Nonlinear Dynamics and Systems Theory, **19** (3) (2019) 416–426

Chaos Synchronization and Anti-Synchronization of Two Fractional-Order Systems via Global Synchronization and Active Control

M. Labid [1] and N. Hamri [2*]

[1] *Department of Mathematics, University Center of Mila, Mila 43000, Algeria*
[2] *Laboratory of Mathematics and Their Interactions, Department of Science and Technology, University Center of Mila, Mila 43000, Algeria.*

Received: December 23, 2018; Revised: July 9, 2019

Abstract: This paper investigates the phenomenon of chaos synchronization and anti-synchronization of two identical chaotic systems of the fractional-order lesser date moth model via the methods of global synchronization and active control. Numerical examples are provided to illustrate the results.

Keywords: *chaos; synchronization; anti-synchronization; active control; fractional-order systems.*

Mathematics Subject Classification (2010): 34H10, 37N35, 93C10, 93C15, 93C95.

1 Introduction

In recent years, the fractional calculus has become an excellent tool in modeling many physical phenomena and engineering problems [16]. One of the very important areas of application of fractional calculus is chaos theory. Chaos is a very interesting nonlinear phenomenon that has been intensively studied over the past two decades. The chaos theory is found to be useful in many areas such as data encryption [14], financial systems [13], biology [17] and biomedical engineering [2], etc. Fractional-order chaotic dynamical systems have begun to attract a lot of attention in recent years and can be seen as a generalization of chaotic dynamic integer-order systems. Recently, the study of the synchronization of fractional-order chaotic systems has become an active area of

* Corresponding author: mailto:mes.laabid@centre-univ-mila.dz

research because of its potential applications in secure communication and cryptography [9, 10]. The synchronization of fractional-order chaotic systems was first studied by Deng and Li [18]. Then the idea of the synchronization is to use the output of the master (drive) system to control the slave (response) system so that the output of the slave system tracks asymptotically the output of the master system. In the past twenty years, various types of synchronization have been proposed and investigated, e.g., complete synchronization [20], lag synchronization [4], phase synchronization [8], project synchronization [15], generalized synchronization [6], etc. As a special case of generalized synchronization, anti-synchronization is achieved when the sum of the states of master and slave systems converges to zero asymptotically with time. In other words, the anti-synchronization is the use of the output of the master system to control the slave system so that the states of the slave system have the same amplitude but opposite signs as the states of the master system. In this paper, we apply global synchronization theory to synchronize two identical chaotic systems, we demonstrate the technique capability on the synchronization of fractional-order lesser date moth model [12] and we apply active control theory to synchronize and anti-synchronize two identical chaotic systems, we demonstrate the technique capability on the synchronization and anti-synchronization of fractional-order lesser date moth model.

The paper is organized as follows. In Section 2, we describe the problem statement and our methodology. In Section 3, a fractional-order lesser date moth model is presented. In Section 4, we discuss the chaos synchronization of two identical fractional-order lesser date moth models using global synchronization. In Section 5, we discuss the chaos synchronization of two identical fractional-order lesser date moth models using active control. In Section 6, we discuss the chaos anti-synchronization of two identical fractional-order lesser date moth models using active control. Section 7 gives the conclusion of this paper.

2 Problem Statement and Our Methodology

Consider the chaotic system described by the dynamics

$$D^{\alpha}x_1 = Ax_1 + g(x_1), \tag{1}$$

where $x_1 \in \mathbb{R}^n$ is the state vector, $A \in \mathbb{R}^{n\times n}$ is a constant matrix, $g(x_1)$ is a continuous nonlinear function, and D^{α} is the Caputo fractional derivative. We consider the system (1) as the master or drive system. As the slave or response system, we consider the following chaotic system described by the dynamics

$$D^{\alpha}x_2 = Ax_2 + g(x_2) + u, \tag{2}$$

where $x_2 \in \mathbb{R}^n$ is the state vector, $A \in \mathbb{R}^{n\times n}$ is a constant matrix, and $g(x_2)$ is a continuous nonlinear function and $u \in \mathbb{R}^n$ is the controller of the slave system.

The global chaos synchronization problem is to design a controller which synchronizes the states of the master system (1) and the slave system (2) for all initial conditions $x(0),\ y(0) \in \mathbb{R}^n$. The synchronization error is defined as

$$e = x_2 - x_1.$$

Then the synchronization error dynamics is obtained as

$$D^{\alpha}e = Ae + g(x_2) - g(x_1) + u. \tag{3}$$

Thus, the global synchronization problem is essentially to find a controller u so as to stabilize the error dynamics (3) for all initial conditions $e(0) \in \mathbb{R}^n$, i.e.,

$$lim_{t\to\infty}\|e(t)\| = 0$$

for all initial conditions $e(0) \in \mathbb{R}^n$.

Theorem 2.1 *[3] The following autonomous system:*

$$D^\alpha x = Ax,\ x(0) = 0,$$

where $0 < \alpha < 1$, $x \in \mathbb{R}^n$, $A \in \mathbb{R}^{n\times n}$, is asymptotically stable if and only if $|arg(eig)A| > \alpha\frac{\pi}{2}$. In this case, each component of the states decays towards 0 like $t^{-\alpha}$. Also, this system is stable if and only if $|arg(eig)A| \geq \alpha\frac{\pi}{2}$ and those critical eigenvalues that satisfy $|arg(eig)A| = \alpha\frac{\pi}{2}$ have geometric multiplicity one.

3 Fractional-Order Lesser Date Moth Model

Following [12], the model of biocontrol of the lesser date moth in palm trees can be written as follows:

$$\begin{cases} \frac{dP}{d\tau} = rP\left(1-\frac{P}{K}\right) - \frac{bPL}{a+P}, \\ \frac{dL}{d\tau} = -dL + \frac{mPL}{a+P} - pLN, \\ \frac{dN}{d\tau} = -\mu N + qLN. \end{cases} \tag{4}$$

The model consists of three populations: the palm tree whose population density at time t is denoted by P; the pest (lesser date moth) whose population density is denoted by L; the predator whose population density is denoted by N. Here all the parameters r, K, b, a, d, m, p, μ, and q are positive. One can reduce the number of parameters in system (4) by using the following transformations: $P = Kx$, $L = \frac{Kr}{b}y$, $N = \frac{r}{p}z$, $\tau = \frac{t}{r}$, then we have the following dimensionless system:

$$\begin{cases} \frac{dx}{dt} = x(1-x) - \frac{xy}{\beta+x}, \\ \frac{dy}{dt} = -\delta y + \frac{\gamma xy}{\beta+x} - yz, \\ \frac{dz}{dt} = -\eta z + \sigma yz, \end{cases} \tag{5}$$

where $\beta = \frac{a}{K}$, $\delta = \frac{d}{r}$, $\gamma = \frac{m}{r}$, $\eta = \frac{\mu}{r}$ and $\sigma = \frac{qK}{b}$.

We introduce fractional order into the ODE model (5). The new system is described by the following set of fractional-order differential equations:

$$\begin{cases} D^\alpha x = x(1-x) - \frac{xy}{\beta+x}, \\ D^\alpha y = -\delta y + \frac{\gamma xy}{\beta+x} - yz, \\ D^\alpha z = -\eta z + \sigma yz, \end{cases} \tag{6}$$

where D^α is the Caputo fractional derivative.

The lesser date moth model is chaotic when the parameter values are taken as $\beta = 1.15$, $\delta = \eta = 1$, $\gamma = 3$, $\sigma = 3$, $\alpha = 0.95$. Figure 1 describes the state portrait of the lesser date moth model.

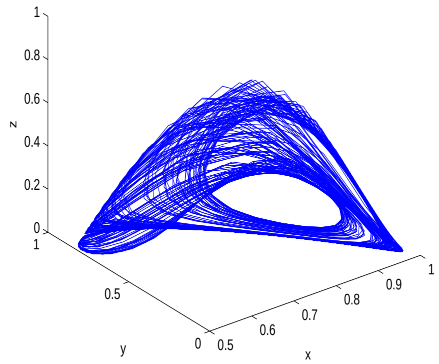

Figure 1: The state portrait of the lesser date moth model.

4 Synchronization of Identical Fractional-Order Lesser Date Moth Systems by Global Synchronization

In this section, we study chaos synchronization between two identical chaotic systems of the fractional-order lesser date moth model using the global synchronization. Thus, the master system is described by

$$\begin{cases} D^{\alpha}x_1 = x_1(1-x_1) - \frac{x_1y_1}{\beta+x_1}, \\ D^{\alpha}y_1 = -\delta y_1 + \frac{\gamma x_1 y_1}{\beta+x_1} - y_1z_1, \\ D^{\alpha}z_1 = -\eta z_1 + \sigma y_1 z_1, \end{cases} \tag{7}$$

the equations of the slave system are

$$\begin{cases} D^{\alpha}x_2 = x_2(1-x_2) - \frac{x_2y_2}{\beta+x_2} + k_1(x_1-x_2), \\ D^{\alpha}y_2 = -\delta y_2 + \frac{\gamma x_2 y_2}{\beta+x_2} - y_2z_2 + k_2(y_1-y_2), \\ D^{\alpha}z_2 = -\eta z_2 + \sigma y_2 z_2 + k_3(z_1-z_2). \end{cases} \tag{8}$$

Consider the case of the integer-order systems (when $\alpha = 1$), and then by subtracting (7) from (8), we obtain

$$\begin{cases} \dot{e}_1 = e_1 - (x_1+x_2)e_1 - (\frac{y_1}{\beta+x_1})e_1 + \frac{x_2y_1}{(\beta+x_1)(\beta+x_2)}e_1 - \frac{x_2}{\beta+x_2}e_2 - k_1e_1, \\ \dot{e}_2 = -\delta e_2 + [\frac{\gamma y_1}{\beta+x_1} - \frac{\gamma x_2 y_1}{(\beta+x_1)(\beta+x_2)}]e_1 - (z_2 - \frac{x_2}{\beta+x_2})e_2 - y_1e_3 - k_2e_2, \\ \dot{e}_3 = -\eta e_3 + \sigma(y_1e_3 + z_2e_2) - k_3e_3, \end{cases} \tag{9}$$

where $e_1 = x_1 - x_2, \quad e_2 = y_1 - y_2, \quad e_3 = z_1 - z_2$. In matrix form, (9) can be rewritten in the form

$$\dot{e} = (A-K)e + M_{x_1,x_2}e, \tag{10}$$

i.e.,

$$\dot{e} = \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & -\delta & 0 \\ 0 & 0 & -\eta \end{bmatrix} - \begin{bmatrix} k_1 & 0 & 0 \\ 0 & k_2 & 0 \\ 0 & 0 & k_3 \end{bmatrix} \right) \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} +$$

$$\begin{bmatrix} -(x_1+x_2) - \frac{y_1}{\beta+x_1} + \frac{x_2 y_1}{(\beta+x_1)(\beta+x_2)} & -\frac{x_2}{\beta+x_2} & 0 \\ \frac{\gamma y_1}{\beta+x_1} - \frac{\gamma x_2 y_1}{(\beta+x_1)(\beta+x_2)} & -z_2 + \frac{\gamma x_2}{\beta+x_2} & -y_1 \\ 0 & \sigma z_2 & \sigma y_2 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_{u_3} \end{bmatrix},$$

where

$$e = [e_1, e_2, e_3]^T, \quad K = \begin{bmatrix} k_1 & 0 & 0 \\ 0 & k_2 & 0 \\ 0 & 0 & k_3 \end{bmatrix},$$

$$M_{x_1,x_2} = \begin{bmatrix} -(x_1+x_2) - \frac{y_1}{\beta+x_1} + \frac{x_2 y_1}{(\beta+x_1)(\beta+x_2)} & -\frac{x_2}{\beta+x_2} & 0 \\ \frac{\gamma y_1}{\beta+x_1} - \frac{\gamma x_2 y_1}{(\beta+x_1)(\beta+x_2)} & -z_2 + \frac{\gamma x_2}{\beta+x_2} & -y_1 \\ 0 & \sigma z_2 & \sigma y_2 \end{bmatrix}, \tag{11}$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\delta & 0 \\ 0 & 0 & -\eta \end{bmatrix}. \tag{12}$$

From (11) and (12), we get
$(A + M_{x_1,x_2}) + (A + M_{x_1,x_2})^T =$

$$\begin{bmatrix} -2(x_1+x_2) - 2\frac{y_1}{\beta+x_1} + 2\frac{x_2 y_1}{(\beta+x_1)(\beta+x_2)} + 2 & -\frac{x_2}{\beta+x_2} + \frac{\gamma y_1}{\beta+x_1} - \frac{\gamma x_2 y_1}{(\beta+x_1)(\beta+x_2)} & 0 \\ \frac{\gamma y_1}{\beta+x_1} - \frac{\gamma x_2 y_1}{(\beta+x_1)(\beta+x_2)} - \frac{x_2}{\beta+x_2} & -2z_2 + \frac{2\gamma x_2}{\beta+x_2} - 2\delta & -y_1 + \sigma z_2 \\ 0 & \sigma z_2 - y_1 & 2\sigma y_1 - 2\eta \end{bmatrix}. \tag{13}$$

By selecting the feedback control gains k_1, k_2 and k_3 that must satisfy the conditions given in [7], the two coupled integer-order lesser date moth systems are asymptotically synchronized, i.e., synchronization is achieved if the feedback control gains satisfy the following inequalities:

$$\begin{cases} k_1 \geq \frac{1}{2}[-2(x_1+x_2) - \frac{2y_1}{\beta+x_1} + \frac{2x_2 y_1}{(\beta+x_1)(\beta+x_2)} + 2 + |-\frac{x_2}{\beta+x_2} + \frac{\gamma y_1}{\beta+x_1} - \frac{\gamma x_2 y_1}{(\beta+x_1)(\beta+x_2)}| - \mu], \\ k_2 \geq \frac{1}{2}[-2z_2 + \frac{2\gamma x_2}{\beta+x_2} - 2\delta + |-\frac{\gamma x_2 y_1}{(\beta+x_1)(\beta+x_2)} - \frac{x_2}{\beta+x_2} + \frac{\gamma y_1}{\beta+x_1}| + |-y_1 + \sigma z_2| - \mu], \\ k_3 \geq \frac{1}{2}[-2\sigma y_1 - 2\eta + |\sigma z_2 - y_1| - \mu]. \end{cases} \tag{14}$$

Now, the coupled fractional-order lesser date moth systems (7) and (8) are integrated numerically with the parameter values $\gamma = 3$, $\delta = \eta = 1$, $\sigma = 3$, $\beta = 1.15$ and the same fractional order $\alpha = 0.95$. By selecting the feedback control gains as $k_1 = 1.2$, $k_2 = 2.45$, $k_3 = 0.7$, which satisfy the inequalities (14), the drive and response lesser date moth systems (7) and (8) are asymptotically synchronized. For the numerical simulations, we use some documented data for some parameters like $\gamma = 3$, $\delta = \eta = 1$, $\sigma = 3$, $\beta = 1.15$, $h = 0.85$, $\alpha = 0.95$, $\mu = -3.5$, then we have $(x_1, y_1, z_1) = (0.7, 0.3, 0.8)$ and $(x_2, y_2, z_2) = (0.98, 0.35, 0.65)$ and $k_1 = 1.2$, $k_2 = 2.45$, $k_3 = 0.7$. The simulation results are illustrated in Figure 2.

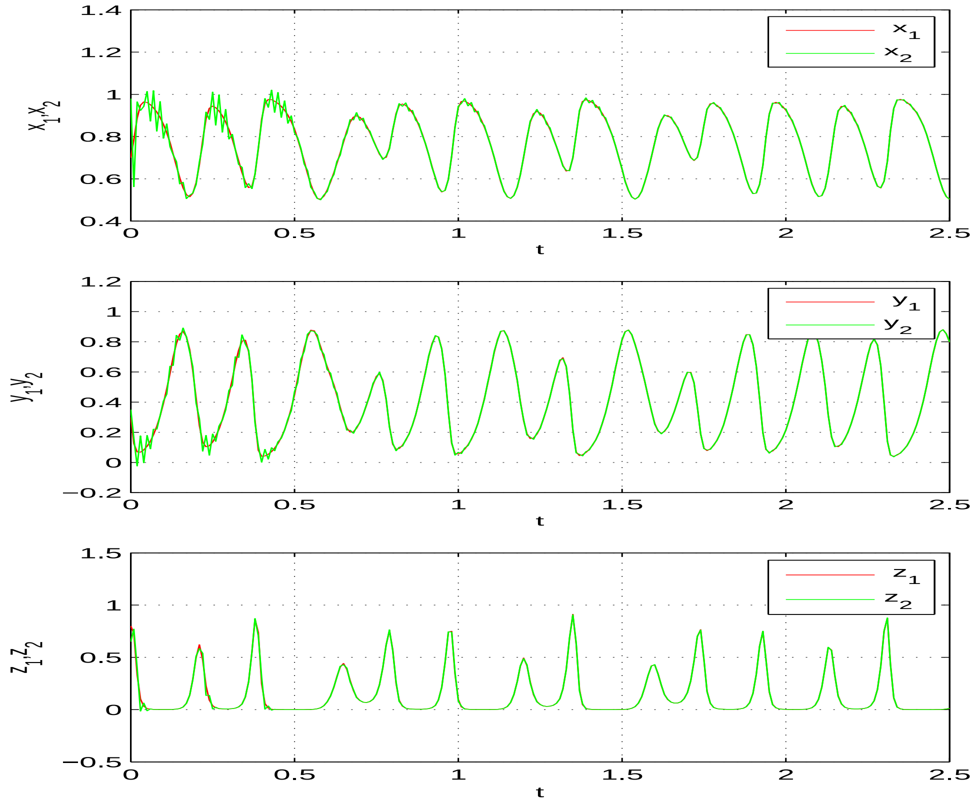

Figure 2: Synchronization of identical fractional-order lesser date moth model for

$$k_1 = 1.2, k_2 = 2.45,\ k_3 = 0.7.$$

5 Synchronization of Identical Fractional-Order Lesser Date Moth Systems by Active Control

In this section, we study chaos synchronization between two identical chaotic systems of the fractional-order lesser date moth model using the technique of active control [5]. Thus, the master system is described by

$$\begin{cases} D^{\alpha}x_1 = x_1(1-x_1) - \frac{x_1 y_1}{\beta + x_1}, \\ D^{\alpha}y_1 = -\delta y_1 + \frac{\gamma x_1 y_1}{\beta + x_1} - y_1 z_1, \\ D^{\alpha}z_1 = -\eta z_1 + \sigma y_1 z_1, \end{cases} \tag{15}$$

the equations of the slave system are

$$\begin{cases} D^{\alpha}x_2 = x_2(1-x_2) - \frac{x_2y_2}{\beta+x_2} + u_1(t), \\ D^{\alpha}y_2 = -\delta y_2 + \frac{\gamma x_2y_2}{\beta+x_2} - y_2z_2 + u_2(t), \\ D^{\alpha}z_2 = -\eta z_2 + \sigma y_2z_2 + u_3(t), \end{cases} \tag{16}$$

where $u_1(t)$, $u_2(t)$, $u_3(t)$ are the active controls.

Subtracting (16) from (15) gives

$$\begin{cases} D^{\alpha}e_1 = e_1 - x_1^2 + x_2^2 - (\frac{y_2}{\beta+x_2})e_1 + \frac{x_1y_2}{(\beta+x_1)(\beta+x_2)}e_1 - \frac{x_1}{\beta+x_1}e_2 + u_1(t), \\ D^{\alpha}e_2 = -\delta e_2 + [\frac{\gamma y_2}{\beta+x_2} - \frac{\gamma x_1y_2}{(\beta+x_1)(\beta+x_2)}]e_1 - (z_2 - \frac{x_1}{\beta+x_1})e_2 - y_1e_3 + u_2(t), \\ D^{\alpha}e_3 = -\eta e_3 + \sigma(y_1e_3 + z_2e_2) + u_3(t), \end{cases} \tag{17}$$

where $e_1 = x_2 - x_1, \quad e_2 = y_2 - y_1, \quad e_3 = z_2 - z_1.$

We introduce a Lyapunov function in terms of the squares of these variables:

$$V(e) = \frac{1}{2}\sum_{i=1}^{3} e_i^2. \tag{18}$$

The fractional-order derivative of the Lyapunov function is given as

$$\begin{aligned} D^{\alpha}V(e) &= e_1[e_1 - (x_2+x_1)e_1 - (\frac{y_2}{\beta+x_2})e_1 + \frac{x_1y_2}{(\beta+x_2)(\beta+x_1)}e_1 - \frac{x_1}{\beta+x_1}e_2] \\ &+ e_2[-\delta e_2 + (\frac{\gamma y_2}{\beta+x_2}e_1 - \frac{\gamma x_1y_2}{(\beta+x_1)(\beta+x_2)})e_1 - (z_2e_2 + \frac{\gamma x_1}{\beta+x_1})e_2 - y_1e_3] \\ &+ e_3[-\eta e_3 + \sigma z_2e_2 + \sigma y_1e_3] + \sum_{i=1}^{3} u_i(t)e_i(t). \end{aligned} \tag{19}$$

From this equation, we conclude that if the active control functions u_i are chosen such that

$$\begin{aligned} u_1(t) &= -[2e_1 - (x_1+x_2)e_1 - (\frac{y_2}{\beta+x_2})e_1 + \frac{x_1y_2}{(\beta+x_1)(\beta+x_2)}e_1 - \frac{x_1}{\beta+x_1}e_2], \\ u_2(t) &= -[(\frac{\gamma y_2}{\beta+x_2} - \frac{\gamma x_1y_2}{(\beta+x_1)(\beta+x_2)})e_1 - (z_2 - \frac{\gamma x_1}{\beta+x_1})e_2 - y_1e_3], \\ u_3(t) &= -[\sigma z_2e_2 + \sigma y_1e_3], \end{aligned}$$

equation (19) becomes

$$D^{\alpha}V(e) = -(e_1^2 + \delta e_2^2 + \eta e_3^2) \leq 0. \tag{20}$$

According to the inequality (18), the system is stable. For the numerical simulations, we use some documented data for some parameters like $\gamma = 3$, $\delta = \eta = 1$, $\sigma = 3$, $\beta = 1.15$, $h = 0.85$, $\alpha = 0.95$, then we have $(x_1, y_1, z_1) = (0.7, 0.3, 0.8)$ and $(x_2, y_2, z_2) = (0.12, 0.21, 0.13)$. The simulation results are illustrated in Figure 3.

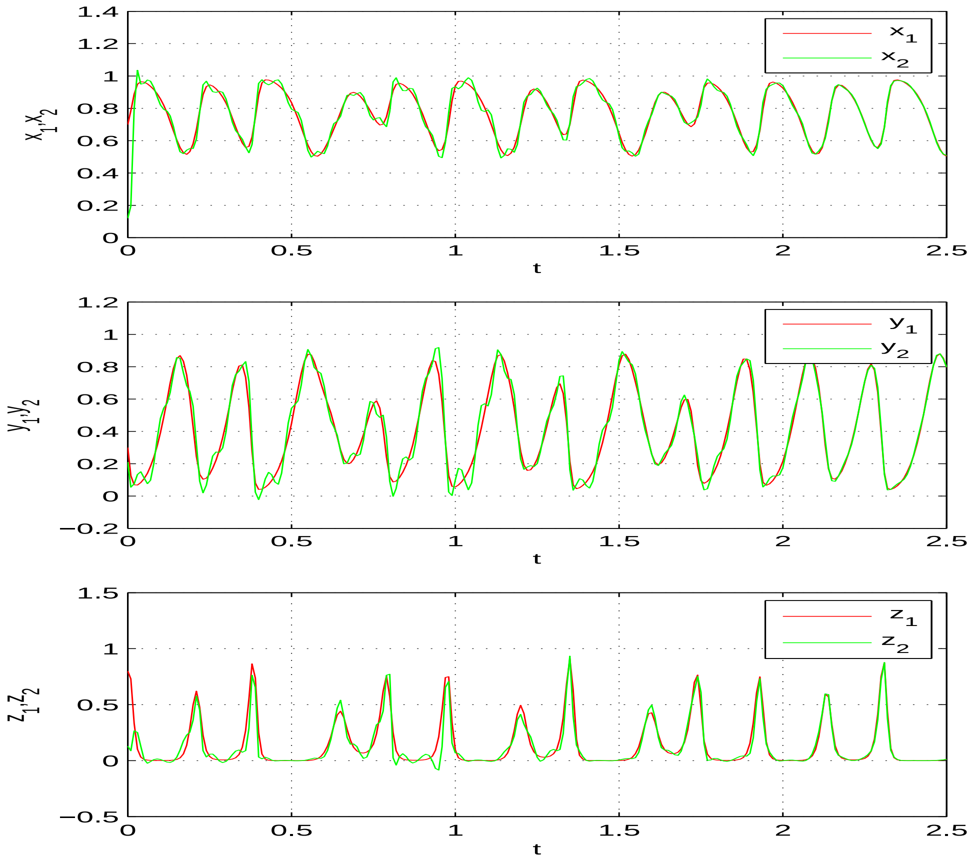

Figure 3: Synchronization of identical fractional-order lesser date moth model.

6 Anti-Synchronization of Identical Fractional-Order Lesser Date Moth Systems by Active Control

In this section, we study chaos anti-synchronization between two identical chaotic systems of the fractional-order lesser date moth model using the technique of active control. Thus, the drive system is described by

$$\begin{cases} D^{\alpha}x_1 = x_1(1-x_1) - \frac{x_1y_1}{\beta+x_1}, \\ D^{\alpha}y_1 = -\delta y_1 + \frac{\gamma x_1y_1}{\beta+x_1} - y_1z_1, \\ D^{\alpha}z_1 = -\eta z_1 + \sigma y_1z_1; \end{cases} \tag{21}$$

the equations of the response system are

$$\begin{cases} D^{\alpha}x_2 = x_2(1-x_2) - \frac{x_2y_2}{\beta+x_2} + u_1(t), \\ D^{\alpha}y_2 = -\delta y_2 + \frac{\gamma x_2y_2}{\beta+x_2} - y_2z_2 + u_2(t), \\ D^{\alpha}z_2 = -\eta z_2 + \sigma y_2z_2 + +u_3(t), \end{cases} \tag{22}$$

where u_1, u_2, u_3 are the active controls.

The anti-synchronization error is defined as

$$\begin{cases} e_1 = x_1 + x_2, \\ e_2 = y_1 + y_2, \\ e_3 = z_1 + z_2. \end{cases} \tag{23}$$

A simple calculation gives the error dynamics

$$\begin{cases} D^\alpha e_1 = e_1 - x_2^2 - x_1^2 - \frac{x_2 y_2}{\beta + x_2} - \frac{x_1 y_1}{\beta + x_1} + u_1(t), \\ D^\alpha e_2 = -\delta e_2 + \frac{\gamma x_1 y_1}{\beta + x_1} + \frac{\gamma x_2 y_2}{\beta + x_2} - z_2 y_2 - y_1 z_2 + u_2(t), \\ D^\alpha e_3 = -\eta e_3 + \sigma y_1 z_1 + \sigma y_2 z_2 + u_3(t). \end{cases} \tag{24}$$

We consider the active nonlinear controller defined by

$$\begin{cases} u_1(t) = x_2^2 + x_1^2 + \frac{x_2 y_2}{\beta + x_2} + \frac{x_1 y_1}{\beta + x_1} - 2e_1, \\ u_2(t) = -\frac{\gamma x_1 y_1}{\beta + x_1} - \frac{\gamma x_2 y_2}{\beta + x_2} + z_2 y_2 + y_1 z_2, \\ u_3(t) = -\sigma y_2 z_2 - \sigma y_1 z_1. \end{cases} \tag{25}$$

Substitution of (25) into (24) yields the linear error dynamics

$$\begin{cases} D^\alpha e_1 = -e_1, \\ D^\alpha e_2 = -\delta e_2, \\ D^\alpha e_3 = -\eta e_3. \end{cases} \tag{26}$$

We consider the quadratic Lyapunov function defined by

$$V(e) = \frac{1}{2} e^T e = \frac{1}{2}(e_1^2 + e_2^2 + e_3^2), \tag{27}$$

which is a positive definite function on $\mathbb{R}^3$. The fractional-order derivative of the Lyapunov function is given as

$$D^\alpha V(e) = -e_1^2 - \delta e_2^2 - \eta e_3^2 \leq 0. \tag{28}$$

According to the inequality (27), the system is stable. For the numerical simulations, we use some documented data for some parameters like $\gamma = 3$, $\delta = \eta = 1$, $\sigma = 3, \beta = 1.15$, $h = 0.85$, $\alpha = 0.95$, then we have $(x_1, y_1, z_1) = (0.7, 0.3, 0.8)$ and $(x_2, y_2, z_2) = (-0.99, -0.11, -0.15)$. The simulation results are illustrated in Figure 4.

7 Conclusion

In this paper, we have studied the phenomenon of chaos synchronization and anti-synchronization between two identical chaotic systems of the fractional-order lesser date moth model. Our results demonstrate that if one uses the technique of global synchronization and the technique of active control, chaos synchronization can be achieved between two identical chaotic systems. On the other hand, if one uses the technique of active control, chaos anti-synchronization can be achieved between two identical chaotic systems.

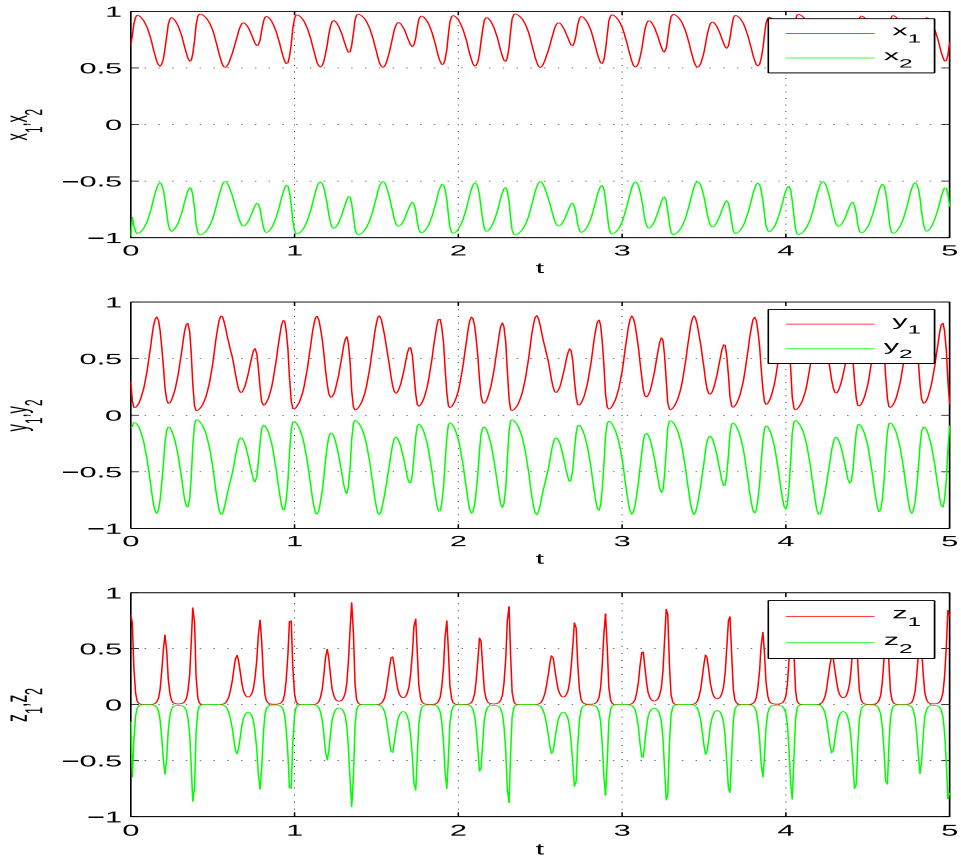

Figure 4: Anti-synchronization of identical fractional-order lesser date moth model.

References

[1] A. E. Matouk. Chaos, feedback control and synchronization of a fractional-order modified autonomous Van der Pol-Duffing circuit. *Commun Nonlinear Sci. Numer. Simulat.* **16** (2011) 975–986.

[2] B. Zsolt. Chaos theory and power spectrum analysis in computerized cardiotocography. *Eur. J. Obstet. Gynecol. Reprod. Biol.* **71** (2) (1997) 163–168.

[3] D. Matignon. Stability result on fractional differential equations with applications to control processing. *Computational Engineering in Systems and Application multi-conference, IMACS, In: IEEE-SMC Proceedings, Lille, France* **2** (1996) 963–968.

[4] D. Pazo, M. A. Zaks and J. Kurths. Role of unstable periodic orbit in phase and lag synchronization between coupled chaotic oscillators. *Chaos* **13** (2003) 309–318.

[5] E. W. Bai and K. E. Lonngren. Synchronization of two Lorenz systems using active control. *Chaos, Solitons and Fractals*, **9** (1998) 1555–1561.

[6] G. Alvarez, S. Li, F. Montoya, G. Pastor and M. Romera. Breaking projective chaos synchronization secure communication using filtering and grneralized synchronization. *Chaos, Solutons and Fractals* **24** (2005) 775–783.

[7] G. P. Jiang, K. S. Tang and G. Chen. A simple global synchronization criterion for coupled chaotic systems. *Chaos, Solitons and Fractals* **15** (2003) 925–935.

[8] H. Targhvafard and G. H. Enjace. Phase and anti-phase synchronization of fractional-order chaotic systems via active control. *Commun. Nonlinear Sci. Numer. Simul.* **16** (2011) 4079–4408.

[9] K. Murali and M. Lakshmanan. Secure communication using a compound signal from generalized synchronizable chaotic system. *Phys. Letters A* **241** (1998) 303–310.

[10] L. Kocarev and U. Parlitz. General approach for chaotic synchronization with applications to communication. *Phys. Rev. Lett.* **74** (1995) 5028–5030.

[11] L. M. Pecora and T. L. Carroll. Synchronization in chaotic systems. *Phys. Rev. Lett.* **64** (1990) 821–824.

[12] M. El-Shahed, Juan J. Nieto, A. M. Ahmed and I. M. E. Abdelstar. Fractional-order model for biocontrol of the lesser date moth in palm trees and its discretization. *Advance in Difference Equations* (2017) 2017–2295.

[13] N. Laskin. Fractional market dynamics. *Physica A* **287** (2000) 482–492.

[14] N. Zhou, Y. Wang, L. Gong, H. He and J. Wu. Novel single-channel color image encryption algorithm based on chaos and fractional Fourier transform. *Optics Communications* **284** (2011) 2789–2796.

[15] R. Mainieri and J. Rehacek . Projective synchronization in three dimensional chaotic systems. *Phys. Rev. Lett.* **82** (1999) 3042–3045.

[16] S. H. Strogatz. *With Applications to Physics, Biology, Chemistry and Engineering.* Nonlinear Dynamics and Chaos, Perseus Books Pub., 1994.

[17] S. Vaidyanathan. Lotka-Volterra two-species mutualistic biology models and their ecological monitoring. *Pharm. Tech. Research* **8** (2015) 199–212.

[18] W. H. Deng and C.P. Li. Chaos synchronization of the fractional Lü system. *Physica A* **353** (2005) 61–72.

[19] W. Laouira and N. Hamri. Feedback Control of Chaotic systems by Using Jacobian Matrix Conditions. *Nonlinear Dynamics and Systems Theory* **18** (3) (2018) 285–295.

[20] Y. Zhang and J. Sun. Chaotic synchronization and anti-synchronization based on suitable separation. *Physics Letters A* **330** (2004) 442–447.