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Estimates of Accuracy for Asymptotic Soliton-Like Solutions to the Singularly Perturbed Benjamin-Bona-Mahony Equation

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Abstract: The paper deals with the singularly perturbed Benjamin-Bona-Mahony equation with variable coefficients. It plays an important role in various applications, in particular, for the description of waves in liquid. The equation appears in mathematical modeling of the wave processes in the media with small dispersion and variable characteristics. In the case of constant coefficients, this equation is known as the regularized long-wave equation or the regularized Korteweg-de Vries equation. We study the problem of estimating the difference between the exact solution and asymptotic soliton-like solution to the Cauchy problem for the singularly perturbed Benjamin-Bona-Mahony equation with variable coefficients. The initial data for the Cauchy problem are defined according to the concept of asymptotic soliton-like solution. It means that the approximate solutions are deformations of the soliton solutions to the Benjamin-Bona-Mahony equation with corresponding constant coefficients.
Asymptotic estimates for the difference between the exact solution to the Benjamin-Bona-Mahony equation and the N-th approximation for the asymptotic soliton-like solution are obtained. In particular, the case of the main term of the solution is considered in detail. Similarly to the case of the singularly perturbed Korteweg-de Vries equation with variable coefficients these estimates are local. Nevertheless, they show that the asymptotic soliton-like solutions constructed through the nonlinear WKB method for the singularly perturbed Benjamin-Bona-Mahony equation with variable coefficients are sufficiently suitable as approximate solutions.

Keywords: *Benjamin-Bona-Mahony equation; asymptotic solutions; soliton-like solutions; Cauchy problem; asymptotic estimates.*

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1 Introduction

The paper deals with asymptotic estimates for the difference between the exact solution and asymptotic soliton-like solution to the singularly perturbed Benjamin-Bona-Mahony equation with variable coefficients

$$a(x,t,\varepsilon)u_t + b(x,t,\varepsilon)u_x + c(x,t,\varepsilon)uu_x - \varepsilon^2 u_{xxt} = 0, \ \ (x,t) \in \mathbf{R} \times (0;T), \tag{1}$$

where $a(x,t,\varepsilon)$, $b(x,t,\varepsilon)$, $c(x,t,\varepsilon)$ are some functions described below, and $\varepsilon > 0$ is a small parameter. At $\varepsilon = 1$ and constant coefficients, equation (1) coincides with the following one:

$$u_t + u_x + uu_x - u_{xxt} = 0, \tag{2}$$

that has been deduced in [1], where it was studied through the numerical methods for the case of the wave form initial data.

In the sequel, Benjamin T.B., Bona J.L., and Mahony J.J. [2] studied the initial value problem for equation (2) whose solution was supposed to be a real smooth non-periodic function. In particular, they pointed the following: "*We shall refer to (2) as the regularized long wave equation, reflecting in this term our view that the Korteweg-de Vries equation is an unsuitably posed model for long waves*". Therefore, at present, equation (2) is known as the regularized long wave equation or the regularized Korteweg-de Vries equation. It is also called the Benjamin-Bona-Mahony equation [3], abbreviated to the BBM equation.

The different properties of equation (2), as well as those of its generalizations, were studied by Eilbeck J.C., and McGruire G.R. [4], [5], Wang B. [6], Wazwaz A.M. [7, 8], Arora R., and Kumar A. [9], Seadway A.R., and Sayed A. [10], El G.A., Hoefer M.A., and Shearer M. [11], and other authors. It was found that the BBM equation possesses soliton solutions [7]

$$u(x,t) = 3(a-1)\cosh^{-2}\left(\frac{1}{2}\sqrt{\frac{a-1}{a}}(x-at)+C\right), \tag{3}$$

where a, C are some real constants, and the inelastic collision of two solitary waves of the BBM equation was discovered [12], but it has neither two- nor multi-soliton solutions [13].

Equation (2), as well as the Korteweg-de Vries equation, describes propagation of soliton waves and cnoidal waves in different media, in particular, in shallow water. Similar waves have also appeared in many areas of science such as solid physics, biology [14], telecommunications [15], etc. Therefore, in the case of the medium with variable characteristics [16] and small dispersion [17, 18] the equation of type (1) should be studied.

One of the most effective methods of constructing approximate solutions to the singularly perturbed equations is the asymptotic analysis [19, 20]. Asymptotic soliton-like solutions to equation (1) were constructed in paper [21] through the approach based on the nonlinear WKB method that has been successfully applied for constructing asymptotic soliton-like solutions to many different problems (see, for example, [22], [23], [24], [25]). In the sequel, the nonlinear WKB technique was used for constructing the asymptotic soliton-like solutions to a number of partial differential equations of integrable type with singular perturbation [23].

Elaboration of algorithms for finding asymptotic expansions of different kinds and their justification consisting of determining asymptotic accuracy with which the solutions satisfy the equation under consideration are the main tasks of the perturbation theory. This traditionally completes the asymptotic analysis of equations with small perturbations.

On the other hand, in many cases it is necessary to examine the question of how much the constructed approximate solution differs from the exact solution to the equation. This problem is usually given much less attention than the previous one [26] since it is necessary to study the equation under additional conditions, for example, under the initial data. Thus, a problem on studying asymptotic solutions to the Cauchy problem for equations with small perturbations appears.

For the case of the asymptotic soliton-like solutions we need to take into account the properties of soliton solutions to the corresponding equation with constant coefficients [21]. Therefore, the initial conditions for the appropriate Cauchy problem should be selected in a special way. In particular, the initial functions must belong to certain functional spaces, for example, the space of quickly decreasing functions.

The problem on estimation of the difference between the exact solution and asymptotic approximation under the same initial condition appears naturally. Namely, this task is considered in the present paper.

The paper is organized as follows. Firstly, the problem under consideration is formulated, then necessary definitions and notations are given. In the sequel, the algorithm of constructing the asymptotic soliton-like solutions to equation (1) is briefly described, and statements on asymptotic estimates for the norm of difference between the exact solution and its constructed asymptotic approximation are finally proved. There is considered the case of the main term of the asymptotic solution as well as the case of the N-th asymptotic approximation.

2 Formulation of the Problem, Preliminary Notes and Definitions

We are facing a problem of constructing the asymptotic soliton-like solution to the Cauchy problem for the singularly perturbed BBM equation with variable coefficients (1) under the initial condition

$$u(x,t,\varepsilon)\big|_{t=0} = f(x,\varepsilon), \quad x \in \mathbf{R}. \tag{4}$$

It should be noted that the choice of the initial condition essentially influences the asymptotic estimate between the exact solution to the Cauchy problem [27] in question and its constructed asymptotic approximation. We consider the problem with the initial function $f(x,\varepsilon)$ obtained from the formulae for asymptotic soliton-like solution [21] to equation (1). The coefficients $a(x,t,\varepsilon)$, $b(x,t,\varepsilon)$, $c(x,t,\varepsilon)$ of equation (1) are supposed to be represented as

$$a(x,t,\varepsilon) = \sum_{k=0}^{N} \varepsilon^k a_k(x,t) + O(\varepsilon^{N+1}), \quad b(x,t,\varepsilon) = \sum_{k=0}^{N} \varepsilon^k b_k(x,t) + O(\varepsilon^{N+1}),$$

$$c(x,t,\varepsilon) = \sum_{k=0}^{N} \varepsilon^k c_k(x,t) + O(\varepsilon^{N+1}), \tag{5}$$

where the functions $a_0(x,t)$, $b_0(x,t)$, $c_0(x,t)$ do not equal zero for all $(x,t) \in \mathbf{R} \times [0;T]$.

Now we recall some notions and definitions.

Let $S = S(\mathbf{R})$ be a space of quickly decreasing functions, i.e., the space of infinitely differentiable on $\mathbf{R}$ functions satisfying, for any integers $m, n \geq 0$, the condition

$$\sup_{x\in\mathbf{R}} \left| x^m \frac{d^n}{dx^n} u(x) \right| < +\infty.$$

By $C^\infty(\mathbf{R}\times(0;T);S)$ we denote a space of infinitely differentiable functions of $(x,t) \in \mathbf{R}\times(0;T)$ such that, for any integers k, m, $n \geq 0$, the following inequality

$$\sup_{t\in[0;T]} \left(\int\limits_{-\infty}^{+\infty} \left(\frac{\partial^{n+k} u}{\partial x^n \partial t^k} \right)^2 dx + \int\limits_{-\infty}^{+\infty} (1+x^2)^m \left(\frac{\partial^k u}{\partial t^k} \right)^2 dx \right) < +\infty$$

holds.

Let $G_1 = G_1(\mathbf{R}\times[0;T]\times\mathbf{R})$ be a space of infinitely differentiable functions $f = f(x,t,\tau)$, $(x,t,\tau) \in \mathbf{R}\times[0;T]\times\mathbf{R}$, for which the following conditions are fulfilled [23]:

1^0. the relation

$$\lim_{\tau\to+\infty} \tau^n \frac{\partial^p}{\partial x^p} \frac{\partial^q}{\partial t^q} \frac{\partial^r}{\partial \tau^r} f(x,t,\tau) = 0, \quad (x,t)\in K,$$

takes place;

2^0. there exists a differentiable function $f^-(x,t)$ such that the condition

$$\lim_{\tau\to-\infty} \tau^n \frac{\partial^p}{\partial x^p} \frac{\partial^q}{\partial t^q} \frac{\partial^r}{\partial \tau^r} \big(f(x,t,\tau) - f^-(x,t)\big) = 0, \quad (x,t)\in K,$$

is satisfied uniformly in $(x,t) \in K$ for any non-negative integers n, p, q, r and every compact set $K \subset \mathbf{R}\times[0;T]$.

Let $G_1^0 = G_1^0(\mathbf{R}\times[0;T]\times\mathbf{R}) \subset G_1$ be a space of functions $f = f(x,t,\tau) \in G_1$, $(x,t,\tau) \in \mathbf{R}\times[0;T]\times\mathbf{R}$, for which the following condition $\lim\limits_{\tau\to-\infty} f(x,t,\tau) = 0$ takes place uniformly in (x,t) on every compact $K \subset \mathbf{R}\times[0;T]$.

Definition 2.1 A function $u = u(x,t,\varepsilon)$, where ε is a small parameter, is called an asymptotic soliton-like function [23] if for any integer $N \geq 0$, it can be represented in the form

$$u(x,t,\varepsilon) = \sum_{j=0}^{N} \varepsilon^j \left[u_j(x,t) + V_j(x,t,\tau)\right] + O(\varepsilon^{N+1}), \;\; \tau = \frac{x-\varphi(t)}{\varepsilon}, \tag{6}$$

where $\varphi(t) \in C^\infty([0;T])$ is a scalar real function; $u_j(x,t) \in C^\infty(\mathbf{R}\times[0;T])$, $j = \overline{0,N}$; $V_0(x,t,\tau) \in G_1^0$; $V_j(x,t,\tau) \in G_1$, $j = \overline{1,N}$.

The function $x - \varphi(t)$ is called a phase of the soliton-like function $u(x,t,\varepsilon)$, and the curve $\Gamma = \{(x,t) : x = \varphi(t),\, t\in[0;T]\}$ is called a discontinuity curve.

Here and below we use the notation $\Psi(x,t,\varepsilon) = O(\varepsilon^N)$. It means that $|\Psi(x,t,\varepsilon)| \leq C_N\varepsilon^N$ for all $\varepsilon \in (0;\varepsilon_0)$, where C_N, ε_0 are some positive values, $(x,t) \in K \subset \mathbf{R}\times[0;T]$ and K is a compact set.

The constant C_N depends only on the number N and the set K.

Remark 2.1 The term "soliton-like solution" reflects the following property of the asymptotic solution to the equations with constant coefficients having soliton solutions. In the case of such a partial differential equation in the presence of variable coefficients, it is expected that its solutions are certain deformations of the soliton-type solutions. Therefore, it is natural to look for asymptotic solutions to the singularly perturbed Benjamin-Bona-Mahony equation with variable coefficients in the form that is similar to the representation of soliton solutions. Moreover, in the case of constant coefficients, the singular part of the asymptotic solution constructed through the nonlinear WKB method coincides with soliton solution (3) to the singularly perturbed Benjamin-Bona-Mahony equation with account of calibrate transformations.

2.1 Scheme of constructing the asymptotic solution

Now we briefly describe the algorithm of constructing the asymptotic soliton-like solution to the BBM equation (1). The asymptotic solution is represented as [21]

$$u(x,t,\varepsilon) = \sum_{j=0}^{N} \varepsilon^j \left[u_j(x,t) + V_j(x,t,\tau)\right] + O(\varepsilon^{N+1}), \ \ \tau = \frac{x-\varphi(t)}{\varepsilon}. \tag{7}$$

Here the function $U_N(x,t,\varepsilon) = \sum_{j=0}^{N} \varepsilon^j u_j(x,t)$ is called a regular part of asymptotic solution (7) and the function $V_N(x,t,\tau,\varepsilon) = \sum_{j=0}^{N} \varepsilon^j V_j(x,t,\tau)$ gives a singular part of asymptotic solution (7). The terms of the regular part solve the equations

$$a_0(x,t)\frac{\partial u_0}{\partial t} + b_0(x,t)\frac{\partial u_0}{\partial x} + c_0(x,t)u_0\frac{\partial u_0}{\partial x} = 0, \tag{8}$$

$$\begin{gathered} a_0(x,t)\frac{\partial u_j}{\partial t} + b_0(x,t)\frac{\partial u_j}{\partial x} + c_0(x,t)\left(u_j\frac{\partial u_0}{\partial x} + u_0\frac{\partial u_j}{\partial x}\right) = \\ = f_j(x,t,u_0,u_1,\ldots,u_{j-1}), \quad j=\overline{1,N}, \end{gathered} \tag{9}$$

and the terms of the singular part satisfy the equations

$$\varphi'\,\frac{\partial^3 V_0}{\partial \tau^3} + \left[b_0(x,t) - a_0(x,t)\,\varphi'(t)\right]\frac{\partial V_0}{\partial \tau} + c_0(x,t)\left[u_0 + V_0\right]\frac{\partial V_0}{\partial \tau} = 0, \tag{10}$$

$$\varphi'\frac{\partial^3 V_j}{\partial \tau^3} + \left(b_0(x,t) - a_0(x,t)\,\varphi'(t)\right)\frac{\partial V_j}{\partial \tau} + c_0(x,t)\left(u_0\frac{\partial V_j}{\partial \tau} + \frac{\partial}{\partial \tau}\left(V_0V_j\right)\right) = F_j(x,t,\tau), \tag{11}$$

where the functions $f_j(x,t,u_0,u_1,\ldots,u_{j-1})$, $j=\overline{1,N}$, are obtained recurrently through the terms $u_0(x,t)$, $u_1(x,t)$, …, $u_{j-1}(x,t)$, $j=\overline{1,N}$, and the functions $F_j(x,t,\tau) = F_j(t, V_0(x,t,\tau),\ \ldots,\ V_{j-1}(x,t,\tau),\ u_0(x,t),\ \ldots,\ u_j(x,t))$, are determined recurrently through the terms $u_0(x,t)$, $u_1(x,t)$, …, $u_j(x,t)$, $V_0(x,t,\tau)$, $V_1(x,t,\tau)$, …, $V_{j-1}(x,t,\tau)$, $j=\overline{1,N}$.

Solutions to equations (8), (9) can be found through the method of characteristics. The singular part of asymptotic solution (7) is constructed in a special way [21]. Firstly, equations (10), (11) are studied on the discontinuity curve Γ that is determined through the solution $\varphi = \varphi(t)$, $t \in [0; T]$, of certain second order ordinary differential equation (see equation (29)). The functions $v_j = v_j(t, \tau) = V_j(x, t, \tau)\big|_\Gamma$, $j = \overline{0, N}$, solve the following partial differential equations:

$$\varphi'(t)\frac{\partial^3 v_0}{\partial \tau^3} + (b_0(\varphi, t) - a_0(\varphi, t)\varphi'(t) + c_0(\varphi, t)u_0(\varphi, t))\,\frac{\partial v_0}{\partial \tau} + c_0(\varphi, t)\,v_0\frac{\partial v_0}{\partial \tau} = 0, \quad (12)$$

$$\varphi'(t)\frac{\partial^3 v_j}{\partial \tau^3} + (b_0(\varphi, t) - a_0(\varphi, t)\varphi'(t) + c_0(\varphi, t)u_0(\varphi, t))\,\frac{\partial v_j}{\partial \tau} + c_0(\varphi, t)\frac{\partial}{\partial \tau}\,(v_0 v_j) = \mathcal{F}_j(t, \tau), \quad (13)$$

where $\mathcal{F}_j(t, \tau) = F_j(x, t, \tau)\big|_\Gamma$. In particular,

$$\mathcal{F}_1(t, \tau) = -a_0(\varphi, t)v_{0\,t} - c_0(\varphi, t)u_{0\,x}(\varphi, t)\,v_0 - \quad (14)$$

$$- [c_{0x}u_0(\varphi, t) + c_0(\varphi, t)u_{0\,x}(\varphi, t) - a_{0x}(\varphi, t)\varphi'(t) + b_{0x}(\varphi, t)]\,\tau\, v_{0\,\tau} -$$
$$- [c_{0x}(\varphi, t)\tau + c_1(\varphi, t)]\,v_0 v_{0\,\tau} - [c_0(\varphi, t)u_1(\varphi, t) + c_1(\varphi, t)u_0(\varphi, t) -$$
$$-a_1(\varphi, t)\varphi'(t) + b_1(\varphi, t)]\,v_{0\,\tau} + v_{0\,\tau\tau t}.$$

Later, an extension of the functions $v_j(t, \tau)$, $j = \overline{0, N}$, is constructed from the curve Γ into its neighborhood.

All details of the algorithm can be found in [21].

3 Principal Results

3.1 Main term of the asymptotic solution (7)

At first, we consider a main term of asymptotic expansion (7). The term is determined through the solution to equation (12) and is given by the formula

$$V_0(x, t, \tau) = V_0(t, \tau) = v_0(t, \tau) = 3\,\frac{A(\varphi, t)}{c_0(\varphi, t)}\,\cosh^{-2}\left(\sqrt{\frac{A(\varphi, t)}{\varphi'(t)}}\,\frac{\tau - \tau_0}{2}\right), \quad (15)$$

where $A(\varphi, t) = a_0(\varphi, t)\,\varphi' - b_0(\varphi, t) - c_0(\varphi, t)u_0(\varphi, t)$, τ_0 is a constant of integration, and function $u_0(x, t)$ is found through the method of characteristics from the Hopf type equation (8). Here the following condition

$$A(\varphi, t)\varphi'(t) > 0, \quad t \in [0; T], \quad (16)$$

is supposed to be satisfied.

Remark 3.1 The function $V_0(t, \tau)$ is an exact solution to equation (12) in the space G_1^0. Its partial derivative $V_{0\,\tau}(t, \tau)$ satisfies equation (13) as the right side function $\mathcal{F}_j(t, \tau) = 0$. The last property can be easily verified by direct calculations.

Now we define the initial data of problem (1), (4) more exactly. Let us put

$$f_0(x, \varepsilon) = 3\,\frac{A(\varphi_0, 0)}{c_0(\varphi_0, 0)}\,\cosh^{-2}\left(\sqrt{\frac{A(\varphi_0, 0)}{\varphi_0'}}\left(\frac{x - \varphi_0}{2\,\varepsilon} - \frac{\tau_0}{2}\right)\right), \quad (17)$$

where $\varphi_0 = \varphi(0)$, $\varphi_0' = \varphi'(0) \neq 0$, $\tau_0 \in \mathbf{R}$ are parameters, and let us denote the set of functions (17) by $\mathcal{M}_0(\varepsilon)$.

The following statements are true.

Theorem 3.1 *Let us suppose the following propositions are fulfilled:*

1^0. *the functions* $a_0(x,t)$, $b_0(x,t)$, $c_0(x,t) \in C^\infty(\mathbf{R}\times[0;T])$ *and they do not equal zero for all* $(x,t) \in \mathbf{R}\times[0;T]$*;*

2^0. *the function* $f(x,\varepsilon)$ *in initial condition (4) can be represented as* $f(x,\varepsilon) = g_0(x) + f_0(x,\varepsilon)$*, where* $g_0(x) \in C^\infty(\mathbf{R})$*,* $f_0(x,\varepsilon) \in \mathcal{M}_0(\varepsilon)$*;*

3^0. *the Cauchy problem for equation (8) with the initial condition* $u_0(x,0) = g_0(x)$ *has the solution* $u_0(x,t) \in C^\infty(\mathbf{R}\times[0;T])$*;*

4^0. *there exists a function* $\varphi(t) \in C^\infty([0;T])$ *satisfying (16) and* $\varphi(0) = \varphi_0$*,* $\varphi'(0) = \varphi_0' \neq 0$.

Then the main term of the asymptotic soliton-like solution to the Cauchy problem (1), (4) is given by the formula

$$Y_0(x,t,\varepsilon) = u_0(x,t) + V_0(t,\tau), \tag{18}$$

where $u_0(x,t)$ *is a solution to equation (8) with the initial condition* $u(x,0) = g_0(x)$ *and* $V_0(t,\tau)$ *is defined through formula (15).*

Function (18) satisfies the Cauchy problem (1), (4) with accuracy $O(1)$ *on the set* $\mathbf{R}\times[0;T]$*. Moreover, as* $\tau \to \pm\infty$*, it satisfies the Cauchy problem (1), (4) with accuracy* $O(\varepsilon)$ *on the set* $\mathbf{R}\times[0;T]$*.*

Proof. It is clear that function (18) satisfies initial condition (4). The other statement of Theorem 3.1 is proved according to the scheme of proof for Theorem 1 in [21]. That is why we omit the details here.

Theorem 3.2 *Let the following propositions hold:*

1^0. *the functions* $a(x,t,\varepsilon)$*,* $b(x,t,\varepsilon)$*,* $c(x,t,\varepsilon)$ *satisfy the assumptions* $a(x,t,\varepsilon) = a(x,\varepsilon) \in C^\infty(\mathbf{R})$*,* $b(x,t,\varepsilon) \in C^\infty(\mathbf{R}\times[0;T])$*,* $c(x,t,\varepsilon) = c(t,\varepsilon) \in C^\infty([0;T])$*;*

2^0. *the inequalities* $r_1 \le a(x,\varepsilon) \le r_2$*,* $|b(x,t,\varepsilon)| < l_1$*,* $|b_x(x,t,\varepsilon)| < l_2$ *take place for all* $x \in \mathbf{R}$*,* $t \in [0;T]$*, where* r_1*,* r_2*,* l_1*,* l_2 *are some positive constants;*

3^0. *the Cauchy problem (1), (4) has a solution* $u(x,t,\varepsilon) \in C^\infty(0,T;S)$*;*

4^0. *the functions* $a_0(x) \in C^\infty(\mathbf{R})$*,* $b_0(x,t) \in C^\infty(\mathbf{R}\times[0;T])$*,* $c_0(t) \in C^\infty([0;T])$ *do not equal zero for all* $x \in \mathbf{R}$*,* $t \in [0;T]$ *and* $a_0(x)$*,* $b_0(x,t)$ *are absolutely bounded for all* $x \in \mathbf{R}$*,* $t \in [0;T]$*;*

5^0. *the function* $f(x,\varepsilon)$ *in initial condition (4) can be represented as* $f(x,\varepsilon) = g_0(x) + f_0(x,\varepsilon)$*, where* $g_0(x) \in S(\mathbf{R})$*,* $f_0(x,\varepsilon) \in \mathcal{M}_0(\varepsilon)$*;*

6^0. *the Cauchy problem for equation (8) with the initial condition* $u_0(x,0) = g_0(x)$*,* $x \in \mathbf{R}$*, has a solution in the space* $C^\infty(0,T;S)$*;*

7^0. *there exists a function* $\varphi(t) \in C^\infty([0;T])$ *satisfying (16) and* $\varphi(0) = \varphi_0$*,* $\varphi'(0) = \varphi_0' \neq 0$.

Then for the exact solution and the asymptotic soliton-like solution to the Cauchy problem (1), (4) the following estimate

$$|||u(x,t,\varepsilon) - Y_0(x,t,\varepsilon)||| \le C\varepsilon, \quad t \in [0;\varepsilon\Theta], \tag{19}$$

is true, where C is a constat not depending on the parameter ε, $\Theta > 0$ is a real number, and

$$|||f|||^2 = ||\sqrt{a(x,\varepsilon)}\, f||^2 + \varepsilon^2 ||f_x||^2, \quad ||f||^2 = \int\limits_{\mathbf{R}} |f(x,t,\varepsilon)|^2 dx.$$

Proof. For proving the theorem let us consider the function

$$\omega(x,t,\varepsilon) = u(x,t,\varepsilon) - Y_0(x,t,\varepsilon), \tag{20}$$

where $u(x,t,\varepsilon)$ is an exact solution to the Cauchy problem (1), (4) and $Y_0(x,t,\varepsilon)$ is given by formula (18). Substituting $u(x,t,\varepsilon) = \omega(x,t,\varepsilon) + Y_0(x,t,\varepsilon)$ into (1), multiplying both sides by $\omega(x,t,\varepsilon)$ and integrating the obtained expression in x from $-\infty$ to $+\infty$, we get

$$-\frac{\varepsilon^2}{2}\frac{d}{dt}\int\limits_{-\infty}^{+\infty} \omega_x^2(x,t,\varepsilon)\, dx = \frac{1}{2}\frac{d}{dt}\int\limits_{-\infty}^{+\infty} a(x,\varepsilon)\omega^2(x,t,\varepsilon) dx - \tag{21}$$

$$-\frac{1}{2}\int\limits_{-\infty}^{+\infty} b_x(x,t,\varepsilon)\omega^2(x,t,\varepsilon)dx - \int\limits_{-\infty}^{+\infty} c(t,\varepsilon)Y_0(x,t,\varepsilon)\omega(x,t,\varepsilon)\omega_x(x,t,\varepsilon)dx +$$

$$+\int\limits_{-\infty}^{+\infty} g(x,t,\varepsilon)\omega(x,t,\varepsilon)dx,$$

where

$$g(x,t,\varepsilon) = -\varepsilon^2 Y_{0xxt}(x,t,\varepsilon) + a(x,\varepsilon)Y_{0t}(x,t,\varepsilon) + \\ + b(x,t,\varepsilon)Y_{0x}(x,t,\varepsilon) + c(t,\varepsilon)Y_0(x,t,\varepsilon)Y_{0x}(x,t,\varepsilon).$$

Taking into account the conditions of Theorem 3.2 and the technique of constructing the asymptotic soliton-like solution to the Cauchy problem (1), (4) we conclude that the function $g(x,t,\varepsilon)$ belongs to the space $C^\infty(0,T;S)$. Moreover, it satisfies the asymptotic relation $g(x,t,\varepsilon) = O(1)$ as $\varepsilon \to 0$.

From equality (21) we find

$$\frac{1}{2}\frac{d}{dt}E^2 \le p\,E^2 + qE, \tag{22}$$

where

$$E^2 = |||\omega(x,t,\varepsilon)|||^2 = ||\sqrt{a(x,\varepsilon)}\,\omega(x,t,\varepsilon)||^2 + \varepsilon^2||\omega_x(x,t,\varepsilon)||^2, \tag{23}$$

$$p = \frac{1}{2}\max_{(x,t)\in\mathbf{R}\times[0;T]}\left|\frac{b_x(x,t,\varepsilon)}{a(x,\varepsilon)}\right| + \max_{(x,t)\in\mathbf{R}\times[0;T]}\left|c(t,\varepsilon)\,\frac{Y_0(x,t,\varepsilon)}{a(x,\varepsilon)}\right| + \tag{24}$$

$$+\frac{1}{\varepsilon}\max_{(x,t)\in\mathbf{R}\times[0;T]}|\,c(t,\varepsilon)\,Y_0(x,t,\varepsilon)\,|,$$

$$q = \max_{t\in[0;T]} \left(\int_{-\infty}^{+\infty} \left| \frac{g(x,t,\varepsilon)}{\sqrt{a(x,\varepsilon)}} \right|^2 dx \right)^{1/2}.$$

According to the algorithm of constructing the main term of the asymptotic solution (7) the values p, q satisfy the following asymptotic relations:

$$p = O\left(\frac{1}{\varepsilon}\right), \quad q = O(1) \quad \text{as} \quad \varepsilon \to 0.$$

To estimate the value $y = E(t,\varepsilon)$ we consider the differential inequality

$$\frac{dy}{dt} \le p\,y + q$$

under the initial condition $y(0) = 0$ according to notations (20) and (23).

Similarly to the proof of the Gronwall-Bellman lemma we find the relation

$$y(t) \le \frac{q}{p}\left(e^{p\,t} - 1\right).$$

As a result, we obtain estimate (19).

3.2 Higher terms of the asymptotic soliton-like solution

Let us describe the initial data of the Cauchy problem (1), (4) corresponding to the higher terms of asymptotic soliton-like solutions (7). As in the previous case, the initial data is a sum of two functions. One of these functions is a sufficiently smooth one connected with the regular part of asymptotic solution (7). The other function belongs to the defined above space G_1 and is associated with the singular part of asymptotic solution (7).

To clarify the type of the last element of the initial data we go back to the algorithm of constructing the singular part of asymptotic solution (7) and recall some results of paper [21]. The terms of the singular part are represented as follows:

$$V_j(x,t,\tau) = u_j^-(x,t)\eta_j(t,\tau) + \psi_j(t,\tau), \quad j = \overline{1,N}, \tag{25}$$

where $u_j^-(x,t)$, $j = \overline{1,N}$, is a solution to the Cauchy problem

$$\Lambda u_j^-(x,t) = f_j^-(x,t), \tag{26}$$

$$u_j^-(x,t)\big|_\Gamma = \nu_j(t), \quad j = \overline{1,N}. \tag{27}$$

Here the differential operator Λ is written as

$$\Lambda = a_0(x,t)\frac{\partial}{\partial t} + [b_0(x,t) + c_0(x,t)u_0(x,t)]\,\frac{\partial}{\partial x} + c_0(x,t)u_{0x}(x,t),$$

the right side functions $f_j^-(x,t)$, $j = \overline{1,N}$, are recursively determined, and, for example, $f_1^-(x,t) = 0$,

$$f_2^-(x,t) = -a_1(x,t)\frac{\partial u_1^-}{\partial t} - b_1(x,t)\frac{\partial u_1^-}{\partial x} - c_1(x,t)u_1^-\frac{\partial u_0}{\partial x} - \tag{28}$$

$$-c_0(x,t)u_1^-\frac{\partial u_1}{\partial x}-c_0(x,t)u_1\frac{\partial u_1^-}{\partial x}-c_0(x,t)u_1^-\frac{\partial u_1^-}{\partial x}-c_1(x,t)u_0\frac{\partial u_1^-}{\partial x};$$

$$\nu_j(t)=\left[-a_0(\varphi,t)\varphi'(t)+b_0(\varphi,t)+c_0(\varphi,t)u_0(\varphi,t)\right]^{-1}\lim_{\tau\to-\infty}\Phi_j(t,\tau),$$

$$\Phi_j(t,\tau)=-\int\limits_{\tau}^{+\infty}\mathcal{F}_j(t,\xi)d\xi,\quad j=\overline{1,N};$$

$\eta_j(t,\tau)\in G_1$ is a function such that $\lim\limits_{\tau\to-\infty}\eta_j(t,\tau)=1$; the function $\psi_j(t,\tau)$ belongs to the space G_1^0, and $u_0(x,t)$ is the main term of the regular part of asymptotic solution (7).

Besides, the function $\varphi=\varphi(t)$, $t\in[0;T]$, is a solution to the second order ordinary differential equation of the following form:

$$\left[A_1\varphi'^{\,2}+A_2\varphi'+A_3\right]\varphi''+A_4\,\varphi'^{\,4}+A_5\,\varphi'^{\,3}+A_6\,\varphi'^{\,2}+A_7\,\varphi'=0,\tag{29}$$

where the coefficients $A_k=A_k(\varphi,t)$, $k=\overline{1,7}$, are given as follows:

$$A_1=24\,a_0^2\,c_0,\ A_2=-8\,a_0\,c_0\,\alpha,\ A_3=-\,c_0\,\alpha^2,\ A_4=-40\,c_{0x}\,a_0^2+30a_0\,a_{0x}\,c_0,$$

$$A_5=60\,a_0\,c_{0x}\,\alpha+20\,a_0\,a_{0t}\,c_0-24\,a_0^2\,c_{0t}-30\,a_0\,c_0\,\alpha_x-15\,a_{0x}\,c_0\,\alpha+20a_0\,c_0^2\,u_{0x},$$

$$A_6=-20\,a_0\,c_0\,\alpha_t\,-5\,a_{0t}\,c_0\alpha+15\,c_0\,\alpha\,\alpha_x+28\,a_0\,c_{0t}\,\alpha-20\,c_0^2u_{0x}\,\alpha-20\,c_{0x}\,\alpha^2,$$

$$A_7=5\,c_0\,\alpha\,\alpha_t-20\,c_{0t}\,\alpha^2,$$

where $\alpha=b_0+c_0u_0$, $a_0=a_0(\varphi,t)$, $b_0=b_0(\varphi,t)$, $c_0=c_0(\varphi,t)$, $u_0=u_0(\varphi,t)$.

Ordinary differential equation (29) is nonlinear and, in general, it possesses a solution on the finite time interval denoted by $[0;T]$.

We suppose that the Cauchy problem (26), (27) has a solution in the domain $\{(x,t):x<\varphi(t),\ t\in[0;T]\}$. In the case, asymptotic solution (7) to the Cauchy problem (1), (4) is written as

$$Y_N(x,t,\varepsilon)=$$

$$=\begin{cases}\sum\limits_{j=0}^{N}\varepsilon^j\left[u_j(x,t)+V_j(x,t,\tau)\right], & (x,t)\in\Omega_\mu(\Gamma),\\ u_0(x,t)+\sum\limits_{j=1}^{N}\varepsilon^j\left[u_j(x,t)+u_j^-(x,t)\right], & (x,t)\in D^-\backslash\Omega_\mu(\Gamma),\\ \sum\limits_{j=0}^{N}\varepsilon^j u_j(x,t), & (x,t)\in D^+\backslash\Omega_\mu(\Gamma),\end{cases}\tag{30}$$

where

$$\Omega_\mu(\Gamma)=\{(x,t)\in\mathbf{R}\times[0;T]:|x-\varphi(t)|<\mu\},$$
$$D^-=\{(x,t)\in\mathbf{R}\times[0;T]:x<\varphi(t)\},$$
$$D^+=\{(x,t)\in\mathbf{R}\times[0;T]:x>\varphi(t)\},$$

μ is a positive number.

Taking into account Remark 3.1 we find the representation of the initial values in (4). So, by substituting $\tau=(x-\varphi(t))/\varepsilon$ and putting $t=0$, we get

$$f_j(x,\varepsilon):=V_j(x,t,\tau)\Big|_{t=0,\,\tau=\frac{x-\varphi_0}{\varepsilon}}=u_j^-(x,0)\eta_j\left(0,\frac{x-\varphi_0}{\varepsilon}\right)+\tag{31}$$

$$+\psi_j\left(0,\frac{x-\varphi_0}{\varepsilon}\right)+\rho_j V_{0\tau}(t,\tau)\Big|_{t=0,\tau=\frac{x-\varphi_0}{\varepsilon}}, \quad j=\overline{1,N},$$

where ρ_j, $j=\overline{1,N}$, are real parameters.

The set of values $f_j(x,\varepsilon)$ is denoted by $\mathcal{M}_j(\varepsilon)$ for any $j=\overline{1,N}$.

So, the following theorem is true.

Theorem 3.3 *Let the following propositions be fulfilled:*

1^0. *the functions $a_k(x,t)$, $b_k(x,t)$, $c_k(x,t)\in C^\infty(\mathbf{R}\times[0;T])$, $k=\overline{0,N}$, and the inequality $a_0(x,t)\,b_0(x,t)\,c_0(x,t)\neq 0$ holds for all $(x,t)\in\mathbf{R}\times[0;T]$;*

2^0. *the function $f(x,\varepsilon)$ in initial condition (4) can be represented as*

$$f(x,\varepsilon)=\sum_{j=0}^{N}\varepsilon^j\,[g_j(x)+f_j(x,\varepsilon)\,],$$

where $g_j(x)\in C^\infty(\mathbf{R})$ and $f_j(x,\varepsilon)\in\mathcal{M}_j(\varepsilon)$, $j=\overline{0,N}$;

3^0. *equation (8) with the initial condition $u_0(x,0)=g_0(x)$, $x\in\mathbf{R}$, as well as equation (9) with the initial condition $u_j(x,0)=g_j(x)$, $x\in\mathbf{R}$, has the solution $u_j(x,t)\in C^\infty(\mathbf{R}\times[0;T])$, $j=\overline{0,N}$;*

4^0. *the function $\mathcal{F}_j(t,\tau)\in G_1^0$, $j=\overline{1,N}$, and the orthogonality condition*

$$\int\limits_{-\infty}^{+\infty}\mathcal{F}_j(t,\tau)v_0(t,\tau)d\tau=0, \quad j=\overline{1,N}; \tag{32}$$

is satisfied;

5^0. *the function $\mathcal{F}_j(t,\tau)$, $j=\overline{1,N}$, is such that the property*

$$\lim_{\tau\to-\infty}\Phi_j(t,\tau)=0, \quad j=\overline{1,N}, \tag{33}$$

takes place;

6^0. *equation (29) has a solution $\varphi(t)\in C^\infty([0;T])$ such that inequality (16) is true and $\varphi(0)=\varphi_0$, $\varphi'(0)={\varphi_0}'\neq 0$ hold.*

Then the asymptotic soliton-like solution to the Cauchy problem (1), (4) can be written as

$$Y_N(x,t,\varepsilon)=\sum_{j=0}^{N}\varepsilon^j\,[u_j(x,t)+V_j(x,t,\tau)]\,. \tag{34}$$

It satisfies the Cauchy problem with accuracy $O(\varepsilon^N)$ for all $(x,t)\in\mathbf{R}\times[0;T]$. Moreover, as $\tau\to\pm\infty$, function (34) satisfies the Cauchy problem (1), (4) with accuracy $O(\varepsilon^{N+1})$, $N\in\mathbf{N}$.

Proof. It is clear that function (34) satisfies initial condition (4). The other part of Theorem 3.3 is proved according to the scheme of proof for Theorem 1 in [21].

Remark 3.2 Condition 5^0 of Theorem 3.3 provides solution (7), singular part of which belongs to the space G_1^0. Therefore, we can put $V_j(x,t,\tau) = V_j(x,t,\tau)\big|_\Gamma = v_j(t,\tau)$, $j = \overline{0,N}$.

In the opposite case, the asymptotic soliton-like solution to the Cauchy problem (1), (4) can be written as (30).

Theorem 3.4 *Suppose the following propositions are fulfilled:*

1^0. *conditions* 1^0 – 4^0, 6^0 *of Theorem 3.3 are true;*

2^0. *the Cauchy problem (26), (27) has a solution in the domain* D^-.

Then the asymptotic soliton-like solution to problem (1), (4) is written as (30) and satisfies the Cauchy problem with accuracy $O(\varepsilon^N)$, $N \in \mathbf{N}$, *for all* $(x,t) \in \mathbf{R} \times [0;T]$. *Moreover, as* $\tau \to \pm\infty$, *function (30) satisfies the Cauchy problem (1), (4) with accuracy* $O(\varepsilon^{N+1})$, $N \in \mathbf{N}$.

Proof. It is obvious that function (30) satisfies initial condition (4). The last part of Theorem 3.4 is proved according to the scheme of proof for Theorem 2 in [21].

Now let us consider the estimate for the difference between the exact solution and asymptotic soliton-like solution to the Cauchy problem (1), (4).

The following theorem is true.

Theorem 3.5 *Suppose the following propositions are satisfied:*

1^0. *the functions* $a(x,t,\varepsilon)$, $b(x,t,\varepsilon)$, $c(x,t,\varepsilon)$ *satisfy the assumptions* $a(x,t,\varepsilon) = a(x,\varepsilon) \in C^\infty(\mathbf{R})$, $b(x,t,\varepsilon) \in C^\infty(\mathbf{R} \times [0;T])$, $c(x,t,\varepsilon) = c(t,\varepsilon) \in C^\infty([0;T])$;

2^0. *the inequalities* $r_1 \le a(x,\varepsilon) \le r_2$, $|b(x,t,\varepsilon)| < l_1$, $|b_x(x,t,\varepsilon)| < l_2$ *take place for all* $x \in \mathbf{R}$, $t \in [0;T]$, *where* r_1, r_2, l_1, l_2 *are some positive constants;*

3^0. *the Cauchy problem (1), (4) has a solution* $u(x,t,\varepsilon) \in C^\infty(0,T;S)$;

4^0. *the functions* $a_k(x) \in C^\infty(\mathbf{R})$, $b_k(x,t) \in C^\infty(\mathbf{R} \times [0;T])$, $c_k(t) \in C^\infty([0;T])$, $k = \overline{0,N}$, *are absolutely bounded for all* $x \in \mathbf{R}$, $t \in [0;T]$, *and the inequality* $a_0(x)\, b_0(x,t)\, c_0(t) \neq 0$ *holds for all* $x \in \mathbf{R}$, $t \in [0;T]$;

5^0. *the function* $f(x,\varepsilon)$ *in initial condition (4) can be represented as*

$$f(x,\varepsilon) = \sum_{j=0}^{N} \varepsilon^j \left[g_j(x) + f_j(x,\varepsilon) \right],$$

where $g_j(x) \in S(\mathbf{R})$, $f_j(x,\varepsilon) \in \mathcal{M}_j(\varepsilon)$, $j = \overline{0,N}$;

6^0. *equation (8) with the initial condition* $u_0(x,0) = g_0(x)$, $x \in \mathbf{R}$, *as well as equation (9) with the initial condition* $u_j(x,0) = g_j(x)$, $x \in \mathbf{R}$, $j = \overline{1,N}$, *has the solution* $u_j(x,t) \in C^\infty(0,T;S)$, $j = \overline{0,N}$;

7^0. *the conditions* 4^0 – 6^0 *of Theorem 3.3 are true.*

Then for the exact solution $u(x,t,\varepsilon)$ and asymptotic solution (34) to the Cauchy problem (1), (4) the following asymptotic estimate

$$|||u(x,t,\varepsilon) - Y_N(x,t,\varepsilon)||| \le C\varepsilon^{N+1}, \quad t \in [0;\varepsilon\Theta], \tag{35}$$

is true, where C is a constant not depending on the parameter ε, and Θ is a positive number.

Proof. Similarly to the proof of Theorem 3.2 let us consider the difference $\omega_N(x,t,\varepsilon) = u(x,t,\varepsilon) - Y_N(x,t,\varepsilon)$. As above, we obtain

$$-\frac{\varepsilon^2}{2}\frac{d}{dt}\int\limits_{-\infty}^{+\infty} |\,\omega_{N\,x}(x,t,\varepsilon)\,|^2\,dx = \frac{1}{2}\frac{d}{dt}\int\limits_{-\infty}^{+\infty} a(x,\varepsilon)\omega_N^2(x,t,\varepsilon)dx - \tag{36}$$

$$-\frac{1}{2}\int\limits_{-\infty}^{+\infty} b_x(x,t,\varepsilon)\omega_N^2(x,t,\varepsilon)dx - \int\limits_{-\infty}^{+\infty} c(t,\varepsilon)Y_N(x,t,\varepsilon)\omega_N(x,t,\varepsilon)\omega_{N\,x}(x,t,\varepsilon)dx +$$

$$+\int\limits_{-\infty}^{+\infty} g_N(x,t,\varepsilon)\omega_N(x,t,\varepsilon)dx,$$

where

$$g_N(x,t,\varepsilon) = -\varepsilon^2 Y_{Nxxt}(x,t,\varepsilon) + a(x,\varepsilon)Y_{Nt}(x,t,\varepsilon) +$$

$$+b(x,t,\varepsilon)Y_{Nx}(x,t,\varepsilon) + c(t,\varepsilon)Y_N(x,t,\varepsilon)Y_{Nx}(x,t,\varepsilon).$$

According to Theorem 3.3 and the technique of constructing the asymptotic soliton-like solution to problem (1), (4), the function $g_N(x,t,\varepsilon)$ belongs to the space $C^\infty(0,T;S)$. Moreover, it satisfies the asymptotic relation $g_N(x,t,\varepsilon) = O(\varepsilon^N)$ as $\varepsilon \to 0$.

From (36) we find

$$\frac{1}{2}\frac{d}{dt}E_N^2 \le pE_N^2 + q\,E_N, \tag{37}$$

where

$$E_N^2 = |||\omega_N(x,t,\varepsilon)|||^2 = ||\sqrt{a(x,\varepsilon)}\,\omega_N(x,t,\varepsilon)||^2 + \varepsilon^2||\omega_{N\,x}(x,t,\varepsilon)||^2, \tag{38}$$

$$p = \frac{1}{2}\max_{(x,t)\in\mathbf{R}\times[0;T]}\left|\frac{b_x(x,t,\varepsilon)}{a(x,\varepsilon)}\right| + \max_{(x,t)\in\mathbf{R}\times[0;T]}\left|c(t,\varepsilon)\frac{Y_N(x,t,\varepsilon)}{a(x,\varepsilon)}\right| + \tag{39}$$

$$+\frac{1}{\varepsilon}\max_{(x,t)\in\mathbf{R}\times[0;T]}|c(t,\varepsilon)Y_N(x,t,\varepsilon)|\,,$$

$$q = \max_{t\in[0;T]}\left(\int\limits_{-\infty}^{+\infty}\left|\frac{g_N(x,t,\varepsilon)}{\sqrt{a(x,\varepsilon)}}\right|^2 dx\right)^{1/2}. \tag{40}$$

It is easy to see that the values p, q satisfy the asymptotic equalities

$$p = O\left(\frac{1}{\varepsilon}\right), \quad q = O\left(\varepsilon^N\right) \quad \text{as} \quad \varepsilon \to 0.$$

Inequality (37) is equivalent to the relation

$$\frac{dy}{dt} \leq py + q, \quad y(0) = 0, \tag{41}$$

where $y = y(t) = E_N(x, t, \varepsilon)$.

It now follows that

$$y(t) \leq \frac{q}{p}\left(e^{p\,t} - 1\right)$$

providing asymptotic estimate (35).

4 Conclusions

The problem of estimating the difference between the exact solution and asymptotic soliton-like solution to the Cauchy problem for the singularly perturbed BBM equation with variable coefficients is considered. The initial data for the Cauchy problem are defined according to the concept of asymptotic soliton-like solution. In other words, it is taken into account that the asymptotic soliton-like solution is a certain deformation of the soliton solution for the corresponding BBM equation with constant coefficients.

We present asymptotic estimates for the difference between the exact solution to the BBM equation and the N-th approximation for the constructed asymptotic soliton-like solution. Similarly to the singularly perturbed Korteweg-de Vries equation, these estimates are local [27,28]. Nevertheless, they show that the asymptotic soliton-like solutions constructed through the nonlinear WKB method for the singularly perturbed BBM equation with variable coefficients are sufficiently suitable as the approximate solutions.

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