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On the Boundedness of a Novel Four-Dimensional Hyperchaotic System

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Abstract: To estimate the ultimate bound and positively invariant set for a dynamical system is an important but quite challenging task in general. This paper attempts to investigate the bounds of a novel four-dimensional hyperchaotic system using a technique combining the generalized Lyapunov function theory and the Lagrange multiplier method. Finally, a numerical example is provided to illustrate the main result.

Keywords: *4D hyperchaotic system; boundedness of solutions; Lyapunov stability; Lagrange multiplier method.*

Mathematics Subject Classification (2010): 65P20, 65P30, 65P40.

1 Introduction

Hyperchaos characterized by more than one positive Lyapunov exponent has attracted an increasing attention of various scientific and engineering communities. It is very important to generate hyperchaos with more complicated dynamics as a model for theoretical research and practical implication. Hyperchaos was firstly reported by Rossler [18] in 1979, and the first circuit implementation of hyperchaos was realized by Matsumoto et al. [10]. Since then, some other hyperchaos generators have also been found. Typical examples are the hyperchaotic Lorenz–Haken system [11], hyperchaotic Chua's circuit [6], hyperchaotic modified Chua's circuit [20], these examples in themselves indicate that hyperchaos has a board range of applications in such fields as nonlinear circuit [2], secure communications [21], lasers [22], neural network [1], control [4], synchronization [5] and so on. In fact, the study of hyperchaos has recently become a central topic of the research in nonlinear sciences.

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In particular, the ultimate boundedness is very important for the study of the qualitative behavior of a chaotic system. If one can show that a chaotic or a hyperchaotic system under consideration has a globally attractive set, one knows that the system cannot have the equilibrium points, periodic or quasi-periodic solutions, or other chaotic or hyperchaotic attractors existing outside the attractive set. This greatly simplifies the analysis of dynamics of a chaotic or hyperchaotic system [9]. The boundedness of a chaotic system also plays an important role in chaos control and chaos synchronization.

Such an estimation is quite difficult to achieve technically, however, several works on this topic were realized for some $3D$ and $4D$ dynamical systems [3], [7], [8], [12], [13], [14], [15], [16], [17], [19], [23], [25].

Furthermore, there are no unified methods for constructing the Lyapunov functions to study the boundedness of the chaotic systems. Therefore, it is necessary to study the boundedness of the hyperchaotic systems.

In the present paper, we study the bounds of solutions of a new of hyperchaotic system based on a technique combining the generalized Lyapunov function theory and optimization. The paper is organized as follows : the problem formulation and main result are presented in Section 2. A numerical example is given in Section 3 to illustrate the main result. Finally, conclusion is made in Section 4.

2 Problem Formulation and Main Result

A novel four-dimensional hyperchaotic system with four nonlinearity terms presented in [24] by Wenjuan, Zengqiang and Zhuzhi can be described by the following system:

$$\begin{cases} x^{'} = ay - ax + eyz - kw, \\ y' = cx - xz - dy, \\ z' = xy - bz, \\ w' = ry + fyz, \end{cases} \tag{1}$$

where a, b, c, d, e, f, k and r are all real constant parameters. For the chosen $a = 56$, $b = 16$, $c = 49$, $d = 9$, $k = 8$, $e = 30$, $f = 40$ and $r = 48$ system (1) exhibits complex hyperchaotic dynamical behaviors. The corresponding three-dimensional phase diagrams in $(x - y - w)$, $(y - z - w)$ spaces are shown in Figure 1.

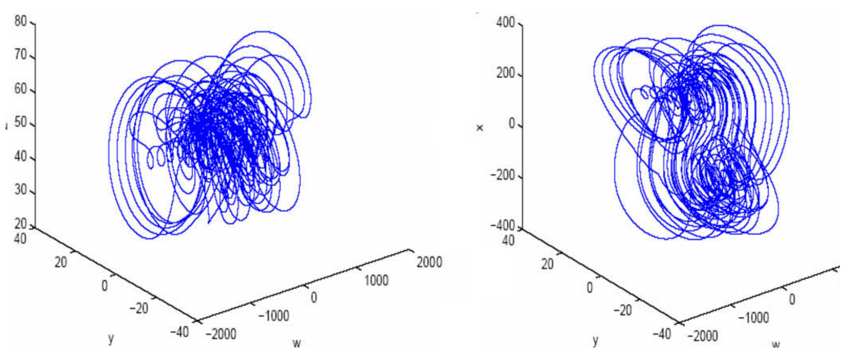

Fig. 1. Phase portrait of the system (1) in the $x - y - z$ space with parameters $\alpha = 5$, $\beta = 0.7$, $\gamma = 26$.

Some basic dynamical properties of the novel four-dimensional hyperchaotic system (1) were studied in [24]. But many properties of the system (1) remain to be uncovered. In the following, we will discuss the boundedness of the novel hyperchaotic system (1).

Lemma 2.1 *Define a set*

$$\Gamma = \left\{ (y,z) / \frac{y^2}{b^2} + \frac{(z-c)^2}{c^2} = 1,\ b > 0,\ c > 0 \right\} \tag{2}$$

and $G = y^2 + z^2$, $H = y^2 + (z-2c)^2$, $(y,z) \in \Gamma$. *Then we have*

$$\max_{(y,z)\in\Gamma} G = \max_{(xy,z)\in\Gamma} H = \left\{ \begin{array}{l} \dfrac{b^4}{b^2-c^2},\ b \geq \sqrt{2}c, \\ 4c^2,\ b < \sqrt{2}c. \end{array} \right. \tag{3}$$

Proof. It can be easily calculated by the Lagrange multiplier method.

Theorem 2.1 *When* $a > 0$, $b > 0$, $c > 0$, $d > 0$, $k > 0$, $e > 0$, $f > 0$, $r > 0$, *the following set defined by*

$$\Omega = \left\{ (x,y,z,w) / y^2 + (z-c)^2 \leq R^2,\ (ax+kw)^2 \leq \left(\frac{aB+kA}{a} \right)^2 \right\} \tag{2}$$

is the bound for system (1), *where*

$$R^2 = \left\{ \begin{array}{l} \dfrac{b^2c^2}{4d(b-d)}, \quad \text{if } b \geq 2d, \\ c^2\ , \quad \text{if } b < 2d, \end{array} \right. \tag{3}$$

$$A = R\left[r + f(R+c)\right],\ B = aR + eR(R+c)\,. \tag{4}$$

Proof. Define the following Lyapunov function

$$V_1(y,z) = y^2 + (z-c)^2\,. \tag{5}$$

Then, its time derivative along the orbits of system (1) is

$$\begin{aligned} \dot{V}_1 &= 2yy' + 2(z-c)z' \\ &= -2dy^2 - 2bz^2 + 2cbz \\ &= -2dy^2 - 2b\left(z - \frac{c}{2}\right)^2 + \frac{bc^2}{2}. \end{aligned} \tag{6}$$

That is to say, for $a > 0$, $b > 0$, $c > 0$, $d > 0$, $k > 0$, $e > 0$, $f > 0$, $r > 0$, the equation $\dot{V}_1 = 0$ holds, that means the surface

$$\Gamma = \left\{ (y,z) / \frac{y^2}{\frac{bc^2}{4d}} + \frac{\left(z - \frac{c}{2}\right)^2}{\frac{c^2}{4}} = 1 \right\} \tag{7}$$

is an ellipsoid in $2D$ space for certain values of a, b, c, d, k, e, f and r. Outside Γ, we have $\dot{V}_1 < 0$, while inside Γ, we have $\dot{V}_1 > 0$. Since the function $V_1 = y^2 + (z-c)^2$ is continuous on the closed set Γ, V_1can reach its maximum on the surface Γ. Denote the maximum value of V as R^2, that is, $R^2 = \max V_{1(y,z)\in\Gamma}$.

By Lemma 1, we can easily get

$$V_1(y,z) \le R^2 = \begin{cases} \dfrac{b^2c^2}{4d(b-d)}, & \text{if } b \ge 2d, \\ c^2\ , & \text{if } b < 2d. \end{cases} \tag{8}$$

From the formula (8), we obtain

$$|y| \le R, \ |z| \le R + c. \tag{9}$$

At the same time, the first equation of formula (1) and (9) yield

$$\begin{aligned} x^{'} &= ay - ax + eyz - kw \\ &\le a|y| + e|y||z| - ax - kw \\ &\le aR + eR(R+c) - ax - kw \\ &= B - ax - kw, \end{aligned}$$

where

$$B = aR + eR(R+c).$$

Also, the fourth equation of formula (1) and (9) yield

$$\begin{aligned} w' &= ry + fyz \le r|y| + f|y||z| \\ &\le rR + fR(R+c) = A. \end{aligned}$$

Let

$$V_2 = ax + kw.$$

Then

$$V_2' = ax' + kw' \le aB + kA - aV_2. \tag{10}$$

Integrating both sides of formula (10), we have

$$V_2(t) \le \frac{aB+kA}{a} + \left(V_2(t_0) - \frac{aB+kA}{a}\right)e^{-a(t-t_0)}. \tag{11}$$

So, we get

$$\lim_{t\to+\infty} V_2(t) \le \frac{aB+kA}{a}. \tag{12}$$

That is to say, the inequality $(ax+kw)^2 \le \left(\dfrac{aB+kA}{a}\right)^2$ holds as $t \to +\infty$.Therefore, we have the conclusion that

$$\Omega = \left\{(x,y,z,w)/y^2 + (z-c)^2 \le R^2,\ (ax+kw)^2 \le \left(\frac{aB+kA}{a}\right)^2\right\} \tag{13}$$

is the bound for the hyperchaotic systems (1). This completes the proof.

3 Example

Consider the system (1), when $a = 56$, $b = 16$, $c = 49$, $d = 9$, $k = 8$, $e = 30$, $f = 40$ and $r = 48$, we have

$$\Omega = \left\{ (x, y, z, w) / y^2 + (z - 49)^2 \leq 49^2,\ (56x + 8w)^2 \leq \left(\frac{56B + 8A}{56} \right)^2 \right\}$$

is the bound for the hyperchaotic system (1).
Consequently, we have

$$\begin{cases} (56x + 8w)^2 \leq 174580^2, \\ |y| \leq 49, \\ 0 \leq z \leq 98. \end{cases}$$

It is obvious that the orbits of system (1) locate in the section where $z \geq 0$.

4 Conclusion

In this paper, we have investigated the boundedness for a novel four-dimensional hyperchaotic system using a combination of the Lyapunov stability theory with optimization. Finally, a numerical example is provided to illustrate the main result.

References

[1] P. Arena, S. Baglio, L. Fortuna and G. Manganaro. Hyperchaos from cellular networks. *Electron. Lett.* **31** (1995) 250–251.

[2] A. Cenys, A. Tamaservicius, A. Baziliauskas, R. Krivickas and E. Lindberg. Hyperchaos in coupled Colpitts oscillators. *Chaos, Solitons & Fract.* **17** (2-3) (2003) 349–353.

[3] Z. Elhadj and J. C. Sprott. About the boundedness of 3D continuous time quadratic systems. *Nonlinear Oscil.* **13** (2-3) (2010) 515–521.

[4] J. Y. Hsieh, C. C. Hwang, A. P. Li and W. J. Li. Controlling hyper-chaos of the Rossler system. *Int. J. Control* **72** (1999) 882–886.

[5] P. Q. Jiang, B. H. Wang, S. L. Bu, Q. H. Xia and X. S. Luo. Hyperchaotic synchronizationin deterministic small-world dynamical networks. Int. J. Modern Phys. B **18** (2004) 2674–2679.

[6] T. Kapitaniak and L. O. Chua. Hyper-chaotic attractor of unidirectionally-coupled Chua's circuit. *Int. J. Bifurcation Chaos* **4** (1994) 477–482.

[7] D. Li, J. A. Lu, X. Wu and G. Chen. Estimating the bounds for the Lorenz family of chaotic systems. *Chaos, Solitons & Fract.* **23** (2005) 529–534.

[8] D. Li, X. Wu and J. Lu. Estimating the ultimating bound and positively invariant set for the hyperchaotic Lorenz-Haken system. *Chaos, Solitons & Fract.* **39** (2009) 1290–1296.

[9] X. Liao, Y. Fu, S. Xie and P. Yu. Globally exponentially attractive sets of the family of Lorenz systems. *Sci. China, Ser. F* **51** (2008) 283–292.

[10] T. Matsumoto, L. O. Chua and K. Kobayashi. Hyperchaos: laboratory exper-iment and numerical confirmation. *IEEE Trans. Circ. Syst.* **33** (1986) 1143–1147.

[11] C. Z. Ning and H. Haken. Detuned lasers and the complex Lorenz equations:subcitical and super-critical Hopf bifurcations. *Phys. Rev. A* **41** (1990) 3826–3837.

[12] A. Y. Pogromsky, G. Santoboni and H. Nijmeijer. An ultimate bound on the trajectories of the Lorenz systems and its applications. *Non-linearity* **16** (2003) 1597–1605.

[13] S. Rezzag. Solution bounds of the hyper-chaotic Rabinovich system. *Nonlinear studies* **24**(4) (2017) 903–909.

[14] S. Rezzag. Boundedness of the new modified hyperchaotic Pan System. *Nonlinear Dyn. Syst. Theory* **17** (3) (2017) 402–408.

[15] S. Rezzag, O. Zehrour and A. Aliouche. Estimating the bounds for the general 4-D hyperchaotic system. *Nonlinear studies* **22** (1) (2015) 41–48.

[16] S. Rezzag, O. Zehrour and A. Aliouche. Estimating the Bounds for the General 4-D Continuous-Time Autonomous System. *Nonlinear Dyn. Syst. Theory* **15** (3) (2015) 313–320.

[17] S. Rezzag. Boundedness Results for a New Hyperchaotic System and Their Application in Chaos Synchronization. *Nonlinear Dyn. Syst. Theory* **18** (4) (2018) 409–417.

[18] O. E. Rosssler. An equation for hyperchaos. *Phys. Lett. A* **71** (1979) 155–157.

[19] Y. J. Sun. Solution bounds of generalized Lorenz chaotic system. *Chaos, Solitons & Fract.* **40** (2009) 691–696.

[20] K. Thamilmaran, M. Lakshmanan and A. Venkatesan. Hyperchaos in a modified Canonical Chua's circuit. *Int. J. Bifurcation Chaos* **14** (2004) 221–243.

[21] V. S. Udaltsov, J. P. Goedgebuer, L. Larger, J. B. Cuenot, P. Levy, J. B. Cuenot, P. Levy and W. T. Rhodes. Communicating with hyperchaos: the dynamics of a DNLFemitter and recovery of transmitted information. *Opt. Spectrosc.* **95** (2003) 114–118.

[22] R. Vicente, J. Dauden, P. Colet and R. Toral. Analysis and characterization of thehyperchaos generated by a semiconductor laser subject to delayed feedbackloop. *IEEE J. Quantum Electron.* **41** (2005) 541–548.

[23] P. Wang, D. Li and Q. Hu. Bounds of the hyper-chaotic Lorenz-Stenflo system. Commun. *Nonlinear Sci. Numer. Simul.* **15** (2010) 2514–2520.

[24] W. Wenjuan, C. Zengqiang and Y. Zhuzhi. The evolution of a novel four-dimensional autonomous system: Among 3-torus, limit cycle, 2-torus, chaos and hyperchaos. *Chaos, Solitons & Fract.* **39 (**2009) 2340–2356.

[25] F. Zhang, Y. Li and C. Mu. Bounds of Solutions of a Kind of Hyper-Chaotic Systems and Application. *Journal of Mathematical Research with Applications* **33** (3) (2013) 345–352.