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Fuzzy Differential Systems and the New Concept of Stability

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Abstract: The study of fuzzy differential systems is initiated and sufficient conditions, in terms of Lyapunov-like functions, are provided for the new concept of stability which unifies Lyapunov and orbital stabilities as well as includes new notions in between.

Keywords: *Fuzzy differential systems; new notion of stability; stability tests*

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1 Introduction

Recently, the theory of fuzzy differential equations has been initiated and the basic results have been systematically investigated, including Lyapunov stability, in [2, 3, 6, 8, 10]. This study of fuzzy differential equations corresponds to scalar differential equations without fuzziness.

A new concept of stability that includes Lyapunov and orbital stabilities as well as leads to new notions of stability in between them is introduced in terms of a given topology of the function space [9] and sufficient conditions in terms of Lyapunov-like functions are provided for such concepts to hold relative to ordinary differential equations [5].

In this paper, we shall extend the notion fuzzy differential system employing the generalized metric space and then develop the new concept of stability theory proving sufficient conditions in terms of vector Lyapunov-like functions in the framework of fuzziness. For this purpose, we develop suitable comparison results to deal with fuzzy differential systems in terms of Lyapunov-like functions and then employing the comparison result offer sufficient conditions for the new concepts to hold. This new approach helps to understand the intricacies involved in incorporating fuzziness in the theory of differential equations.

2 Preliminaries

Let $P_k(R^n)$ denote the family of all nonempty compact, convex subsets of R^n. If α, $\beta \in R$ and A, $B \in P_k(R^n)$, then

$$\alpha(A+B) = \alpha A + \alpha B, \quad \alpha(\beta A) = (\alpha\beta)A, \quad 1A = A$$

and if α, $\beta \geq 0$, then $(\alpha+\beta)A = \alpha A + \beta A$. Let $I = [t_0, t_0+a]$, $t_0 \geq 0$ and $a > 0$ and denote by $E^n = [u\colon R^n \to [0,1]$ such that u satisfies (i) to (iv) mentioned below]:

(i) u is normal, that is, there exists an $x_0 \in R^n$ such that $u(x_0) = 1$;

(ii) u is fuzzy convex, that is, for x, $y \in R^n$ and $0 \leq \lambda \leq 1$,

$$u(\lambda x + (1-\lambda)y) \geq \min[u(x), u(y)];$$

(iii) u is upper semicontinuous;

(iv) $[u]^0 = \overline{[x \in R^n\colon u(x) > 0]}$ is compact.

For $0 < \alpha \leq 1$, we denote $[u]^\alpha = [x \in R^n\colon u(x) \geq \alpha]$. Then from (i) to (iv), it follows that the α-level sets $[u]^\alpha \in P_k(R^n)$ for $0 \leq \alpha \leq 1$.

Let $d_H(A,B)$ be the Hausdorff distance between the sets A, $B \in P_k(R^n)$. Then we define

$$d[u,v] = \sup_{0 \leq \alpha \leq 1} d_H[[u]^\alpha, [v]^\alpha],$$

which defines a metric in E^n and (E^n, d) is a complete metric space. We list the following properties of $d[u,v]$:

$$d[u+w, v+w] = d[u,v] \quad \text{and} \quad d[u,v] = d[v,u],$$

$$d[\lambda u, \lambda v] = |\lambda| d[u,v],$$

$$d[u,v] \leq d[u,w] + d[w,v],$$

for all u, v, $w \in E^n$ and $\lambda \in R$.

For x, $y \in E^n$ if there exists a $z \in E^n$ such that $x = y + z$, then z is called the H-difference of x and y and is denoted by $x - y$. A mapping $F\colon I \to E^n$ is differentiable at $t \in I$ if there exists a $F'(t) \in E^n$ such that the limits

$$\lim_{h \to 0^+} \frac{F(t+h) - F(t)}{h} \quad \text{and} \quad \lim_{h \to 0^+} \frac{F(t) - F(t-h)}{h}$$

exist and are equal to $F'(t)$. Here the limits are taken in the metric space (E^n, d).

Moreover, if $F\colon I \to E^n$ is continuous, then it is integrable and

$$\int_a^b F = \int_a^c F + \int_c^b F.$$

Also, the following properties of the integral are valid. If F, $G\colon I \to E^n$ are integrable, $\lambda \in R$, then the following hold:

$$\int (F+G) = \int F + \int G;$$

$$\int \lambda F = \lambda \int F, \quad \lambda \in R;$$

$$d[F,G] \text{ is integrable;}$$

$$d\Big[\int F, \int G\Big] \leq \int d[F,G].$$

Finally, let $F\colon I \to E^n$ be continuous. Then the integral $G(t) = \int_a^t F$ is differentiable and $G'(t) = F(t)$. Furthermore,

$$F(t) - F(t_0) = \int_a^t F'(t).$$

See [2, 3, 8, 10] for details.

We need the following known [4] results from the theory of ordinary differential inequalities. Hereafter, the inequalities between vectors in R^d are to be understood componentwise.

Theorem 2.1 *Let $g \in C[R_+ \times R_+^d \times R_+^d, R^d]$, $g(t,w,\xi)$ be quasimonotone nondecreasing in w for each (t,ξ) and monotone nondecreasing in ξ for each (t,w). Suppose further that $r(t) = r(t,t_0,w_0)$ is the maximal solution of*

$$w' = g(t,w,w), \qquad w(t_0) = w_0 \ge 0, \tag{2.1}$$

existing on $[t_0,\infty)$. Then the maximal solution $R(t) = R(t,t_0,w_0)$ of

$$w' = g(t,w,r(t)), \qquad w(t_0) = w_0 \ge 0, \tag{2.2}$$

exists on $[t_0,\infty)$ and

$$r(t) \equiv R(t), \quad t \ge t_0. \tag{2.3}$$

Theorem 2.2 *Assume that the function $g(t,w,\xi)$ satisfies the conditions of Theorem 2.1. Then $m \in C[R_+, R_+^d]$ and*

$$D^+ m(t) \le g(t,m(t),\xi), \quad t \ge t_0. \tag{2.4}$$

Then for all $\xi \le r(t)$, it follows that

$$m(t) \le r(t), \quad t \ge t_0.$$

3 Fuzzy Differential System

We have been investigating so far the fuzzy differential equation

$$u' = f(t,u), \quad u(t_0) = u_0, \tag{3.1}$$

where $f \in C[R_+ \times E^n, E^n]$, which corresponds to, without fuzziness, scalar differential equation [2, 3, 6, 8]. To consider the situation analogous to differential system, we need to prepare suitable notation. Let $u = (u_1, u_2, \dots, u_N)$ with $u_i \in E^n$ for each $1 \le i \le N$ so that $u \in E^{nN}$, where

$$E^{nN} = (E^n \times E^n \times \cdots \times E^n), \quad N - \text{times}.$$

Let $f \in C[R_+ \times E^{nN}, E^{nN}]$ and $u_0 \in E^{nN}$. Then consider the fuzzy differential system

$$u' = f(t,u), \quad u(t_0) = u_0. \tag{3.2}$$

We have two possibilities to measure the new fuzzy variables u, u_0, f, that is,

(1) we can define $d_0[u,v] = \sum_{i=1}^{N} d[u_i, v_i]$, where u_i, $v_i \in E^n$ for each $1 \le i \le N$ and employ the metric space (E^{nN}, d_0), or
(2) we can define the generalized metric space (E^{nN}, D), where

$$D[u,v] = (d[u_1,v_1], d[u_2,v_2], \ldots, d[u_N,v_N]).$$

In any of the foregoing set-ups, one can prove existence and uniqueness results for (3.2) using the appropriate contraction mapping principles. See [1] for the details of generalized spaces and generalized contraction mapping principle.

We can now prove the needed comparison result in terms of suitable Lyapunov-like functions. For this purpose, we let

$$\Omega = [\sigma \in C^1[R_+, R_+] \colon \sigma(t_0) = t_0 \text{ and } w(t,\sigma,\sigma') \le r(t),\ t \ge t_0], \tag{3.3}$$

where $w \in C[R_+^2 \times R, R_+^d]$ and $r(t)$ is the maximal solution of (2.1).

Theorem 3.1 *Assume that for some $\sigma \in \Omega$, there exists a V such that $V \in C[R_+^2 \times E^{nN} \times E^{nN}, R_+^d]$ and*

$$|V(t,\sigma,u_1,v_1) - V(t,\sigma,u_2,v_2)| \le A[D[u_1,u_2] + D[v_1,v_2]],$$

where A is an $N \times N$ positive matrix. Moreover,

$$\begin{aligned} &D^+V(t,\sigma,u,v)\\ &= \limsup_{h\to 0^+} \frac{[V(t+h,\sigma(t+h),u+hf(t,u),v+hf(\sigma,v)\sigma') - V(t,\sigma,u,v)]}{h}\\ &\le g(t,V(t,\sigma,u,v),w(t,\sigma,\sigma')), \end{aligned}$$

where $g(t,w,\xi)$ satisfies the conditions of Theorem 2.1.

Then $V(t_0,\sigma(t_0),u_0,v_0) \le w_0$ implies

$$V(t,\sigma(t),u(t,t_0,u_0),v(\sigma(t),t_0,v_0)) \le r(t,t_0,w_0), \quad t \ge t_0.$$

Proof Let $u(t) = u(t,t_0,u_0)$, $v(t) = v(t,t_0,v_0)$ be the solutions of (3.2) and set $m(t) = V(t,\sigma(t),u(t),v(\sigma(t))$ so that $m(t_0) = V(t_0,\sigma(t_0),u_0,v_0)$. Let $w_0 = m(t_0)$. Then for small $h > 0$, we have, in view of the Lipschitz condition given in (i),

$$\begin{aligned} m(t+h) - m(t) &= V(t+\sigma,\sigma(t+h),u(t+h),v(\sigma(t+h)))\\ &- V(t,\sigma(t)u(t),v(\sigma(t))) + V(t+h,\sigma(t+h),u(t)+hf(t,u(t)),\\ &v(\sigma(t)) + hf(\sigma(t),v(\sigma)))\sigma'(t)))\\ &\le A[D[u(t+h),u(t)+hf(t,u(t))] + D[v(\sigma(t+h)),\\ &v(\sigma(t)) + hf(\sigma(t),v(\sigma(t)))\sigma'(t)]] + V(t+h,\sigma(t+h),u(t)+hf(t,u(t)),\\ &v(\sigma(t)) + hf(\sigma(t),v(\sigma(t)))\sigma'(t)) - V(t,\sigma(t),u(t),v(\sigma(t))). \end{aligned}$$

It therefore follows that

$$\begin{aligned} D^+m(t) = \limsup_{h\to 0^+} \frac{1}{h}\,[m(t+h) - m(t)] &\le D^+V(t,\sigma(t),u(t),v(t)) \\ + A \limsup_{h\to 0^+} \frac{1}{h}\,&[D[u(t+h), u(t) + hf(t,u(t))] \\ + D[v(\sigma(t+h), v(\sigma(t)&+hf(\sigma(t), v(\sigma(t))\sigma')]]. \end{aligned}$$

Since $u'(t)$, $v'(t)$ is assumed to exist, we see that $u(t+h) = u(t) + z(t)$, $v(\sigma(t+h)) = v(\sigma(t)) + \xi(\sigma(t))$, where $z(t)$, $\xi(\sigma(t))$ are the H-differences for small $h > 0$. Hence utilizing the properties of $D[u,v]$, we obtain

$$\begin{aligned} D[u(t+h), u(t) + hf(t,u(t))] &= D[u(t)+z(t), u(t)+hf(t,u(t)))] \\ = D[z(t), hf(t,u(t))] &= D[u(t+h) - u(t), hf(t,u(t))]. \end{aligned}$$

As a result, we get

$$\frac{1}{h}\,D[u(t+h), u(t) + hf(t,u(t))] = D\left[\frac{u(t+h)-u(t)}{h},\ f(t,u(t))\right]$$

and consequently

$$\begin{aligned} &\limsup_{h\to 0^+} \frac{1}{h}\,D[u(t+h, u(t) + h(f(t,u(t))] \\ &= \limsup_{h\to 0^+} D\left[\frac{u(t+h)-u}{h},\ f(t,u(t))\right] = D[u'(t), f(t,u(t))] = 0, \end{aligned}$$

since $u(t)$ is the solution of (3.2). Similarly, we can obtain

$$\begin{aligned} &\limsup_{h\to 0^+} \frac{1}{h}\,D[v(\sigma(t+h), v(\sigma(t)) + hf(\sigma(t), v(\sigma(t))\sigma'] \\ &\qquad = D[v'(\sigma(t)), f(\sigma(t), v(\sigma(t))\sigma'(t)] = 0, \end{aligned}$$

since $v(t)$ is the solution of (3.2). We have therefore the vector differential inequality

$$D^+m(t) \le g(t, m(t), w(t,\sigma(t),\sigma'(t))), \quad t \ge t_0.$$

Since $\sigma \in \Omega$, we then get

$$D^+m(t) \le g(t, m(t), r(t)), \quad t \ge t_0,$$

where $r(t)$ is the maximal solution of (2.1). By the theory of differential inequalities for systems [4] the claimed estimate (3.4) follows and the proof is complete.

Let us next introduce the new concept of stability. Let $v(t,t_0,v_0)$ bet the given unperturbed solution of (3.2) on $[t_0,\infty)$ and $u(t,t_0,u_0)$ be any perturbed solution of (3.2) on $[t_0,\infty)$ and $u(t,t_0,u_0)$ be any perturbed solution of (3.2) on $[t_0,\infty)$. Then the Lyapunov stability (LS) compares the phase space positions of the unperturbed and perturbed solutions at exactly simultaneous instants, namely

$$d_0[u(t, t_0, u_0), v(t, t_0, v_0)] < \epsilon, \quad t \geq t_0, \tag{LS}$$

which is a too restrictive requirement from the physical point of view. The orbital stability (OS), on the other hand, compares phase space positions of the same solutions at any two unrelated times, namely,

$$\inf_{s \in [t_0, \infty)} d_0[u(t, t_0, u_0), v(s, t_0, v_0)] < \epsilon, \quad t \geq t_0.$$

In this case, the measurement of time is completely irregular and therefore (OS) is too loose a demand.

We therefore need a new notion unifying (LS) and (OS) which would lead to concepts between them that could be physically significant. This is precisely what we plan to do below.

Let E denote the space of all functions from $R_+ \to R_+$, each function $\sigma(t) \in E$ representing a clock. Let us call $\sigma(t) = t$, the perfect clock. Let τ-be any topology in E. Given the solution $v(t, t_0, v_0)$ of (3.2) existing on $[t_0, \infty)$, we define following Massera [9], the new concept of stability as follows.

Definition 3.1 The solution $u(t, t_0, v_0)$ of (3.2) is said to be

(1) *τ-stable*, if, given $\epsilon > 0$, $t_0 \in R_+$, there exist a $\delta = \delta(t_0, \epsilon) > 0$ and an τ-neighborhood of N of the perfect clock satisfying $d_0[u_0, v_0] < \delta$ implies

$$d_0[u(t, t_0, u_0), v(\sigma(t), t_0, v_0)] < \epsilon, \quad t \geq t_0,$$

where $\sigma \in N$;

(2) *τ-uniformly stable*, if δ in (1) is independent of t_0.

(3) *τ-asymptotically stable*, if (1) holds and given $\epsilon > 0$, $t_0 \in R_+$, there exist a $\delta_0 = \delta_0(t_0) > 0$, a τ-neighborhood N of the perfect clock and a $T = T(t_0, \epsilon) > 0$ such that

$$d_0[u_0, v_0] < \delta_0 \quad \text{implies} \quad d_0[u(t, t_0, u_0), v(\sigma(t), t_0, v_0)] < \epsilon, \quad t \geq t_0 + T,$$

where $\sigma \in N$;

(4) *τ-uniformly asymptotically stable*, if δ_0 and T are independent of t_0.

We note that a partial ordering of topologies induces a corresponding partial ordering of stability concepts.

Let us consider the following topologies of E:

(τ_1) the discrete topology, where every set in E is open;

(τ_2) the chaotic topology, where the open sets are only the empty set and the entire clock space E;

(τ_3) the topology generated by the base

$$U_{\sigma_0, \epsilon} = [\sigma \in E \colon \sup_{t \in [t_0, \infty)} |\sigma(t) - \sigma_0(t)| < \epsilon];$$

(τ_4) the topology defined by the base

$$U_{\sigma_0, \epsilon} = [\sigma \in C^1[R_+, R_+] \colon |\sigma(t_0) - \sigma_0(t_0)| < \epsilon \quad \text{and}$$
$$\sup_{t \in [t_0, \infty)} |\sigma'(t) - \sigma_0'(t)| < \epsilon].$$

It is easy to see that (τ_3), (τ_4) topologies lie between (τ_1) and (τ_2). Also, an obvious conclusion is that if the unperturbed motion $v(t, t_0, v_0)$ is the trivial solution, then (OS) implies (LS).

4 Stability Criteria

In τ_1-topology, one can use the neighborhood consisting of solely the perfect clock $\sigma(t) = t$ and therefore, Lyapunov stability follows immediately from the existing results.

Define $B = B[t_0, v_0] = v([t_0, \infty), t_0, v_0)$ and suppose that B is closed. Then the stability of the set B can be considered the usual way in terms of Lyapunov functions [4, 7] since

$$\rho[u(t, t_0, u_0), B] = \inf_{s \in [t_0, \infty)} d_0[u(t, t_0, u_0), v(s, t_0, v_0)],$$

denoting the infimum for each t by s_t and defining $\sigma(t) = s_t$ for $t > t_0$, we see that $\sigma \in E$ in τ_2-topology. We therefore obtain orbital stability of the given solution $v(t, t_0, v_0)$ in terms of τ_2-topology.

To investigate the results corresponding to (τ_3) and (τ_4) topologies, we shall utilize the comparison Theorem 3.1 and modify suitably the proofs of standard stability results [4, 7].

Theorem 4.1 *Let the condition (i) of Theorem 3.1 be satisfied. Suppose further that*

(a) $b(d_0[u, v]) \le \sum_{i=1}^{d} v_i(t, \sigma, u, v) \le a(t, \sigma, d_0[u, v])$,

(b) $d(|t - \sigma(t)|) \le \sum_{i=1}^{d} w_i(t, \sigma, \sigma')$,

where $a(t, \sigma, \cdot)$, $b(\cdot)$ *and* $d(\cdot) \in \mathcal{K} = [a \in C[R_+, R_+],\ a(0) = 0$ *and* $a(\eta)$ *is increasing in* $\eta]$.

Then the stability properties of the trivial solution of (2.1) imply the corresponding τ_3*-stability properties of fuzzy differential system (3.2) relative to the given solution* $v(t, t_0, v_0)$.

Proof Let $v(t) = v(t, t_0, v_0)$ be the given solution of (3.2) and let $0 < \epsilon$ and $t_0 \in R_+$ be given. Suppose that the trivial solution of (2.1) is stable. Then given $b(\epsilon) > 0$ and $t_0 \in R_+$, there exists a $\delta_1 = \delta_1(t_0, \epsilon) > 0$ such that

$$0 \le \sum_{i=1}^{d} w_{i0} < \delta_1 \quad \text{implies} \quad \sum_{i=1}^{d} w_i(t, t_0, w_0) < b(\epsilon), \quad t \ge t_0, \tag{4.1}$$

where $w(t, t_0, w_0)$ is any solution of (2.1). We set $w_0 = V(t_0, \sigma(t_0), u_0, v_0)$ and choose $\delta = \delta(t_0, \epsilon)$, $\eta = \eta(\epsilon)$ satisfying

$$a(t_0, \sigma(t_0), \delta) < \delta_1 \quad \text{and} \quad \eta = d^{-1}(b(\epsilon)). \tag{4.2}$$

Using (b) and the fact $\sigma \in \Omega$, we have

$$d(|t - \sigma|) \le \sum_{i=1}^{d} w_i(t, \sigma, \sigma') \le \sum_{i=1}^{d} r_i(t, t_0, w_0) \le \sum_{i=1}^{d} r_i(t, t_0, \delta_1) < b(\epsilon).$$

It then follows that $|t-\sigma(t)|<\eta$ and hence $\sigma \in N$. We claim that whenever

$$d_0[u_0, v_0] < \delta \quad \text{and} \quad \sigma \in N,$$

it follows that

$$d_0[u(t,t_0,u_0), v(\sigma(t),t_0,v_0] < \epsilon, \quad t \geq t_0.$$

If this is not true, there would exist a solution $u(t,t_0,u_0)$ and a $t_1 > t_0$ such that

$$\begin{aligned} &d_0[u(t_1,t_0,u_0), v(\sigma(t_1),t_0,v_0] = \epsilon \quad \text{and} \\ &d_0[u(t,t_0,u_0,v(\sigma(t),t_0,v_0] \leq \epsilon \end{aligned} \tag{4.3}$$

for $t_0 \leq t \leq t_1$. Then by Theorem 3.1, we get for $t_0 \leq t \leq t_1$,

$$V(t,\sigma(t),u(t,t_0,u_0),v(t,t_0,v_0)) \leq r(t,t_0,V(t_0,\sigma(t_0,u_0,v_0)),$$

where $r(t,t_0,w_0)$ is the maximal solution of (2.1). It then follows from (4.1), (4.3), using (a), that

$$b(\epsilon) = b(d_0[u(t_1), v(\sigma(t_1))]) \leq \sum_{i=1}^{d} V_i(t_1,\sigma(t_1),u(t_1),v(\sigma(t_1))]$$

$$\leq \sum_{i=1}^{d} r_i(t_1,t_0,V(t_0,\sigma(t_0),u_0,v_0)) \leq \sum_{i=1}^{d} r_i(t_1,t_0,a(t_0,\sigma(t_0),\delta_1)) < b(\epsilon),$$

a contradiction, which proves τ_3-stability.

Suppose next that the trivial solution of (2.1) is asymptotically stable. Then it is stable and given $b(\epsilon) > 0$, $t_0 \in R_+$, there exist $\delta_{01} = \delta_{01}(t_0) > 0$ and $T = T(t_0,\epsilon) > 0$ satisfying

$$0 \leq \sum_{i=1}^{d} w_{0i} < \delta_{10} \quad \text{implies} \quad \sum_{i=1}^{d} w_i(t,t_0,w_0) < b(\epsilon), \quad t \geq t_0 + T. \tag{4.4}$$

The τ_3-stability yields taking $\epsilon = \rho > 0$ and designating $\delta_0(t_0) = \delta(t_0,\rho)$

$$d_0[u_0,v_0] < \delta_0 \quad \text{implies} \quad d_0[u(t),v(\sigma(t))] < \rho, \quad t \geq t_0$$

for every σ such that $|t-\sigma| < \eta(\rho)$. This means that by Theorem 3.1

$$V(t,\sigma(t),u(t),v(t)) \leq r(t,t_0,\delta_{10}), \quad t \geq t_0. \tag{4.5}$$

In view of (4.4), we find that

$$\sum_{i=1}^{d} r_i(t,t_0,\delta_{10}) < b(\epsilon), \quad t \geq t_0 + T,$$

which in turn implies

$$d[|(t-\sigma(t)|] \leq \sum_{i=1}^{d} w_i(t,\sigma,\sigma') \leq \sum_{i=1}^{d} r_i(t,t_0,\delta_{10}) < b(\epsilon), \quad t \geq t_0 + T.$$

Thus $|t-\sigma(t)| < d^{-1}b(\epsilon) = \eta(\epsilon)$, $t \geq t_0 + T$. Hence there exists a $\sigma \in N$ satisfying

$$d_0[u(t), v(\sigma(t))] \leq \sum_{i=1}^{d} V_i(t, \sigma(t), u(t), v(\sigma(t)))$$

$$\leq \sum_{i=1}^{d} r_i(t, t_0, \delta_{10}) < b(\epsilon), \quad t \geq t_0 + T,$$

which yields

$$d_0[u(t), v(\sigma(t))] < \epsilon, \quad t \geq t_0 + T,$$

whenever $d_0[u_0, v_0] < \delta_0$ and $\sigma \in N$. This proves τ_3-asymptotic stability of (3.2) and the proof is complete.

To obtain sufficient conditions for τ_4-stability, we need to replace (b) in Theorem 4.1 by

(c) $\quad d[|1-\sigma'(t)|] \leq \sum\limits_{i=1}^{d} w_i(t, \sigma, \sigma'),$

and then mimic the proof with suitable modifications. We leave the details to avoid monotony.

It would be interesting to obtain different sets of sufficient conditions as well as discover other topologies that would be of interest.

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