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A Variety of New Solitary-Solutions for the Two-mode Modified Korteweg-de Vries Equation

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Abstract: In this paper, we studied the nonlinear two-mode modified Korteweg-de Vries (TMmKdV) equation. We derived multiple singular soliton solutions to this new version of KdV equation by using the simplified form of Hirota's direct method. Also, kink and periodic solutions are extracted by using the tanh-expansion and the sine-cosine function methods. Finally, graphical analysis is conducted to show some physical features regarding TMmKdV equation.

Keywords: *two-mode mKdV; Hirota bilinear method; sine-cosine function method; multiple singular solutions; kink and periodic solutions.*

Mathematics Subject Classification (2010): 35C08, 74J35.

1 Introduction

Sergei V. Korsunsky [1] was the first who established the nonlinear two-mode Korteweg-de Vries (TMKdV) equation which reads

$$\begin{aligned} & w_{tt} + (a_1 + a_2)w_{xt} + a_1a_2w_{xx} + ((\lambda_1 + \lambda_2)\frac{\partial}{\partial t} + (\lambda_1 a_2 + \lambda_2 a_1)\frac{\partial}{\partial x})ww_x \\ & + ((\mu_1 + \mu_2)\frac{\partial}{\partial t} + (\mu_1 a_2 + \mu_2 a_1)\frac{\partial}{\partial x})w_{xxx}, \end{aligned} \tag{1}$$

where $w(x,t)$ is a field function representing the height of the free water surface above a flat bottom, a_1 and a_2 are the phase velocities, μ_1 and μ_2 are the dispersion parameters,

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λ_1 and λ_2 are the parameters of nonlinearity.

The modified Korteweg-de Vries (mKdV) equation for the one-dimensional propagation of solitary waves in a fluid is given by

$$w_t + \alpha w_{xxx} + \beta w^2 w_x = 0, \tag{2}$$

which is a generalized model in ocean dynamics, nonlinear lattice and plasma physics. In this paper we reconstruct and study the two-mode modified Korteweg-de Vries equation which describes the propagation of two wave modes of the same orientation. Now, the two-mode modified Korteweg-de Vries (TMmKdV) equation in a scaled-form reads

$$\begin{aligned} &w_{tt} + (a_1 + a_2)w_{xt} + a_1 a_2 w_{xx} + (\beta(\lambda_1 + \lambda_2)\frac{\partial}{\partial t} + \beta(\lambda_1 a_2 + \lambda_2 a_1)\frac{\partial}{\partial x})w^2 w_x \\ &+ (\alpha(\mu_1 + \mu_2)\frac{\partial}{\partial t} + \alpha(\mu_1 a_2 + \mu_2 a_1)\frac{\partial}{\partial x})w_{xxx}, \end{aligned} \tag{3}$$

where $a_1, a_2, \lambda_1, \lambda_2, \mu_1, \mu_2$ are some real numbers, $w(x,t)$ is a field function, a_1 and a_2 are the phase velocities, μ_1 and μ_2 are the dispersion parameters, λ_1 and λ_2 are the parameters of nonlinearity. Note that a_1, a_2 are considered to be distinct and $x, t \in (-\infty, \infty)$. Now we suggest the changes of variable by using the transformations [1–5]:

$$\begin{aligned} T &= (\mu_1 + \mu_2)^{\frac{-1}{2}} t, \\ X &= (\mu_1 + \mu_2)^{\frac{-1}{2}} (x - a_0 t), \\ a_0 &= \frac{a_1 + a_2}{2}, \\ W &= (\lambda_1 + \lambda_2)^{\frac{1}{2}} w. \end{aligned}$$

Therefore, equation (3) reduces to TMmKdV equation in a scaled form as

$$W_{TT} - a^2 W_{XX} + (\beta\frac{\partial}{\partial T} - \beta\lambda a\frac{\partial}{\partial X})W^2 W_x + (\alpha\frac{\partial}{\partial T} - \alpha\mu a\frac{\partial}{\partial X})W_{XXX}, \tag{4}$$

where

$$\begin{aligned} a &= \frac{a_1 - a_2}{2}, \\ \lambda &= \frac{\lambda_2 - \lambda_1}{\lambda_2 + \lambda_1}, |\lambda| \le 1, \\ \mu &= \frac{\mu_2 - \mu_1}{\mu_2 + \mu_1}, |\mu| \le 1, \end{aligned}$$

where $|\lambda| \le 1, |\mu| \le 1$ and a is defined above. Note that when $a = 0$, by integrating with respect to t, the two-mode modified Korteweg-de Vries equation (4) is reduced to the standard modified Korteweg-de Vries equation (2).

Finally, for more details about generating two-mode equations and physical features such models possess, we recommend for the readers the following references [6–14].

2 Multiple Soliton Solutions

In this section, we apply the simplified bilinear method [15–20], to find single soliton solutions and multiple soliton solutions for TMmKdV equation. First, we substitute

$$W(X,T) = e^{\varepsilon_i}, \qquad \varepsilon_i(X,T) = h_i X - \omega_i T$$

into the linear terms of (4) and solve the resulting equation to obtain the dispersion relation

$$\omega_i = \frac{\alpha h_i^3 \pm h_i \sqrt{\alpha^2 h_i^4 + 4\alpha\mu a h_i^2 + 4a^2}}{2}. \tag{5}$$

As a result ε_i becomes

$$\varepsilon_i(X,T) = h_i X - \frac{\alpha h_i^3 \pm h_i \sqrt{\alpha^2 h_i^4 + 4\alpha\mu a h_i^2 + 4a^2}}{2} T, \qquad i = 1, 2, \ldots. \tag{6}$$

Second, we propose the solutions of (4) in the form

$$W(X,T) = R\left(\arctan\left(\frac{h(X,T)}{k(X,T)}\right)\right)_X = R\frac{h_X k - k_X h}{h^2 + k^2}. \tag{7}$$

The auxiliary functions $h(X,T)$ and $k(X,T)$ for single-soliton solution are given by

$$\begin{cases} h(X,T) = e^{\varepsilon_1(X,T)} = e^{h_1 X - \frac{\alpha h_1^3 \pm h_1 \sqrt{\alpha^2 h_1^4 + 4\alpha\mu a h_1^2 + 4a^2}}{2} T}, \\ k(X,T) = 1. \end{cases} \tag{8}$$

Substituting (7) and (8) into (4) and solving for R, we get

$$R = \pm 2\sqrt{\frac{6\alpha}{\beta}}. \tag{9}$$

Under the constraint condition $\lambda = \mu$, the single soliton solution is given by

$$\begin{aligned} W(X,T) &= 2h_1 \sqrt{\frac{6\alpha}{\beta}} \frac{e^{h_1 X - \frac{\alpha h_1^3 \pm h_1 \sqrt{\alpha^2 h_1^4 + 4\alpha\mu a h_1^2 + 4a^2}}{2} T}}{1 + e^{2h_1 X - (\alpha h_1^3 \pm h_1 \sqrt{\alpha^2 h_1^4 + 4\alpha\mu a h_1^2 + 4a^2}) T}} \\ &= h_1 \sqrt{\frac{6\alpha}{\beta}} \operatorname{sech}\left(\varepsilon_1(X,T)\right), \end{aligned} \tag{10}$$

where

$$\varepsilon_1(X,T) = h_1 X - \frac{\alpha h_1^3 \pm h_1 \sqrt{\alpha^2 h_1^4 + 4\alpha\mu a h_1^2 + 4a^2}}{2} T.$$

To find the two-soliton solution, we assume

$$\begin{aligned} h(X,T) &= e^{\varepsilon_1} + e^{\varepsilon_2} \\ &= e^{h_1 X - \frac{\alpha h_1^3 \pm h_1 \sqrt{\alpha^2 h_1^4 + 4\alpha\mu a h_1^2 + 4a^2}}{2} T} + e^{h_2 X - \frac{\alpha h_2^3 \pm h_2 \sqrt{\alpha^2 h_2^4 + 4\alpha\mu a h_2^2 + 4a^2}}{2} T}, \\ k(X,T) &= 1 - c_{12} e^{h_1 X - \frac{\alpha h_1^3 \pm h_1 \sqrt{\alpha^2 h_1^4 + 4\alpha\mu a h_1^2 + 4a^2}}{2} T + h_2 X - \frac{\alpha h_2^3 \pm h_2 \sqrt{\alpha^2 h_2^4 + 4\alpha\mu a h_2^2 + 4a^2}}{2} T}. \end{aligned} \tag{11}$$

Substituting (7) and (11) into (4) and solving for c_{12}, we see that the constraint condition of two soliton solutions exists only if $\lambda = \mu = \pm 1$ and the phase shift c_{12} is obtained by

$$c_{12} = \frac{(h_1 - h_2)^2}{(h_1 + h_2)^2} \tag{12}$$

and this can be generalized as

$$c_{ij} = \frac{(h_i - h_j)^2}{(h_i + h_j)^2}, \qquad 1 \le i < j \le 3. \tag{13}$$

To get the two-soliton solutions for (4), we substitute (11) and (12) into (7) and use $\lambda = \mu = 1$. As a result, we get

$$\begin{aligned} U(X,T) &= \frac{h_1 e^{h_1 X - r_1 T}\left(1 + \frac{(h_1-h_2)^2}{(h_1+h_2)^2} e^{2h_2 X - (\alpha h_2^3 \pm (\alpha h_2^3 + 2a))T}\right)\sqrt{\frac{6\alpha}{\beta}}}{(\frac{(h_1-h_2)^2}{(h_1+h_2)^2} e^{h_1 X - r_1 T + h_2 X - r_2 T} - 1)^2 + (e^{h_1 X - r_1 T} + e^{h_2 X - r_2 T})^2} \\ &+ \frac{h_2 e^{h_2 X - r_2 T}\left(1 + \frac{(h_1-h_2)^2}{(h_1+h_2)^2} e^{2h_1 X - (\alpha h_1^3 \pm (\alpha h_1^3 + 2a))T}\right)\sqrt{\frac{6\alpha}{\beta}}}{(\frac{(h_1-h_2)^2}{(h_1+h_2)^2} e^{h_1 X - r_1 T + h_2 X - r_2 T} - 1)^2 + (e^{h_1 X - r_1 T} + e^{h_2 X - r_2 T})^2}, \end{aligned} \tag{14}$$

where

$$\begin{aligned} r_1 &= \frac{\alpha h_1^3 \pm (\alpha h_1^3 + 2a)}{2}, \\ r_2 &= \frac{\alpha h_2^3 \pm (\alpha h_2^3 + 2a)}{2}. \end{aligned}$$

For the three-soliton solutions, we use

$$\begin{cases} h(X,T) = e^{\varepsilon_1} + e^{\varepsilon_2} + e^{\varepsilon_3} + c_{123} e^{\epsilon_1 + \epsilon_2 + \varepsilon_3}, \\ k(X,T) = 1 - c_{12} e^{\epsilon_1 + \epsilon_2} - c_{13} e^{\epsilon_1 + \varepsilon_3} - c_{23} e^{\epsilon_2 + \varepsilon_3}, \end{cases} \tag{15}$$

where c_{ij} are given in (13). Substituting (7) and (15) into (4) and solving for c_{123} under the constraint condition $\lambda = \mu = \pm 1$, we find

$$c_{123} = c_{12} c_{13} c_{23}.$$

Finally, we reach to the fact that TMmKdV equation given in (4) has N-soliton solutions under the constraint condition $\lambda = \mu = \pm 1$ which can be obtained for finite N, where $N \ge 3$.

3 Singular Soliton Solutions

In this section we construct a multiple singular-soliton solution for (4) where the solution is assumed to be of the form

$$W(X,T) = R \ln\left(\frac{h(X,T)}{k(X,T)}\right)_X = R \frac{k h_X - h k_X}{kh}. \tag{16}$$

The dispersion relation as in the previous section is given by

$$\omega_i = \frac{\alpha h_i^3 \pm h_i \sqrt{\alpha^2 h_i^4 + 4\alpha\mu a h_i^2 + 4a^2}}{2},$$

and hence $\varepsilon_i(X,T) = h_i X - \dfrac{\alpha h_i^3 \pm h_i \sqrt{\alpha^2 h_i^4 + 4\alpha\mu a h_i^2 + 4a^2}}{2} T$, $i = 1, 2, \ldots$.

For the singular one-soliton solution, we consider

$$h(X,T) = 1 + e^{\varepsilon_1(X,T)}, \quad k(X,T) = 1 - e^{\varepsilon_1(X,T)}. \tag{17}$$

Substituting (17) and (16) into (4) and solving for R, we get

$$R = \pm\sqrt{-6\alpha/\beta}. \tag{18}$$

Under the constraint condition $\lambda = \mu$, the single-soliton solution is given by

$$\begin{aligned} W(X,T) &= -2h_1\sqrt{\frac{-6\alpha}{\beta}}\frac{e^{h_1X-\frac{\alpha h_1^3 \pm h_1\sqrt{\alpha^2h_1^4+4\alpha\mu a h_1^2+4a^2}}{2}T}}{e^{2h_1X-(\alpha h_1^3\pm h_1\sqrt{\alpha^2h_1^4+4\alpha\mu a h_1^2+4a^2})T}-1} \\ &= -h_1\sqrt{\frac{-6\alpha}{\beta}}\operatorname{csch}\left(\varepsilon_1(X,T)\right). \end{aligned} \tag{19}$$

To obtain singular two-soliton solution, we set

$$\begin{cases} h(X,T) = 1 + e^{\varepsilon_1(X,T)} + e^{\varepsilon_2(X,T)} + c_{12}e^{\varepsilon_1(X,T)+\varepsilon_2(X,T)}, \\ k(x,t) = 1 - e^{\varepsilon_1(X,T)} - e^{\varepsilon_2(X,T)} + c_{12}e^{\varepsilon_1(X,T)+\varepsilon_2(X,T)}. \end{cases} \tag{20}$$

Substituting (18), (20) and (16) into (4) and solving for c_{12} lead to the two soliton solutions only if $\lambda = \mu = \pm 1$ and the same phase shift c_{12} obtained in (12) and hence c_{ij} given by (13).

To construct the singular three-soliton solution, we set

$$\begin{aligned} h(X,T) &= 1 + e^{\varepsilon_1(X,T)} + e^{\varepsilon_2(X,T)} + e^{\epsilon_3(X,T)} + c_{12}e^{\varepsilon_1(X,T)+\varepsilon_1(X,T)} + c_{23}e^{\varepsilon_2(X,T)+\varepsilon_3(X,T)} \\ &+ c_{13}e^{\varepsilon_1(X,T)+\varepsilon_3(X,T)} + c_{123}e^{\varepsilon_1(X,T)+\varepsilon_2(X,T)+\varepsilon_3(X,T)}, \\ k(X,T) &= 1 - e^{\varepsilon_1(X,T)} - e^{\varepsilon_2(X,T)} - e^{\epsilon_3(X,T)} + c_{12}e^{\varepsilon_1(X,T)+\varepsilon_1(X,T)} + c_{23}e^{\varepsilon_2(X,T)+\varepsilon_3(X,T)} \\ &+ c_{13}e^{\varepsilon_1(X,T)+\varepsilon_3(X,T)} - c_{123}e^{\varepsilon_1(X,T)+\varepsilon_2(X,T)+\varepsilon_3(X,T)}. \end{aligned} \tag{21}$$

Repeating the same previous steps, we reach to the same fact that the three single-soliton solutions exists only under the constraint condition $\lambda = \mu = \pm 1$.

4 Solitary Ansatze Methods

In this part, we introduce in brief two methods, the tanh-technique and the sine-cosine function method to solve the problem (4).

4.1 The tanh method

The tanh technique [21–26] suggests the following solution

$$W(\zeta) = S(Y) = \sum_{i=0}^{M} b_i Y^i, \tag{22}$$

where $Y = \tanh(\delta\zeta)$. The index M can be determined by a balance procedure. Once we have M, we collect all coefficients of powers of Y in the resulting equation and set them to zero. Finally, we solve the obtained algebraic system to retrieve the values of the required coefficients b_i.

Now, we consider a new variable $\zeta = X - \gamma T$ to reduce (4) into the following differential equation

$$(\gamma^2 - a^2)W - \frac{\beta}{3}(\gamma + \lambda a)W^3 - \alpha(\gamma + \mu a)W'' = 0, \tag{23}$$

where $W = W(\zeta)$ and the prime denotes the ordinary derivative. By a blanching procedure for equation (23), the value of the parameter M is equal to 1 and thus $W(\zeta) = A + B\ \tanh(\delta\zeta)$. Substituting this proposed solution in (23) yields the following algebraic system:

$$\begin{aligned}
0 &= -Aa^2 - \frac{1}{3}\lambda A^3 a\beta - \frac{1}{3}A^3\beta\gamma + A\gamma^2,\\
0 &= -Ba^2 - \lambda A^2 Ba\beta - A^2 B\beta\gamma + B\gamma^2 + 2\mu Ba\alpha\delta^2 + 2B\alpha\gamma\delta^2,\\
0 &= -\lambda AB^2 a\beta - AB^2\beta\gamma,\\
0 &= -\frac{1}{3}\lambda B^3 a\beta - \frac{1}{3}B^3\beta\gamma - 2\mu Ba\alpha\delta^2 - 2B\alpha\gamma\delta^2.
\end{aligned} \tag{24}$$

Solving the above system produces the following two-wave solution

$$W(X,T) = \pm\ \frac{\sqrt{-6\alpha\delta^2\left((-1+\lambda\mu)a + (\lambda-\mu)\gamma\right)}}{\sqrt{\beta\left((-1+\lambda^2)a + 2(-\lambda+\mu)\alpha\delta^2\right)}}\ \tanh\left(\delta\left(X - \gamma T\right)\right), \tag{25}$$

with $\gamma = (-\alpha\delta^2 \pm \sqrt{a^2 - 2\mu a\alpha\delta^2 + \alpha^2\delta^4})$. If the tanh-function is replaced by coth-function in (25), a new solution will be obtained.

4.2 The sine-cosine method

The sine-cosine technique [24, 25, 27–31] assumes the solution of (23) in the form of

$$W(\zeta) = A\sin^B(\delta\zeta), \tag{26}$$

or

$$W(\zeta) = A\cos^B(\delta\zeta), \tag{27}$$

To determine the values of $A,\ B,\ \gamma$ and δ, we substitute (26) in (23) to get

$$\begin{aligned}
0 &= \left(A\mu Ba\alpha\delta^2 - A\mu B^2 a\alpha\delta^2 + AB\alpha\gamma\delta^2 - AB^2\alpha\gamma\delta^2\right)\sin^{B-2}(\delta z)\\
&\quad - \left(Aa^2 + A\gamma^2 + A\mu B^2 a\alpha\delta^2 + AB^2\alpha\gamma\delta^2\right)\sin^B(\delta z) - \left(\frac{1}{3}\lambda A^3 a\beta - \frac{1}{3}A^3\beta\gamma\right)\sin^{3B}(\delta z).
\end{aligned} \tag{28}$$

Now, equating the exponents $B - 2$ and $3B$ in (28) and setting the coefficients of same power to zero, produce the following two-wave solution

$$W(X,T) = \ \frac{\sqrt{-6\alpha\delta^2\left((-1+\lambda\mu)a + (\lambda-\mu)\gamma\right)}}{\sqrt{\beta\left((-1+\lambda^2)a + (-\lambda+\mu)\alpha\delta^2\right)}}\ \csc\left(\delta\left(X - \gamma T\right)\right), \tag{29}$$

with $\gamma = \frac{1}{2}(-\alpha\delta^2 \pm \sqrt{4a^2 - 4\mu a\alpha\delta^2 + \alpha^2\delta^4})$.

Finally, by using the cosine-function method (27), another two-wave solution will be obtained being the same as given in (29) but with csc replaced by sec.

5 Numerical Example

In this section, we study some physical features of the solution of TMmKdV equation given in (25). In Figure 1, increasing the phase velocity a leads to a gradual increase in the space between the two-waves of TMmKdV equation. In Figure 2, decreasing of the nonlinearity parameter λ leads to interaction of the two-waves of the TMmKdV equation.

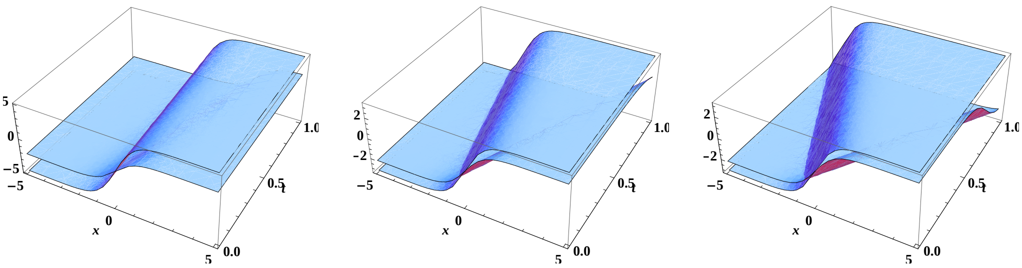

Figure 1: Behaviors of two-waves in (25) at the increasing phase velocity: $a = 1,\ 3,\ 5$ respectively. The assigned values for the other parameters are $\delta = \gamma = 1$, $\lambda = \frac{1}{2}$, $\mu = \frac{1}{4}$, $\alpha = -1$, $\beta = 1$.

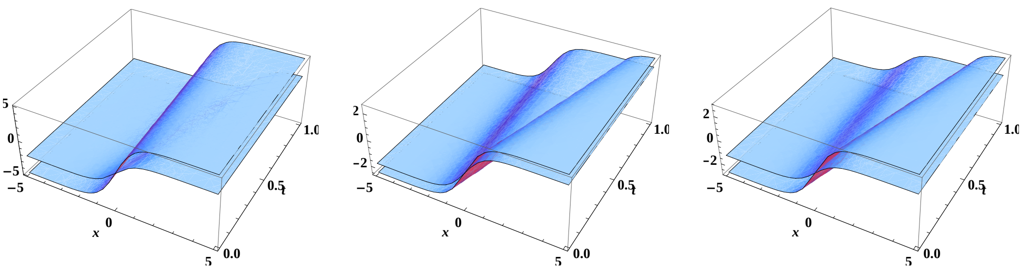

Figure 2: Behaviors of two-waves in (25) at the decreasing nonlinearity parameter: $\lambda = -\frac{1}{2},\ 0,\ \frac{1}{2}$ respectively. The assigned values for the other parameters are $\delta = \gamma = 1$, $s = 1$, $\mu = \frac{1}{4}$, $\alpha = -1$, $\beta = 1$.

6 Conclusion

In this paper we studied the solutions of the scaled TMmKdV equation which reads

$$W_{TT} - a^2 W_{XX} + (\beta\frac{\partial}{\partial T} - \beta\lambda a\frac{\partial}{\partial X})W^2 W_x + (\alpha\frac{\partial}{\partial T} - \alpha\mu a\frac{\partial}{\partial X})W_{XXX}.$$

We used three different methods, the simplified bilinear method, the tanh-technique and the sine-cosine function method. The following findings are observed in this work.

- When $\lambda = \mu = \pm 1$, TMmKdV equation admits multiple-soliton solutions by means of the simplified bilinear method.
- For arbitrary λ and μ, periodic solutions are obtained for TMmKdV equation by using the sine-cosine method.
- For arbitrary λ and μ, kink solutions are obtained for TMmKdV equation by using the tanh-expansion method.

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