstein polynomials are applied to solve equation(1). Using the procedures mentioned in Section 2, we approximate functions $(u(x, y))^p$, $k(x, y, s, t)$ and $f(x, y)$ as follows:

$$\begin{aligned} (u(x, y))^p &= \varphi^T(x, y)\overline{U}_p = \overline{U}_p^T\varphi(x, y), \\ f(x, y) &= \varphi^T(x, y)F = F^T\varphi(x, y), \\ k(x, y, s, t) &= \varphi^T(x, y)K\varphi(x, y), \end{aligned} \tag{25}$$

where the $(m + 1)(n + 1) \times 1$ vectors $\overline{U}_p$, F and $(m + 1)(n + 1) \times (m + 1)(n + 1)$ matrix K are 2D-BPs coefficients of $(u(x, y))^p$, $f(x, y)$ and $k(x, y, s, t)$ respectively. Substituting equations(25) in equation(1), we have:

$$\varphi^T(x, y)U - \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^x\int_0^y (x-s)^{\alpha-1}(y-t)^{\beta-1}\varphi^T(x, y)K\varphi(s, t)\varphi^T(s, t)\overline{U}_p\,dt\,ds = \varphi^T(x, y)F.$$

By using (23), we get

$$\varphi^T(x, y)U - \frac{\varphi^T(x, y)K\,\hat{\overline{U}}_p^T}{\Gamma(\alpha)\Gamma(\beta)}\int_0^x\int_0^y (x-s)^{\alpha-1}(y-t)^{\beta-1}\varphi(s, t)\,dt\,ds = \varphi^T(x, y)F.$$

From equation(20) and the above equation, we obtain

$$\varphi^T(x,y)U - \varphi^T(x,y)K\,\hat{\hat{U}}_p^T P^r\varphi(x,y) = \varphi^T(x,y)F,$$

or

$$U - K\,\hat{\hat{U}}_p^T P^r\varphi(x,y) = F. \tag{26}$$

Now, we collocate equation(26) in $(m+1)(n+1)$ Newton-Cotes nodes as

$$x_i = \frac{2i-1}{2(m+1)}\ ,\quad y_j = \frac{2j-1}{2(n+1)}\ ,\quad i=1,2,\dots,m+1,\quad j=1,2,\dots,n+1.$$

So, we have a linear($p=1$) or nonlinear($p\geq 1$) algebraic system

$$U - B\psi = F, \tag{27}$$

where $B = K\,\hat{\hat{U}}_p^T P^r$, and

$$\psi = [\varphi(x_1,y_1),\varphi(x_1,y_2),\dots,\varphi(x_1,y_{n+1}),\dots\varphi(x_{m+1},y_1),\dots,\varphi(x_{m+1},y_{n+1})]^T.$$

4 Error analysis

Theorem 4.1 *Suppose $u(x,y)$ is an exact solution of the equation (1) and $\hat{u}(x,y)$ shows its approximate solution by Bernstein polynomials, and*

1. $|(x-\xi)^{\alpha-1}(y-\tau)^{\beta-1}k(x,y,\xi,\tau)| < C,$

2. $(u(x,y))^p$ *is a Lipschitz continuous function, i.e.*

$$|(u(x,y))^p - (\hat{u}(x,y))^p| \leq L|u(x,y)-\hat{u}(x,y)|,$$

where L is a Lipschitz constant

3. $m_1 = m_2 = m.$

Then $\hat{u}(x,y)$ converges to $u(x,y)$, if $0 < \dfrac{LC}{\Gamma(\alpha)\Gamma(\beta)} < 1.$

Proof.

$$\parallel u(x,y)-\hat{u}(x,y)\parallel_\infty = \max_{0\leq x,y\leq 1}|u(x,y)-\hat{u}(x,y)|$$

$$= \max_{0\leq x,y\leq 1}\left|\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^x\int_0^y (x-\xi)^{\alpha-1}(y-\tau)^{\beta-1}k(x,y,\xi,\tau)((u(\xi,\tau))^p-(\hat{u}(\xi,\tau))^p)d\xi d\tau\right|$$

$$\leq \max_{0\leq x,y\leq 1}\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^x\int_0^y |(x-\xi)^{\alpha-1}(y-\tau)^{\beta-1}k(x,y,\xi,\tau)||(u(\xi,\tau))^p-(\hat{u}(\xi,\tau))^p|d\xi d\tau$$

$$\leq \max_{0\leq x,y\leq 1}\frac{CL}{\Gamma(\alpha)\Gamma(\beta)}\int_0^x\int_0^y |u(\xi,\tau)-\hat{u}(\xi,\tau)|\,d\xi d\tau$$

$$\leq \frac{CLxy}{\Gamma(\alpha)\Gamma(\beta)}\parallel u(\xi,\tau)-\hat{u}(\xi,\tau)\parallel_\infty \leq \frac{CL}{\Gamma(\alpha)\Gamma(\beta)}\parallel u(\xi,\tau)-\hat{u}(\xi,\tau)\parallel_\infty.$$

Therefore we get

$$\| u(x,y) - \hat{u}(x,y) \|_\infty \leq \frac{CL}{\Gamma(\alpha)\Gamma(\beta)} \| u(\xi,\tau) - \hat{u}(\xi,\tau) \|_\infty . \quad (28)$$

Equation (28) shows that if $0 < \frac{LC}{\Gamma(\alpha)\Gamma(\beta)} < 1$, then $\| u(\xi,\tau) - \hat{u}(\xi,\tau) \|_\infty \longrightarrow 0$.

5 Numerical Examples

To demonstrate the validity and applicability of this scheme, we use the present method for the following four examples. In view of (2), we rewrite (1) in the following form of 2DFOVIE:

$$u(x,y) - \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^x \int_0^y (x-\xi)^{\alpha-1}(y-\tau)^{\beta-1} k(x,y,\xi,\tau)\, u^p(\xi,\tau)\, d\xi\, d\tau = g(x,y) \quad (29)$$

Now, for different values of α, β, $k(x,y,\xi,\tau)$, p and $g(x,y)$, we solve (29).

Example 5.1 Let $\alpha = \frac{5}{3}$, $\beta = \frac{7}{3}$, $k(x,y,\xi,\tau) = \xi\tau\sqrt{xy}$, $p = 1$ and $g(x,y) = x^3(y^2 - y) - \frac{x^{\frac{17}{3}} y^{\frac{13}{3}} \sqrt{xy}(9y-16)}{5000}$ The exact solution is $u(x,y) = x^3(y^2 - y)$. We applied the proposed method to solve this example for various values of m and n. Also, we compare the numerical results with the exact solution. The results are tabulated in Table 1.

$x = y$	$m = n = 1$	$m = n = 2$	$m = n = 3$
0.0	6.292×10^{-6}	3.091×10^{-6}	7.394×10^{-6}
0.1	7.702×10^{-5}	3.942×10^{-4}	4.401×10^{-5}
0.2	1.261×10^{-3}	2.814×10^{-3}	9.460×10^{-5}
0.3	5.645×10^{-3}	4.417×10^{-3}	1.492×10^{-4}
0.4	1.533×10^{-2}	3.212×10^{-3}	2.022×10^{-4}
0.5	3.121×10^{-2}	1.926×10^{-4}	2.515×10^{-4}
0.6	5.180×10^{-2}	3.579×10^{-3}	2.919×10^{-4}
0.7	7.198×10^{-2}	4.819×10^{-3}	3.020×10^{-4}
0.8	8.187×10^{-2}	2.975×10^{-3}	2.257×10^{-4}
0.9	6.556×10^{-2}	5.010×10^{-4}	5.289×10^{-5}

Table 1: The maximum absolute errors in Example 5.1.

Example 5.2 Let $\alpha = \beta = \frac{5}{2}$, $k(x,y,\xi,\tau) = \sqrt{xy\xi}$, $p = 2$ and $f(x,y) = x\sqrt{y} - \frac{1}{420} x^{\frac{11}{2}} y^4$ with the exact solution $u(x,y) = x\sqrt{y}$. The maximum absolute errors are shown in Table 2.

Example 5.3 Let $\alpha = \frac{5}{2}$, $\beta = \frac{7}{2}$, $k(x,y,\xi,\tau) = (y+\xi)e^{-2\tau}$, $p = 2$ and $f(x,y) = xe^y - \frac{1024 x^{\frac{9}{2}} y^{\frac{7}{2}} (6x+11y)}{1091475\pi}$ with the exact solution $u(x,y) = xe^y$. The maximum absolute errors are shown in Table 3.

$x=y$	$m=n=1$	$m=n=2$	$m=n=3$
0.0	5.603×10^{-5}	2.592×10^{-6}	1.327×10^{-5}
0.1	3.064×10^{-3}	1.273×10^{-3}	1.611×10^{-3}
0.2	4.032×10^{-3}	4.748×10^{-3}	1.497×10^{-3}
0.3	1.220×10^{-2}	4.694×10^{-3}	1.253×10^{-3}
0.4	1.818×10^{-2}	1.276×10^{-3}	3.797×10^{-3}
0.5	2.009×10^{-2}	3.948×10^{-3}	4.052×10^{-3}
0.6	1.666×10^{-2}	8.789×10^{-3}	1.451×10^{-3}
0.7	6.937×10^{-3}	1.070×10^{-2}	2.806×10^{-3}
0.8	9.786×10^{-3}	6.921×10^{-3}	5.652×10^{-3}
0.9	3.410×10^{-2}	5.490×10^{-3}	2.132×10^{-3}

Table 2: The maximum absolute errors in Example 5.2.

$x=y$	$m=n=1$	$m=n=2$	$m=n=4$
0.0	9.890×10^{-5}	3.578×10^{-4}	7.921×10^{-4}
0.1	6.034×10^{-3}	6.324×10^{-4}	9.468×10^{-4}
0.2	1.666×10^{-3}	3.307×10^{-4}	1.104×10^{-3}
0.3	9.537×10^{-3}	1.114×10^{-3}	1.266×10^{-3}
0.4	2.339×10^{-2}	8.532×10^{-4}	1.424×10^{-3}
0.5	3.514×10^{-2}	6.995×10^{-4}	1.566×10^{-3}
0.6	3.935×10^{-2}	3.095×10^{-3}	1.673×10^{-3}
0.7	2.987×10^{-2}	5.105×10^{-3}	1.675×10^{-3}
0.8	3.150×10^{-4}	4.622×10^{-3}	1.370×10^{-3}
0.9	5.916×10^{-2}	1.445×10^{-3}	2.511×10^{-4}

Table 3: The maximum absolute errors in Example 5.3.

Example 5.4 As the last example, let $\alpha=\frac{3}{2}$, $\beta=\frac{5}{2}$, $k(x,y,\xi,\tau)=\sqrt{xy\tau}$, $p=2$ and $f(x,y)=\sqrt{y}(\frac{-1}{180}x^3y^{\frac{7}{2}}+\sqrt{\frac{x}{3}})$ The exact solution of this example is $u(x,y)=\frac{\sqrt{3xy}}{3}$. The maximum absolute errors are shown in Table 4. Also, the obtained numerical results are compared with the method of block pulse operational matrix (BPOM) proposed in [23, 30].

6 Conclusion

A new approach to obtain numerical solution of 2DFOVIE based on the operational matrices of Bernstein polynomials has been presented. With the help of the operational matrix of fractional integration P^r and the collocation method, the given 2DFOVIE is reduced to a linear or nonlinear system of algebraic equations. Illustrative examples show that the proposed method can be a suitable method for solving these equations. All of computations are done by *Mathematica* 9.

$x=y$	$m=n=1$	$m=n=2$	$m=n=3$	$\frac{m_1=m_2=32}{BPOM}$
0.0	4.091×10^{-2}	1.701×10^{-2}	9.354×10^{-3}	9.386×10^{-3}
0.1	1.171×10^{-2}	4.572×10^{-3}	5.766×10^{-3}	1.561×10^{-2}
0.2	1.017×10^{-2}	1.183×10^{-2}	3.740×10^{-3}	8.812×10^{-3}
0.3	2.472×10^{-2}	9.513×10^{-3}	2.911×10^{-3}	1.630×10^{-2}
0.4	3.196×10^{-2}	1.934×10^{-3}	7.428×10^{-3}	8.239×10^{-3}
0.5	3.186×10^{-2}	7.003×10^{-3}	7.270×10^{-3}	1.410×10^{-2}
0.6	2.444×10^{-2}	1.382×10^{-2}	2.893×10^{-3}	7.665×10^{-3}
0.7	9.702×10^{-3}	1.545×10^{-2}	3.149×10^{-3}	1.430×10^{-2}
0.8	1.236×10^{-2}	9.258×10^{-3}	6.781×10^{-3}	7.091×10^{-3}
0.9	4.176×10^{-2}	6.980×10^{-3}	2.666×10^{-3}	1.260×10^{-2}

Table 4: The maximum absolute errors in Example 5.4.

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