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Existence of Renormalized Solutions for Some Strongly Parabolic Problems in Musielak-Orlicz-Sobolev Spaces

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Abstract: In this work, we prove an existence result of renormalized solutions in Musielak-Orlicz-Sobolev spaces for a class of nonlinear parabolic equations with two lower order terms and L^1-data.

Keywords: *parabolic problems, Musielak-Orlicz space, renormalized solutions.*

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1 Introduction

We consider the following nonlinear parabolic problem:

$$(\mathcal{P})\begin{cases}\frac{\partial u}{\partial t} - \operatorname{div}\big(a(x,t,u,\nabla u) + \Phi(u)\big) + g(x,t,u,\nabla u) = f \quad \text{in } Q,\\ u = 0 \quad \text{on } \partial Q = \partial\Omega \times [0,T],\\ u(x,0) = u_0 \quad \text{on} \quad \Omega,\end{cases}$$

where $A(u) = -\operatorname{div}(a(x,t,u,\nabla u))$ is an operator of Leray-Lions type, the lower order term $\Phi \in C^0(\mathbb{R},\mathbb{R}^N)$, g is a nonlinearity term which satisfies the growth and the sign condition and the data f belong to $L^1(Q)$. Under these assumptions the term $\operatorname{div}(\Phi(u))$ may not exist in the distributions sense, since the function $\Phi(u)$ does not belong to $(L^1_{\text{loc}}(Q))^N$.

In the setting of classical Sobolev spaces, the existence of a weak solution for the problem $(\mathcal{P})$ has been proved in [10] in the case of $\Phi \equiv g \equiv 0$. It is well known that this weak solution is not unique in general (see [16] for a counter-example in the stationary case).

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In order to obtain well-posedness for this type of problems the notion of renormalized solution has been introduced by Lions and DiPerna [12] for the study of Boltzmann equation (see also Lions [13] for a few applications to fluid mechanics models). This notion was then adapted to the elliptic version by Boccardo et al. [11]. At the same time, the equivalent notion of entropy solutions has been developed independently by Bénilan et al. [5] for the study of nonlinear elliptic problems.

The existence and uniqueness of a renormalized solution has been proved by D. Blanchard and F. Murat [8] in the case where $a(x,t,s,\xi)$ is independent of s, with $\Phi \equiv 0$ and $g \equiv 0$, by D. Blanchard, F. Murat and H. Redwane [9] with the large monotonicity on a. For measure data, $u = b(x,u)$ and $\Phi \equiv 0$, the existence of renormalized solution for the problem $(\mathcal{P})$ has been proved by Y. Akdim et al.[3] in the framework of weighted Sobolev space, by L. Aharouch, J. Bennouna and A. Touzani [1], and by A. Benkirane and J. Bennouna [6] in the Orlicz spaces and degenerated spaces.

In the Musielak framework, the existence of a weak solution for the problem $(\mathcal{P})$ has been proved by M.L. Ahmed Oubeid, A. Benkirane and M. Sidi El Vally in [2] where $\Phi \equiv 0$, the existence of entropy solutions for the problem $(\mathcal{P})$ has been studied by A. Talha, A. Benkirane and M.S.B. Elemine Vall in [19].

As an example of equations to which the present result can be applied, we give

$$\frac{\partial u}{\partial t} - \operatorname{div}\Big(\frac{m(x,|\nabla u|)}{|\nabla u|}.\nabla u + u|u|^{\sigma}\Big) + \frac{\operatorname{sign}(u)}{1+u^2}\,\varphi(x,|\nabla u|) = f \in L^1(Q),$$

where m is the derivative of φ with respect to t.

2 Preliminaries

2.1 Musielak-Orlicz-Sobolev spaces.

Let Ω be an open set in $\mathbb{R}^N$ and let φ be a real-valued function defined in $\Omega \times \mathbb{R}_+$, and satisfying the following conditions:
a) $\varphi(x,\cdot)$ is an N-function,
b) $\varphi(\cdot,t)$ is a measurable function.

The function φ is called a Musielak–Orlicz function. For a Musielak-orlicz function φ we put $\varphi_x(t) = \varphi(x,t)$ and we associate its nonnegative reciprocal function φ_x^{-1}, with respect to t that is $\varphi_x^{-1}(\varphi(x,t)) = \varphi(x,\varphi_x^{-1}(t)) = t$. The Musielak-orlicz function φ is said to satisfy the Δ_2-condition if for some $k > 0$ and a non negative function h integrable in Ω, we have

$$\varphi(x,2t) \le k\varphi(x,t) + h(x) \text{ for all } x \in \Omega \text{ and } t \ge 0. \tag{1}$$

When (1) holds only for $t \ge t_0 > 0$; then φ is said to satisfy the Δ_2-condition near infinity.

Let φ and γ be two Musielak-orlicz functions. We say that γ grows essentially less rapidly than φ at 0 (resp. near infinity), and we write $\gamma \prec\prec \varphi$, if for every positive constant c we have

$$\lim_{t\longrightarrow 0}\left(\sup_{x\in\Omega}\frac{\gamma(x,ct)}{\varphi(x,t)}\right) = 0, \quad (\text{resp. } \lim_{t\longrightarrow \infty}\left(\sup_{x\in\Omega}\frac{\gamma(x,ct)}{\varphi(x,t)}\right) = 0).$$

We define the functional $\rho_{\varphi,\Omega}(u) = \displaystyle\int_\Omega \varphi(x,|u(x)|)dx$, where $u : \Omega \longrightarrow \mathbb{R}$ is a Lebesgue measurable function.

We define the Musielak-Orlicz space (the generalized Orlicz spaces) by

$$L_\varphi(\Omega) = \Big\{ u : \Omega \longrightarrow \mathbb{R} \ \text{ measurable} \Big/ \rho_{\varphi,\Omega}\Big(\frac{|u(x)|}{\lambda}\Big) < +\infty, \text{ for some } \lambda > 0 \Big\}.$$

For a Musielak-Orlicz function we put: $\psi(x,s) = \sup_{t\geq 0}\{st - \varphi(x,t)\}$. ψ is called the Musielak-Orlicz function complementary to φ in the sense of Young with respect to the variable s. In the space $L_\varphi(\Omega)$ we define the following two norms:

$$\|u\|_{\varphi,\Omega} = \inf\Big\{\lambda > 0/ \int_\Omega \varphi\Big(x, \frac{|u(x)|}{\lambda}\Big)dx \leq 1\Big\},$$

which is called the Luxemburg norm and the so-called Orlicz norm by

$$|||u|||_{\varphi,\Omega} = \sup_{\|v\|_\psi \leq 1} \int_\Omega |u(x)v(x)|dx,$$

where ψ is the Musielak-Orlicz function complementary to φ. These two norms are equivalent [14]. The closure in $L_\varphi(\Omega)$ of the bounded measurable functions with compact support in $\overline{\Omega}$ is denoted by $E_\varphi(\Omega)$.

We say that a sequence of functions $u_n \in L_\varphi(\Omega)$ is modular convergent to $u \in L_\varphi(\Omega)$ if there exists a constant $k > 0$ such that $\lim_{n\to\infty} \rho_{\varphi,\Omega}\Big(\frac{u_n - u}{k}\Big) = 0$.

For any fixed nonnegative integer m we define

$$W^m L_\varphi(\Omega) = \Big\{ u \in L_\varphi(\Omega) : \forall |\alpha| \leq m, \ D^\alpha u \in L_\varphi(\Omega)\Big\}$$

and

$$W^m E_\varphi(\Omega) = \Big\{ u \in E_\varphi(\Omega) : \forall |\alpha| \leq m, \ D^\alpha u \in E_\varphi(\Omega)\Big\},$$

where $\alpha = (\alpha_1, ..., \alpha_n)$ with nonnegative integers α_i, $|\alpha| = |\alpha_1| + ... + |\alpha_n|$ and $D^\alpha u$ denote the distributional derivatives. The space $W^m L_\varphi(\Omega)$ is called the Musielak-Orlicz-Sobolev space. Let

$$\overline{\rho}_{\varphi,\Omega}(u) = \textstyle\sum_{|\alpha|\leq m} \rho_{\varphi,\Omega}\Big(D^\alpha u\Big) \text{ and } \|u\|^m_{\varphi,\Omega} = \inf\Big\{\lambda > 0 : \overline{\rho}_{\varphi,\Omega}\Big(\frac{u}{\lambda}\Big) \leq 1\Big\}.$$

For $u \in W^m L\varphi(\Omega)$ these functionals are a convex modular and a norm on $W^m L_\varphi(\Omega)$, respectively, and the pair $\Big(W^m L\varphi(\Omega), \|\|^m_{\varphi,\Omega}\Big)$ is a Banach space if φ satisfies the following condition [14] :

$$\text{there exists a constant } c > 0 \text{ such that } \inf_{x\in\Omega} \varphi(x,1) \geq c. \tag{2}$$

The space $W^m L_\varphi(\Omega)$ will always be identified with a subspace of the product $\prod_{|\alpha|\leq m} L_\varphi(\Omega) = \Pi L_\varphi$, this subspace is $\sigma(\Pi L_\varphi, \Pi E_\psi)$ closed. We denote by $\mathcal{D}(\Omega)$ the space of infinitely smooth functions with compact support in Ω and by $\mathcal{D}(\overline{\Omega}))$ the restriction of $\mathcal{D}(\mathbb{R}^N)$ on Ω. Let $W^m_0 L_\varphi(\Omega)$ be the $\sigma(\Pi L_\varphi, \Pi E_\psi)$ closure of $\mathcal{D}(\Omega)$ in $W^m L_\varphi(\Omega)$. Let $W^m E_\varphi(\Omega)$ be the space of functions u such that u and its distribution derivatives up to order m lie in $E_\varphi(\Omega)$, and $W^m_0 E_\varphi(\Omega)$ is the (norm) closure of $\mathcal{D}(\Omega)$ in $W^m L_\varphi(\Omega)$.

The following spaces of distributions will also be used:

$$W^{-m} L_\psi(\Omega) = \Big\{ f \in \mathcal{D}'(\Omega); \ f = \textstyle\sum_{|\alpha|\leq m} (-1)^{|\alpha|} D^\alpha f_\alpha \text{ with } f_\alpha \in L_\psi(\Omega)\Big\}$$

and

$$W^{-m}E_\psi(\Omega) = \Big\{ f \in \mathcal{D}'(\Omega);\ f = \textstyle\sum_{|\alpha|\le m} (-1)^{|\alpha|} D^\alpha f_\alpha \text{ with } f_\alpha \in E_\psi(\Omega) \Big\}.$$

We say that a sequence of functions $u_n \in W^m L_\varphi(\Omega)$ is modular convergent to $u \in W^m L_\varphi(\Omega)$ if there exists a constant $k > 0$ such that $\lim_{n\to\infty} \overline{\rho}_{\varphi,\Omega}\Big(\frac{u_n - u}{k}\Big) = 0$.

The inhomogeneous Musielak-Orlicz-Sobolev spaces of order 1 are defined as follows:

$$W^{1,x} L_\varphi(Q) = \Big\{ u \in L_\varphi(Q) : \forall |\alpha| \le 1\ D_x^\alpha u \in L_\varphi(Q) \Big\}$$

and

$$W^{1,x} E_\varphi(Q) = \Big\{ u \in E_\varphi(Q) : \forall |\alpha| \le 1\ D_x^\alpha u \in E_\varphi(Q) \Big\}.$$

The last space is a subspace of the first one, and both are Banach spaces under the norm $\|u\| = \sum_{|\alpha|\le m} \|D_x^\alpha u\|_{\varphi,Q}$. We have the following complementary system

$$\begin{pmatrix} W_0^{1,x} L_\varphi(Q) & F \\ W_0^{1,x} E_\varphi(Q) & F_0 \end{pmatrix},$$

F being the dual space of $W_0^{1,x} E_\varphi(Q)$. It is also, except for an isomorphism, the quotient of ΠL_ψ by the polar set $W_0^{1,x} E_\varphi(Q)^\perp$, and will be denoted by $F = W^{-1,x} L_\psi(Q)$ and it is shown that

$$W^{-1,x} L_\psi(Q) = \Big\{ f = \textstyle\sum_{|\alpha|\le 1} D_x^\alpha f_\alpha : f_\alpha \in L_\psi(Q) \Big\}.$$

This space will be equipped with the usual quotient norm $\|f\| = \inf \sum_{|\alpha|\le 1} \|f_\alpha\|_{\psi,Q}$, where the inf is taken on all possible decompositions $f = \sum_{|\alpha|\le 1} D_x^\alpha f_\alpha, \quad f_\alpha \in L_\psi(Q)$.

The space F_0 is then given by

$$F_0 = \Big\{ f = \textstyle\sum_{|\alpha|\le 1} D_x^\alpha f_\alpha : f_\alpha \in E_\psi(Q) \Big\} = W^{-1,x} E_\psi(Q).$$

Let us give the following lemma which will be needed later.

Lemma 2.1 *[7]. Let Ω be a bounded Lipschitz domain in $\mathbb{R}^N$ and let φ and ψ be two complementary Musielak-Orlicz functions which satisfy the following conditions:*
i) *There exists a constant $c > 0$ such that $\inf_{x\in\Omega} \varphi(x,1) \ge c$,*
ii) *There exists a constant $A > 0$ such that for all $x, y \in \Omega$ with $|x-y| \le \frac{1}{2}$ we have*

$$\frac{\varphi(x,t)}{\varphi(y,t)} \le t^{\left(\frac{A}{log(\frac{1}{|x-y|}}\right)}, \quad \forall t \ge 1. \tag{3}$$

iii)

$$\textit{If } D \subset \Omega \textit{ is a bounded measurable set, then } \int_D \varphi(x,1)dx < \infty. \tag{4}$$

iv) *There exists a constant $C > 0$ such that $\psi(x,1) \le C$ a.e in Ω.*
Under these assumptions, $\mathcal{D}(\Omega)$ is dense in $L_\varphi(\Omega)$ with respect to the modular topology, $\mathcal{D}(\Omega)$ is dense in $W_0^1 L_\varphi(\Omega)$ for the modular convergence and $\mathcal{D}(\overline{\Omega})$ is dense in $W^1 L_\varphi(\Omega)$ for the modular convergence.

Consequently, the action of a distribution S in $W^{-1} L_\psi(\Omega)$ on an element u of $W_0^1 L_\varphi(\Omega)$ is well defined. It will be denoted by $< S, u >$.

Lemma 2.2 (Poincaré inequality) *[18] Let φ be a Musielak-Orlicz function which satisfies the assumptions of Lemma 2.1, suppose that $\varphi(x,t)$ decreases with respect to one* *of coordinates of x. Then, there exists a constant $c > 0$ depending only on Ω such that*

$$\int_\Omega \varphi(x, |u(x)|)\, dx \leq \int_\Omega \varphi(x, c|\nabla u(x)|)\, dx, \quad \forall u \in W_0^1 L_\varphi(\Omega). \tag{5}$$

3 Assumptions and Main Result

Let Ω be a bounded open set on $\mathbb{R}^N$ satisfying the segment property and $T > 0$, we denote $Q = \Omega \times [0, T]$, and let φ and γ be two Musielak-Orlicz functions such that $\gamma \prec\prec \varphi$ and φ satisfies the conditions of Lemma 2.2. Let $A : D(A) \subset W_0^{1,x} L_\varphi(Q) \longrightarrow W^{-1,x} L_\psi(Q)$ be a mapping given by $A(u) = -\mathrm{div}(a(x, t, u, \nabla u))$, where $a : a(x, t, s, \xi) : \Omega \times [0, t] \times \mathbb{R} \times \mathbb{R}^N \longrightarrow \mathbb{R}^N$ is a Carathéodory function satisfying, for a.e $(x, t) \in Q$ and for all $s \in \mathbb{R}$ and all $\xi, \xi' \in \mathbb{R}^N$, $\xi \neq \xi'$,

$$|a(x, t, s, \xi)| \leq \beta\Big(c(x, t) + \psi_x^{-1} \varphi(x, \nu|\xi|)\Big), \tag{6}$$

$$\Big(a(x, t, s, \xi) - a(x, t, s, \xi')\Big)(\xi - \xi') > 0, \tag{7}$$

$$a(x, t, s, \xi).\xi \geq \alpha \varphi(x, |\xi|), \tag{8}$$

where $c(x, t)$ is a positive function, $c(x, t) \in E_\psi(Q)$ and $\beta, \nu, \alpha \in \mathbb{R}_+^*$.
Let $g : \Omega \times [0, t] \times \mathbb{R} \times \mathbb{R}^N \longrightarrow \mathbb{R}^N$ be a Caratheodory function satisfying for a.e. $(x, t) \in \Omega \times [0, t]$ and $\forall s \in \mathbb{R}$, $\xi \in \mathbb{R}^N$,

$$|g(x, t, s, \xi)| \leq b(|s|)(c_2(x, t) + \varphi(x, |\xi|)), \tag{9}$$

$$g(x, t, s, \xi)s \geq 0, \tag{10}$$

where $c_2(x, t) \in L^1(Q)$ and $b : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is a continuous and nondecreasing function. Furthermore, let

$$\Phi \in C^0(\mathbb{R}, \mathbb{R}^N), \tag{11}$$

$$f \in L^1(Q) \text{ and } u_0 \text{ is an element of } L^1(Q). \tag{12}$$

For $\ell > 0$ we define the truncation at height ℓ: $T_\ell : \mathbb{R} \longrightarrow \mathbb{R}$ by

$$T_\ell(s) = \begin{cases} s & \text{if } |s| \leq \ell, \\ \ell \frac{s}{|s|} & \text{if } |s| > \ell. \end{cases} \tag{13}$$

The definition of a renormalized solution for problem $(\mathcal{P})$ can be stated as follows.

Definition 3.1 A measurable function u defined on Q is a renormalized solution of Problem $(\mathcal{P})$ if

$$T_\ell(u) \in W_0^{1,x} L_\varphi(Q), \tag{14}$$

$$\int_{\{(x,t)\in Q; m \leq |u(x,t)| \leq m+1\}} a(x, t, u, \nabla u) \cdot \nabla u\, dxdt \longrightarrow 0 \text{ as } m \longrightarrow \infty, \tag{15}$$

and if, for every function S in $W^{2,\infty}(\mathbb{R})$ which is piecewise C^1 and such that S' has a compact support, we have

$$\begin{aligned} &\frac{\partial S(u)}{\partial t} - \mathrm{div}\, (a(x, t, u, \nabla u) S'(u)) + S''(u) a(x, t, u, \nabla u) \cdot \nabla u \\ &- \mathrm{div}\, (\Phi(u) S'(u)) + S''(u) \Phi(u) \cdot \nabla u + g(x, t, u, \nabla u) S'(u) = f S'(u) \quad \text{in } \mathcal{D}'(Q), \end{aligned}$$

$$S(u)(t = 0) = S(u_0) \text{ in } \Omega. \tag{16}$$

We will prove the following existence theorem.

Theorem 3.1 *Assume that (6) to (11) hold true. Then, there exists a renormalized solution u of problem $(\mathcal{P})$ in the sense of Definition 3.1.*

Proof. The proof of Theorem 3.1 is divided into five steps.

Step 1: Approximate problem. Let consider us the following approximate problem

$$(\mathcal{P}_n)\begin{cases}\frac{\partial u_n}{\partial t}-\operatorname{div}\Big(a(x,t,u_n,\nabla u_n)+\Phi_n(u_n)\Big)+g_n(x,t,u_n,\nabla u_n)=f_n \quad \text{in } \mathcal{D}'(Q),\\ u_n=0 \quad \text{on} \quad \partial\Omega\times(0,T),\\ u_n(t=0)=u_{0n} \quad \text{on } \Omega,\end{cases}$$

where $(f_n)\in L^1(Q)$ is a sequence of smooth functions such that $f_nf_n\to f$ in $L^1(Q)f$ in $L^1(Q)$, $\Phi_n(s)=\Phi(T_n(s))$ and $g_n(x,t,s,\xi)=T_n(g(x,t,s,\xi))$. Note that $g_n(x,t,s,\xi)s\geq 0$, $|g_n(x,t,s,\xi)|\leq|g(x,t,s,\xi)|$ and $|g_n(x,t,s,\xi)|\leq n$. Since Φ is continuous, we have $\Phi(T_n(s))\leq c_n$, then the problem $(\mathcal{P}_n)$ has at least one solution $u_n\in W_0^{1,x}L_\varphi(Q)$ (see e.g. [2]).

Step 2: A priori estimates. We take $T_\ell(u_n)\chi_{(0,\tau)}$ as a test function in $(\mathcal{P}_n)$, we get for every $\tau\in(0,T)$

$$\int_\Omega \widehat{T}_\ell(u_n(\tau))\,dx+\int_{Q_\tau}a(x,t,T_\ell(u_n),\nabla T_\ell(u_n))\cdot\nabla T_\ell(u_n)\,dxdt+\int_{Q_\tau}\Phi_n(u_n)\cdot\nabla T_\ell(u_n)dxdt$$

$$=\int_{Q_\tau}f_nT_\ell(u_n)\,dxdt-\int_{Q_\tau}g_n(x,t,u_n,\nabla u_n)T_\ell(u_n)\,dxdt+\int_\Omega\widehat{T}_\ell(u_{0n})\,dx, \tag{17}$$

where

$$\widehat{T}_\ell(s)=\int_0^s T_\ell(\sigma)d\sigma=\begin{cases}\frac{s^2}{2}, & \text{if } |s|\leq\ell,\\ \ell|s|-\frac{s^2}{2}, & \text{if } |s|>\ell.\end{cases} \tag{18}$$

The Lipshitz character of Φ_n and the Stokes formula together with the boundary condition $u_n=0$ on $(0,T)\times\partial\Omega$ make it possible to obtain

$$\int_{Q_\tau}\Phi_n(u_n)\cdot\nabla T_\ell(u_n)\,dxdt=0. \tag{19}$$

Due to the definition of $\widehat{T}_\ell$ and (12) we have

$$0\leq\int_\Omega\widehat{T}_\ell(u_{0n})\,dx\leq\ell\int_\Omega|u_{0n}|\,dx\leq\ell||u_0||_{L^1(\Omega)}. \tag{20}$$

Using the same argument as in [15], we can see that

$$\int_Q g_n(x,t,u_n,\nabla u_n)\,dxdt\leq C_g. \tag{21}$$

Here and below C_i denotes positive constants not depending on n and ℓ. By using (12), (19), (20), (21) we can deduce from (17) that

$$\int_\Omega\widehat{T}_\ell(u_n(\tau))\,dx+\int_{Q_\tau}a(x,t,T_\ell(u_n),\nabla T_\ell(u_n))\cdot\nabla T_\ell(u_n)\,dxdt\leq\ell C_0. \tag{22}$$

By using (22), (7) and the fact that $\widehat{T}_\ell(u_n) \geq 0$, we deduce that

$$\int_{Q_\tau} \varphi(x, |\nabla T_\ell(u_n)|)\, dxdt \leq \frac{1}{\alpha} \int_{Q_\tau} a(x,t,T_\ell(u_n), \nabla T_\ell(u_n)) \cdot \nabla T_\ell(u_n)\, dxdt \leq \ell C_1, \quad (23)$$

we deduce from the above inequality (22) that

$$\int_\Omega \widehat{T}_\ell(u_n(\tau))\, dx \leq \ell C_0, \text{ for almost any } \tau \text{ in } (0,T). \quad (24)$$

On the other hand, thanks to Lemma 2.2, there exists a constant $\lambda > 0$ depending only on Ω such that

$$\int_{Q_\tau} \varphi(x, |v|)\, dxdt \leq \int_{Q_\tau} \varphi(x, \lambda |\nabla v|)\, dxdt, \quad \forall v \in W_0^1 L_\varphi(\Omega). \quad (25)$$

Taking $v = \frac{T_\ell(u_n)}{\lambda}$ in (25) and using (23), one has

$$\int_{Q_\tau} \varphi(x, \frac{|T_\ell(u_n)|}{\lambda})\, dxdt \leq \ell C_1. \quad (26)$$

Then we deduce by using (26), that

$$\begin{aligned} meas\{|u_n| > \ell\} &\leq \frac{1}{\inf\limits_{x\in\Omega} \varphi(x, \frac{\ell}{\lambda})} \int_{Q_\tau} \varphi(x, \frac{1}{\lambda}|T_\ell(u_n)|)\, dxdt \\ &\leq \frac{C_1 \ell}{\inf\limits_{x\in\Omega} \varphi(x, \frac{\ell}{\lambda})} \quad \forall n, \quad \forall \ell \geq 0. \end{aligned} \quad (27)$$

By using the definition of φ, we can deduce

$$\lim_{\ell \longrightarrow \infty} (\text{meas}\{(x,t) \in Q_\tau : |u_n| > \ell\}) = 0 \quad (28)$$

uniformly with respect to n. Moreover, we have from (26) that $T_\ell(u_n)$ is bounded in $W_0^{1,x} L_\varphi(Q)$ for every $\ell > 0$. Consider now in $C^2(\mathbb{R})$ a nondecreasing function $\zeta_\ell(s) = s$ for $|s| \leq \frac{\ell}{2}$ and $\zeta_\ell(s) = \ell$ sign (s). Multiplying the approximating equation by $\zeta'_\ell(u_n)$, we obtain

$$\begin{aligned} &\frac{\partial(\zeta_\ell(u_n))}{\partial t} = \text{div}\, \big(a(x,t,u_n,\nabla u_n)\zeta'_\ell(u_n)\big) - \zeta''_\ell(u_n) a(x,t,u_n,\nabla u_n) \cdot \nabla u_n \\ &+ \text{div}\, \big(\Phi_n(u_n)\zeta'_\ell(u_n)\big) - \zeta''_\ell(u_n)\Phi_n(u_n) \cdot \nabla u_n - g_n(x,t,u_n,\nabla u_n)\zeta'_\ell(u_n) + f_n \zeta'_\ell(u_n) \end{aligned}$$

in the sense of distributions. Thanks to (26) and the fact that ζ'_ℓ has a compact support, $\zeta'_\ell(u_n)$ is bounded in $W_0^{1,x} L_\varphi(Q)$ while its time derivative $\frac{\partial(\zeta_\ell(u_n))}{\partial t}$ is bounded in $W_0^{-1,x} L_\varphi(Q) + L^1(Q)$, hence Corollary 4.5 of [2] allows us to conclude that $\zeta_\ell(u_n)$ is compact in $L^1(Q)$. Due to the choice of ζ_ℓ, we conclude that for each ℓ, the sequence $T_\ell(u_n)$ converges almost everywhere in Q. Therefore, following [8,9,15], we can see that there exists a measurable function $u \in L^\infty(0,T; L^1(\Omega))$ such that for every $\ell > 0$ and a subsequence, not relabeled,

$$u_n \to u \text{ a. e. in } Q, \quad (29)$$

and

$$T_\ell(u_n) \rightharpoonup T_\ell(u) \text{ weakly in } W_0^{1,x}L_\varphi(Q) \text{ for } \sigma(\Pi L_\varphi, \Pi E_\psi), \qquad (30)$$
$$\text{strongly in } L^1(Q) \text{ and a. e. in } Q.$$

Now we shall to prove the boundness of $(a(x,t,T_\ell(u_n),\nabla T_\ell(u_n)))_n$ in $(L_\psi(Q))^N$.
Let $\phi \in (E_\varphi(Q))^N$ with $||\phi||_{\varphi,Q} = 1$. In view of the monotonicity of a one easily has,

$$\int_Q \big[a(x,t,T_\ell(u_n),\nabla T_\ell(u_n)) - a(x,t,T_\ell(u_n),\phi)\big]\big[\nabla T_\ell(u_n) - \phi\big]\, dxdt \geq 0, \qquad (31)$$

which gives

$$\begin{aligned}\int_Q a(x,t,T_\ell(u_n),\nabla T_\ell(u_n))\cdot\phi\, dxdt \leq& \int_Q a(x,t,T_\ell(u_n),\nabla T_\ell(u_n))\cdot\nabla T_\ell(u_n)\, dxdt \\ &+ \int_Q a(x,t,T_\ell(u_n),\phi)\cdot\Big[\nabla T_\ell(u_n) - \phi\Big]\, dxdt.\end{aligned} \qquad (32)$$

Using (6) and (23), we easily see that

$$\int_Q a(x,t,T_\ell(u_n),\nabla T_\ell(u_n))\cdot\phi\, dxdt \leq C_3. \qquad (33)$$

And so, we conclude that $(a(x,t,T_\ell(u_n),\nabla T_\ell(u_n)))_n$ is a bounded sequence in $(L_\psi(Q))^N$.
Now, we prove that

$$\lim_{m\longrightarrow\infty}\lim_{n\longrightarrow\infty}\int_{\left\{m\leq|u_n|\leq m+1\right\}} a(x,t,u_n,\nabla u_n)\cdot\nabla u_n\, dxdt = 0. \qquad (34)$$

Using in $(\mathcal{P}_n)$ the test function $v = T_1(u_n - T_m(u_n))$, we obtain

$$\begin{aligned}&\langle\frac{\partial u_n}{\partial t}, v\rangle + \int_{\{m\leq u_n\leq m+1\}} a(x,t,u_n,\nabla u_n)\cdot\nabla u_n\, dxdt + \int_Q g_n(x,t,u_n,\nabla u_n)v\, dxdt \\ &+ \int_Q \mathrm{div}\Big[\int_0^{u_n}\Phi_n(r)T_1'(u_n - T_m(u_n))dr\Big]\, dxdt = \int_Q f_n v\, dxdt.\end{aligned} \qquad (35)$$

By using $\int_0^{u_n}\Phi_n(r)T_1'(u_n - T_m(u_n))dr \in W_0^{1,x}L_\varphi(Q)$ and the Stokes formula, we get

$$\begin{aligned}&\int_\Omega U_n^m(u_n(T))\, dx + \int_{\{m\leq u_n\leq m+1\}} a(x,t,u_n,\nabla u_n)\cdot\nabla u_n\, dxdt \\ &\leq \int_Q (|f_n + g_n(x,t,u_n,\nabla u_n)|)|T_1(u_n - T_m(u_n))|\, dxdt + \int_\Omega U_n^m(x,u_{0n})\, dx,\end{aligned} \qquad (36)$$

where $U_n^m(r) = \int_0^{u_n}\frac{\partial u_n}{\partial t}T_1(s - T_m(s))ds$. In order to pass to the limit as n tends to $+\infty$ in (36), we use $U_n^m(u_n(T)) \geq 0$, (12) and (21), we obtain that

$$\begin{aligned}&\lim_{n\longrightarrow\infty}\int_{\{m\leq u_n\leq m+1\}} a(x,t,u_n,\nabla u_n)\cdot\nabla u_n\, dxdt \\ &\qquad\leq \int_{\{|u_n|>m\}}(|f| + C_g)\, dxdt + \int_{\{|u_0|>m\}}|u_0|\, dx.\end{aligned} \qquad (37)$$

Finally, by(12) and (37) we obtain (34).

Step 3: Almost everywhere convergence of the gradients. Fix $\ell > 0$ and let $\phi(s) = s\exp(\delta s^2), \delta > 0$. It is well known that when $\delta \geq (\frac{b(\ell)}{2\alpha})^2$ one has

$$\phi'(s) - \frac{b(\ell)}{\alpha}|\phi(s)| \geq \frac{1}{2} \text{ for all } s \in \mathbb{R}. \tag{38}$$

Let $v_j \in \mathcal{D}(Q)$ be a sequence which converges to u for the modular convergence in $W_0^{1,x}L_\varphi(Q)$ and let $\omega_i \in \mathcal{D}(Q)$ be a sequence which converges strongly to u_0 in $L^2(\Omega)$. Set $\omega^\mu_{i,j} = T_\ell(v_j)_\mu + \exp(-\mu t)T_\ell(w_i)$, where $T_\ell(v_j)_\mu$ is the mollification with respect to time of $T_\ell(v_j)$. Note that $\omega^\mu_{i,j}$ is a smooth function having the following properties:

$$\frac{\partial}{\partial t}(\omega^\mu_{i,j}) = \mu(T_\ell(v_j) - \omega^\mu_{i,j}), \omega^\mu_{i,j}(0) = T_\ell(\omega_i), |\omega^\mu_{i,j}| \leq \ell, \tag{39}$$

$$\omega^\mu_{i,j} \to T_\ell(u)_\mu + \exp(-\mu t)T_\ell(w_i) \text{ in } W_0^{1,x}L_\varphi(Q) \tag{40}$$

for the modular convergence as $j \to \infty$,

$$T_\ell(u)_\mu + \exp(-\mu t)T_\ell(w_i) \to T_\ell(u) \text{ in } W_0^{1,x}L_\varphi(Q) \tag{41}$$

for the modular convergence as $\mu \to \infty$. Let now the function ρ_m on $\mathbb{R}$ with $m \geq \ell$ be defined by

$$\rho_m(s) = \begin{cases} 1, & \text{if } |s| \leq m, \\ m+1-|s|, & \text{if } m \leq |s| \leq m+1, \\ 0, & \text{if } |s| \geq m+1. \end{cases} \tag{42}$$

We set $\theta^{\mu,n}_{i,j} = T_\ell(u_n) - \omega^\mu_{i,j}$. Using the admissible test function $Z^{\mu,m}_{i,j,n} = \phi(\theta^{\mu,n}_{i,j})\rho_m(u_n)$ as test function in $(\mathcal{P}_n)$ and since $g_n(x,t,u_n,\nabla u_n)\phi(\theta^{\mu,n}_{i,j})\rho_m(u_n) \geq 0$ on $\{|u_n| > \ell\}$, we arrive at

$$\begin{aligned} &\langle \frac{\partial u_n}{\partial t}, Z^{\mu,m}_{i,j,n}\rangle + \int_Q a(x,t,u_n,\nabla u_n)\cdot(\nabla T_\ell(u_n) - \nabla\omega^\mu_{i,j})\phi'(\theta^{\mu,n}_{i,j})\rho_m(u_n)\,dxdt \\ &+ \int_{\{m\leq|u_n|\leq m+1\}} a(x,t,u_n,\nabla u_n)\cdot\nabla u_n\phi(\theta^{\mu,n}_{i,j})\rho'_m(u_n)\,dxdt \\ &+ \int_{\{m\leq|u_n|\leq m+1\}} \Phi_n(u_n)\cdot\nabla u_n\phi(\theta^{\mu,n}_{i,j})\rho'_m(u_n)\,dxdt \\ &+ \int_Q \Phi_n(u_n)\cdot(\nabla T_\ell(u_n) - \nabla\omega^\mu_{i,j})\phi'(\theta^{\mu,n}_{i,j})\rho_m(u_n)\,dxdt \\ &+ \int_{\{|u_n|\leq\ell\}} g_n(x,t,u_n,\nabla u_n)\phi(\theta^{\mu,n}_{i,j})\rho_m(u_n)\,dxdt \leq \int_Q f_n Z^{\mu,m}_{i,j,n}\,dxdt. \end{aligned} \tag{43}$$

Denote by $\epsilon(n,j,\mu,i)$ any quantity such that $\lim\limits_{i\to\infty}\lim\limits_{\mu\to\infty}\lim\limits_{j\to\infty}\lim\limits_{n\to\infty}\epsilon(n,j,\mu,i) = 0$.

The very definition of the sequence $\omega^\mu_{i,j}$ makes it possible to establish the following lemma.

Lemma 3.1 *(cf.[2]) Let $Z^{\mu,m}_{i,j,n} = \phi(\theta^{\mu,n}_{i,j})\rho_m(u_n)$, we have for any $\ell \geq 0$*

$$\langle \frac{\partial u_n}{\partial t}, Z^{\mu,m}_{i,j,n}\rangle \geq \epsilon(n,j,i). \tag{44}$$

Concerning the right-hand of (43), by the almost everywhere convergence of u_n, we have $\phi(T_\ell(u_n)-\omega_{i,j}^\mu)\rho_m(u_n) \rightharpoonup \phi(T_\ell(u)-\omega_{i,j}^\mu)\rho_m(u)$ weakly-* in $L^\infty(Q)$ as $n\to\infty$, and then

$$\int_Q f_n\phi(T_\ell(u_n)-\omega_{i,j}^\mu)\rho_m(u_n)\,dxdt \to \int_Q f\phi(T_\ell(u)-\omega_{i,j}^\mu)\rho_m(n)\,dxdt,$$

so that $\phi(T_\ell(u)-\omega_{i,j}^\mu)\rho_m(u) \rightharpoonup \phi(T_\ell(u)-T_\ell(u)_\mu-\exp(-\mu t)T_\ell(w_i))\rho_m(u)$ weakly star in $L^\infty(Q)$ as $j\to\infty$, and finally,

$$\phi(T_\ell(u)-T_\ell(u)_\mu-\exp(-\mu t)T_\ell(w_i))\rho_m(u) \rightharpoonup 0 \text{ weakly star as } \mu\to\infty.$$

Then, we deduce that

$$\langle f_n,\phi(T_\ell(u_n)-\omega_{i,j}^\mu)\rho_m(u_n)\rangle = \epsilon(n,j,\mu). \tag{45}$$

Similarly, Lebesgue's convergence theorem shows that

$$\Phi_n(u_n)\rho_m(u_n)\to\Phi(u)\rho_m(u) \text{ strongly in } (E_\psi(Q)^N) \text{ as } n\to\infty,$$

and

$$\Phi_n(u_n)\chi_{\{m\le|u_n|\le m+1\}}\phi'(T_\ell(u_n)-\omega_{i,j}^\mu)\to\Phi(u)\chi_{\{m\le u\le m+1\}}\phi'(T_\ell(u)-\omega_{i,j}^\mu)$$

strongly in $(E_\psi(Q)^N)$. Then by virtue of $\nabla T_\ell(u_n)\rightharpoonup\nabla T_\ell(u)$ weakly star in $(L_\varphi(Q)^N)$, and $\nabla u_n\chi_{\{m\le|u_n|\le m+1\}}=\nabla T_{m+1}(u_n)\chi_{\{m\le|u_n|\le m+1\}}$ a. e. in Q, one has

$$\int_Q\Phi_n(u_n)\cdot(\nabla T_\ell(u_n)-\nabla\omega_{i,j}^\mu)\phi'(T_\ell(u_n)-\omega_{i,j}^\mu)\rho_m(u_n)\,dxdt$$

$$\to\int_Q\Phi(u)\nabla(\nabla T_\ell(u)-\nabla\omega_{i,j}^\mu)\phi'(T_\ell(u)-\omega_{i,j}^\mu)\rho_m(u)\,dxdt$$

as $n\to\infty$, and

$$\int_{\{m\le|u_n|\le m+1\}}\Phi_n(u_n)\phi(T_\ell(u_n)-\omega_{i,j}^\mu)\nabla u_n\rho_m'(u_n)\,dxdt$$

$$\to\int_{\{m\le|u_n|\le m+1\}}\Phi(u)\phi(T_\ell(u_n)-\omega_{i,j}^\mu)\nabla u\rho_m'(u)\,dxdt$$

as $n\to+\infty$. Thus, by using the modular convergence of $\omega_{i,j}^\mu$ as $j\to+\infty$ and letting μ tend to infinity, we get

$$\int_Q\Phi_n(u_n)\cdot(\nabla T_\ell(u_n)-\nabla\omega_{i,j}^\mu)\phi'(\theta_{i,j}^{\mu,n})\rho_m(u_n)\,dxdt=\epsilon(n,j,\mu) \tag{46}$$

and

$$\int_{\{m\le|u_n|\le m+1\}}\Phi_n(u_n)\cdot\nabla u_n\phi(\theta_{i,j}^{\mu,n})\rho_m'(u_n)\,dxdt=\epsilon(n,j,\mu). \tag{47}$$

Concerning the third term of the right-hand side of (43) we obtain that

$$\left|\int_{\{m\le|u_n|\le m+1\}}a(x,t,u_n,\nabla u_n)\cdot\nabla u_n\phi(\theta_{n,j}^{\mu,i})\rho_m'(u_n)\,dxdt\right|$$

$$\le\phi(2k)\int_{\{m\le|u_n|\le m+1\}}a(x,t,u_n,\nabla u_n)\cdot\nabla u_n\,dxdt.$$

Then by (34) we deduce that

$$|\int_Q a(x,t,u_n,\nabla u_n)\cdot\nabla u_n\phi(\theta_{n,j}^{\mu,i})\rho'_m(u_n)\,dxdt|\le\epsilon(n,\mu,m). \tag{48}$$

Using the same technics as in the proof of Proposition 5.6 in [4], we obtain

$$\begin{aligned}&\int_Q\Big(a(x,t,T_k(u_n),\nabla T_k(u_n))-a(x,t,T_k(u_n),\nabla T_k(u)\chi^s)\Big)\\&\times\Big(\nabla T_k(u_n)-\nabla T_k(u)\chi^s\Big)\,dxdt\le\varepsilon(n,j,\mu,i,s,m).\end{aligned} \tag{49}$$

To pass to the limit in (49) as n j, m, s tend to infinity, we obtain

$$\begin{aligned}&\lim_{s\to\infty}\lim_{n\to\infty}\int_Q\Big(a(x,t,T_k(u_n),\nabla T_k(u_n))-a(x,t,T_k(u_n),\nabla T_k(u)\chi^s)\Big)\\&\times\Big(\nabla T_k(u_n)-\nabla T_k(u)\chi^s\Big)\,dxdt=0.\end{aligned} \tag{50}$$

And thus, as in the elliptic case (see [18]), there exists a subsequence also denoted by u_n such that

$$\nabla u_n\to\nabla u \text{ a.e. in } Q. \tag{51}$$

Then, for all $k>0$, one has

$$\begin{aligned}&a(x,t,T_k(u_n),\nabla T_k(u_n))\rightharpoonup a(x,t,T_k(u),\nabla T_k(u))\\&\qquad\text{weakly star in } (L_\psi(Q))^N \text{ for } \sigma(\Pi L_\psi,\Pi E_\varphi).\end{aligned} \tag{52}$$

Step 4: In this step we prove that** u **satisfies (15). According to (50), one can pass to the limit as n tends to $+\infty$ for fixed $m\ge0$ to obtain

$$\begin{aligned}&\lim_{n\longrightarrow\infty}\int_{\left\{m\le|u_n|\le m+1\right\}}a(x,t,u_n,\nabla u_n)\nabla u_n\,dxdt\\&=\int_Q a(x,t,T_{m+1}(u),\nabla T_{m+1}(u))\nabla T_{m+1}(u)\,dxdt\\&\quad-\int_Q a(x,t,T_m(u),\nabla T_m(u))\nabla T_m(u)\,dxdt\\&\quad=\int_{\left\{m\le|u|\le m+1\right\}}a(x,t,u,\nabla u)\cdot\nabla u\,dxdt.\end{aligned} \tag{53}$$

Taking the limit as $m\to+\infty$ in (53) and using the estimate (34) show that u satisfies (15). Following the same technique as that used in [2], and by using (29), (50) and Vitali's theorem, we have

$$g_n(x,t,u_n,\nabla u_n)\to g(x,t,u,\nabla u) \text{ strongly in } L^1(Q). \tag{54}$$

Step 5 : Passing to the limit. Let S be a function in $W^{2,\infty}(\mathbb{R})$ such that S' has a compact support. Let K be a positive real number such that $\mathrm{supp}(S') \subset [-K, K]$. Pointwise multiplication of the approximate equation $(\mathcal{P}_n)$ by $S'(u_n)$ leads to

$$\begin{aligned}\frac{\partial S(u_n)}{\partial t} &- \operatorname{div}\Big(a(x,t,u_n,\nabla u_n)S'(u_n)\Big) + S''(u_n)a(x,t,u_n,\nabla u_n)\cdot\nabla u_n \\ &- \operatorname{div}\Big(S'(u_n)\Phi(u_n)\Big) + S''(u_n)\Phi(u_n)\cdot\nabla u_n \\ &+ g_n(x,t,u_n,\nabla u_n)S'(u_n) \\ &= f_nS'(u_n). \end{aligned} \tag{55}$$

In what follows we pass to the limit as n tends to $+\infty$ in each term of (55).
• Since S is bounded and continuous, then the fact that $u_n \longrightarrow u$ a.e. in Q, implies that $S(u_n)$ converges to $S(u)$ a.e. in Q and L^∞ weakly-$*$. Consequently,

$$\frac{\partial S(u_n)}{\partial t} \longrightarrow \frac{\partial S(u)}{\partial t} \quad \text{in } \mathcal{D}'(Q) \text{ as } n \text{ tends to } +\infty.$$

• Since $\mathrm{supp}(S') \subset [-K, K]$, we have for $n \geq K$,

$$a(x,t,u_n,\nabla u_n)S'(u_n) = a(x,t,T_K(u_n),\nabla T_K(u_n))S'(u_n) \quad \text{a.e. in } Q.$$

The pointwise convergence of u_n to u and (52) as n tends to ∞ and the bounded character of S' permit us to conclude that

$$a(x,t,T_K(u_n),\nabla T_K(u_n))S'(u_n) \longrightarrow a(x,t,T_K(u),\nabla T_K(u))S'(u) \text{ weakly star in } (L_\psi(Q))^N \tag{56}$$

as n tends to infinity.
• Regarding the 'energy' term, we have for $n \geq K$

$$S''(u_n)a(x,t,u_n,\nabla u_n)\cdot\nabla u_n = S''(u_n)a(x,t,T_K(u_n),\nabla T_K(u_n))\cdot\nabla T_K(u_n) \text{ a.e. in } Q.$$

The pointwise convergence of $S'(u_n) \longrightarrow S'(u)$ and (52) as n tends to $+\infty$ and the bounded character of S'' permit us to conclude that

$$S''(u_n)a(x,t,u_n,\nabla u_n)\cdot\nabla u_n \rightharpoonup S''(u)a(x,t,T_K(u),\nabla T_K(u))\cdot\nabla T_K(u) \text{ weakly star in } L^1(Q). \tag{57}$$

Recall that $S''(u)a(x,t,T_K(u),\nabla T_K(u))\cdot\nabla T_K(u) = S''(u)a(x,t,u,\nabla u)\cdot\nabla u$ a.e. in Q.
• Since $\mathrm{supp}(S') \subset [-K, K]$, we have

$$S'(u_n)\Phi_n(u_n) = S'(u_n)\Phi_n(T_K(u_n)) \quad \text{a.e. in } Q. \tag{58}$$

As a consequence of (11) and (29), it follows that

$$S'(u_n)\Phi_n(u_n) \to S'(u)\Phi(T_K(u)) \quad \text{a.e. in } (E_\varphi(Q))^N, \tag{59}$$

we have $\nabla S''(u_n)$ converges to $\nabla S''(u)$ weakly in $(L_\varphi(Q))^N$ as n tends to $+\infty$, while $\Phi_n(T_K(u_n))$ is uniformly bounded with respect to n and converges a. e. in Q to $\Phi(T_K(u))$ as n tends to $+\infty$. Therefore

$$S''(u_n)\Phi_n(u_n)\nabla u_n \rightharpoonup S''(u)\Phi(u)\nabla u \quad \text{weakly in } L_\varphi(Q), \tag{60}$$

• Since $suppS' \subset [-K, K]$ and from (54), we have

$$S'(u_n)g_n(x,t,u_n,\nabla u_n) \longrightarrow g(x,t,u,\nabla u)S'(u) \quad \text{strongly in } L^1(Q). \tag{61}$$

• Due to $f_n \longrightarrow f$ in $L^1(Q)$ and the fact that $u_n \longrightarrow u$ a.e. in Q, we have

$$S'(u_n)f_n \longrightarrow S'(u)f \quad \text{strongly in } L^1(Q). \tag{62}$$

As a consequence of the above convergence results, we are in a position to pass to the limit as n tends to $+\infty$ in equation (55) and to conclude that

$$\begin{aligned}\frac{\partial S(u)}{\partial t} \quad &- \quad \operatorname{div}\Big(a(x,t,u,\nabla u)S'(u)\Big) + S''(u)a(x,t,u,\nabla u)\cdot\nabla u \\ &- \quad \operatorname{div}\Big(S'(u)\Phi(u)\Big) + S''(u)\Phi(u)\cdot\nabla u \\ &+ \quad g(x,t,u,\nabla u)S'(u) \\ &= \quad fS'(u).\end{aligned} \tag{63}$$

It remains to show that $S(u)$ satisfies the initial condition.

To this end, firstly note that, S being bounded, $S(u_n)$ is bounded in $L^\infty(Q)$. Secondly, (55) and the above considerations on the behavior of the terms of this equation show that $\frac{\partial S(u_n)}{\partial t}$ is bounded in $L^1(Q) + V^*$. As a consequence, an Aubin's type lemma (see, e.g, [17]) implies that $S(u_n)$ lies in a compact set of $C^0([0,T], L^1(\Omega))$. It follows that, on the one hand, $S(u_n)(t=0) = S(u_{0n})$ converges to $S(u)(t=0)$ strongly in $L^1(\Omega)$.

On the other hand, the smoothness of S implies that

$$S(u)(t=0) = S(u_0) \quad \text{in } \Omega.$$

As a conclusion of step 1 to step 6, the proof of Theorem 3.1 is complete.

Example 3.1 Let Ω be a bounded Lipschitz domain of $\mathbb{R}^N$ and $T > 0$, we denote by $Q = \Omega \times [0,T]$, and let φ and ψ be two complementary Musielak functions. Moreover, we assume that $\varphi(x,t)$ decreases with respect to one of coordinates of x (for example, $\varphi(x,t) = |t|^{p(x)}\log(1+t^3)$, $p(x) = e^{(-x_1^2+x_2^2+\cdots+x_N^2)}$. We set

$$a(x,t,s,\zeta) = (3+\cos^2(\varphi(x,s)))\psi_x^{-1}(\varphi(x,|\zeta|))\frac{\zeta}{|\zeta|},$$

$$g(x,t,s,\zeta) = \tfrac{\varphi(x,|\zeta|)}{1+s^2},\ \Phi(s) = (|s|^{r_1-1}s, ..., |s|^{r_N-1}s), \quad 1 \le r_1, ..., R_N < \infty.$$

It is easy to show that $a(x,t,s,\zeta)$ is the Caratheodory function satisfying the growth condition (6), the coercivity (8) and the monotonicity condition, while the Caratheodory function $g(x,t,s,\zeta)$ satisfies the condition (9) and (10), Finally, the hypotheses of Theorem 3.1 are satisfied. Therefore, the following problem

$$\begin{cases} \lim\limits_{m\longrightarrow\infty}\displaystyle\int_{\{(x,t)\in Q; m\le|u(x,t)|\le m+1\}} a(x,t,u,\nabla u)\cdot\nabla u \quad dxdt = 0, \\ \frac{\partial S(u)}{\partial t} - \operatorname{div}\Big(a(x,t,u,\nabla u)S'(u)\Big) + S''(u)a(x,t,u,\nabla u)\cdot\nabla u \\ -\operatorname{div}\Big(S'(u)\Phi(u)\Big) + S''(u)\Phi(u)\cdot\nabla u + g(x,t,u,\nabla u)S'(u) = fS'(u), \\ S(u)(t=0) = S(u_0) \text{ in } \Omega, \\ \text{for every function } S \text{ in } W^{2,\infty}(\mathbb{R}) \text{ and such that } S' \text{ has a compact support in } \mathbb{R} \end{cases} \tag{64}$$

has at least one renormalised solution for any $f \in L^1(Q)$.

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