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Existence and Stability of Equilibrium Points in the Problem of a Geo-Centric Satellite Including the Earth's Equatorial Ellipticity

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Abstract: This paper deals with the existence and stability of the equilibrium points in the problem of a geo-centric satellite including the earth's equatorial ellipticity. We have determined the equations of motion of the geo-centric satellite which include the earth's equatorial ellipticity parameter Γ (the satellite's angular position relative to the minor axis of the earth's equatorial section) and then we have investigated the existence and stability of equilibrium points. It is observed that there exists an infinite number of equilibrium points which lie on a circle for different values of Γ. It is shown that the effect of the earth's equatorial ellipticity parameter Γ on the location of equilibrium points is very small (i.e., the coordinates of the equilibrium points are different after the fifth decimal places). Further, we have observed that the collinear points are unstable for different values of Γ. The non-collinear points lying on the y-axis are unstable for different values of Γ. We have also found that some of the non-collinear points lying on the circle are stable and others are unstable for different values of Γ.

Keywords: *geo-centric satellite; earth's equatorial ellipticity; equilibrium points and stability.*

Mathematics Subject Classification (2010): 70F07, 70F10, 70F15.

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1 Introduction

The motion of an artificial satellite is affected by various forces, some of which are the earth's gravitational field, atmospheric drag, solar radiation pressure, lunar and solar gravitational fields, relativistic effect and Poynting-Robertson drag. In recent years, the idea of establishing an artificial satellite in a synchronous equatorial orbit about the earth has become increasingly attractive. Since such a satellite would remain above the same position on the earth's equator, it could be used as a communication relay station between any two points on the earth which are within its field of view.

Sehnal [8] discussed the influence of the equatorial ellipticity of the earth's gravitational field on the motion of a close satellite. The method of variation of constants is applied to discuss the perturbation of angular elements. Blitzer [4] discussed the motion of a satellite under the influence of the longitude-dependent terms of the geopotential in a frame of reference rotating with the mean motion of the spacecraft. Allan [1] investigated the motion in longitude of a nominally geostationary satellite due to the tesseral harmonics. He further developed the corrective impulses required for the principal $J_{2,2}$ term. Wagner [9] investigated the motion of 24-hour near equatorial earth satellites in an earth gravity field through the 4^{th} order. Bhatnagar and Mehra [3] discussed the motion of a satellite under the gravitational forces of the sun, moon, earth (including the ellipticity of the earth's equator) and solar radiation pressure. They studied the orientation of the orbital plane of a geosynchronous satellite. It is shown that the significant effect of the earth's equatorial ellipticity is to produce a change in the relative angular position Γ of the satellite as seen from the earth. Bhatnagar and Kaur [2] studied the in-plane perturbation of the satellite caused by the attraction of the sun, moon and oblate earth including the earth's equatorial ellipticity. Gilthorpe and Moore [6] developed a theory for the motion of a satellite in a nearly circular orbit perturbed by zonal harmonic terms in the earth's gravity field. Mark [7] developed a first-order analytical theory of the tesseral harmonic J_2^2 effects on satellite orbits. Correa et al. [13] investigated two models of the restricted three-body and four-body problems. They determined the transfer orbits from a parking orbit around the Earth to the halo orbit in both the dynamical models. They also compared the total velocity increment to both the models. Prado [14] studied space trajectories in the circular restricted three-body problem. He assumed that the spacecraft moves under the gravitational forces of two massive bodies which are in circular orbits. He also determined orbits which can be used to transfer a spacecraft from one body back to the same body or to transfer a spacecraft from one body to the respective Lagrangian points L4 and L5. Yadav and Aggarwal [10] investigated resonances resulting from the commensurability between the mean motion of a geo-centric satellite and the earth's equatorial ellipticity parameter. Kumari and Kushvah [11] studied the stability regions of equilibrium points in the restricted four-body problem with oblateness effects. Camargo et al. [12] studied the attitude synchronization of two dumbbell shaped satellites by using a generalized Hamiltonian systems approach. They presented the numerical results of the synchronization behavior of the satellites.

In this paper, we aim to investigate the impact of the earth equatorial ellipticity parameter Γ on the location and stability of the equilibrium points, which exist in the problem of a geo-centric satellite. The effect of the earth's equatorial ellipticity parameter Γ is also analyzed on the zero-velocity curves by taking different values of the Jacobi constants.

This paper is organized as follows. We write the equations of motion of geo-centric satellite and find the Jacobi integral of the system in Section 2. In Section 3, we determine equilibrium points and describe the zero-velocity curves whereas, in Section 4, we examine the stability of the equilibrium points. Finally, Section 5 includes the discussion and conclusions of the paper.

2 Configuration and the Equation of Motion

The equations of motion of the geo-centric satellite $P(r, \theta, \phi)$ moving around the earth E in the equatorial plane are given in [5]:

$$M_s\left(\ddot{r} - r\dot{\theta}^2\cos^2\phi - r\dot{\phi}^2\right) = \frac{\partial U}{\partial r}, \tag{1}$$

$$M_s\left(\frac{1}{r\cos\phi}\frac{d}{dt}\left(r^2\dot{\theta}\cos\phi\right)\right) = \frac{1}{r\cos\phi}\frac{\partial U}{\partial\theta}, \tag{2}$$

$$M_s\left(\frac{1}{r}\frac{d}{dt}\left(r^2\dot{\phi}\right) + r\dot{\theta}^2\cos\phi\sin\phi\right) = \frac{1}{r}\frac{\partial U}{\partial\theta}. \tag{3}$$

Here U is known as the earth's gravitational potential which can be written as

$$\mathrm{U} = \frac{g_0{R_0}^2}{r}\left\{1 - \frac{J_2{R_0}^2}{r^2}\left(\frac{3\sin^2\phi - 1}{2}\right)\right\} + \frac{3g_0{R_0}^2}{r}\left(\frac{{J_2}^{(2)}{R_0}^2}{r^2}\cos^2\phi\cos 2\Gamma\right), \tag{4}$$

where:

$g_0 = 9.8\mathrm{m/sec}^2$ = gravitational acceleration on the earth's surface,

r = radial distance of the satellite from the centre of the earth,

M_s = mass of the satellite,

$J_2 = 1.08219 \times 10^{-3}$ = coefficient due to the oblateness of the earth,

$R_0 = 6367.4 \times 10^5\mathrm{cm}$ = mean radius of the earth,

$J_2^{(2)} = 2.32 \times 10^{-6}$ = coefficient due to the earth's equatorial ellipticity,

$\phi = \angle PEM$ = latitude of the satellite (Fig.1(a)),

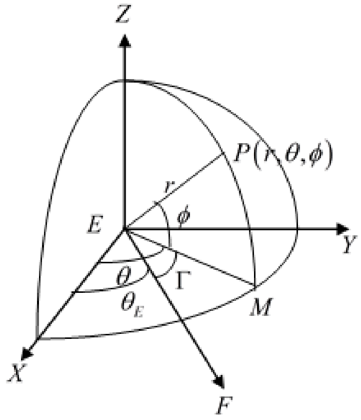
$\theta = \angle XEF$ = longitude of the satellite (Fig.1(a)),

$\Gamma = \angle MEF = \theta - \theta_E$ = satellite angular position relative to minor axis of the earth's equatorial section (Fig. 1(a)),

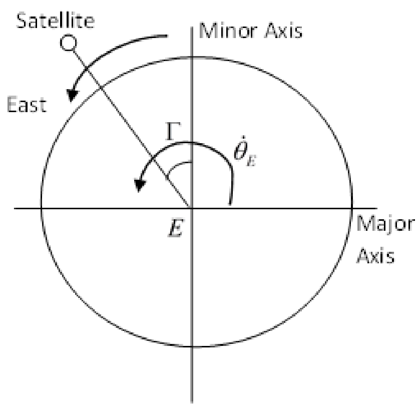
$\theta_E = \angle XEF$ = angular position of the minor axis of the earth's equatorial section ,

$\dot{\theta}_E$ = angular rate of rotation of the earth (Fig. 1(b)),

X, Y, Z=inertial coordinate system with the origin at the centre of the earth and XY plane in the earth's equatorial plane.



(a) Configuration of the satellite P



(b) The earth's equitorial ellipticity parameter Γ

Figure 1: Configuration of the geo-centric satellite.

Substituting the value of U from (4) in equations (1), (2) and (3), we obtain

$$\ddot{r} - r\dot{\theta}^2\cos^2\phi - r\dot{\phi}^2 = -\frac{g_0{R_0}^2}{r^2} + 3\frac{J_2 g_0 {R_0}^4}{2r^4}\left(3\sin^2\phi - 1\right) - 9\frac{J_2^{(2)} g_0 {R_0}^4}{r^4}\cos^2\phi\cos 2\Gamma, \tag{5}$$

$$\frac{1}{r\cos\phi}\frac{d}{dt}\left(r^2\dot{\theta}\cos\phi\right) = -6\frac{J_2^{(2)} g_0 {R_0}^4}{r^4}\cos\phi\sin 2\Gamma, \tag{6}$$

$$\frac{1}{r}\frac{d}{dt}\left(r^2\dot{\phi}\right) + r\dot{\theta}^2\cos\phi\sin\phi = -3\frac{J_2 g_0 {R_0}^4}{2r^4}\sin\phi\cos\phi - 6\frac{J_2^{(2)} g_0 {R_0}^4}{r^4}\sin\phi\cos\phi\cos 2\Gamma. \tag{7}$$

We assume that the satellite P lies in the equatorial plane i.e., $\phi = 0$.
Equations (5) and (6) become

$$\ddot{r} - r\dot{\theta}^2 = -\frac{g_0{R_0}^2}{r^2} - 3\frac{J_2 g_0 {R_0}^4}{2r^4} - 9\frac{J_2^{(2)} g_0 {R_0}^4}{r^4}\cos 2\Gamma, \tag{8}$$

$$\frac{1}{r}\frac{d}{dt}\left(r^2\dot{\theta}\right) = r\ddot{\theta} + 2\dot{r}\dot{\theta} = -6\frac{J_2^{(2)} g_0 {R_0}^4}{r^4}\sin 2\Gamma. \tag{9}$$

In the synodic coordinate system, we have

$$\ddot{\bar{r}} = (\ddot{r} - r\dot{\theta}^2)\hat{e_r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{e_\theta} + \ddot{z}\hat{k},$$

where

$$\hat{e_r} = \cos\theta\ \hat{i} + \sin\theta\ \hat{j} \quad \hat{e_\theta} = -\sin\theta\ \hat{i} + \cos\theta\ \hat{j},$$

$$\ddot{\bar{r}} = \frac{d}{dt}\left(\frac{d\bar{r}}{dt}\right) = \frac{d}{dt}\left(\frac{\partial\bar{r}}{\partial t} + \hat{w}\times\hat{r}\right) = \frac{\partial^2\bar{r}}{\partial t^2} + 2\bar{w}\times\frac{\partial\bar{r}}{\partial t} + \bar{w}\times(\bar{w}\times\bar{r}),$$

$$\bar{w} = n\hat{k},$$

$$\bar{r} = r\hat{e_r} = r\cos\theta\ \hat{i} + r\sin\theta\ \hat{j} = x\ \hat{i} + y\ \hat{j}.$$

We take the origin of coordinates at the centre of mass of the earth, the plane of motion of the infinitesimal satellite P is in the xy-plane orthogonal to the line of motion of the centre of mass of the earth and the motion of the earth takes place on the z-axis. The equations of the motion of $P(x, y)$ in the synodic coordinate system and dimensionless variables, i.e., the distance between the synchronous satellite and the earth is unity, the mass of the earth is unity and choose time t such that the universal gravitational constant G is unity, are

$$\ddot{x} - 2n\dot{y} - n^2x = \frac{1}{r}\left[(\ddot{r} - r\dot{\theta}^2)x - (r\ddot{\theta} + 2\dot{r}\dot{\theta})y\right],$$
$$\ddot{y} + 2n\dot{x} - n^2y = \frac{1}{r}[(\ddot{r} - r\dot{\theta}^2)y + (r\ddot{\theta} + 2\dot{r}\dot{\theta})x].$$

We may take

$$r^2\dot{\theta} = h \text{ (constant)}.$$

Differentiating with respect to t, we get

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0.$$

The equations of the motion of P in the synodic coordinate system and dimensionless variables are

$$\ddot{x} - 2n\dot{y} - n^2x = \frac{1}{r}[(\ddot{r} - r\dot{\theta}^2)x],$$
$$\ddot{y} + 2n\dot{x} - n^2y = \frac{1}{r}[(\ddot{r} - r\dot{\theta}^2)y].$$

Using equation (8), we get

$$\ddot{x} - 2n\dot{y} = n^2x + \frac{1}{r}\left(-\frac{g_0{R_0}^2}{r^2} - 3\frac{J_2g_0{R_0}^4}{2r^4} - 9\frac{J_2^{(2)}g_0{R_0}^4}{r^4}\cos 2\Gamma\right)x, \tag{10}$$

$$\ddot{y} + 2n\dot{x} = n^2y + \frac{1}{r}\left(-\frac{g_0{R_0}^2}{r^2} - 3\frac{J_2g_0{R_0}^4}{2r^4} - 9\frac{J_2^{(2)}g_0{R_0}^4}{r^4}\cos 2\Gamma\right)y. \tag{11}$$

Now, we define a function F such that

$$F = \frac{n^2}{2}(x^2 + y^2) + \frac{g_0{R_0}^2}{r} + \frac{g_0{R_0}^4}{3r^3}\left(\frac{3}{2}J_2 + 9J_2^{(2)}\cos 2\Gamma\right).$$

Hence, equations (10) and (11) become

$$\ddot{x} - 2n\dot{y} = F_x, \tag{12}$$
$$\ddot{y} + 2n\dot{y} = F_y, \tag{13}$$

where F_x and F_y are the partial derivatives of F with respect to x and y, respectively. The integral analogous to the Jacobi integral is

$$\dot{x}^2 + \dot{y}^2 = 2F - C. \tag{14}$$

The perturbed mean motion of the earth is governed by

$$n(\Gamma) = \sqrt{g_0{R_0}^2\left(1 + \frac{3}{2}{R_0}^2J_2 + 9J_2^{(2)}\cos 2\Gamma\right)}.$$

3 Location of Equilibrium Points

The points described by $F_x = 0$ and $F_y = 0$ are called the equilibrium points. In fact, all the derivatives of the co-ordinates with respect to time are zero at these points. Therefore the satellite P placed at the equilibrium points with zero velocity, will stay there. The terms "Libration points" and "Lagrangian points" are also used in place of equilibrium points.

Thus, we have

$$F_x = 0 \text{ implies } \quad x \times f(x, y) = 0,$$
$$F_y = 0 \text{ implies } \quad y \times f(x, y) = 0,$$

where

$$f(x, y) = n^2 - \frac{g_0 {R_0}^2}{(x^2 + y^2)^{\frac{3}{2}}} + \frac{g_0 {R_0}^4}{3(x^2 + y^2)^{\frac{5}{2}}} \left(\frac{3}{2} J_2 + 9 J_2^{(2)} \cos 2\Gamma \right).$$

3.1 Collinear points

Solving the above equations for $f(x, y) = 0$ when $y = 0$ and by taking different values of the earth's equatorial ellipticity parameter Γ, we obtained two collinear points on the x-axis. In Table 1, we have shown the coordinates of these collinear points for different values of the earth's equatorial ellipticity parameter Γ. We noticed the one equilibrium point is on the positive side of the x-axis, while the other equilibrium point lies on the negative side of the x-axis for different values of the earth's equatorial ellipticity parameter Γ. Also, the effect of the earth's equatorial ellipticity parameter Γ on the location of equilibrium points on the x-axis is very small (i.e., the coordinates of the equilibrium points are different after the fifth decimal places) and the number of equilibrium points remains same for different values of Γ.

3.2 Non-collinear points lying on the y-axis

The non-collinear points lying on the y-axis are the solution of the equations $f(x, y) = 0$ when $x = 0$. We have found that there exist two non-collinear points lying on the y-axis for different values of the earth's equatorial ellipticity parameter Γ (Table 2). Also, the effect of the earth's equatorial ellipticity parameter Γ on the location of non-equilibrium points lying on the y-axis is very small and the number of equilibrium points remains same for different values of Γ.

3.3 Non-collinear points lying on the circle

The non-collinear points lying on the y-axis are the solution of the equations $f(x, y) = 0$ when $x \neq 0$ and $y \neq 0$. We observed that there exists an infinite number of non-collinear points lying on the circle. We have shown the location of some of the non-collinear points lying on the circle in Table 3.

3.4 Zero-velocity curves

Equation (14) represents the relation between the positions and square of the velocity of the satellite P in the rotating coordinate system. Using initial conditions, the Jacobi constant C can be found numerically. Therefore the contour curves describing the

Γ	Collinear Equilibrium Points	Stability
$0°$	$(-0.9994650500230987, 0)$ $(0.9994650500230987`, 0)$	Unstable
$5°$	$(-0.9994751560349406`, 0)$ $(0.9994751560349406`, 0)$	Unstable
$10°$	$(-0.9994753049325603, 0)$ $(0.9994753049325603, 0)$	Unstable
$15°$	$(-0.999475391830673, 0)$ $(0.999475391830673, 0)$	Unstable
$20°$	$(-0.9994754533738673, 0)$ $(0.9994754533738673, 0)$	Unstable
$25°$	$(-0.9994750068339644, 0)$ $(0.9994750068339644, 0)$	Unstable
$30°$	$(-0.9994752431809284, 0)$ $(0.9994752431809284, 0)$	Unstable
$35°$	$(-0.9994753527773587, 0)$ $(0.9994753527773587, 0)$	Unstable
$40°$	$(-0.9994754248202805, 0)$ $(0.9994754248202805, 0)$	Unstable
$45°$	$(-0.9994754785409711, 0)$ $(0.9994754785409711, 0)$	Unstable

Table 1: Location and stability of collinear equilibrium points.

boundaries of the permitted region within the infinitesimal satellite P move freely and can be found by using equation (14). These curves obtained in the XY-plane by taking $\dot{x} = \dot{y} = 0$ are known as zero-velocity curves and are given by $2F = C$. Fig. 3 shows zero-velocity curves at $\Gamma = 0°$, for different values of the Jacobi constant C taken in increasing order. Fig. 4 indicates zero-velocity curves at $\Gamma = 15°$, for different values of the Jacobi constant C taken in increasing order. Fig. 5 shows zero-velocity curves at $\Gamma = 30°$, for different values of the Jacobi constant C taken in increasing order. Fig. 6 indicates zero-velocity curves at $\Gamma = 45°$, for different values of the Jacobi constant C taken in increasing order. It is observed that at a fixed value of the earth's equatorial ellipticity parameter Γ, on increasing the values of the Jacobi constant C, the represented possible boundary regions decrease, where the satellite can move freely. We also noticed that the possible boundary regions depend on the Jacobi constant, while the effect of the earth's equatorial ellipticity parameter Γ on the possible boundary regions is minimal.

4 Stability of Equilibrium Points

To study the stability of equilibrium points, we denote the location of equilibrium points by (x_0, y_0) and consider a small displacement (ξ, η) from the point such that

$$x = x_0 + \xi, \; y = y_0 + \eta.$$

Γ	Non-Collinear Equilibrium Points lying on the y-axis	Stability
0°	$(0, -0.9994650500230987)$ $(0, 0.9994650500230987)$	Unstable
5°	$(0, -0.9994751560349406)$ $(0, 0.9994751560349406)$	Unstable
10°	$(0, -0.9994753049325603)$ $(0, 0.9994753049325603)$	Unstable
15°	$(0, -0.999475391830673)$ $(0, 0.999475391830673)$	Unstable
20°	$(0, -0.9994754533738673)$ $(0, 0.9994754533738673)$	Unstable
25°	$(0, -0.9994750068339644)$ $(0, 0.9994750068339644)$	Unstable
30°	$(0, -0.9994752431809284)$ $(0, 0.9994752431809284)$	Unstable
35°	$(0, -0.9994753527773587)$ $(0, 0.9994753527773587)$	Unstable
40°	$(0, -0.9994754248202805)$ $(0, 0.9994754248202805)$	Unstable
45°	$(0, -0.9994754785409711)$ $(0, -0.9994754785409711)$	Unstable

Table 2: Location and stability of non-collinear equilibrium points lying on the y-axis.

Substituting these values in equations of motion (12) and (13), we obtain the variational equations as

$$\ddot{\xi} - 2n\dot{\eta} = \left(F^0_{xx}\right)\xi + \left(F^0_{xy}\right)\eta, \tag{15}$$

$$\ddot{\eta} + 2n\dot{\xi} = \left(F^0_{yx}\right)\xi + \left(F^0_{yy}\right)\eta, \tag{16}$$

where the superscript '0' indicates that the partial derivatives are evaluated at the equilibrium point (x_0, y_0). Let the solution of the variational equations (15) and (16) be

$$\xi = Ae^{\lambda t}, \eta = Be^{\lambda t},$$

where A, B and λ are constants. Then equations (15) and (16) will have a non-trivial solution for A and B when

$$\begin{vmatrix} \lambda^2 - F^0_{xx} & -2n\lambda - F^0_{xy} \\ 2n\lambda - F^0_{yx} & \lambda^2 - F^0_{xx} \end{vmatrix} = 0.$$

On expanding the determinant, we obtain the characteristic equation corresponding to the variational equations (15) and (16) as

$$\lambda^4 - (F^0_{xx} + F^0_{yy} - 4n^2)\lambda^2 + F^0_{xx}F^0_{yy} - \left(F^0_{xy}\right)^2 = 0. \tag{17}$$

The four roots of characteristic equation (17) play an important role for determining the stability of equilibrium points. An equilibrium point will be stable if the above equation

Γ	Non-Collinear Equilibrium Points lying on the circle	Stability
$0°$	$(0.027075043161954526, -0.9990982575580011)$ $(0.950631409170543, 0.3085940863271852)$	Unstable Stable
$5°$	$(0.02706787135141031, -0.9991084123488571)$ $(0.9506385809810823, 0.3086042411180343)$	Unstable Stable
$10°$	$(0.027067763883445894, -0.9991085645161091)$ $(-0.446978878143267, 0.893957756286534)$	Stable Unstable
$15°$	$(0.446978974966415, -0.8939578053071354)$ $(0.9506387512193974, 0.3086044821637891)$	Stable Unstable
$20°$	$(0.027067656634011672, -0.9991087163739373)$ $(0.9506387956984799, 0.3086045451431129)$	Unstable Stable
$25°$	$(0.02706762217188341, -0.9991087651699426)$ $(-0.4469789651947475, -0.893957932725505)$	Stable Unstable
$30°$	$(0.9945151859441937, 0.09945151482739278)$ $(0.95063885829031, 0.3086046337688429)$	Unstable Stable
$35°$	$(0.027067570280104203, -0.9991088386451358)$ $(-0.9945152184903229, 0.0994515209075532)$	Stable Unstable
$40°$	$(0.9945152473685147, -0.09945151908563067)$ $(0.9506389026192351, 0.3086046965355544)$	Unstable Stable
$45°$	$(0.027067531585647564, -0.999108893433827)$ $(0.9506389207468378, 0.30860472220299384)$	Stable Unstable

Table 3: Location and stability of non-collinear equilibrium points lying on the circle.

evaluated at the equilibrium point has four pure imaginary roots or complex roots with negative real parts.

4.1 Stability of collinear points

At the collinear point $(-0.9994650500230987,\ 0)$, at $(\Gamma = 0°)$, the characteristic roots are given by $\lambda_1 = -0.00125276$ $\lambda_2 = -0.12927\iota, \lambda_3 = 0.12927\iota, \lambda_4 = 0.00125276$. At the collinear point $(0.9994650500230987, 0)$ at $(\Gamma = 0°)$, the characteristic roots are given by $\lambda_1 = -0.00125276, \lambda_2 = -0.12927\iota, \lambda_3 = 0.12927\iota, \lambda_4 = 0.00125276$. Thus both the collinear points are unstable. For $\Gamma = 45°$, at the non-collinear point $(0.9994754785409711,\ 0)$ the characteristic roots are given by $\lambda_1 = -1.81113 \times 10^{-9}\iota, \lambda_2 = 1.81113 \times 10^{-9}\iota, \lambda_3 = -0.129266\iota, \lambda_4 = 0.129266\iota$. For $\Gamma = 45°$, at the non-collinear point $(-0.9994754785409711,\ 0)$ the characteristic roots are given by $\lambda_1 = -1.81113 \times 10^{-9}\iota, \lambda_2 = 1.81113 \times 10^{-9}\iota, \lambda_3 = -0.129266\iota, \lambda_4 = 0.129266\iota$. Similarly, we have also examined that both the collinear points are unstable for other values of the earth's equatorial ellipticity parameter Γ (Table 1).

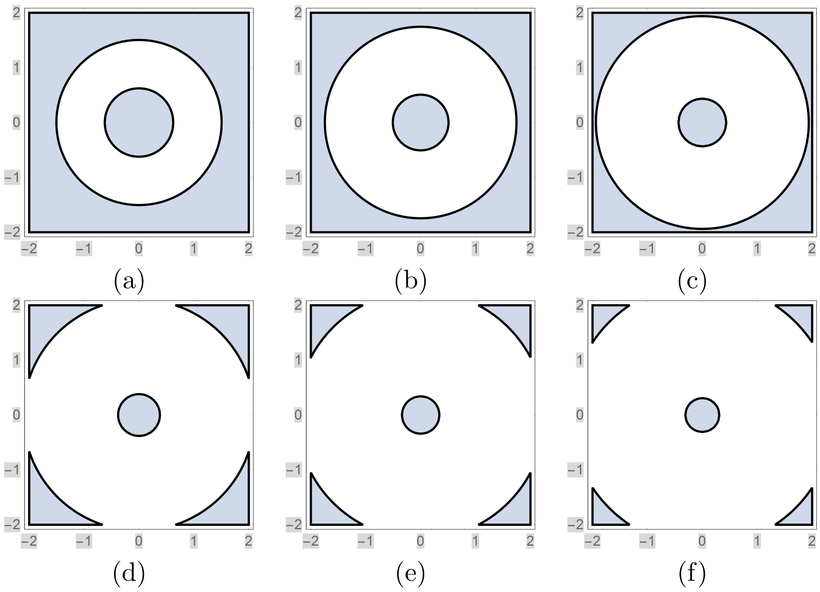


Figure 2: Effect on zero-velocity curves at the ellipticity parameter $\Gamma = 0^\circ$ and different values of the Jacobi constants: (a) $\Gamma = 0^\circ$ and C=0.06; (b) $\Gamma = 0^\circ$ and C=0.07; (c) $\Gamma = 0^\circ$ and C=0.08; (d) $\Gamma = 0^\circ$ and C=0.09; (e) $\Gamma = 0^\circ$ and C=0.10; (f) $\Gamma = 0^\circ$ and C=0.11.

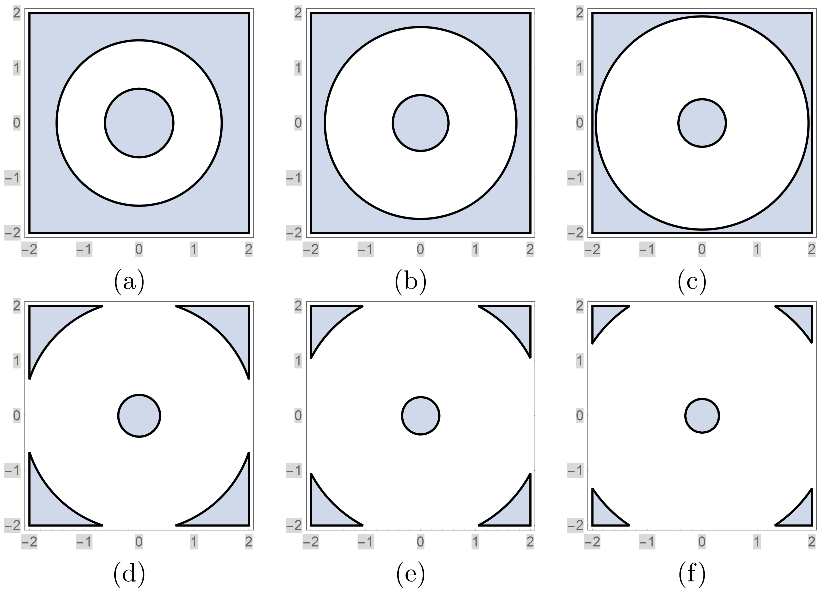


Figure 3: Effect on zero-velocity curves at the ellipticity parameter $\Gamma = 15^\circ$ and different values of the Jacobi constants: (a) $\Gamma = 15^\circ$ and C=0.06; (b) $\Gamma = 15^\circ$ and C=0.07; (c) $\Gamma = 15^\circ$ and C=0.08; (d) $\Gamma = 15^\circ$ and C=0.09; (e) $\Gamma = 15^\circ$ and C=0.10; (f) $\Gamma = 15^\circ$ and C=0.11.

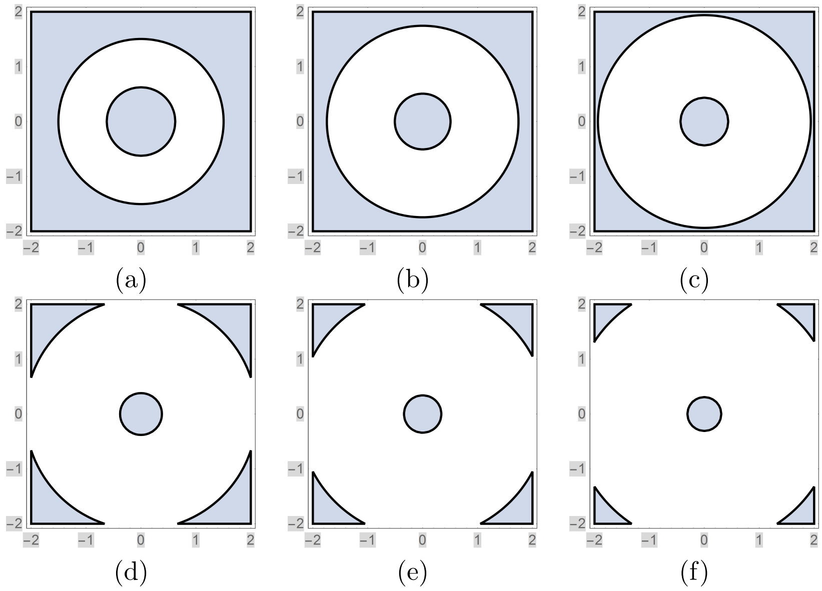


Figure 4: Effect on zero-velocity curves at the ellipticity parameter $\Gamma = 30^\circ$ and different values of the Jacobi constants: (a) $\Gamma = 30^\circ$ and C=0.06; (b) $\Gamma = 30^\circ$ and C=0.07; (c) $\Gamma = 30^\circ$ and C=0.08; (d) $\Gamma = 30^\circ$and C=0.09; (e) $\Gamma = 30^\circ$ and C=0.10; (f) $\Gamma = 30^\circ$ and C=0.11.

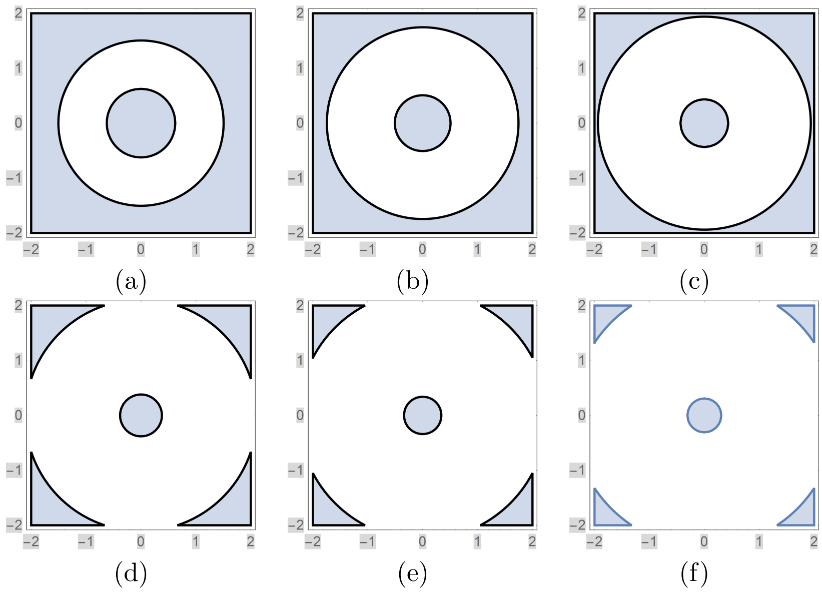


Figure 5: Effect on zero-velocity curves at the ellipticity parameter $\Gamma = 45^\circ$and different values of the Jacobi constants: (a) $\Gamma = 45^\circ$and C=0.06, (b) $\Gamma = 45^\circ$and C=0.07, (c) $\Gamma = 45^\circ$and C=0.08, (d) $\Gamma = 45^\circ$and C=0.09, (e) $\Gamma = 45^\circ$and C=0.10, (f) $\Gamma = 45^\circ$and C=0.11.

4.2 Stability of non-collinear points lying on the y-axis

At the non-collinear point $(0, 0.9994650500230987)$ for $\Gamma = 0^\circ$ (i.e., in the case of a geo synchronous satellite), the characteristic roots are given by $\lambda_1 = -0.00125276, \lambda_2 =$

$-0.12927\iota, \lambda_3 = 0.12927\iota, \lambda_4 = 0.00125276$. At the non-collinear point $(0, -0.9994650500230987)$ (at $\Gamma = 0°$), the characteristic roots are given by $\lambda_1 = -0.00125276, \lambda_2 = -0.12927\iota, \lambda_3 = 0.12927\iota, \lambda_4 = 0.00125276$. Thus both the non-collinear points lying on the y-axis are unstable for $\Gamma = 0°$. For $\Gamma = 45°$, at the non-collinear point $(0\ , 0.9994754785409711)$ the characteristic roots are given by $\lambda_1 = -1.81113 \times 10^{-9}\iota, \lambda_2 = 1.81113 \times 10^{-9}\iota, \lambda_3 = -0.129266\iota, \lambda_4 = 0.129266\iota$. For $\Gamma = 45°$, at the non-collinear point $(0\ , -0.9994754785409711)$ the characteristic roots are given by $\lambda_1 = -1.81113\times10^{-9}\iota, \lambda_2 = 1.81113\times10^{-9}\iota, \lambda_3 = -0.129266\iota, \lambda_4 = 0.129266\iota$. Similarly, we have examined that both the non-collinear points lying on the y-axis are unstable for other values of Γ (Table 2).

4.3 Stability of non-collinear points lying on the circle

From the roots of the characteristic equation (17), we have noted that some of the non-collinear points lying on the circle are stable and others are unstable for different values of the earth's equatorial ellipticity parameter Γ. In Table 3, we have shown the stability of two non-collinear points for different values of Γ.

4.4 Stability regions of equilibrium points

From characteristic equation (17), an equilibrium point will be stable if the above equations evaluated at the equilibrium points has purely imaginary roots or complex roots with negative real parts. This happens if the following three conditions

$$\begin{aligned}(F^0_{xx} + F^0_{yy} - 4n^2)^2 - \left(F^0_{xx}F^0_{yy} - (F^0_{xx})^2\right) > 0,\\ F^0_{xx} + F^0_{yy} - 4n^2 > 0,\\ F^0_{xx}F^0_{yy} - (F^0_{xx})^2 > 0,\end{aligned}$$

evaluated at the equilibrium point are satisfied simultaneously.

We have plotted the stability regions of the equilibrium points for different values of the earth's equatorial ellipticity parameter Γ (Fig. 6). We observed that there is a very small change in the stability region as the value of Γ increases.

5 Discussion and Conclusion

We have studied the locations and stability of the equilibrium points in the problem of a geo-centric satellite including the earth's equatorial ellipticity parameter Γ. First, we write the equations of motion of the geo-centric satellite P moving around the earth in the equatorial plane. We assume that the satellite P lies in the equatorial plane. We choose the origin of coordinates at the centre of mass of the earth. The plane of motion of the infinitesimal satellite P is in the XY-plane orthogonal to the line of motion of the centre of mass of the earth, and the motion of the earth takes place on the z-axis. We write the Jacobi integral of the system, and then we calculate the perturbed mean motion n which is a function of Γ. The possible boundary regions for the motion of an infinitesimal satellite P are obtained with the help of zero-velocity curves at different values of the Jacobi constant by fixing the values of the earth's equatorial ellipticity parameter Γ. In Figs. 2–5, we observed that at a fixed value of the earth's equatorial ellipticity parameter Γ, on increasing the values of the Jacobi constant C, the possible boundary regions decrease. We also observed that the possible boundary regions depend

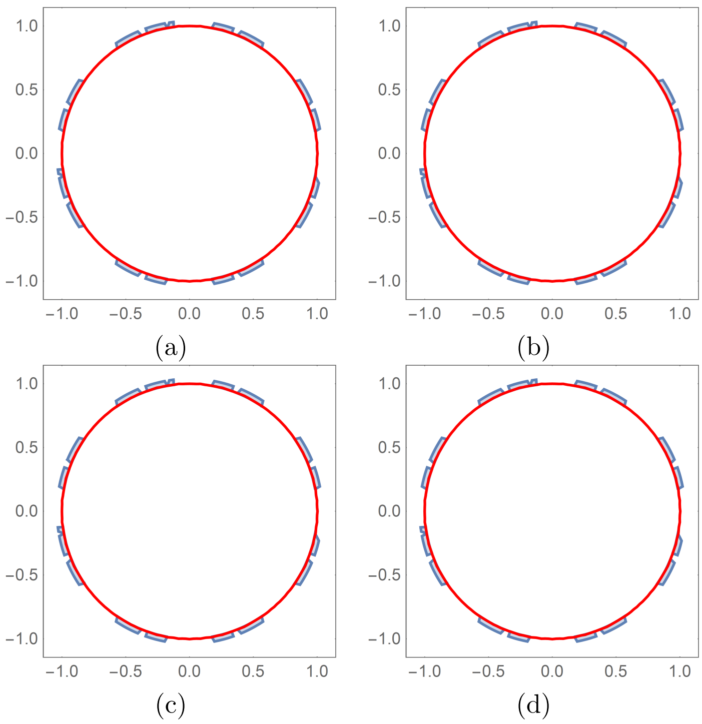


Figure 6: Stability regions of equilibrium points for different values of Γ: (a) $\Gamma = 0°$; (b) $\Gamma = 15°$; (c) $\Gamma = 30°$; (d) $\Gamma = 45°$.

on the Jacobi constant, while the effect of the earth's equatorial ellipticity parameter Γ on possible boundary regions is very small. We have also investigated the existence and stability of the equilibrium points of the system for different values of Γ. We observed that there exist two collinear points and both of them are unstable for different values of Γ (Table 1). It is shown that the effect of earth's equatorial ellipticity parameter Γ on the location of equilibrium points is very small and the number of equilibrium points remains the same for different values of Γ. We also observe that there exist non-collinear points lying on the y-axis and both of them are unstable for different values of Γ (Table 2). Further, we have found that there exist an infinite number of non-collinear points lying on the circle. Some of them are stable, and others are unstable. Two non-collinear points for different values of Γ and their stability are shown in Table 3. Finally, we have plotted the stability regions of the equilibrium points for different values of Γ (Fig. 6). We notice that there is a minimal change in the stability regions of the equilibrium points as the value of Γ increases.

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