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Increased and Reduced Synchronization between Discrete-Time Chaotic and Hyperchaotic Systems

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Abstract: In this paper, by combining generalized synchronization (GS) and inverse generalized synchronisation (IGS), new schemes for increased and reduced synchronization between different dimensional discrete-time systems are proposed. Based on the Lyapunov stability theory, two control laws are proposed to prove the co-existence of GS and IGS between the general three-dimensional drive map and the two-dimensional response map in 3D and 2D, respectively. Numerical simulation has confirmed the findings of the paper.

Keywords: *discrete chaos; generalized synchronization; inverse generalized synchronisation; co-existence; Lyapunov stability.*

Mathematics Subject Classification (2010): 93C10, 93C55, 93D05.

1 Introduction

Chaotic discrete–time systems have received a considerable attention over the last two decades due to their many applications in secure communications [1]. One of the most studied aspects in discrete-time chaotic systems is the synchronization of chaotic systems. Synchronization refers to the addition of a set of control parameters to the controlled chaotic system and adaptively updating the controls so that the states become synchronized [2–4]. Throughout the years, many studies have considered the synchronization of discrete–order chaotic and hyperchaotic systems including [5–7]. One of the most exciting synchronization types is the generalized synchronization (GS). It refers to the existence of a functional relationship between the drive states and the response states. Instead of the conventional definition of synchronization, which stipulates that

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the difference between the drive and response trajectories tends to zero as $t \to +\infty$, GS forces the difference between the response states and a function of the drive states to zero. IGS is the natural reversal of GS, i.e. the error is the difference between the master states and a function of the slave states. The importance of GS and IGS stems from the fact that they can enrich the behavior of chaotic systems [8].

Naturally, curiosity grew as to the possibility of multiple synchronization types being achieved simultaneously for the states of the response system. This phenomenon is commonly referred to as the coexistence of synchronization types [9–11]. The present research work focuses on the coexistence of GS and IGS between chaotic and hyperchaotic systems. The next section of this paper describes the model for the drive and response systems. Section 3 presents the control law that guarantees the coexistence of GS and IGS in 3D. Section 4 presents numerically the control laws that establish the coexistence of GS and IGS in 2D. Finally, Section 5 summarizes the work carried out in this paper.

2 Drive–Response Model

We consider the following drive chaotic system:

$$\begin{aligned} x_1(k+1) &= f_1(x_1(k), x_2(k)), \\ x_2(k+1) &= f_2(x_1(k), x_2(k)), \end{aligned} \tag{1}$$

where $(x_1(k), x_2(k))^T$ is the state vector of the drive system and $f_i : \mathbb{R}^2 \longrightarrow \mathbb{R}$, $1 \le i \le 2$.

As the response system, we consider the following hyperchaotic system:

$$\begin{aligned} y_1(k+1) &= \sum_{j=1}^{3} b_{1j} y_j(k) + g_1(y_1(k), y_2(k), y_3(k)) + u_1, \\ y_2(k+1) &= \sum_{j=1}^{3} b_{2j} y_j(k) + g_2(y_1(k), y_2(k), y_3(k)) + u_2, \\ y_3(k+1) &= \sum_{j=1}^{3} b_{3j} y_j(k) + g_3(y_1(k), y_2(k), y_3(k)) + u_3, \end{aligned} \tag{2}$$

where $(y_1(k), y_2(k), y_3(k))^T$ is the state vector of the response system, $(b_{ij}) \in \mathbb{R}^{3\times 3}$ is the linear part of the response system, $g_i : \mathbb{R}^3 \longrightarrow \mathbb{R}$, $1 \le i \le 3$, are the nonlinear functions and u_i, $1 \le i \le 3$, are the controllers to be designed.

3 Synchronization in 3D

The problem of increased synchronization in 3D between the drive system (1) and the response system (2) is to find controllers u_i, $i = 1, 2, 3$, and functions $\phi, \chi : \mathbb{R}^2 \longrightarrow \mathbb{R}$, $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$, such that the synchronization errors

$$\begin{aligned} e_1(k) &= y_1(k) - \phi(x_1(k), x_2(k)), \\ e_2(k) &= x_2(k) - \varphi(y_2(k)), \\ e_3(k) &= y_3(k) - \chi(x_1(k), x_2(k)) \end{aligned} \tag{3}$$

satisfy the condition $\lim_{n\to+\infty} e_i(k) = 0$, for $i = 1, 2, 3$.

Remark 3.1 From the error system (3), it is clear that y_1 and $(x_1, x_2)^T$ are generalized synchronized, x_2 and y_2 are inverse generalized synchronized and y_3 is in generalized synchronization with x_1 and x_2 so that generalized synchronization and inverse generalized synchronization coexist in the synchronization of the systems (1) and (2) in 3D.

We assumed that φ is an invertible function and its inverse is noted by φ^{-1}. Hence, we have proved the following result.

Theorem 3.1 *Increased synchronization in 3D between systems (1) and (2) is achieved under the following controllers:*

$$\begin{aligned}
u_1 &= -\sum_{j=1}^{3} b_{1j}y_j(k) - g_1(Y(k)) + \phi\left(f_1(X(k)), f_2(X(n))\right) + \tfrac{1}{2}e_1(k) + \tfrac{2}{5}e_2(k) - \tfrac{2}{3}e_3(k), \\
u_2 &= -\sum_{j=1}^{3} b_{2j}y_j(k) - g_2(Y(k)) + \varphi^{-1}\left(\tfrac{1}{2}e_1(k) + \tfrac{2}{5}e_2(k) + \tfrac{2}{3}e_3(k) - f_2(X(k))\right), \\
u_3 &= -\sum_{j=1}^{3} b_{3j}y_j(k) - g_3(Y(k)) + \chi\left(f_1(X(k)), f_2(X(k))\right) + \tfrac{1}{2}e_1(k) - \tfrac{4}{5}e_2(k),
\end{aligned} \tag{4}$$

where $X(k) = (x_1(k), x_2(k))^T$ *and* $Y(k) = (y_1(k), y_2(k), y_3(k))^T$.

Proof. The error system (3) can be derived as

$$\begin{aligned}
e_1(k+1) &= \sum_{j=1}^{3} b_{1j}y_j(k) + g_1(Y(k)) + u_1 - \phi\left(f_1(X(k)), f_2(X(k))\right), \\
e_2(k+1) &= f_2(X(k)) - \varphi\left(\sum_{j=1}^{3} b_{2j}y_j(k) + g_2(Y(k)) + u_2\right), \\
e_3(k+1) &= \sum_{j=1}^{3} b_{3j}y_j(k) + g_3(Y(k)) - \chi\left(f_1(X(k)), f_2(X(k))\right).
\end{aligned} \tag{5}$$

Substituting the control law (4) into (5), one can get

$$\begin{aligned}
e_1(k+1) &= \frac{1}{2}e_1(k) + \frac{2}{5}e_2(k) - \frac{2}{3}e_3(k), \\
e_2(k+1) &= \frac{1}{2}e_1(k) + \frac{2}{5}e_2(k) + \frac{2}{3}e_3(k), \\
e_3(k+1) &= \frac{1}{2}e_1(k) - \frac{4}{5}e_2(k).
\end{aligned} \tag{6}$$

We construct the Lyapunov function in the form $V(e_1(k), e_2(k), e_3(k)) = e_1^2(k) + e_2^2(k) + e_3^2(k)$, so

$$\begin{aligned}
\Delta V &= e_1^2(k+1) + e_2^2(k+1) + e_3^2(k+1) - e_1^2(k) - e_2^2(k) - e_3^2(k) \\
&= -\left(\frac{1}{4}e_1^2(k) + \frac{17}{25}e_2^2(k) + \frac{1}{9}e_3^2(k)\right) < 0.
\end{aligned}$$

It is immediate that all solutions of error system (6) go to zero as $k \to +\infty$. Therefore, systems (1) and (2) are globally synchronized in 3D.

The result of the numerical simulation of the error system (6) is plotted in (Figure 1).

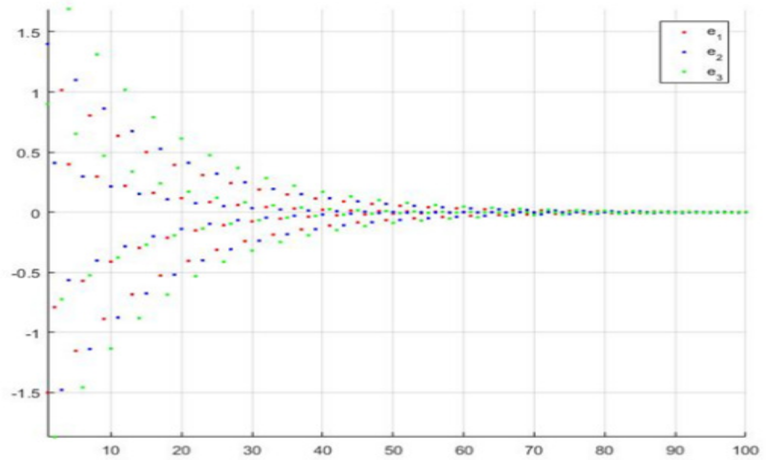

Figure 1: Time evolution of the synchronization errors 6.

4 Synchronization in 2D

The problem of reduced synchronization in 2D between the drive system (1) and the response system (2) is to find controllers u_i, $i = 1, 2, 3$, and functions $\psi : \mathbb{R}^2 \longrightarrow \mathbb{R}$, $\lambda, \omega : \mathbb{R} \longrightarrow \mathbb{R}$, such that the synchronization errors

$$
\begin{aligned}
e_1(k) &= y_1(k) - \psi(x_1(k), x_2(k)), \\
e_2(k) &= x_2(k) - \lambda(y_2(k)) - \omega(y_3(k))
\end{aligned} \tag{7}
$$

satisfy the condition $\lim_{n \to +\infty} e_i(k) = 0$, for $i = 1, 2$. We assume that the functions λ and ω are invertible.

Remark 4.1 From the error system (7), it is clear that y_1 is generalized synchronized with x_1 and x_2, and x_2 is inverse generalized synchronized with y_2 and y_3, so that generalized synchronization and inverse generalized synchronization coexist in the synchronization of the systems (1) and (2) in 2D.

The error system (7) can be described as

$$
\begin{aligned}
&e_1(k+1) = \sum_{j=1}^{3} b_{1j} y_j(k) + g_1(Y(k)) + u_1 - \psi(f_1(X(k)), f_2(X(k))), \\
&e_2(k+1) = f_2(X(k)) - \lambda\left(\sum_{j=1}^{3} b_{2j} y_j(k) + g_2(Y(k)) + u_2\right) \\
&-\omega\left(\sum_{j=1}^{3} b_{3j} y_j(k) + g_3(Y(k)) + u_3\right).
\end{aligned} \tag{8}
$$

In this case, the controllers can be constructed as follow:

$$\begin{aligned} u_1 &= -\sum_{j=1}^{3} b_{1j} y_j(k) - g_1(Y(k)) - u_1 + \psi\left(f_1(X(k)), f_2(X(k))\right) - e_1(k) - e_2(k), \quad (9) \\ u_2 &= -\sum_{j=1}^{3} b_{1j} y_j(k) - g_1(Y(k)) + \lambda^{-1}\left(e_1(k) - e_2(k)\right), \\ u_3 &= -\sum_{j=1}^{3} b_{3j} y_j(k) - g_3(Y(k)) + \omega^{-1}\left(f_2(X(k))\right), \end{aligned}$$

where λ^{-1} and ω^{-1} are the inverse functions of λ and ω, respectively. By substituting the control law (9) into (8), the error system can be described as

$$\begin{aligned} e_1(k+1) &= \frac{1}{2} e_1(k) + \frac{1}{2} e_2(k), \quad (10) \\ e_2(k+1) &= \frac{1}{2} e_1(k) - \frac{1}{2} e_2(k). \end{aligned}$$

We construct a Lyapunov function in the form $V(e_1(k), e_2(k)) = e_1^2(k) + e_2^2(k)$, so

$$\begin{aligned} \Delta V &= e_1^2(k+1) + e_2^2(k+1) - e_1^2(k) - e_2^2(k) \\ &= -\frac{1}{2}\left(e_1^2(k) + e_2^2(k)\right) < 0. \end{aligned}$$

Thus, from the Lyapunov stability theory, it is immediate that $\lim_{n\to+\infty} e_i(k) = 0$, $(i = 1, 2)$. Hence, we have proved the following result.

Theorem 4.1 *The drive system (1) and the response system (2) are reduced synchronization in 2D under the control law (9).*

The result of the numerical simulation of the error system (10) is plotted in Figure 2.

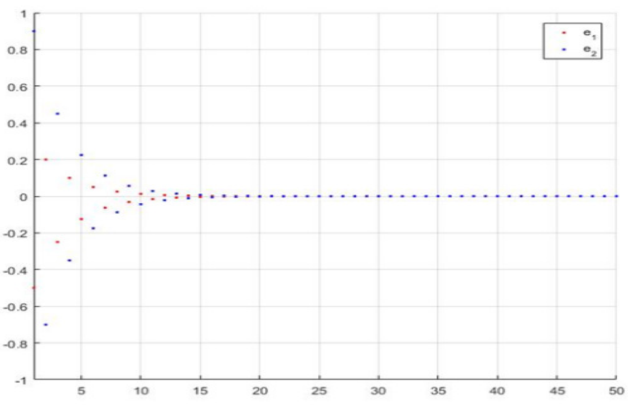

Figure 2: Time evolution of the synchronization errors (10).

5 Conclusion

In this work, we have shown that different types of synchronization can co-exist for different dimensional discrete-time chaotic systems. We assumed a two dimensional drive system and a three dimensional response system. The main results of the study were two-fold. First, we presented a control scheme whereby GS and IGS are achieved simultaneously in 3D. The stability of the zero solutions, and, consequently, the convergence of the synchronization errors were established by means of the Lyapunov stability theory. The second main result concerns the co-existence of GS and IGS in 2D. Simulations were carried out on Matlab to ensure that the errors converge to zero subject to the proposed control laws.

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