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Decentralized Stabilization for a Class of Nonlinear Interconnected Systems Using SDRE Optimal Control Approach

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Abstract: This paper presents a new approach to assure the decentralized optimal control of interconnected nonlinear systems based on the decentralized state-dependent riccati equation (SDRE). To remedy the problem of persistent stability in other works, we based our approach on the foundations of the Lyapunov theory. It allows developing a new sufficient condition to guarantee the global asymptotic stability of the systems under study. We conducted a simulation of this new control method on a numerical example. It demonstrated its efficiency and the sufficiency of the new stability conditions.

Keywords: *decentralized optimal control; state-dependent Riccati equation (SDRE); interconnected nonlinear systems; Lyapunov theory; Kronecker product.*

Mathematics Subject Classification (2010): 93D15, 34D23, 93A14.

1 Introduction

In recent years, the modern dynamical systems are getting more complex, highly interconnected, and mutually interdependent. This change is caused either by physical attributes, and/or a multitude of information and communication network constraints [1–3]. The important dimension and complexity of these large-scale systems often require a hierarchical decentralized architecture to analyze and control these systems [4–10]. Since these complex dynamic systems can be characterized by an interconnection between many subsystems, possible control strategies are generally based on a decentralized approach. The

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advantage of such method is to reduce the complexity and therefore make the implementation of the control law more feasible.

In fact, the decentralized control refers to a control design with local decisions. These decisions are based only on local information of the subsystems. This method is given considerable interest because it brings up significant solutions for the traditional control approach limitations such as the implementation constraints, cost and reliability considerations especially for large-scale systems.

Optimal control of nonlinear systems is one of the most challenging subjects in control theory. Indeed, the classical problems of optimal control are based on the solution of the Hamilton-Jacobi equation (HJE) [11, 12]. The solution to the HJE is a function of the state of the nonlinear system which makes it possible to characterize the quadratic optimal law of control sought under some hypotheses. However, in most cases it is impossible to solve it analytically, and despite recent progress, unsolved problems still exist and researchers often complain about the very limited applicability of contemporary theories because of conditions imposed on the system. This has led to numerous methods proposed in the literature for obtaining a suboptimal state feedback control law for the general case of nonlinear dynamic systems [13, 14].

The SDRE approach is one of the methods applied in the determination of a suboptimal quadratic control based on the solution of a state-dependent Riccati equation. This strategy provides an efficient algorithm for nonlinear state feedback control synthesis while retaining the nonlinearities of the complex dynamic system, thanks to the flexibility of the state-dependent weighting matrices [15, 16]. This approach, proposed by Pearson [17] and later extended by Wernli and Cook [18], was studied independently by Mracek and Cloutier [19]. It should be pointed out that, although it is a relatively simplified and practical technique for controlling nonlinear systems, the SDRE approach involves problems that deserve to be treated with great attention, in particular the stability problem of the system controller [20,21]. Elloumi and Benhadj Braiek [22,23] have developed a sufficient condition for the stability of nonlinear system with optimal control based on SDRE approach. In this paper, we extend this work to the case of large scale interconnected systems. In this direction we carried out the synthesis of decentralized optimal control law based on the SDRE technique. This approach aims to minimize a performance criterion in order to compute decentralized optimal control gains when some sufficient conditions developed using the Lyapunov theory are verified.

The rest of the paper is organized as follows: the second section is devoted to the description of the systems under study and the formulation of the problem. In the third section, we present the decentralized optimal control law based on the SDRE approach. The fourth section treats the stability of the system in question using the quadratic Lyapunov function. The simulation results are set out in the fifth section to illustrate the applicability of the developed approach. Finally, conclusions are drawn and future scope of study is outlined.

2 Description of the System Under Study and Problem Formulation

A nonlinear system can be described by the interconnection of subsystems as follows:

$$\begin{cases} \dot{x}_i = f_i\left(\left(x_i, x_j\right), u_i\left(t\right), t\right), \quad i \neq j, \\ y_i = h_i\left(x_i\right), \quad i = 1, ..., n, \ j = 1, ..., n, \end{cases} \tag{1}$$

where $x_i(t) \in \mathbb{R}^{n_i}$, $u_i \in \mathbb{R}^{m_i}$ and $y_i \in \mathbb{R}^{p_i}$ are, respectively, the state, the control and the output of the i^{th} subsystem.

$f_i(x_i, x_j)$ and $h_i(x_i)$ are nonlinear functions of the state. Through the state-dependent coefficient (SDC) factorization, system designers can represent the nonlinear equations of the system under consideration as linear structures with state-dependent coefficients. Thus, the following procedure is similar to the optimal linear control (LQR) method, except that all matrices may depend on the states. Based on this concept, the state space equation for the nonlinear interconnected subsystem can be expressed as a linear-like state-space equation using direct SDC factorization as:

$$\begin{cases} \dot{x}_i(t) = A_i(x_i)\, x_i(t) + B_i(x_i)\, u_i(t) + \sum\limits_{j=1, j\neq i}^{n} H_{ij}(x_i, x_j) x_j(t), \\ y_i(t) = C_i(x_i)\, x_i(t), \quad i = 1, \ldots, n, \end{cases} \tag{2}$$

where $A_i(x_i)$ is the characteristic matrix that depends on the state of the i^{th} subsystem, $B_i(x_i)$ is the control vector of the i^{th} subsystem, $C_i(x_i)$ is the state-dependent observation matrix of the i^{th} subsystem and $H_{ij}(x_i, x_j)$ is the state -dependent interconnection matrix between the i^{th} and the j^{th} subsystem.

The global interconnected system can be defined by the following compact form:

$$\begin{cases} \dot{x} = A(x)\, x + B(x)\, u + H(x)\, x, \\ y = C(x)\, x, \end{cases} \tag{3}$$

with
$x^T = \left[x_1^T,\ x_2^T,\ \ldots,\ x_n^T\right]$ being the state vector of the overall system; $x \in \mathbb{R}^n$, $n = \sum\limits_{i=1}^{n} n_i$;
$u^T = \left[u_1^T,\ u_2^T,\ \ldots,\ u_n^T\right]$ being the control vector of the overall system ,
$A(x) = diag\left[A_i(x_i)\right], B(x) = diag\left[B_i(x_i)\right]$ and $C(x) = diag\left[C_i(x_i)\right]$.
$H(x)$ is the global interconnection matrix given as follows:

$$H(x) = \begin{pmatrix} 0 & H_{12}(x) & \cdots & H_{1n}(x) \\ H_{21}(x) & 0 & \cdots & H_{2n}(x) \\ \vdots & \vdots & \ddots & \vdots \\ H_{n1}(x) & \cdots & \cdots & 0 \end{pmatrix}. \tag{4}$$

Our contribution consists in the application of a decentralized optimal control via the SDRE approach to nonlinear interconnected systems. We based on solving the decentralized state-dependent Riccati equations to obtain the local control gains. The synthesis of a decentralized control for the system in question is detailed in the following section.

3 Decentralized State-Dependent Riccati Regulation Theory

The decentralized state-dependent Riccati equation technique is a nonlinear control design method for the direct construction of nonlinear sub-optimal feedback controllers. The determination of such decentralized control is based on considering the decoupled subsystem, expressed as follows:

$$\begin{cases} \dot{x}_i = A_i(x_i)\, x_i + B_i(x_i)\, u_i\,, \quad i = 1, \ldots, n \\ y_i = C_i(x_i)\, x_i. \end{cases} \tag{5}$$

Note that $A_i(x_i)$ is not a unique matrix because there could be many possible choices in the direct (SDC) factorization. For this subsystem, the SDRE technique finds an input $u_i(t)$ that approximately minimizes the following performance criterion:

$$J_i = \frac{1}{2}\int_0^{\infty} \left(x_i^T Q_i(x_i)\, x_i + u_i^T R_i(x_i)\, u_i\right) dt, \tag{6}$$

where $Q_i(x_i) \in \mathbb{R}^{(n_i \times n_i)}$ and $R_i(x_i) \in \mathbb{R}^{(m_i \times m_i)}$ are symmetric, positive definite matrices. $x_i^T Q_i(x_i)\, x_i$ is a measure of the control accuracy and $u_i^T R_i(x_i)\, u_i$ is a measure of the control effort.

3.1 Existence of a control solution

The SDRE feedback control provides a similar approach as the algebraic Riccati equation (ARE) for LQR problems to the nonlinear regulation problem for the decoupled nonlinear subsystem (5) with cost functional (6). Indeed, once a SDC form has been found, the SDRE approach is reduced to solving a LQR problem at each sampling instant.

To guarantee the existence of such controller, the conditions in the following definitions must be satisfied [19].

- **Definition 3.1**: $A_i(x_i)$ is a controllable (stabilizable) parametrization of the nonlinear subsystem for a given region if $[A_i(x_i), B_i(x_i)]$ are pointwise controllable (stabilizable) in the linear sense for all x_i in that region.

- **Definition 3.2**: $A_i(x_i)$ is an observable (detectable) parametrization of the nonlinear subsystem for a given region if $[C_i(x_i), A_i(x_i)]$ are pointwise observable (detectable) in the linear sense for all x_i in that region.

Given these standing assumption, the state feedback decentralized controller is obtained in the following form:

$$u_i(x_i) = -K_i(x_i)\, x_i \tag{7}$$

and the state feedback decentralized gain for minimizing (6) is

$$K_i(x_i) = R_i^{-1}(x_i)\, B_i^T(x_i)\, P_i(x_i), \tag{8}$$

where $P_i(x_i)$ is the unique symmetric positive-definite solution of the decentralized state dependent Riccati equation (SDRE)

$$\begin{aligned} &A_i^T(x_i)\, P_i(x_i) + P_i(x_i)\, A_i(x_i) \\ &\qquad - P_i(x_i)\, B_i(x_i)\, R_i^{-1}(x_i)\, B_i^T(x_i)\, P_i(x_i) + C_i^T(x_i)\, Q_i(x_i)\, C_i(x_i) = 0. \end{aligned} \tag{9}$$

Remark 3.1: It is important to note that the existence of the decentralized optimal control for a particular parametrization of the subsystem is not guaranteed. Furthermore, there may be an infinite number of parametrizations of the subsystem, therefore the choice of parametrization is very important. The other factor which may determine the existence of a solution is the $Q_i(x_i)$ and $R_i(x_i)$ weighting matrices in the state dependent Riccati equation (9).

Remark 3.2. The greatest advantage of the state-dependent Riccati equation approach is that physical intuition is always present and the designer can directly control the performance by tuning the weighting matrices $Q_i(x_i)$ and $R_i(x_i)$. In other words, via the SDRE, the design flexibility of LQR formulation is directly translated to control the nonlinear interconnected systems. Moreover, $Q_i(x_i)$ and $R_i(x_i)$ are not only allowed to be constant, but can also vary as functions of states. In this way, different modes of behavior can be imposed in different regions of the state-space [21].

3.2 Optimality of the SDRE regulation

As $x_i \to 0$, $A_i(x_i) \to \partial f_i(0)/\partial x_i$ which implies that $P_i(x_i)$ approaches the linear ARE at the origin. Furthermore, the SDRE control solution asymptotically approaches the optimal control as $x_i \to 0$ and away from the origin the SDRE control is arbitrarily close to the optimal feedback. Hence the SDRE approach yields an asymptotically optimal feedback solution.

Let the Hamiltonian be defined by the following expression:

$$H_i(x_i, u_i, \lambda_i) = \tfrac{1}{2}\left[x_i^T Q_i(x_i)\, x_i + u_i^T R_i(x_i)\, u_i\right] + \lambda_i^T \left[A_i(x_i)\, x_i + B_i(x_i)\, u_i\right]. \tag{10}$$

Mracek and Cloutier developed the necessary conditions for the optimality of a general nonlinear regulator, that is the regulator governed by (5) and (6), and then extend these results to determine the optimality of the SDRE approach [19].

Theorem 1. *For the general multivariable nonlinear SDRE control case (i.e., $n > 1$), the SDRE nonlinear feedback solution and its associated state satisfy the first necessary condition for optimality $\partial H_i/\partial u_i = 0$ of the nonlinear optimal regulator problem defined by (5) and (6). Additionally, the second necessary condition for optimality $\dot{\lambda}_i = -\partial H_i \big/ \partial x_i$ is asymptotically satisfied at a quadratic rate.*

Proof. Pontryagin'S maximum principle states that necessary conditions for optimality are

$$\frac{\partial H_i}{\partial u_i} = 0, \quad \dot{\lambda}_i = -\frac{\partial H_i}{\partial x_i}, \quad \dot{x}_i = \frac{\partial H_i}{\partial \lambda_i}, \tag{11}$$

where H_i is the Hamiltonian. Using (7) yields

$$\frac{\partial H_i}{\partial u_i} = B_i^T(x_i)\left[\lambda_i - P_i(x_i)\, x_i\right] \tag{12}$$

and λ_i, the adjoint vector for the system, satisfies

$$\lambda_i = P_i(x_i)\, x_i, \tag{13}$$

and the first optimality condition (12) is satisfied identically for the nonlinear regulator problem. With the Hamiltonian defined in (10), the second necessary condition becomes

$$\begin{aligned} \dot{\lambda}_i = -x_i^T \left(\frac{\partial A_i(x_i)}{\partial x_i}\right)^T \lambda_i - u_i^T \left(\frac{\partial B_i(x_i)}{\partial x_i}\right)^T \lambda_i - Q_i(x_i)\, x_i \\ -\frac{1}{2} x_i^T \frac{\partial Q_i(x_i)}{\partial x_i} x_i - \frac{1}{2} u_i^T \frac{\partial R_i(x_i)}{\partial x_i} u_i. \end{aligned} \tag{14}$$

Taking the time derivative of (13) yields

$$\dot{\lambda}_i = \dot{P}_i(x_i)\,x_i + P_i(x_i)\,\dot{x}_i. \tag{15}$$

Substituting this result, along with (5), (7) and (14) into (9) leads to the SDRE necessary condition for optimality

$$\begin{aligned}
&\dot{P}_i(x_i)\,x_i + \frac{1}{2}x_i^T P_i(x_i)\,B_i(x_i)\,R_i^{-1}(x_i)\,\frac{\partial R_i(x_i)}{\partial x_i}R_i^{-1}(x_i)\,B_i^T(x_i)\,P_i(x_i)\,x_i\\
&\qquad + x_i^T\left(\tfrac{\partial A_i(x_i)}{\partial x_i}\right)^T P_i(x_i)\,x_i + \frac{1}{2}x_i^T\frac{\partial Q_i(x_i)}{\partial x_i}x_i\\
&\qquad - x_i^T P_i(x_i)\,B_i(x_i)\,R_i^{-1}(x_i)\left(\frac{\partial B_i(x_i)}{\partial x_i}\right)^T P_i(x_i)\,x_i = 0.
\end{aligned} \tag{16}$$

Hence, whenever (16) is satisfied, the closed-loop SDRE solution satisfies all the first-order necessary conditions for an extremum of the cost functional.

4 Stability Study

In this section, we study the asymptotic stability of interconnected system based on the Lyapunov theory [10]. We begin with the stability study of each subsystem, thereafter we deal with the development of a sufficient condition to assure the asymptotic stability of the overall interconnected nonlinear system.

4.1 Stability of a decoupled nonlinear subsystem

Stability of SDRE systems is still an open problem. Local stability results are presented by Cloutier, D'souza and Mracek in the case when the closed-loop coefficient matrix is assumed to have a special structure.

The authors in [22,23] presented the optimal control solution for nonlinear subsystem using the SDRE method. The asymptotic stability of decoupled subsystem (5) with SDRE feedback control is guaranteed provided that

$$\begin{aligned}
&M_i(x_i) = -C_i^T(x_i)\,Q_i(x_i)\,C_i(x_i) - P_i(x_i)\,B_i(x_i)\,R_i^{-1}(x_i)\,B_i^T(x_i)\,P_i(x_i)\\
&- \left(I_n \otimes x_i^T P_i(x_i)\,B_i(x_i)\,R_i^{-1}(x_i)\,B_i^T(x_i)\right)\frac{\partial P_i(x_i)}{\partial x_i} + \left(I_n \otimes (x_i^T A_i^T(x_i)\right)\frac{\partial P_i(x_i)}{\partial x_i}
\end{aligned} \tag{17}$$

is negative definite for all $x_i \in \mathbb{R}^{n_i}$.

Now, to guarantee the asymptotic stability of the overall interconnected system (3), we carry out a stability study of interconnected system (2) with the decentralized control (7) as depicted in the following subsection.

4.2 Stability of a global interconnected system

In this paragraph, we present our contribution which consists in developing a sufficient condition to assure the asymptotic stability of the overall interconnected nonlinear system (3) with the decentralized control law (7). This study is based on the quadratic Lyapunov function

$$V(x) = x^T P(x)\,x, \tag{18}$$

where $P(x) = diag\left[P_i(x_i)\right]$.

The global asymptotic stability of the equilibrium state $(x = 0)$ of system (3) is ensured when the time derivative $\dot{V}(x)$ of $V(x)$ is negative define for all $x \in \mathbb{R}^n$,

$$\dot{V}(x) = \dot{x}^T P(x)\,x + x^T \frac{dP(x)}{dt} x + x^T P(x)\,\dot{x}. \tag{19}$$

The use of expression (19) and the following equality:

$$\frac{dP(x)}{dt} = \left(I_n \otimes \dot{x}^T\right) \frac{\partial P(x)}{\partial x} \tag{20}$$

yields

$$\begin{aligned} \dot{V}(x) &= x^T\left[A^T(x)\,P(x) + P(x)\,A(x)\right]x + x^T\left[H^T(x)\,P(x) + P(x)\,H(x)\right]x \\ &-2x^T\left[P(x)\,B(x)\,R^{-1}(x)\,B^T(x)\,P(x)\right]x + x^T\left(I_n \otimes \dot{x}^T\right)\frac{\partial P(x)}{\partial x}x, \end{aligned} \tag{21}$$

then

$$\begin{aligned} \dot{V}(x) &= x^T\left[A^T(x)\,P(x) + H^T(x)\,P(x)\right. \\ &\left.+P(x)\,A(x) + P(x)\,H(x) - 2P(x)\,B(x)\,R^{-1}(x)\,B^T(x)\,P(x)\right]x \\ &+x^T\left[\left(I_n \otimes \left(x^T A^T(x) + x^T H^T(x) - x^T P(x)\,B(x)\,R^{-1}(x)\,B^T(x)\right)\right)\frac{\partial P(x)}{\partial x}\right], \end{aligned} \tag{22}$$

where $\otimes$ is the Kronecker product notation whose definition and properties are detailed in the appendix. Using the state-dependent Riccati equation (9), expression (22) can be simplified as follows:

$$\begin{aligned} \dot{V}(x) &= x^T\left[-C^T(x)\,Q(x)\,C(x) - P(x)\,L(x)\,P(x)\right]x \\ &+x^T\left[H^T(x)\,P(x) + P(x)\,H(x)\right]x + x^T\left[\left(I_n \otimes \left(x^T A^T(x) + x^T H^T(x)\right)\right)\frac{\partial P(x)}{\partial x}\right]x \\ &-x^T\left[\left(I_n \otimes \left(x^T P(x)\,B(x)\,R^{-1}(x)\,B^T(x)\right)\right)\frac{\partial P(x)}{\partial x}\right]x, \end{aligned} \tag{23}$$

where $L(x) = B(x)\,R^{-1}(x)\,B^T(x)\,, \forall\, x \in \mathbb{R}^n$.

To ensure the asymptotic stability of the overall systems (3) with the decentralized optimal control law (7), $\dot{V}(x)$ should be negative definite, which is equivalent to $M(x)$ being negative definite, with

$$\begin{aligned} M(x) &= -C^T(x)\,Q(x)\,C(x) - P(x)\,B(x)\,R^{-1}(x)\,B^T(x)\,P(x) \\ &\quad - \left(I_n \otimes x^T P(x)\,B(x)\,R^{-1}(x)\,B^T(x)\right)\frac{\partial P(x)}{\partial x} \\ &\quad + \left(I_n \otimes x^T A^T(x) + x^T H^T(x)\right)\frac{\partial P(x)}{\partial x} + P(x)\,H(x) + H^T(x)\,P(x)\,. \end{aligned} \tag{24}$$

We need to simplify the manipulation of matrix $M(x)$ by expressing $\partial P(x)/\partial x$ in terms of $P(x) > 0$, $\forall\, x \in \mathbb{R}^n$. When deriving the SDRE (9) with respect to the state vector $x \in \mathbb{R}^n$, we get the following expression:

$$\begin{aligned}&\frac{\partial P(x)}{\partial x} A(x) + (I_n \otimes P(x)) \frac{\partial A(x)}{\partial x} + \frac{\partial A^T(x)}{\partial x} P(x) + \left(I_n \otimes A^T(x)\right) \frac{\partial P(x)}{\partial x} + \frac{\partial \Phi(x)}{\partial x} \\ &- \frac{\partial P(x)}{\partial x} L(x) P(x) - (I_n \otimes P(x) L(x)) \frac{\partial P(x)}{\partial x} - (I_n \otimes P(x)) \frac{\partial L(x)}{\partial x} P(x) = 0\end{aligned} \tag{25}$$

with $\Phi(x) = C^T(x) Q(x) C(x)$, which gives

$$\left[I_n \otimes A^T(x) - I_n \otimes (P(x) L(x))\right] \frac{\partial P(x)}{\partial x} + \frac{\partial P(x)}{\partial x} [A(x) - L(x) P(x)] = W(x) \tag{26}$$

with

$$W(x) = (I_n \otimes P(x)) \frac{\partial L(x)}{\partial x} P(x) - (I_n \otimes P(x)) \frac{\partial A(x)}{\partial x} - \frac{\partial A^T(x)}{\partial x} P(x) - \frac{\partial \Phi(x)}{\partial x}. \tag{27}$$

To simplify the partial derivative expression $\partial P(x)/\partial x$ we use the functions Vec and mat and their properties defined in this paper appendix; so (26) becomes

$$\begin{aligned} Vec\left(\frac{\partial P(x)}{\partial x}\right) &= \left[I_n \otimes \left(I_n \otimes A^T(x) + I_n \otimes P(x) L(x)\right)\right. \\ &\quad \left. + (A(x) - L(x) P(x)) \otimes I_n\right]^{-1} Vec(W(x)) \end{aligned} \tag{28}$$

which leads to

$$\begin{aligned} \frac{\partial P(x)}{\partial x} &= mat_{(n^2,n)} \left[\left(I_n \otimes \left[I_n \otimes A(x) + I_n \otimes A^T(x)\right.\right.\right. \\ &\quad \left.\left. -2 (I_n \otimes L(x) P(x))\right]\right)^{-1} Vec(W(x))\Big]. \end{aligned} \tag{29}$$

Therefore, we can state the following result.

Theorem 2. *The overall system (3) is globally asymptotically stabilizable by the optimal decentralized control law (7), with the cost function (6) if the matrix $M(x)$ defined by (24) is negative definite for all $x \in \mathbb{R}^n$.*

5 Simulation Results

In this section we will illustrate the performance of the decentralized SDRE approach, discussed in the previous paragraph, by a numerical example. We consider a nonlinear interconnected system defined by the following two subsystems of state equations:

$$\begin{cases} \sum 1: \begin{cases} \dot{x}_{11} = -2x_{11} + x_{11}x_{12}, \\ \dot{x}_{12} = x_{13} + x_{12}x_{11} + x_{22}^2 x_{21}, \\ \dot{x}_{13} = u_1 + x_{13}^2 \left(x_{12}x_{11} + x_{11}^2\right) + x_{22}x_{21}, \end{cases} \\ \sum 2: \begin{cases} \dot{x}_{21} = -x_{21} + x_{22}^2, \\ \dot{x}_{22} = x_6 + \left(x_{12}^2 x_{11} + x_{22}^2 x_{21}\right), \\ \dot{x}_{23} = u_2 + x_{23}^2 \left(x_{12}x_{11}^2 + x_{22}x_{21}^2\right) + x_{23}^2 x_{21}^2, \end{cases} \end{cases} \tag{30}$$

with

- $x_1 = [x_{11}\ x_{12}\ x_{13}]^T, x_2 = [x_{21}\ x_{22}\ x_{23}]^T$ being the state vectors of subsystems $\sum 1$ and $\sum 2$,
- $u = \begin{bmatrix} u_1 & u_2 \end{bmatrix}^T$ being the inputs of the interconnected nonlinear system.

We solve equation (9) with

$$Q_1(x_1) = Q_2(x_2) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{31}$$

$$R_1(x_1) = R_2(x_2) = 0.1. \tag{32}$$

For interconnected nonlinear systems (30), we choose the following (SDC) parametrization:

$$A_1(x_1) = \begin{pmatrix} -2 & x_{11} & 0 \\ x_{12} & 0 & 1 \\ x_{13}^2 x_{12} & 0 & x_{13}x_{11}^2 \end{pmatrix}, A_2(x_2) = \begin{pmatrix} -1 & x_{22} & 0 \\ 0 & x_{22}x_{21} & 1 \\ x_{23}^2 x_{22} x_{21} + x_{23}^2 x_{21}^2 & 0 & 0 \end{pmatrix}.$$

The control matrices are given as follows:

$$B_1(x_1) = B_2(x_2) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

The interconnection matrices between subsystem 1 and subsystem 2 are expressed as follows:

$$H_{12}(x_2) = \begin{pmatrix} 0 & 0 & 0 \\ x_{22}^2 & 0 & 0 \\ x_{22} & 0 & 0 \end{pmatrix}, \quad H_{21}(x_1, x_2) = \begin{pmatrix} 0 & 0 & 0 \\ x_{12}^2 & 0 & 0 \\ x_{23}^2 x_{12} & 0 & 0 \end{pmatrix}.$$

The controllability matrices, respectively, for subsystem 1 and subsystem 2 are given as follows:

$$\begin{aligned} \zeta_1(x_1) &= \left[B_1(x_1)\ \ A_1(x_1)\, B_1(x_1)\ \ A_1^2(x_1)\, B_1(x_1)\right] \\ &= \begin{pmatrix} 0 & 0 & x_{11} \\ 0 & 1 & x_{13}x_{11}^2 \\ 1 & x_{13}x_{11}^2 & x_{13}^2 x_{11}^4 \end{pmatrix}, \end{aligned} \tag{33}$$

$$\begin{aligned} \zeta_2(x_2) &= \left[B_2(x_2)\ \ A_2(x_2)\, B_2(x_2)\ \ A_2^2(x_2)\, B_2(x_2)\right] \\ &= \begin{pmatrix} 0 & 0 & x_{22} \\ 0 & 1 & x_{22}x_{21} \\ 1 & 0 & 0 \end{pmatrix}. \end{aligned} \tag{34}$$

$\zeta_1(x_1), \zeta_2(x_2)$ have a full order rank for all x_i, which can justify the good choice of (SDC) parametrization. Now, we referring to equation (9), we can write the following decentralized state-dependent Riccati equations:

$$\begin{cases} P_1(x_1) A_1(x_1) + A_1^T(x_1) P_1(x_1) + Q_1(x_1) \\ -P_1(x_1) B_1(x_1) R_1^{-1}(x_1) B_1^T(x_1) P_1(x_1) = 0, \\ \\ P_2(x_2) A_2(x_2) + A_2^T(x_2) P_2(x_2) + Q_2(x_2) \\ -P_2(x_2) B_2(x_2) R_2^{-1}(x_2) B_2^T(x_2) P_2(x_2) = 0. \end{cases} \tag{35}$$

The decentralized optimal control are expressed as follows:

$$u_1(x_1) = -0.1 \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} P_1(x_1) \begin{pmatrix} x_{11} \\ x_{12} \\ x_{13} \end{pmatrix} \tag{36}$$

and

$$u_2(x_2) = -0.1 \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} P_2(x_2) \begin{pmatrix} x_{21} \\ x_{22} \\ x_{23} \end{pmatrix}. \tag{37}$$

• **Numerical simulation:**

Figure 1 (respectively Figure 2) shows the behavior of the first states variables x_{11}, x_{12} and x_{13}, (respectively, the second states variables x_{21}, x_{21} and x_{23} of interconnected system (30) controlled by the decentralized control laws illustrated in Figure 3. Initial conditions were taken as follows: $x_{11}(0) = x_{13}(0) = x_{21}(0) = x_{22}(0) = 0.1$, $x_{12}(0) = x_{23}(0) = 0$.

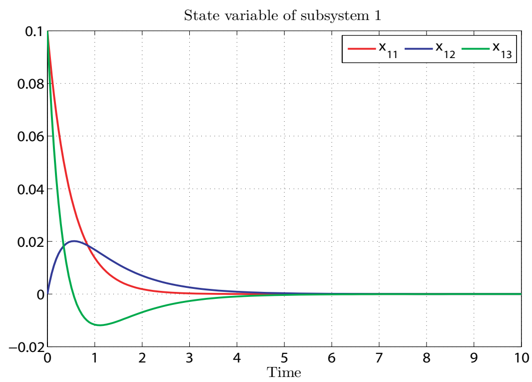

Figure 1: Closed loop reponses of x_1.

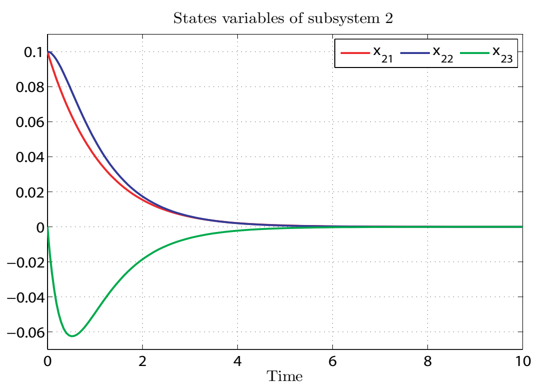

Figure 2: Closed loop reponses of x_2.

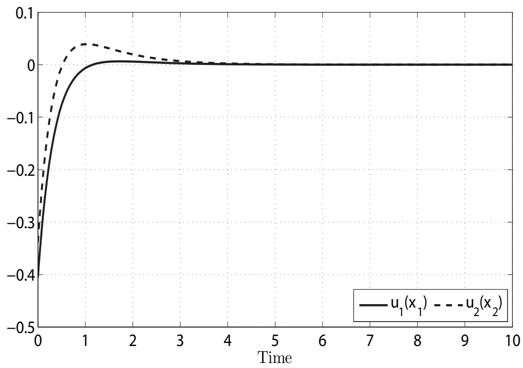

Figure 3: Decentralized control signals evolution.

We can note a satisfactory stabilization of state variables which converge into the origin point confirming the asymptotic stability of the controlled interconnected system using the decentralized SDRE approach.

6 Conclusion

In this paper, we have considered the method for feedback control of nonlinear interconnected systems using the decentralized state-dependent Riccati equation. This decentralized optimal approach is based on the solution of algebraic Riccati equation. Our first result was to determine and prove sufficient conditions that guarantee the global asymptotic stability of the overall interconnected system. We have then run some numerical simulations on a third order system. As expected, these simulations have shown the aptitude of the SDRE approach to be implemented easily and to give satisfactory result in terms of performance for a wide class of nonlinear interconnected systems. One of the possible perspectives that we can consider as a continuity of this research would be to investigate an optimal control for interconnected nonlinear systems via approximate methods.

Appendix

We recall hereafter the useful mathematical notations and properties concerning the Kronecker tensor product used in this paper.

A.1. Kronecker product:

The Kronecker product of $A\,(p \times q)$ and $B\,(r \times s)$ denoted by $A \otimes B$ is the $(pr \times qs)$ matrix defined by [24, 25]

$$A \otimes B = \begin{pmatrix} a_{11}B & \dots & a_{1q}B \\ \vdots & \ddots & \vdots \\ a_{p1}B & \cdots & a_{pq}B \end{pmatrix}. \tag{38}$$

A.2. Vec-function:

Vec-function is a linear algebra tool which is important in the multidimensional regression matrix representation. This operator is defined as follows [24, 25] :

$$A = (A_1\ A_2\ \ldots\ A_n)\,; \quad Vec\,(A) = \begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_n \end{pmatrix}, \tag{39}$$

where $\forall\, i \in \{1, \ldots, n\}$, A_i is a vector of $\mathbb{R}^m$.

We recall the following useful rule of this function, given as follows:

$$Vec(E.A.C) = (C^T \otimes E)Vec(A). \tag{40}$$

A.3. Mat function :

An important matrix-valued linear function of a vector, denoted by $mat_{(n,m)}\,(\,.\,)$, was defined in [24, 25] as follows: if V is a vector of dimension $p = n.m$, then $M = mat_{(n,m)}\,(V)$ is the $(n \times m)$ matrix verifying $V = Vec\,(M)$.

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