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Degenerate Bogdanov-Takens Bifurcations in the Gray-Scott Model

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Abstract: In this paper, we show that for a wide range of parameter values, the Gray-Scott model of families of traveling wave solutions posses two degenerate Bogdanov-Takens points. Furthermore, we explicitly define a unique compact form for the critical normal form coefficients of order 3 and 4. This is guaranteed by applying suitable solvability conditions to singular linear systems coming from the center manifold reduction combined with a normalization technique.

Keywords: *Gray-Scott model; travelling waves; degenerate Bogdanov-Takens bifurcation.*

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1 Introduction

One of the most important contributions to the bifurcation theory has been developed independently and simultaneously by Bogdanov [3,4] and Takens [19], where the topological normal form of the so-called "Bogdanov-Takens (BT) bifurcation" is derived. This bifurcation plays an important role in the analysis of dynamical systems because it gives the appearance of local bifurcations (Saddle-node bifurcation and Hopf bifurcation) and global bifurcations (homoclinic orbits to saddle equilibria) near the critical parameter values [12].

The exact bifurcation scenario near a BT point is determined by an unfolding of the critical ODE on the 2D center manifold, with as many unfolding parameters as the codimension of the bifurcation. More precisely, the bifurcation diagram of the unfolding depends on the coefficients of the critical normal form on the center manifold. The

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restriction of a system of ODEs to any center manifold at the critical parameter values can be transformed by formal smooth coordinate changes to the form [2, 13]

$$\begin{cases} \dot{w}_0 &= w_1, \\ \dot{w}_1 &= \displaystyle\sum_{k\geq 2} a_k w_0^k + b_k w_0^{k-1} w_1^k, \end{cases} \tag{1}$$

where $w = (w_0, w_1) \in \mathbb{R}^2$ are the center manifold coordinates and a_k, b_k are the critical normal form coefficients.

The Gray-Scott model consists of the following coupled pair of reaction-diffusion equations:

$$\begin{cases} \dfrac{\partial U}{\partial t} &= D_u \nabla^2 U - UV^2 + \alpha_1(1-U), \\ \dfrac{\partial V}{\partial t} &= D_v \nabla^2 V + UV^2 - \alpha_2 V, \end{cases} \tag{2}$$

where α_1 and α_2 are the rate constants, D_u and D_v are the diffusivities, $U = U(x,t)$ and $V = V(x,t)$ are the concentration of the chemical species U (the inhibitor of the reaction) and V (the catalyst or the activator). A standard notation, ∇^2 is the Laplacian operator. Equation (1) was proposed by P. Gray and S. K. Scott in 1983 [8]; that's why it's called the Gray-Scott model. We refer the interested reader to [9, 10] more physical and chemical backgrounds of the model. Motivated by the experiments and simulations of Pearson [17] (see also [16, 21]), attention is primarily focused on the case in which the diffusivity of the inhibitor U is greater than that of the activator V. In this case, U is able to rapidly reach the localized regions of high V concentration and hence sustain the reaction, while the relatively slow diffusion of V makes it possible for these localized regions to persist. We thus set $D_u = 1$ and introduce the small parameter ε by setting $D_v = \varepsilon$, with $0 < \varepsilon << 1$. This choice of the diffusion coefficient D_v will enable us to explore a wide region of the parameter space. The existence of the saddle-node, Hopf, and nondegenrate BT bifurcations in (1) was studied by many authors, see [6, 14–16, 18, 20]. They pointed out that the homoclinic bifurcation occurs.

The purpose of this study is to derive conditions for the appearance of degenerate BT bifurcation (where the nondegeneracy condition $a_2 b_2 \neq 0$ is no longer satisfied). For a range of parameter values, we show that the Gray-Scott model of families of traveling wave solutions posses two degenerate BT points. Under certain conditions, we explicitly define a unique formula from the critical normal form coefficients, namely $\{a_k, b_k\}$ for $k = 3, 4$. This is guaranteed by defining a unique compact form explicitly for the Taylor expansion of the center manifold near the critical parameter values under reasonable conditions. To this end, we apply suitable solvability conditions to singular linear systems coming from the center manifold reduction combined with the normalization technique.

The paper is organized as follows. In Section 2, we describe the model to be studied in the paper and its equilibria. A traveling wave ansatz will be introduced, such that one variable will describe both the spatial and the temporal behavior. This reduces the system of PDEs to a system of ODEs. Also, we provide explicit formulas for the equilibrium points. In Section 3, a unique explicit formula for the Taylor expansion of the 2D center manifold up to order 4 and the critical normal form coefficients of order 3 and 4 will then be derived for not only the Gray-Scott model, but also for any n-dimensional ODEs using the combined reduction-normalization technique. Numerical examples and discussions are given in Section 4. All the computations shown in this paper have been performed using the symbolic algebra system MAPLE.

2 The Model under Study

Consider the traveling wave ansatz $U = u(x - ct)$, $V = v(x - ct)$ [6],

$$\begin{aligned} \frac{\partial U}{\partial t} &= -cu', \quad \nabla^2 U = \tfrac{\partial^2}{\partial x^2}\left(u(x-ct)\right) = u'', \\ \frac{\partial V}{\partial t} &= -cv', \quad \nabla^2 V = \tfrac{\partial^2}{\partial x^2}\left(v(x-ct)\right) = v'', \end{aligned}$$

where $c \in \mathbb{R}$ is the wave speed and $c = 0$ corresponds to stationary states, $'$ is the derivative with respect to the independent variable $x - ct$. Substituting this traveling wave ansatz in (2) and by assuming that $u' = p$ and $\varepsilon v' = q$, we obtain the following wave system:

$$\begin{cases} u' &= p, \\ p' &= -cp + uv^2 - \alpha_1(1-u), \\ \varepsilon v' &= q, \\ \varepsilon q' &= \frac{-c}{\varepsilon} p - uv^2 + \alpha_2 v. \end{cases} \tag{3}$$

This system possesses fast-slow time scales. As $\varepsilon \to 0$, the system (3) reduces into a fast subsystem

$$\begin{cases} u' &= p, \\ p' &= -cp + uv^2 - \alpha_1(1-u). \end{cases}$$

On the other hand, introducing γ by $\gamma = \frac{c}{\varepsilon}$ and rescaling the independent variable $x - ct = \varepsilon\eta$ yield

$$\begin{cases} \dot{u} &= \varepsilon p, \\ \dot{p} &= \varepsilon\left(-\varepsilon\gamma p + uv^2 - \alpha_1(1-u)\right), \\ \dot{v} &= q, \\ \dot{q} &= -\gamma q - uv^2 + \alpha_2 v, \end{cases} \tag{4}$$

where $\dot{}$ denotes the derivative with respect to the new independent variable η. Hence, as $\varepsilon \to 0$, the systems (4) reduces into a slow subsystem

$$\begin{cases} \dot{v} &= q, \\ \dot{q} &= -\gamma q - uv^2 + \alpha_2 v. \end{cases}$$

Any bounded orbit of (3) corresponds to a traveling wave solution of the model (2) at the parameter value $(\alpha_1, \alpha_2, \varepsilon)$ propagating with wave velocity c. For $c \geq 0$, the system (3) has equilibrium points $(u_e, 0, v_e, 0)$ with the solution sets of

$$u_e v_e^2 - \alpha_1(1 - u_e) = 0, \quad -u_e v_e^2 + \alpha_2 v_e = 0.$$

Therefore, the system (3) has the following equilibria $E_1 = (1, 0, 0, 0)$ for all $(\alpha_1, \alpha_2, \varepsilon, c)$, and

$$E_2 = \left(\frac{\alpha_1 \pm \tau}{2\alpha_1}, 0, \frac{\alpha_1}{\alpha_2}\left(1 - \frac{\alpha_1 \pm \tau}{2\alpha_1}\right), 0\right), \quad \tau = \sqrt{\alpha_1^2 - 4\alpha_1\alpha_2^2},$$

for all $\alpha_1 \geq 4\alpha_2^2$. The Jacobian matrix of the system (3) is given by

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ v_e^2 + \alpha_1 & -c & 2u_e v_e & 0 \\ 0 & 0 & 0 & \frac{1}{\varepsilon} \\ \frac{1}{\varepsilon}(-v_e^2) & \frac{-c}{\varepsilon^2} & \frac{1}{\varepsilon}(-2u_e v_e + \alpha_2) & 0 \end{pmatrix}. \tag{5}$$

The eigenvalues of the Jacobian matrix corresponding to the equilibrium E_1 are given by $\lambda = \left\{\pm\frac{\sqrt{\alpha_2}}{\varepsilon},\ \frac{1}{2}\left(-1 \pm \sqrt{c^2 + 4\alpha_1}\right)\right\}$. It is clear that for the case $\alpha_2 = 0$, the Jacobian matrix (5) has a double zero eigenvalue. On the other hand, at the equilibrium point E_2, the characteristic polynomial is

$$P(\lambda) \quad := \lambda^4 + c\lambda^3 - \left(\frac{\alpha_1(\alpha_1 \mp k)}{2\alpha_2^2} - \frac{\alpha_2}{\varepsilon^2}\right)\lambda^2 + \frac{c\alpha_2(\varepsilon+2)}{\varepsilon^3}\lambda - \frac{(\alpha_1 \mp k)k}{2\varepsilon^2\alpha_2},$$

where $k = \sqrt{\alpha_1^2 - 4\alpha_1\alpha_2^2}$. If we consider the case when $\alpha_1 = 4\alpha_2^2$ and $c = 0$, then

$$P(\lambda) = \lambda^4 - \frac{\alpha_2\left(8\alpha_2\varepsilon^2 - 1\right)}{\varepsilon^2}\lambda^2,$$

which has a double-zero root given by $\lambda = 0,\ 0,\ \frac{\sqrt{8\alpha_2^2\varepsilon^2 - \alpha_2}}{\varepsilon},\ -\frac{\sqrt{8\alpha_2^2\varepsilon^2 - \alpha_2}}{\varepsilon}$.

3 Center Manifold Reduction Combined with Normalization

In this section, we discuss the computation of normal form coefficients a_k and b_k of the critical normal form (1). First, suppose that at $x_0 = 0$, the Jacobian matrix $A = f_x(x_0)$ of a generic smooth family of autonomous ODEs

$$\dot{x} = f(x), \quad f : \mathbb{R}^n \to \mathbb{R}^n, \tag{6}$$

has a double (and not semi-simple) zero eigenvalue, i.e. x_0 is a BT point. Then, there exist two real linearly independent (generalized) eigenvectors $q_{0,1} \in \mathbb{R}^n$, of A, and two adjoint eigenvectors $p_{0,1} \in \mathbb{R}^n$, of A^{T}, such that

$$\begin{pmatrix} A & 0 \\ -I_n & A \end{pmatrix}\begin{pmatrix} q_0 \\ q_1 \end{pmatrix} = 0, \quad \begin{pmatrix} A^{\mathrm{T}} & 0 \\ -I_n & A^{\mathrm{T}} \end{pmatrix}\begin{pmatrix} p_1 \\ p_0 \end{pmatrix} = 0. \tag{7}$$

Assume that the vectors $\{q_0, q_1, p_0, p_1\}$ satisfy

$$p_0^{\mathrm{T}} q_0 = p_1^{\mathrm{T}} q_1 = 1, \quad p_0^{\mathrm{T}} q_1 = p_1^{\mathrm{T}} q_0 = 0. \tag{8}$$

If we impose the conditions (see [13])

$$q_0^{\mathrm{T}} q_0 = 1, \quad q_1^{\mathrm{T}} q_0 = 0, \tag{9}$$

then the vectors $\{q_0, q_1, p_0, p_1\}$ are uniquely defined up to a $\pm$ sign. We can parametrize the critical center manifold for (6) with respect to $w = (w_0, w_1) \in \mathbb{R}^2$ as

$$x = H(w), \quad H : \mathbb{R}^2 \to \mathbb{R}^n. \tag{10}$$

The invariance of the center manifold implies the *homological equation* [5, 7, 11]

$$\frac{\partial H}{\partial w_0}\dot{w}_0 + \frac{\partial H}{\partial w_1}\dot{w}_1 = f(H(w)). \tag{11}$$

We write the Taylor expansions of H and f as

$$f(x) = Ax + \frac{1}{2}B(x,x) + \frac{1}{6}C(x,x,x) + \frac{1}{24}D(x,x,x,x) + \mathcal{O}\left(||x||^5\right), \tag{12a}$$

$$H(w) = q_0w_0 + q_1w_1 + \sum_{2\le j+k\le 4}\frac{1}{j!k!}H_{jk}w_0^jw_1^k + \mathcal{O}(||w||^5), \tag{12b}$$

where B, C, D, and E are vector-valued functions with n-components. The i^{th} component of these functions are defined by

$$B_i(x,y) = \sum_{j,k=1}^{n}\frac{\partial^2 f_i(\xi)}{\partial\xi_j\partial\xi_k}\Big|_{\xi=0}x_jy_k, \quad C_i(x,y,z) = \sum_{j,k,l=1}^{n}\frac{\partial^3 f_i(\xi)}{\partial\xi_j\partial\xi_k\partial\xi_l}\Big|_{\xi=0}x_jy_kz_l,$$

$$D_i(x,y,z,w) = \sum_{j,k,l,m=1}^{n}\frac{\partial^4 f_i(\xi)}{\partial\xi_j\partial\xi_k\partial\xi_l\partial\xi_m}\Big|_{\xi=0}x_jy_kz_lw_m.$$

We insert the expansions (12a) and (12a) into (11) together with $(\dot{w}_0, \dot{w}_1)$ as we defined in (1). Then, collecting the terms with equal components in w^{j+k} at the homological equation gives a linear systems that can be solved for the coefficients $H_{jk} \in \mathbb{R}^n$ by a recursive procedure based on Fredholm's solvability condition.

The quadratic w-terms in the homological equation (11) lead to (see [1, 11, 13]):

$$a_2 = \frac{1}{2}p_1^{\mathrm{T}}B(q_0,q_0), \tag{13}$$

$$b_2 = p_0^{\mathrm{T}}B(q_0,q_0) + p_1^{\mathrm{T}}B(q_0,q_1), \tag{14}$$

$$H_{20} = -A^{\mathrm{INV}}\left(B(q_0,q_0) - 2a_2q_1\right) + \gamma_0q_0, \tag{15}$$

$$H_{11} = -A^{\mathrm{INV}}\left(B(q_0,q_1) - H_{20} - b_2q_1\right), \tag{16}$$

$$H_{02} = -A^{\mathrm{INV}}\left(B(q_1,q_1) - 2H_{11}\right), \tag{17}$$

where $\gamma_0 := \frac{1}{2}p_1^{\mathrm{T}}B(q_1,q_1) + p_0^{\mathrm{T}}\left(B(q_0,q_1) - H_{20}\right)$. Note that γ_0q_0 is added to H_{20} to ensure that the right-hand side of the system for H_{02} is in the range of A (see Section 8.7 in [12] for more details).

Collecting the terms with equal components in w of order three at the homological equation gives the following equations.

$$w_0^3\colon AH_{30} + C(q_0,q_0,q_0) + 3B(q_0,H_{20}) - 6a_2H_{11} - 6a_3q_1 = 0, \tag{18}$$

$$\begin{aligned} w_0^2w_1\colon AH_{21} + C(q_0,q_0,q_1) + 2B(q_0,H_{11}) + B(q_1,H_{20}) - 2a_2H_{02}& \\ - 2b_2H_{11} - H_{30} - 2b_3q_1 = 0,& \end{aligned} \tag{19}$$

$$w_0w_1^2\colon AH_{12} + C(q_0,q_1,q_1) + 2B(q_1,H_{11}) + B(q_0,H_{02}) - 2b_2H_{02} - 2H_{21} = 0, \tag{20}$$

$$w_1^1\colon AH_{03} + C(q_1,q_1,q_1) + 3B(q_1,H_{02}) - 3H_{12} = 0. \tag{21}$$

The solvability condition implies the following expressions for the cubic coefficients (see [13]):

$$a_3 = \frac{1}{6} p_1^{\mathrm{T}} \left(C(q_0, q_0, q_0) + 3B(q_0, H_{20}) - 6a_2 H_{11} \right), \tag{22}$$

$$H_{30} = -A^{\mathrm{INV}} \left(C(q_0, q_0, q_0) + 3B(q_0, H_{20}) - 6a_2 H_{11} - 6a_3 q_1 \right), \tag{23}$$

$$b_3 = \frac{1}{2} p_1^{\mathrm{T}} \left(C(q_0, q_0, q_1) + 2B(q_0, H_{11}) + B(q_1, H_{20}) - 2a_2 H_{02} - 2b_2 H_{11} - H_{30} \right), \tag{24}$$

$$\begin{aligned} H_{21} = -A^{\mathrm{INV}} (&C(q_0, q_0, q_1) + 2B(q_0, H_{11}) + B(q_1, H_{20}) - 2a_2 H_{02} \\ &- 2b_2 H_{11} - H_{30} - 2b_3 q_1), \end{aligned} \tag{25}$$

$$H_{12} = -A^{\mathrm{INV}} \left(C(q_0, q_1, q_1) + 2B(q_1, H_{11}) + B(q_0, H_{02}) - 2b_2 H_{02} - 2H_{21} \right), \tag{26}$$

$$H_{03} = -A^{\mathrm{INV}} \left(C(q_1, q_1, q_1) + 3B(q_1, H_{02}) - 3H_{12} \right). \tag{27}$$

However, given a_3 and b_3, the solutions to the singular linear system (23), (25)-(27) are not unique. The uniqueness of the solutions can be guaranteed by requiring that (20) and (21) are solvable for H_{12} and H_{03}, respectively, *i.e.* H_{12} and H_{03} are in the range of A. Multiply the equations (20) and (21) by p_1^{T}, then the solvability condition requires that

$$p_1^{\mathrm{T}} H_{21} - \frac{1}{2} p_1^{\mathrm{T}} \left(C(q_0, q_1, q_1) + 2B(q_1, H_{11}) + B(q_0, H_{02}) - 2b_2 H_{02} \right) = 0, \tag{28}$$

$$p_1^{\mathrm{T}} H_{12} - \frac{1}{3} p_1^{\mathrm{T}} \left(C(q_1, q_1, q_1) + 3B(q_1, H_{02}) \right) = 0. \tag{29}$$

Multiply the equation (19) by p_0^{T}, Then using the substitution

$$H_{30} \mapsto H_{30} + \gamma_1 q_0 \tag{30}$$

gives

$$p_1^{\mathrm{T}} H_{21} = -p_0^{\mathrm{T}} \left(C(q_0, q_0, q_1) + 2B(q_0, H_{11}) + B(q_1, H_{20}) - 2a_2 H_{02} - 2b_2 H_{11} - H_{30} \right) + \gamma_1.$$

Substituting this into (28) with

$$\begin{aligned} \gamma_1 := &\, p_0^{\mathrm{T}} \left(C(q_0, q_0, q_1) + 2B(q_0, H_{11}) + B(q_1, H_{20}) - 2a_2 H_{02} - 2b_2 H_{11} - H_{30} \right) \\ &+ \frac{1}{2} p_1^{\mathrm{T}} \left(C(q_0, q_1, q_1) + 2B(q_1, H_{11}) + B(q_0, H_{02}) - 2b_2 H_{02} \right) \end{aligned} \tag{31}$$

makes the left-hand side of (28) equal to zero. So, the substitution for H_{30} implies that (20) is solvable for H_{12}. Note that adding a scalar multiple of q_0 to H_{30} does not affect the coefficient b_3 given by (24), since $\langle p_1, q_0 \rangle = 0$. On the other hand, to ensure that (21) is solvable for H_{03}, one can use the substitution

$$H_{21} \mapsto H_{21} + \gamma_2 q_0, \tag{32}$$

then multiplying the equation (20) by p_0^{T} gives

$$p_1^{\mathrm{T}} H_{12} = -p_0^{\mathrm{T}} \left(C(q_0, q_1, q_1) + 2B(q_1, H_{11}) + B(q_0, H_{02}) - 2b_2 H_{02} - 2H_{21} \right) + 2\gamma_2. \tag{33}$$

Substitute this into (29) with

$$\begin{aligned}\gamma_2 := \frac{1}{6}p_1^{\mathrm{T}}\left(C(q_1,q_1,q_1)+3B(q_1,H_{02})\right)+\frac{1}{2}p_0^{\mathrm{T}}(C(q_0,q_1,q_1)\\ +2B(q_1,H_{11})+B(q_0,H_{02})-2b_2H_{02}-2H_{21});\end{aligned} \tag{34}$$

makes the left-hand side of (29) is equal to zero. So, this substitution implies that (21) is solvable for H_{03}. Note that the substitution for H_{21} does not affect the coefficients a_4 and b_4, as we will see in equations (35) and (37).

Finally, the homological equation implies the following expressions for the coefficients of the w^4-terms [13]:

$$\begin{aligned}a_4 = \frac{1}{24}p_1^{\mathrm{T}}(D(q_0,q_0,q_0,q_0)+6C(q_0,q_0,H_{20})+4B(q_0,H_{30})+3B(H_{20},H_{20})\\ -12a_2H_{21}-24a_3H_{11}),\end{aligned} \tag{35}$$

$$\begin{aligned}H_{40} = -A^{\mathrm{INV}}\left(D(q_0,q_0,q_0,q_0)+6C(q_0,q_0,H_{20})+4B(q_0,H_{30})\right.\\ \left.+3B(H_{20},H_{20})-12a_2H_{21}-24a_3H_{11}-24a_4q_1\right),\end{aligned} \tag{36}$$

$$\begin{aligned}b_4 = \frac{1}{6}p_1^{\mathrm{T}}\left(D(q_0,q_0,q_0,q_1)+3C(q_0,q_1,H_{20})+3C(q_0,q_0,H_{11})\right.\\ +3B(q_0,H_{21})+3B(H_{20},H_{11})+B(q_1,H_{30})-H_{40}-3b_2H_{21}\\ \left.-6a_2H_{12}-6a_3H_{02}-6b_3H_{11}\right).\end{aligned} \tag{37}$$

In systems (15)-(17), (23), (25), (26), (27) and (36), the expression $x = A^{\mathrm{INV}}y$ is defined by using the non-singular bordered system

$$\begin{pmatrix} A & p_1 \\ q_0^{\mathrm{T}} & 0 \end{pmatrix}\begin{pmatrix} x \\ s \end{pmatrix} = \begin{pmatrix} y \\ 0 \end{pmatrix},$$

where y is in the range of A.

4 Degenerate Bogdanov-Takens Bifurcation in the Gray-Scott Model

In this section, we will use the analytical results obtained in Section 3 to prove that the Gray-Scott model has two degenerate BT points at its equilibria E_1 and E_2. Recall that the Gray-Scott model (3) exhibits a BT bifurcation of the equilibrium E_2 occurring at the parameter values $(\alpha_1,\alpha_2,\varepsilon,c) = (4\alpha_2^2,\alpha_2,\varepsilon,0)$. First, we apply the change of variables $(u,p,v,q) = E_2 + (x_1,x_2,x_3,x_4)$, which brings the equilibrium point E_2 to the origin $(0,0,0,0)$. Then the Jacobian matrix evaluated at $(0,0,0,0)$ is

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 8\alpha_2^2 & 0 & 2\alpha_2 & 0 \\ 0 & 0 & 0 & \dfrac{1}{\delta} \\ -\dfrac{4\alpha_2^2}{\delta} & 0 & -\dfrac{\alpha_2}{\delta} & 0 \end{pmatrix}.$$

The following vectors

$$q_0 = m_1\left(-1,\ 0,\ 4|\alpha_2|,\ 0\right)^{\mathrm{T}},\quad q_1 = m_1\left(0,\ -1,\ 0,\ 4|\alpha_2|\varepsilon\right)^{\mathrm{T}},$$

$$p_1 = n_1\left(1,\ 0,\ 2\varepsilon^2,\ 0\right)^{\mathrm{T}},\quad p_0 = n_1\left(0,\ 1,\ 0,\ 2\varepsilon\right)^{\mathrm{T}},$$

with $m_1 = \dfrac{\frac{\alpha_2}{|\alpha_2|}}{\sqrt{16\alpha_2^2+1}}$, $n_1 = \dfrac{1}{m_1}\left(\dfrac{1}{8\varepsilon^2\alpha_2-1}\right)$, satisfy (7)-(9). The vector-valued function $B : \mathbb{R}^4 \times \mathbb{R}^4 \to \mathbb{R}^4$ can be defined for arbitrary vectors $z,\ r \in \mathbb{R}^4$ as follows:

$$B(z,r) = \frac{4\alpha_2}{\varepsilon}\left(0,\ -z_1r_3 - z_1r_3 - z_3r_3,\ 0,\ z_1r_3 + z_1r_3 + z_3r_3\right)^{\mathrm{T}}.$$

Therefore, the vectors $B(q_0,q_0)$ and $B(q_0,q_1)$ can be expressed as

$$B(q_0,q_0) = \frac{16\alpha_2^2 m_1^2}{\varepsilon}\left(0,\ -\varepsilon,\ 0,\ 1\right)^{\mathrm{T}},\quad B(q_0,q_1) = \left(0,\ 0,\ ,\ 0,\ 0\right)^{\mathrm{T}}.$$

Thus, the formulas (13) and (14) give the values of the critical normal form coefficients

$$a_2 = \frac{8\alpha_2^2}{8\delta^2\alpha_2 - 1} \quad \text{and} \quad b_2 = 0.$$

These values confirm that the Gray-Scott model has a degenerate BT point of codimension ≥ 3. Similarly, one can compute the normal form coefficients at the BT of the equilibrium E_1 occurring at the parameter values $\alpha_2 = 0$, and satisfy $a_2 = -\dfrac{1}{\varepsilon^2}$ and $b_2 = \dfrac{2c\left(\varepsilon^4 + c^2\varepsilon + c^2\right)}{\varepsilon^3\left(\varepsilon^4 + c^2\right)\alpha_1}$. Therefore, the case $c = 0$ indicates a degenerate BT point for the parameter values $(\alpha_1,\alpha_2,\varepsilon,c) = (\alpha_1,0,\varepsilon,0)$.

5 Example

Based on the analysis carried out in Section 2, we are going to perform numerical studies of the degenerate BT bifurcation of the equilibrium E_2 [(1)] which occurs at $(\alpha_1,\alpha_2,\varepsilon,c) = (4\alpha_2^2,\alpha_2,\varepsilon,0)$. To simplify, we fix the variables $\alpha_2 = 1$ and $\varepsilon = 0.1$. At the bifurcation parameter, we compute the following expressions for (13)-(17): $a_2 = \dfrac{14356}{6807}$, $b_2 = 0$ and

$$H_{20} = \left(\frac{21151}{990682},\ 0,\ \frac{-9600}{8993},\ 0\right)^{\mathrm{T}},\quad H_{11} = \left(0,\ \frac{21151}{990682},\ 0,\ \frac{-960}{8993}\right)^{\mathrm{T}},$$

$$H_{02} = \left(\frac{5423}{1079523},\ 0,\ \frac{5807}{4623854},\ 0\right)^{\mathrm{T}}$$

It is clear that the system (17) is solvable for H_{02}; this can be easily shown by multiplying both sides of (17) by p_1^{T} which is indeed

$$p_1^{\mathrm{T}}\left(2H_{11} - B(q_1,q_1)\right) = 0.$$

(1) Similar results can be derived for the BT point of the equilibrium point E_1.

Later, we compute the value of the system (22)-(27): $a_3 = \dfrac{-6441}{5329}, \quad b_3 = 0$ and

$$H_{30} = \left(\frac{-4608}{88283},\ 0,\ \frac{73432}{36131},\ 0\right)^{\mathrm{T}}, \quad H_{21} = \left(0,\ \frac{-4358}{140551},\ 0,\ \frac{12276}{60245}\right)^{\mathrm{T}},$$

$$H_{12} = \left(\frac{-1266}{253087},\ 0,\ \frac{-633}{506174},\ 0\right)^{\mathrm{T}}, \quad H_{03} = \left(0,\ \frac{-3798}{253087},\ 0,\ \frac{-2539}{6767645}\right)^{\mathrm{T}}.$$

The systems H_{21}, H_{12} and H_{03} are uniquely defined such that the solvability conditions (28) and (28) are satisfied. Finally, (35)-(37) is given by

$$a_4 = \frac{100212}{120473}, \quad H_{40} = \left(\frac{-31070}{25263},\ 0,\ \frac{-15535}{50526},\ 0\right)^{\mathrm{T}}, \quad b_4 = 0.$$

Thus, our unique normal form for the Gray-Scott model is

$$\begin{cases} \dot{w}_0 &= w_1, \\ \dot{w}_1 &= \dfrac{14356}{6807} w_0^2 - \dfrac{6441}{5329} w_0^3 + \dfrac{100212}{120473} w_0^4. \end{cases}$$

6 Conclusion

In the present paper, we consider the Gray-Scott model (2), where a travel wave ansatz is introduced for which one variable describes both the spatial and the temporal behavior. This reduces the system of PDEs (2) into a system of ODEs (3). For a wide range of parameter values, the Gray-Scott model (3) possesses two degenerate BT points. The main aim of this paper is to define a unique explicit formula for the Taylor expansion of the 2D center manifold up to a term of order 4. The uniqueness of the Taylor expansion is guaranteed by applying the variables transformations (30) and (32) with the suitable choice for γ_1 and γ_2 as shown in (31) and (34), respectively. The results of this paper can be applied also for any n-dimensional system of ODEs. The theoretical results of the paper are illustrated by an example in Section 5. Natural directions for future research include developing a robust predictor for the homoclinic orbits bifurcating from a BT point in generic n-dimensional ODEs. Such a predictor needs a unique expression for the vectors H_{jk} in the Taylor expansion (12b).

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