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Solution of the Problem of Constructing Liapunov Matrix Function for a Class of Large Scale Systems

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Abstract: New sufficient conditions for the Liapunov stability of a class of large scale systems described by ordinary differential equations are established. In all cases we proposed a new construction for matrix-valued Liapunov function and the objective is the same: to analyze the stability of large scale systems (nonautonomous and autonomous) in terms of sign definiteness of specific matrices. In order to demonstrate the usefulness of the presented results several examples are considered.

Keywords: *Large scale systems; Liapunov function construction; stability; asymptotic stability; nonautonomous oscillator.*

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1 Introduction

The methods of stability analysis of large-scale dynamical systems via one-level decomposition of the system and a vector Liapunov functions were summarized in a series of monographs. The necessity of further development of the known approaches for the mentioned class of dynamical systems and creation of new ones is caused by the fact that the methods of qualitative analysis based on vector Liapunov function yield, as a rule, "super-sufficient" stability conditions.

The aim of this paper is to present a new method of constructing the matrix-valued function and then to obtain efficient stability conditions for one class of large scale systems admitting one-level decomposition.

2 A Class of Large Scale System

We consider a system with finite number of degrees of freedom whose motion is described by the equations (2.1)

$$\frac{dx_i}{dt} = f_i(x_i) + g_i(t, x_1, \dots, x_m), \qquad i = 1, 2, \dots, m \tag{2.1}$$

where $x_i \in R^{n_i}$, $t \in \mathcal{T}_\tau$, $\mathcal{T}_\tau = [\tau, +\infty)$, $f_i \in C(R^{n_i}, R^{n_i})$, $g_i \in C(\mathcal{T}_\tau \times R^{n_1} \times \dots \times R^{n_m}, R^{n_i})$.

Introduce the designation

$$G_i(t, x) = g_i(t, x_1, \dots, x_m) - \sum_{j=1,\, j\neq i}^{m} g_{ij}(t, x_i, x_j), \tag{2.2}$$

where $g_{ij}(t, x_i, x_j) = g_i(t, 0, \dots, x_i, \dots, x_j, \dots, 0)$ for all $i \neq j$; $i, j = 1, 2, \dots, m$. Taking into consideration (2.2) system (2.1) is rewritten as

$$\frac{dx_i}{dt} = f_i(x_i) + \sum_{j=1,\, j\neq i}^{m} g_{ij}(t, x_i, x_j) + G_i(t, x). \tag{2.3}$$

Actually equations (2.3) describe the class of large-scale nonlinear nonautonomously connected systems. It is of interest to extend the method of matrix Liapunov functions to this class of equations in view of the new method of construction of nondiagonal elements of matrix-valued functions.

3 On Construction of Nondiagonal Elements of Matrix-Valued Function

In order to extend the method of matrix Liapunov functions to systems (2.3) it is necessary to estimate variation of matrix-valued function elements and their total derivatives along solutions of the corresponding systems. Such estimates are provided by the assumptions below.

Assumption 3.1 There exist open connected neighborhoods $\mathcal{N}_i \subseteq R^{n_i}$ of the equilibriums state $x_i = 0$, functions $v_{ii} \in C^1(R^{n_i}, R_+)$, the comparison functions φ_{i1}, φ_{i2} and ψ_i of class $K(KR)$ and real numbers $\underline{c}_{ii} > 0$, $\bar{c}_{ii} > 0$ and γ_{ii} such that

(1) $v_{ii}(x_i) = 0$ for all $(x_i = 0) \in \mathcal{N}_i$;
(2) $\underline{c}_{ii}\varphi_{i1}^2(\|x_i\|) \leq v_{ii}(x_i) \leq \bar{c}_{ii}\varphi_{i2}^2(\|x_i\|)$;
(3) $(D_{x_i} v_{ii}(x_i))^{\mathrm{T}} f_i(x_i) \leq \gamma_{ii}\psi_i^2(\|x_i\|)$ for all $x_i \in \mathcal{N}_i$, $i = 1, 2, \dots, m$.

It is clear (see [3, 5]) that under conditions of Assumption 3.1 the equilibrium states $x_i = 0$ of nonlinear isolated subsystems

$$\frac{dx_i}{dt} = f_i(x_i), \qquad i = 1, 2, \dots, m \tag{3.1}$$

are

(a) uniformly asymptotically stable in the whole, if $\gamma_{ii} < 0$ and $(\varphi_{i1}, \varphi_{i2}, \psi_i) \in KR$-class;
(b) stable, if $\gamma_{ii} = 0$ and $(\varphi_{i1}, \varphi_{i2}) \in K$-class;
(c) unstable, if $\gamma_{ii} > 0$ and $(\varphi_{i1}, \varphi_{i2}, \psi_i) \in K$-class.

The approach proposed in this section takes large scale systems (2.3) into consideration, subsystems (3.1) having various dynamical properties specified by conditions of Assumption 3.1.

Assumption 3.2 There exist open connected neighborhoods $\mathcal{N}_i \subseteq R^{n_i}$ of the equilibrium states $x_i = 0$, functions $v_{ij} \in C^{1,1,1}(\mathcal{T}_\tau \times R^{n_i} \times R^{n_j}, R)$, comparison functions $\varphi_{i1}, \varphi_{i2} \in K(KR)$, positive constants $(\eta_1, \dots, \eta_m)^{\mathrm{T}} \in R^m$, $\eta_i > 0$ and arbitrary constants $\underline{c}_{ij}$, $\bar{c}_{ij}$, $i, j = 1, 2, \dots, m$, $i \neq j$ such that

(1) $v_{ij}(t, x_i, x_j) = 0$ for all $(x_i, x_j) = 0 \in \mathcal{N}_i \times \mathcal{N}_j$, $t \in \mathcal{T}_\tau$, $i, j = 1, 2, \dots, m$, $(i \neq j)$;

(2) $\underline{c}_{ij}\varphi_{i1}(\|x_i\|)\varphi_{j1}(\|x_j\|) \le v_{ij}(t, x_i, x_j) \le \bar{c}_{ij}\varphi_{i2}(\|x_i\|)\varphi_{j2}(\|x_j\|)$ for all $(t, x_i, x_j) \in \mathcal{T}_\tau \times \mathcal{N}_i \times \mathcal{N}_j$, $i \neq j$;

(3)
$$\begin{aligned} &D_t v_{ij}(t, x_i, x_j) + (D_{x_i} v_{ij}(t, x_i, x_j))^{\mathrm{T}} f_i(x_i) \\ &+(D_{x_j} v_{ij}(t, x_i, x_j))^{\mathrm{T}} f_j(x_j) + \tfrac{\eta_i}{2\eta_j}(D_{x_i} v_{ii}(x_i))^{\mathrm{T}} g_{ij}(t, x_i, x_j) \\ &+\tfrac{\eta_j}{2\eta_i}(D_{x_j} v_{jj}(x_j))^{\mathrm{T}} g_{ji}(t, x_i, x_j) = 0; \end{aligned} \tag{3.2}$$

It is easy to notice that first order partial equations (3.2) are a somewhat variation of the classical Liapunov equation proposed in [8] for determination of auxiliary function in the theory of his direct method of motion stability investigation. In a particular case these equations are transformed into the systems of algebraic equations whose solutions can be constructed analytically.

Assumption 3.3 There exist open connected neighbourhoods $\mathcal{N}_i \subseteq R^{n_i}$ of the equilibrium states $x_i = 0$, comparison functions $\psi \in K(KR)$, $i = 1, 2, \dots, m$, real numbers α^1_{ij}, α^2_{ij}, α^3_{ij}, ν^1_{ki}, ν^1_{kij}, μ^1_{kij} and μ^2_{kij}, $i, j, k = 1, 2, \dots, m$, such that

(1) $(D_{x_i} v_{ii}(x_i))^{\mathrm{T}} G_i(t, x) \le \psi_i(\|x_i\|) \sum\limits_{k=1}^{m} \nu^1_{ki}\psi(\|x_k\|) + R_1(\psi)$
for all $(t, x_i, x_j) \in \mathcal{T}_\tau \times \mathcal{N}_i \times \mathcal{N}_j$;

(2) $(D_{x_i} v_{ij}(t, \cdot))^{\mathrm{T}} g_{ij}(t, x_i, x_j) \le \alpha^1_{ij}\psi^2_i(\|x_i\|) + \alpha^2_{ij}\psi_i(\|x_i\|)\psi_j(\|x_j\|) + \alpha^3_{ij}\psi^2_j(\|x_j\|)$
$+R_2(\psi)$ for all $(t, x_i, x_j) \in \mathcal{T}_\tau \times \mathcal{N}_i \times \mathcal{N}_j$;

(3) $(D_{x_i} v_{ij}(t, \cdot))^{\mathrm{T}} G_i(t, x) \le \psi_j(\|x_j\|) \sum\limits_{k=1}^{m} \nu^2_{ijk}\psi_k(\|x_k\|) + R_3(\psi)$
for all $(t, x_i, x_j) \in \mathcal{T}_\tau \times \mathcal{N}_i \times \mathcal{N}_j$;

(4) $(D_{x_i} v_{ij}(t, \cdot))^{\mathrm{T}} g_{ik}(t, x_i, x_k) \le \psi_j(\|x_j\|)(\mu^1_{ijk}\psi_k(\|x_k\|) + \mu^2_{ijk}\psi_i(\|x_i\|)) + R_4(\psi)$
for all $(t, x_i, x_j) \in \mathcal{T}_\tau \times \mathcal{N}_i \times \mathcal{N}_j$.

Here $R_s(\psi)$ are polynomials in $\psi = (\psi_1(\|x_1\|, \dots, \psi_m(\|x_m\|))$ in a power higher than three, $R_s(0) = 0$, $s = 1, \dots, 4$.

Under conditions (2) of Assumptions 3.1 and 3.2 it is easy to establish for function

$$v(t, x, \eta) = \eta^{\mathrm{T}} U(t, x)\eta = \sum_{i,j=1}^{m} v_{ij}(t, \cdot)\eta_i\eta_j \tag{3.3}$$

the bilateral estimate (cf. [4])

$$u_1^{\mathrm{T}} H^{\mathrm{T}} \underline{C} H u_1 \le v(t, x, \eta) \le u_2^{\mathrm{T}} H^{\mathrm{T}} \bar{C} H u_2, \tag{3.4}$$

where

$$u_1 = (\varphi_{11}(\|x_1\|, \dots, \varphi_{m1}(\|x_m\|))^{\mathrm{T}},$$
$$u_2 = (\varphi_{12}(\|x_1\|, \dots, \varphi_{m2}(\|x_m\|))^{\mathrm{T}},$$

which holds true for all $(t,x) \in \mathcal{T}_\tau \times \mathcal{N}$, $\mathcal{N} = \mathcal{N}_1 \times \cdots \times \mathcal{N}_m$.

Based on conditions (3) of Assumptions 3.1, 3.2 and conditions (1)–(4) of Assumption 3.3 it is easy to establish the inequality estimating the auxiliary function variation along solutions of system (2.3). This estimate reads

$$Dv(t,x,\eta)\big|_{(2.1)} \le u_3^{\mathrm{T}} M u_3, \tag{3.5}$$

where $u_3 = (\psi_1(\|x_1\|), \dots, \psi_m(\|x_m\|)$ and holds for all $(t,x) \in \mathcal{T}_\tau \times \mathcal{N}$.

Elements σ_{ij} of matrix M in the inequality (3.8) have the following structure

$$\sigma_{ii} = \eta_i^2 \gamma_{ii} + \eta_i^2 \nu_{ii} + \sum_{k=1,\, k\neq i}^{m} (\eta_k \eta_i \nu_{kii}^2 + \eta_i^2 \nu_{kii}^2) + 2 \sum_{j=1,\, j\neq i}^{m} \eta_i \eta_j (\alpha_{ij}^1 + \alpha_{ji}^3);$$

$$\sigma_{ij} = \frac{1}{2} (\eta_i^2 \nu_{ji}^1 + \eta_j^2 \nu_{ij}^1) + \sum_{k=1,\, k\neq j}^{m} \eta_k \eta_j \nu_{kij}^2 + \sum_{k=1,\, k\neq i}^{m} \eta_i \eta_j \nu_{kij}^2$$

$$+ \eta_i \eta_j (\alpha_{ij}^2 + \alpha_{ji}^2) + \sum_{\substack{k=1,\, k\neq i,\\ k\neq j}}^{m} (\eta_k \eta_j \mu_{kji}^1 + \eta_i \eta_j \mu_{ijk}^2 + \eta_i \eta_k \mu_{kij}^1 + \eta_i \eta_j \mu_{jik}^2),$$

$$i = 1, 2, \dots, m, \quad i \neq j.$$

4 Test for Stability Analysis

Sufficient criteria of various types of stability of the equilibrium state $x = 0$ of system (2.3) are formulated in terms of the sign definiteness of matrices $\underline{C}$, $\bar{C}$ and M from estimates (3.4), (3.5). We shall show that the following assertion is valid.

Theorem 4.1 *Assume that the perturbed motion equations are such that all conditions of Assumptions 3.1–3.3 are fulfilled and moreover*

(1) *matrices $\underline{C}$ and $\bar{C}$ in estimate (3.4) are positive definite;*
(2) *matrix M in inequality (3.5) is negative semi-definite (negative definite).*

Then the equilibrium state $x = 0$ of system (2.1) is uniformly stable (uniformly asymptotically stable).

If, additionally, in conditions of Assumptions 3.1–3.3 all estimates are satisfied for $\mathcal{N}_i = R^{n_i}, R_k(\psi) = 0, k = 1, ...4$ and comparison functions $(\varphi_{i1}, \varphi_{i2}) \in KR$-class, then the equilibrium state of system (2.1) is uniformly stable in the whole (uniformly asymptotically stable in the whole).

Proof If all conditions of Assumptions 3.1–3.2 are satisfied, then it is possible for system (2.1) to construct function $v(t,x,\eta)$ which together with total derivative $Dv(t,x,\eta)$ satisfies the inequalities (3.4), and (3.5). Condition (1) of Theorem 4.1 implies that function $v(t,x,\eta)$ is positive definite and decreasing for all $t \in \mathcal{T}_\tau$. Under condition (2) of Theorem 4.1 function $Dv(t,x,\eta)$ is negative semi-definite (definite). Therefore all conditions of Theorem 2.3.1, 2.3.3 from [9] are fulfilled. The proof of the second part of Theorem 4.1 is based on Theorem 2.3.4 from the same monograph [9].

5 Nonautonomous Oscillator

We shall study the motion of two non-autonomously connected oscillators whose behaviour is described by the equations

$$\begin{aligned} \frac{dx_1}{dt} &= \gamma_1 x_2 + v\cos\omega t y_1 - v\sin\omega t y_2, \\ \frac{dx_2}{dt} &= -\gamma_1 x_1 + v\sin\omega t y_1 + v\cos\omega t y_2, \\ \frac{dy_1}{dt} &= \gamma_2 y_2 + v\cos\omega t x_1 + v\sin\omega t x_2, \\ \frac{dy_2}{dt} &= -\gamma_2 y_2 + v\cos\omega t x_2 - v\sin\omega t x_1, \end{aligned} \tag{5.1}$$

where $\gamma_1,\ \gamma_2,\ v,\ \omega,\ \omega+\gamma_1-\gamma_2\neq 0$ are some constants.

For the independent subsystems

$$\begin{aligned} \frac{dx_1}{dt} &= \gamma_1 x_2, \qquad \frac{dx_2}{dt} = -\gamma_1 x_1 \\ \frac{dy_1}{dt} &= \gamma_2 y_2, \qquad \frac{dy_2}{dt} = -\gamma_2 y_1 \end{aligned} \tag{5.2}$$

the auxiliary functions $v_{ii},\ i=1,2$, are taken in the form

$$\begin{aligned} v_{11}(x) &= x^{\mathrm{T}}x, \qquad x=(x_1,x_2)^{\mathrm{T}}, \\ v_{22}(y) &= y^{\mathrm{T}}y, \qquad y=(y_1,y_2)^{\mathrm{T}}. \end{aligned} \tag{5.3}$$

We use the equation (3.2) (see Assumption 3.2) to determine the non-diagonal element $v_{12}(x,y)$ of the matrix-valued function $U(t,x,y)=[v_{ij}(\cdot)],\ i,\ j=1,2$. To this end set $\eta=(1,1)^{\mathrm{T}}$ and $v_{12}(x,y)=x^{\mathrm{T}}P_{12}y$, where $P_{12}\in C^1(\mathcal{T}_\tau,R^{2\times 2})$. For the equation

$$\begin{aligned} &\frac{dP_{12}}{dt} + \begin{pmatrix} 0 & -\gamma_1 \\ \gamma_1 & 0 \end{pmatrix} P_{12} \\ &+ P_{12}\begin{pmatrix} 0 & \gamma_2 \\ -\gamma_2 & 0 \end{pmatrix} + 2v\begin{pmatrix} \cos\omega t & -\sin\omega t \\ \sin\omega t & \cos\omega t \end{pmatrix} = 0, \end{aligned} \tag{5.4}$$

the matrix

$$P_{12} = -\frac{2v}{\omega+\gamma_1-\gamma_2}\begin{pmatrix} \sin\omega t & \cos\omega t \\ -\cos\omega t & \sin\omega t \end{pmatrix}$$

is a partial solution bounded for all $t\in\mathcal{T}_\tau$.

Thus, for the function $v(t,x,y)=\eta^{\mathrm{T}}U(t,x,y)\eta$ it is easy to establish the estimate of (3.4) type with matrices $\underline{C}$ and $\bar{C}$ in the form

$$\underline{C} = \begin{pmatrix} \underline{c}_{11} & \underline{c}_{12} \\ \underline{c}_{12} & \underline{c}_{22} \end{pmatrix}, \qquad \bar{C} = \begin{pmatrix} \bar{c}_{11} & \bar{c}_{12} \\ \bar{c}_{12} & \bar{c}_{22} \end{pmatrix},$$

where $\bar{c}_{11}=\underline{c}_{11}=1$; $\bar{c}_{22}=\underline{c}_{22}=1$, $\bar{c}_{12}=-\underline{c}_{12}=\frac{|2v|}{|\omega+\gamma_1-\gamma_2|}$. Besides, the vector $u_1^{\mathrm{T}}=(\|x\|,\|y\|)=u_2^{\mathrm{T}}$ since the system (5.1) is linear.

For system (5.1) the estimate (3.5) becomes

$$Dv(t,x,y)\big|_{(5.1)} = 0$$

for all $(x,y) \in R^2 \times R^2$ because $M = 0$.

Due to (5.4) the motion stability conditions for system (5.1) are established basing on the analysis of matrices $\underline{C}$ and $\bar{C}$ property of having fixed sign.

It is easy to verify that the matrices $\underline{C}$ and $\bar{C}$ are positive definite, if

$$1 - \frac{4v^2}{(\omega + \gamma_1 - \gamma_2)^2} > 0.$$

Consequently, the motion of nonautonomously connected oscillators is uniformly stable in the whole, if

$$|v| < \frac{1}{2}\,|\omega + \gamma_1 - \gamma_2|.$$

6 Large Scale Linear System

Assume that in the system

$$\begin{aligned}
\frac{dx_1}{dt} &= A_{11}x_1 + A_{12}x_2 + A_{13}x_3,\\
\frac{dx_2}{dt} &= A_{21}x_1 + A_{22}x_2 + A_{23}x_3,\\
\frac{dx_3}{dt} &= A_{31}x_1 + A_{32}x_2 + A_{33}x_3,
\end{aligned} \tag{6.1}$$

the state vectors $x_i \in R^{n_i}$, $i = 1,2,3$, and $A_{ij} \in R^{n_i \times n_j}$ are constant matrices for all i, $j = 1,2,3$.

For the independent systems

$$\frac{dx_i}{dt} = A_{ii}x_i, \quad i = 1,2,3 \tag{6.2}$$

we construct auxiliary functions $v_{ii}(x_i)$ as the quadratic forms

$$v_{ii}(x_i) = x_i^{\mathrm{T}} P_{ii} x_i, \quad i = 1,2,3 \tag{6.3}$$

whose matrices P_{ii} are determined by

$$A_{ii}^{\mathrm{T}} P_{ii} + P_{ii} A_{ii} = -G_{ii}, \quad i = 1,2,3, \tag{6.4}$$

where G_{ii} are prescribed matrices of definite sign. In order that to construct non-diagonal elements $v_{ij}(x_i,x_j)$ of matrix-valued function $U(x)$ we employ equation (3.2) from Assumption 3.2. Note that for system (6.1)

$$\begin{gathered}
f_i(x_i) = A_{ii}x_i, \quad f_j(x_j) = A_{jj}x_j,\\
g_{ij}(x_i,x_j) = A_{ij}x_j, \quad G_i(t,x) = 0, \quad i = 1,2,3.
\end{gathered}$$

Since for the bilinear forms

$$v_{ij}(x_i, x_j) = v_{ji}(x_j, x_i) = x_i^{\mathrm{T}} P_{ij} x_j, \tag{6.5}$$

the correlations

$$D_{x_i} v_{ij}(x_i, x_j) = x_j^{\mathrm{T}} P_{ij}^{\mathrm{T}}, \quad D_{x_j} v_{ij}(x_i, x_j) = x_i^{\mathrm{T}} P_{ij},$$

are true, equation (3.2) becomes

$$x_i^{\mathrm{T}} \left(A_{ii}^{\mathrm{T}} P_{ij} + P_{ij} A_{jj} + \frac{\eta_i}{\eta_j} P_{ii} A_{ij} + \frac{\eta_j}{\eta_i} A_{ji}^{\mathrm{T}} P_{ii} \right) x_j = 0.$$

From this correlation for determining matrices P_{ij} we get the system of algebraic equations

$$\begin{gathered} A_{ii} P_{ij} + P_{ij} A_{jj} = -\frac{\eta_i}{\eta_j} P_{ii} A_{ij} - \frac{\eta_j}{\eta_i} A_{ji}^{\mathrm{T}} P_{ii}, \\ i \neq j, \quad i,\, j = 1, 2, 3. \end{gathered} \tag{6.6}$$

Since for (6.3), and (6.5) the estimates (see [4, 6])

$$\begin{aligned} v_{ii}(x_i) &\geq \lambda_m(P_{ii}) \|x_i\|^2, && x_i \in R^{n_i}; \\ v_{ij}(x_i, x_j) &\geq -\lambda_M^{1/2}(P_{ij} P_{ij}^{\mathrm{T}}) \|x_i\| \|x_j\|, && (x_i, x_j) \in R^{n_i} \times R^{n_j}, \end{aligned}$$

hold true, for function $v(x, \eta) = \eta^{\mathrm{T}} U(x) \eta$ the inequality

$$w^{\mathrm{T}} H^{\mathrm{T}} C H w \leq v(x, \eta) \tag{6.7}$$

is satisfied for all $x \in R^n$, $w = (\|x_1\|, \|x_2\|, \|x_3\|)^{\mathrm{T}}$ and the matrix

$$C = \begin{pmatrix} \lambda_m(P_{11}) & -\lambda_M^{1/2}(P_{12} P_{12}^{\mathrm{T}}) & -\lambda_M^{1/2}(P_{13} P_{13}^{\mathrm{T}}) \\ -\lambda_M^{1/2}(P_{12} P_{12}^{\mathrm{T}}) & \lambda_m(P_{22}) & -\lambda_M^{1/2}(P_{23} P_{23}^{\mathrm{T}}) \\ -\lambda_M^{1/2}(P_{13} P_{13}^{\mathrm{T}}) & -\lambda_M^{1/2}(P_{23} P_{23}^{\mathrm{T}}) & \lambda_m(P_{33}) \end{pmatrix}.$$

For system (6.1) the constants from Assumption 3.3 are:

$$\begin{gathered} \alpha_{ij}^1 = \alpha_{ij}^2 = 0; \quad \alpha_{ij}^3 = \lambda_M(A_{ij}^{\mathrm{T}} P_{ij} + P_{ij}^{\mathrm{T}} A_{ij}), \\ \nu_{ki}^1 = \nu_{ijk}^2 = 0; \quad \nu_{ijk}^1 = \lambda_M^{1/2}[(P_{ij}^{\mathrm{T}} A_{ik})(P_{ij}^{\mathrm{T}} A_{ik})], \quad \mu_{ijk}^2 = 0. \end{gathered}$$

Therefore the elements σ_{ij} of matrix M in (3.5) for system (6.1) have the structure

$$\sigma_{ii} = -\eta_i^2 \lambda_m(G_{ii}) + 2 \sum_{j=1,\, j\neq i}^{3} \eta_i \eta_j \alpha_{ij}^3, \quad i = 1, 2, 3,$$

$$\sigma_{ij} = \sum_{\substack{k=1,\, k\neq i, \\ k\neq j}}^{3} (\eta_k \eta_j \nu_{ijk}^1 + \eta_i \eta_k \nu_{kij}^1), \quad i,\, j = 1, 2, 3, \quad i \neq j.$$

Consequently, the function $Dv(x, \eta)$ variation along solutions of system (6.1) is estimated by the inequality

$$Dv(x, \eta)\big|_{(6.1)} \leq w^{\mathrm{T}} M w \tag{6.8}$$

for all $(x_1, x_2, x_3) \in R^{n_1} \times R^{n_2} \times R^{n_3}$.

We summarize our presentation as follows.

Corollary 6.1 *Assume for system (6.1) the following conditions are satisfied:*

(1) *algebraic equations (6.4) have the sign-definite matrices P_{ii}, $i = 1, 2, 3$ as their solutions;*
(2) *algebraic equations (6.6) have constant matrices P_{ij}, for all i, $j = 1, 2, 3$, $i \neq j$ as their solutions;*
(3) *matrix C in (6.7) is positive definite;*
(4) *matrix M in (6.8) is negative semi-definite (negative definite).*

Then the equilibrium state $x = 0$ of system (6.1) is uniformly stable (uniformly asymptotically stable).

This corollary follows from Theorem 4.1 and hence its proof is obvious.

7 Discussion and Numerical Example

To start to illustrate the possibilities of the proposed method of Liapunov function construction we consider a system of two connected equations that was studied earlier by the Bellman-Bailey approach (see [7, 8], etc.).

Partial case of system (6.1) is the system

$$\begin{aligned} \frac{dx_1}{dt} &= Ax_1 + C_{12}x_2, \\ \frac{dx_2}{dt} &= Bx_2 + C_{21}x_1, \end{aligned} \tag{7.1}$$

where $x_1 \in R^{n_1}$, $x_2 \in R^{n_2}$, and A, B, C_{12} and C_{21} are constant matrices of corresponding dimensions. For independent subsystems

$$\begin{aligned} \frac{dx_1}{dt} &= Ax_1, \\ \frac{dx_2}{dt} &= Bx_2 \end{aligned} \tag{7.2}$$

the functions $v_{11}(x_1)$ and $v_{22}(x_2)$ are constructed as the quadratic forms

$$v_{11} = x_1^{\mathrm{T}} P_{11} x_1, \qquad v_{22} = x_2^{\mathrm{T}} P_{22} x_2, \tag{7.3}$$

where P_{11} and P_{22} are sign-definite matrices.

Function $v_{12} = v_{21}$ is searched for as a bilinear form $v_{12} = x_1^{\mathrm{T}} P_{12} x_2$ whose matrix is determined by the equation

$$A^{\mathrm{T}} P_{12} + P_{12} B = -\frac{\eta_1}{\eta_2} P_{11} C_{12} - \frac{\eta_2}{\eta_1} C_{21}^{\mathrm{T}} P_{22}, \quad \eta_1 > 0, \quad \eta_2 > 0. \tag{7.4}$$

According to Lankaster [7, p.240] equation (7.4) has a unique solution, provided that matrices A and $-B$ have no common eigenvalues.

Matrix C in (6.7) for system (7.1) reads

$$C = \begin{pmatrix} \lambda_m(P_{11}) & -\lambda_M^{1/2}(P_{12}P_{12}^{\mathrm{T}}) \\ -\lambda_M^{1/2}(P_{12}P_{12}^{\mathrm{T}}) & \lambda_m(P_{22}) \end{pmatrix}. \tag{7.5}$$

Here $\lambda_m(\cdot)$ are minimal eigenvalues of matrices P_{11}, P_{22}, and $\lambda_M^{1/2}(\cdot)$ is the norm of matrix $P_{12}P_{12}^{\mathrm{T}}$.

Estimate (6.7) for function $Dv(x,\eta)$ by virtue of system (7.1) is

$$Dv(x,\eta)\mid_{(7.1)} \leq w^{\mathrm{T}}\Xi w, \tag{7.6}$$

where $w = (\|x_1\|, \|x_2\|)^{\mathrm{T}}$, $\Xi = [\sigma_{ij}]$, $i,\, j = 1,2$;

$$\sigma_{11} = \lambda_1\eta_1^2 + \eta_1\eta_2\alpha_{22},$$

$$\sigma_{22} = \lambda_2\eta_2^2 + \eta_1\eta_2\beta_{22},$$

$$\sigma_{12} = \sigma_{21} = 0.$$

The notations are

$$\lambda_1 = \lambda_M(A^{\mathrm{T}}P_{11} + P_{11}A),$$

$$\lambda_2 = \lambda_M(B^{\mathrm{T}}P_{22} + P_{22}B),$$

$$\alpha_{22} = \lambda_M(C_{12}^{\mathrm{T}}P_{12} + P_{12}^{\mathrm{T}}C_{12}),$$

$$\beta_{22} = \lambda_M(C_{21}^{\mathrm{T}}P_{12}^{\mathrm{T}} + P_{12}C_{21}),$$

$\lambda(\cdot)$ is maximal eigenvalue of matrix $(\cdot)$. Partial case of Assumption 3.1 is as follows.

Corollary 7.1 *For system (7.1) let functions $v_{ij}(\cdot)$, $i,\, j = 1,2$ be constructed so that matrix C for system (7.1) is positive definite and matrix Ξ in inequality (7.6) is negative definite. Then the equilibrium $x = 0$ of system (7.1) is uniformly asymptotically stable.*

We consider the numerical example. Let the matrices from system (7.1) be of the form

$$A = \begin{pmatrix} -2 & 1 \\ 3 & -2 \end{pmatrix}, \qquad B = \begin{pmatrix} -4 & 1 \\ 2 & -1 \end{pmatrix}, \tag{7.7}$$

$$C_{12} = \begin{pmatrix} -0.5 & -0.5 \\ 0.8 & -0.7 \end{pmatrix}, \qquad C_{21} = \begin{pmatrix} 1.1 & 0.5 \\ -0.6 & -0.3 \end{pmatrix}. \tag{7.8}$$

Functions v_{ii} for subsystems

$$\dot{x} = Ax, \quad x = (x_1, x_2)^{\mathrm{T}},$$

$$\dot{y} = By, \quad y = (y_1, y_2)^{\mathrm{T}}$$

are taken as the quadratic forms

$$\begin{aligned} v_{11} &= 1.75x_1^2 + x_1x_2 + 1.5x_2^2, \\ v_{22} &= 0.35y_1^2 + 0.9y_1y_2 + 0.95y_2^2. \end{aligned} \tag{7.9}$$

Let $\eta = (1,1)^{\mathrm{T}}$. Then $\lambda_1 = \lambda_2 = -1$,

$$P_{12} = \begin{pmatrix} -0.011 & 0.021 \\ -0.05 & -0.022 \end{pmatrix},$$

$$\alpha_{22} = 0.03, \qquad \beta_{22} = -0.002.$$

It is easy to verify that $\sigma_{11} < 0$, and $\sigma_{22} < 0$, and hence all conditions of Corollary 7.1, are fulfilled in view that

$$\lambda_M^{1/2}(P_{12}P_{12}^{\mathrm{T}}) \le (\lambda_m(P_{11})\lambda_m(P_{22}))^{1/2},$$

for the values of $\lambda_M^{1/2}(P_{12}P_{12}^{\mathrm{T}}) = 0.06$, $\lambda_m(P_{11}) = 1.08$, $\lambda_m(P_{22}) = 0.115$. This implies uniform asymptotic stability in the whole of the equilibrium state of system (7.1) with matrices (7.7), and (7.8).

Let us show now that stability of system (7.1) with matrices (7.7), and (7.8) can not be studied in terms of the Bailey [2] theorem.

We recall that in this theorem the conditions of exponential stability of the equilibrium state are

(1) for subsystems (7.2) there must exist functions (7.3) satisfying estimates
   (a) $c_{i1}\|x_i\|^2 \le v_i(t, x_i) \le c_{i2}\|x_i\|^2$,
   (b) $Dv_i(t, x_i) \le -c_{i3}\|x_i\|^2$,
   (c) $\|\partial v_i/\partial x_i\| \le c_{i4}\|x_i\|$ for $x_i \in R^{n_i}$,
   where c_{ij} are some positive constants, $i = 1, 2$, $j = 1, 2, 3, 4$;
(2) the norms of matrices C_{ij} in system (2.4.17) must satisfy the inequality (see Abdullin, *et al.* [1, p. 106])

$$\|C_{12}\|\|C_{21}\| < \left(\frac{c_{11}c_{21}}{c_{12}c_{22}}\right)^{1/2}\left(\frac{c_{13}c_{23}}{c_{14}c_{24}}\right). \tag{7.10}$$

We note that this inequality is refined as compared with the one obtained firstly by Bailey [2].

The constants $c_{11}, \dots, c_{24}$ for functions (7.9) and system (7.1) with matrices (7.7), and (7.8) take the values

$$c_{11} = 1.08, \quad c_{21} = 0.115, \quad c_{12} = 2.14,$$
$$c_{22} = 2.14, \quad c_{22} = 1.135, \quad c_{13} = c_{23} = 1, \quad c_{14} = 4.83, \quad c_{24} = 2.4.$$

Condition (7.10) requires that $\|C_{12}\|\|C_{21}\| < 0.0184$ whereas for system (7.1), (7.7), and (7.8) we have

$$\|C_{12}\|\|C_{21}\| = 1.75.$$

Thus, the Bailey theorem turns out to be nonapplicable to this system and the condition (7.10) is "super-sufficient" for the property of stability.

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